MINIMAL SURFACES IN S^3 FOLIATED BY CIRCLES

NIKOLAI KUTEV AND VELICHKA MILOUSHEVA

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We study minimal surfaces in the unit sphere S^3 that are one-parameter families of circles. Minimal surfaces in $\mathbb{R}^3$ foliated by circles were first investigated by Riemann, and a hundred years later Lawson constructed examples of such surfaces in S^3. We prove that in S^3 only two types of minimal surfaces are foliated by circles crossing the principal lines at a constant angle. The first type of surfaces are foliated by great circles that are bisectrices of the principal lines, and we show that these are the examples of Lawson. The second type, which are new in the literature, are families of small circles, and the circles are principal lines. We give a constructive formula for these surfaces and an application to the theory of minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$.

1. Introduction

We study minimal surfaces in the unit sphere S^3 in the four-dimensional Euclidean space $\mathbb{R}^4$, equipped with the standard Euclidean metric $\langle \cdot, \cdot \rangle$. A surface M^2 in S^3 is given by a unit vector-valued function $l(u, v)$ in $\mathbb{R}^4$ defined in a domain $\mathcal{D} \subset \mathbb{R}^2$, that is,

$$l(u, v) = \big(l^1(u, v), l^2(u, v), l^3(u, v), l^4(u, v)\big) \quad \text{for } (u, v) \in \mathcal{D},$$

where $\langle l(u, v), l(u, v) \rangle = 1$ for $(u, v) \in \mathcal{D}$. Since our considerations are local, we assume that the parameters (u, v) are isothermal (conformal), which means that $\langle l_u, l_u \rangle = \langle l_v, l_v \rangle$ and $\langle l_u, l_v \rangle = 0$.

The minimal surfaces in S^3 are determined by the solutions $l = l(u, v)$ of the system

(1-1)
$$\Delta l + |\nabla l|^2 l = 0,$$
$$\langle l_u, l_u \rangle = \langle l_v, l_v \rangle, \quad \langle l_u, l_v \rangle = 0, \quad \langle l, l \rangle = 1,$$

where ∇ and Δ denote the gradient and the Laplacian operators, respectively, with respect to the Euclidean metric in $\mathbb{R}^4$.

The system (1-1) is an Euler–Lagrange system of harmonic maps and has been studied intensively in the last decades by variational methods [Giaquinta et al. 1998a, 1998b; Hildebrandt 1982; Jost 1984, 1991; Struwe 1988].

Our aim is to find the minimal surfaces in S^3 that satisfy the geometric property that locally they are one-parameter families of circles. Variational methods cannot be applied for studying the geometric structure of these minimal surfaces. Hence, we use a method based on the differential geometry of surfaces in $\mathbb{R}^4$ rather than one based on PDEs.

It is well known that the only minimal rotational surface in $\mathbb{R}^3$ is the catenoid, which is a surface fibered by circles in parallel planes. Another class of minimal surfaces in $\mathbb{R}^3$ foliated by circles in parallel planes consists of the *Riemann examples* [Riemann 1868]. Enneper [1869] proved that catenoids and Riemann examples are the only minimal surfaces in $\mathbb{R}^3$ foliated by circles. A surface in $\mathbb{R}^3$ that is determined by a smooth one-parameter family of circles is also called a cyclic surface. Cyclic surfaces of constant mean curvature and cyclic surfaces of constant Gauss curvature in $\mathbb{R}^3$ are described in [Nitsche 1989] and [López 2001].

Our idea to find the minimal surfaces in S^3 that are one-parameter families of circles is motivated by what happens for cyclic minimal surfaces in $\mathbb{R}^3$.

A well-known example of a minimal surface in S^3 is the Clifford torus (the standard flat torus), which consists of two orthogonal families of circles. This is generalized by the Lawson tori [1970], which have two orthogonal families of parametric lines: one of them consists of circles, the other of curves with constant Frenet curvatures in $\mathbb{R}^4$.

The circles on the Lawson torus cross the principal lines at an angle $\pi/4$. Here we will find all minimal surfaces in S^3 that are one-parameter families of circles crossing the principal lines at a constant angle. We call these surfaces *generalized tori*. In Theorem 2.2, we prove that there are only two types of generalized tori in S^3: The *generalized tori of first type* are those whose circles are bisectrices of the principal lines. The *generalized tori of second type* are those whose circles are principal lines. In Theorem 2.3, we show that all generalized tori of first type are Lawson tori. In Theorem 2.4, we give a constructive formula for the generalized tori of second type.

In Section 3, we will point out the relationship between the theory of minimal surfaces in S^3 and the theory of minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$. Each minimal surface in S^3 generates a minimal foliated semisymmetric hypersurface in $\mathbb{R}^4$ according to a special construction given in [Ganchev and Milousheva 2007a]. We will illustrate how this construction can be applied to two examples of minimal surfaces in S^3 to obtain first and second type helicoids, which

are special minimal foliated semisymmetric hypersurfaces. We will also apply the construction to the class of generalized tori of second type, and thus obtain new minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$.

2. Generalized tori in S^3

Let $M^2 : l = l(u, v)$ for $(u, v) \in \mathscr{D}$ be a surface, parametrized by isothermal parameters and lying on the unit sphere S^3 in $\mathbb{R}^4$. In other words, the vector-valued function $l(u, v)$ satisfies the equalities

$$\langle l_u, l_u \rangle = \langle l_v, l_v \rangle = E(u, v),$$
(2-1)
$$\langle l_u, l_v \rangle = 0,$$
$$\langle l, l \rangle = 1.$$

Since l, l_u, and l_v are mutually orthogonal, there exists a unique (up to a sign) unit vector field $n(u, v)$ such that $\{l, l_u, l_v, n\}$ form an orthogonal basis in $\mathbb{R}^4$. Differentiating the equalities (2-1), we get the derivative formulas

$$l_{uu} = \frac{E_u}{2E} l_u - \frac{E_v}{2E} l_v - El + a_{11}n,$$

$$l_{uv} = \frac{E_v}{2E} l_u + \frac{E_u}{2E} l_v \qquad + a_{12}n,$$

$$l_{vv} = -\frac{E_u}{2E} l_u + \frac{E_v}{2E} l_v - El + a_{22}n,$$

where $a_{ij}(u, v)$ for $i, j = 1, 2$ are functions defined in $\mathscr{D}$. Hence

(2-2)
$$l_{uu} + l_{vv} + 2E\, l = (a_{11} + a_{22})n.$$

M^2 is a minimal surfaces in S^3 if and only if $a_{11} + a_{22} = 0$. Equation (2-2) implies that M^2 is minimal if and only if

$$l_{uu} + l_{vv} + 2E\, l = 0.$$

Consequently, the problem of finding the minimal surfaces in S^3 is equivalent to the solvability of the system

(2-3)
$$\begin{cases} \Delta l(u, v) + 2E(u, v)\, l(u, v) = 0, \\ \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = E(u, v), \quad \langle l_u, l_v \rangle = 0, \quad \langle l, l \rangle = 1 \end{cases}$$

with an appropriate $\mathscr{C}^\infty$ smooth scalar function $E(u, v) > 0$ in a small neighborhood $\mathscr{D}$ of the origin.

According to the theorem of Hélein [Giaquinta et al. 1998b, page 346], the solutions of system (1-1) (and hence of (2-3)) are $\mathscr{C}^\infty$ smooth because $\mathscr{D}$ is a two-dimensional domain.

Let $M^2 : l = l(u, v)$ be a minimal surface in S^3. Then the derivative formulas of M^2 are

(2-4)
$$l_{uu} = \frac{E_u}{2E} l_u - \frac{E_v}{2E} l_v - El + an,$$
$$l_{uv} = \frac{E_v}{2E} l_u + \frac{E_u}{2E} l_v \qquad + bn,$$
$$l_{vv} = -\frac{E_u}{2E} l_u + \frac{E_v}{2E} l_v - El - an,$$

where $a(u, v)$ and $b(u, v)$ are functions in $\mathcal{D}$. The derivatives n_u and n_v of $n(u, v)$ satisfy

(2-5)
$$n_u = -\frac{a}{E} l_u - \frac{b}{E} l_v \quad \text{and} \quad n_v = -\frac{b}{E} l_u + \frac{a}{E} l_v.$$

Using the Gauss and Codazzi equations (or equivalently, the identities of the mixed third derivatives of $l(u, v)$ and mixed second derivatives of $n(u, v)$) from (2-4) and (2-5), we see that the functions $a(u, v)$ and $b(u, v)$ are harmonic and satisfy the Cauchy–Riemann conditions

$$b_u(u, v) = a_v(u, v) \quad \text{and} \quad b_v(u, v) = -a_u(u, v).$$

The Gauss and Codazzi equations for M^2 also imply the identity

(2-6)
$$a^2 + b^2 = \tfrac{1}{2}\Delta E - \frac{E_u^2 + E_v^2}{2E} + E^2.$$

The Gauss curvature K of M^2 is given by

(2-7)
$$K = \frac{E_u^2 + E_v^2}{2E^3} - \frac{\Delta E}{2E^2} = -\frac{1}{2E}\Delta \ln E.$$

Hence, equalities (2-6) and (2-7) imply that the Gauss curvature K is expressed in terms of the functions a and b as

(2-8)
$$K = 1 - \frac{a^2 + b^2}{E^2}.$$

The simplest case in which problem (2-3) can be solved completely is the case $K = \text{const}$. Lawson [1969] proved that if M^2 is a minimal surface in S^3 of constant Gauss curvature K, then either $K = 1$ and M^2 is totally geodesic, or $K = 0$ and M^2 is an open piece of the Clifford torus.

From (2-8) it follows that the case M^2 is totally geodesic in S^3 (that is, M^2 is a sphere with radius 1) corresponds to $a = b = 0$. Further on we shall consider only the case $(a, b) \neq (0, 0)$.

Let us recall that the Clifford torus is a surface in $\mathbb{R}^4$ parametrized as

$$\mathcal{M} : l(u, v) = (\cos u \cos v; \cos u \sin v; \sin u \cos v; \sin u \sin v).$$

A direct computation shows that $l(u, v)$ satisfies the equality $l_{uu} + l_{vv} + 2l = 0$, and $\langle l_u, l_u \rangle = \langle l_v, l_v \rangle = 1$, $\langle l_u, l_v \rangle = 0$ and $\langle l, l \rangle = 1$. The parametric lines $u = $ const and $v = $ const of $\mathcal{M}$ are circles.

H. B. Lawson [1970] found other minimal surfaces in S^3 that generalize the Clifford torus: He proved that every "ruled" minimal surface in S^3 is an open submanifold of one of the surfaces $\mathcal{M}_\alpha$ given by

$$(2\text{-}9) \qquad \mathcal{M}_\alpha : l(x, y) = (\cos x \cos \alpha y; \cos x \sin \alpha y; \sin x \cos y; \sin x \sin y)$$

for some constant $\alpha > 0$. By "ruled", Clifford means it belongs to a one-parameter family of great circles in S^3. The surface $\mathcal{M}_1$ is the Clifford torus, and it is the only surface $\mathcal{M}_\alpha$ with constant Gauss curvature [Lawson 1969]. We call the surfaces $\mathcal{M}_\alpha$ with $\alpha \neq 1$ *Lawson tori*.

The tangent space of $\mathcal{M}_\alpha$ is spanned by the vector fields

$$(2\text{-}10) \quad \begin{aligned} l_x(x, y) &= (-\sin x \cos \alpha y; -\sin x \sin \alpha y; \cos x \cos y; \cos x \sin y), \\ l_y(x, y) &= (-\alpha \cos x \sin \alpha y; \alpha \cos x \cos \alpha y; -\sin x \sin y; \sin x \cos y), \end{aligned}$$

and the coefficients E, F, and G of the first fundamental form of $\mathcal{M}_\alpha$ are given by $E = 1$, $F = 0$ and $G = G(x) = \alpha^2 \cos^2 x + \sin^2 x$.

Using (2-9) and (2-10), we find the unit normal vector field $n(u, v)$ of $\mathcal{M}_\alpha$ that is orthogonal to $\{l, l_u, l_v\}$:

$$n(x, y) = \frac{(\sin x \sin \alpha y; -\sin x \cos \alpha y; -\alpha \cos x \sin y; \alpha \cos x \cos y)}{\sqrt{\alpha^2 \cos^2 x + \sin^2 x}}.$$

A direct computation shows that the Gauss curvature of $\mathcal{M}_\alpha$ is given by

$$K = 1 - \frac{\alpha^2}{(\alpha^2 \cos^2 x + \sin^2 x)^2},$$

and obviously $K \neq $ const when $\alpha \neq 1$.

Let us consider the Lawson torus $\mathcal{M}_\alpha$ for $\alpha \neq 1$. In such case the parametric lines $y = y_0 = $ const of $\mathcal{M}_\alpha$ are circles, while the parametric lines $x = x_0 = $ const are curves in $\mathbb{R}^4$ with constant Frenet curvatures.

The parametrization (2-9) of $\mathcal{M}_\alpha$ is not isothermal. We shall find isothermal parameters for $\mathcal{M}_\alpha$ that are also principal parameters of the surface. Consider the change of parameters

$$\bar{u} = \int_0^x \frac{1}{\sqrt{G(\tau)}} \, d\tau \quad \text{and} \quad \bar{v} = v.$$

Then we obtain $\bar{E} = \bar{G} = G(x(\bar{u}))$ and $\bar{F} = 0$, that is, the parameters $(\bar{u}, \bar{v})$ are isothermal. A direct computation shows that the vector function $l(\bar{u}, \bar{v})$ satisfies

the system

$$l_{\bar{u}\bar{u}} = \frac{\overline{E}_{\bar{u}}}{2\overline{E}} l_{\bar{u}} - \overline{E}l, \quad l_{\bar{u}\bar{v}} = \frac{\overline{E}_{\bar{u}}}{2\overline{E}} l_{\bar{v}} + \alpha n, \quad l_{\bar{v}\bar{v}} = -\frac{\overline{E}_{\bar{u}}}{2\overline{E}} l_{\bar{u}} - \overline{E}l.$$

Hence, for the Lawson torus $\mathcal{M}_\alpha$, the functions a and b in formulas (2-4) are $a = 0$, $b = \alpha = \text{const}$, and $\overline{E} = \overline{E}(\bar{u})$. The circles on $\mathcal{M}_\alpha$ are the parametric $\bar{u}$-lines.

If we change the isothermal parameters $(\bar{u}, \bar{v})$ to isothermal parameters (u, v) via

$$u = (\bar{u} + \bar{v})/\sqrt{2} \quad \text{and} \quad v = (\bar{u} - \bar{v})/\sqrt{2},$$

then $\widetilde{E}(u, v) = \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = \widetilde{G}(u, v)$, and $l(u, v)$ satisfies (2-4) with $a = \alpha$, $b = 0$ and $\widetilde{E}$ instead of E.

With respect to (u, v), the surface $\mathcal{M}_\alpha$ is parametrized by principal lines, that is, the shape operator corresponding to the normal vector field $n(u, v)$ is in diagonal form. The circles on $\mathcal{M}_\alpha$ are bisectrices of the principal lines.

Now we shall find all minimal surfaces in S^3 that are one-parameter families of circles crossing the principal lines at a constant angle. We call these surfaces *generalized tori* in S^3. They generalize the Lawson tori.

Proposition 2.1. *Suppose M^2 is a minimal surface in S^3 with nonconstant Gauss curvature. Then M^2 can locally be parametrized by principal lines, and the new parameters are isothermal.*

Proof. Let $M^2 : l = l(u, v)$ for $(u, v) \in \mathcal{D}$ be a minimal surface in S^3 parametrized by isothermal parameters. Then the derivative formulas (2-4) of M^2 hold, and the functions $a(u, v)$ and $b(u, v)$ are harmonic functions satisfying the Cauchy–Riemann conditions. In case of $b(u, v) \equiv 0$, the parameters (u, v) are principal. Let $b(u_0, v_0) \neq 0$ for $(u_0, v_0) \in \mathcal{D}$. Then there exists $\mathcal{D}_0 \subset \mathcal{D}$ such that $b(u, v) \neq 0$ for all $(u, v) \in \mathcal{D}_0$. We shall prove that there exist isothermal parameters (x, y) such that $\bar{b}(x, y) = \langle l_{xy}, n \rangle = 0$. If $x = x(u, v)$ and $y = y(u, v)$ is a holomorphic change of the parameters (so that $x(u, v)$ and $y(u, v)$ satisfy the Cauchy–Riemann conditions), then $\bar{b}(x, y) = 2au_x u_y - b(u_x^2 - u_y^2)$. Hence,

$$\bar{b} = 0 \quad \text{if and only if} \quad b\left(\frac{u_x}{u_y}\right)^2 - 2a\left(\frac{u_x}{u_y}\right) - b = 0.$$

From the inverse change of parameters, using the Cauchy–Riemann conditions we have $x_u = u_x/(u_x^2 + u_y^2)$ and $x_v = -u_y/(u_x^2 + u_y^2)$, and hence we obtain that $\bar{b} = 0$ if and only if

$$b\left(\frac{x_u}{x_v}\right)^2 + 2a\left(\frac{x_u}{x_v}\right) - b = 0,$$

that is, $x_u/x_v = (-a \pm \sqrt{a^2 + b^2})/b$. We write $\beta(u, v) = (-a + \sqrt{a^2 + b^2})/b$ and $\gamma(u, v) = (-a - \sqrt{a^2 + b^2})/b$.

Now let us consider the equations

(2-11) $$\frac{dv}{du} = \beta(u, v) \quad \text{and} \quad \frac{dv}{du} = \gamma(u, v).$$

For each point $(u_0, v_0) \in \mathcal{D}_0$, there exist $\mathcal{D}_1 \subset \mathcal{D}_0$ and functions $\Phi(u, v) \neq 0$ and $\Psi(u, v) \neq 0$ in $\mathcal{D}_1$, such that the integral curves of the first equation in (2-11) are given by $\Phi(u, v) = \text{const}$, while the integral curves of the second equation in (2-11) are $\Psi(u, v) = \text{const}$. Hence,

(2-12) $$\Phi_u = -\beta \Phi_v \quad \text{and} \quad \Psi_u = -\gamma \Psi_v.$$

We consider the following smooth change of parameters:

(2-13) $$x = \Phi(u, v) \quad \text{and} \quad y = \Psi(u, v) \quad \text{for } (u, v) \in \mathcal{D}_1.$$

When (u, v) ranges over in $\mathcal{D}_1$, the parameters (x, y) describe a domain $\overline{\mathcal{D}} \subset \mathbb{R}^2$. Now $x_u = \Phi_u$, $x_v = \Phi_v$, $y_u = \Psi_u$ and $y_v = \Psi_v$. Using $\beta\gamma = -1$ and equalities (2-12), we get $\langle l_x, l_y \rangle = 0$, that is, the parametrization (x, y) is orthogonal. We shall prove that this parametrization is isothermal. With respect to the new parameters, the coefficients of the first fundamental form are $E = \langle l_x, l_x \rangle$, $F = \langle l_x, l_y \rangle = 0$, and $G = \langle l_y, l_y \rangle$. Then for the surface M^2, we have the derivative formulas

(2-14) $$\begin{cases} l_{xx} = \dfrac{E_x}{2E} l_x - \dfrac{E_y}{2E} l_y - El + an, \\[2mm] l_{xy} = \dfrac{E_y}{2E} l_x + \dfrac{G_x}{2G} l_y, \\[2mm] l_{yy} = -\dfrac{G_x}{2G} l_x + \dfrac{G_y}{2G} l_y - Gl - an, \end{cases}$$

(2-15) $$n_x = -\frac{a}{E} l_x \quad \text{and} \quad n_y = \frac{a}{G} l_y.$$

Taking into account the second fundamental form of M^2 as a surface in $\mathbb{R}^4$, from (2-14) we calculate that the Gauss curvature K is given by

(2-16) $$K = 1 - \frac{a^2}{EG}.$$

Using $n_{xy} = n_{yx}$, $l_{xxy} = l_{xyx}$, and $l_{xyy} = l_{yyx}$, from (2-14) and (2-15) we obtain

(2-17) $$a_x = 0, \quad a_y = 0, \quad (E - G)E_y = 0, \quad (E - G)G_x = 0,$$

(2-18) $$\frac{E_{yy}}{2E} + \frac{G_{xx}}{2G} - \frac{3E_y^2}{4E^2} - \frac{G_x^2}{4G^2} - \frac{E_x G_x - E_y G_y}{4EG} + E - \frac{a^2}{G} = 0,$$
$$\frac{E_{yy}}{2E} + \frac{G_{xx}}{2G} - \frac{3G_x^2}{4G^2} - \frac{E_y^2}{4E^2} + \frac{E_x G_x - E_y G_y}{4EG} + G - \frac{a^2}{E} = 0.$$

If we assume that $E(x_0, y_0) - G(x_0, y_0) \neq 0$ at some point $(x_0, y_0) \in \overline{\mathcal{D}}$, and hence $E - G \neq 0$ in a neighborhood $\overline{\mathcal{D}}_0 \subset \overline{\mathcal{D}}$ of (x_0, y_0), then from (2-17) we get $E_y = G_x = 0$ in $\overline{\mathcal{D}}_0$. Now equalities (2-16), (2-17), and (2-18) imply that $K = 0$ in $\overline{\mathcal{D}}_0$, which contradicts the assumption in the theorem. Hence, $E - G \equiv 0$ in $\overline{\mathcal{D}}$. Consequently, the parameters (x, y) defined by change (2-13) are principal. □

Now let $M^2 : l = l(u, v)$ be a minimal surface in S^3 parametrized locally by isothermal principal parameters, that is, $b = 0$. Using $b_u = a_v$ and $b_v = -a_u$, we get $a = $ const. Without loss of generality, we assume that $a = 1$ (if $a \neq 1$, we multiply the parameters by $\sqrt{|a|}$). Hence the derivative formulas (2-4) and (2-5) hold with $a = 1$ and $b = 0$.

Theorem 2.2. *Suppose M^2 is a minimal surface in S^3 with nonconstant Gauss curvature. If on M^2 there exists a family of circles crossing the principal lines at a constant angle θ, then the circles are either principal lines ($\theta = 0$ or $\theta = \pi/2$) or bisectrices of the principal lines ($\theta = \pi/4$ or $\theta = 3\pi/4$).*

Proof. Let $M^2 : l = l(u, v)$ be parametrized locally by principal parameters. Suppose that on M^2 there exists a family of circles crossing the principal lines at a constant angle θ. Let

$$(2\text{-}19) \qquad \begin{aligned} x &= \cos\theta\, u + \sin\theta\, v, \\ y &= -\sin\theta\, u + \cos\theta\, v, \quad \theta = \text{const} \quad \text{for } \theta \in [0; 2\pi). \end{aligned}$$

Then from (2-19) we get

$$(2\text{-}20) \qquad E_x = \cos\theta\, E_u + \sin\theta\, E_v \quad \text{and} \quad E_y = -\sin\theta\, E_u + \cos\theta\, E_v.$$

Using (2-4) with $a = 1$, $b = 0$, and (2-20), we calculate

$$(2\text{-}21) \qquad \begin{aligned} l_{xx} &= \frac{E_x}{2E} l_x - \frac{E_y}{2E} l_y - El + \cos 2\theta\, n, \\ l_{xy} &= \frac{E_y}{2E} l_x + \frac{E_x}{2E} l_y \qquad - \sin 2\theta\, n, \\ l_{yy} &= -\frac{E_x}{2E} l_x + \frac{E_y}{2E} l_y - El - \cos 2\theta\, n. \end{aligned}$$

Let us write $a = \cos 2\theta$ and $b = \sin 2\theta$ ($a, b - $ const).

Consider an arbitrary x-line $c : l(x) = l(x, y_0)$ for constant y_0. It is a circle if and only if its Frenet curvatures are $\varkappa = $ const and $\tau = \sigma = 0$. Using (2-21) we calculate the tangent vector t_c and the principal normal vector n_c of c:

$$t_c = \frac{l_x}{\sqrt{E}}, \quad n_c = \frac{1}{\varkappa}\left(-\frac{E_y}{2E^2} l_y - l + \frac{a}{E} n\right), \quad \text{where } \varkappa^2 = \frac{E_y^2}{4E^3} + 1 + \frac{a^2}{E^2}.$$

The derivatives of t_c and n_c are

$$t_c' = \varkappa\, n_c,$$

$$(2\text{-}22) \qquad n_c' = -\varkappa\, t_c - \left(\left(\frac{1}{\varkappa}\right)' \frac{E_y}{2E^2} + \frac{1}{\varkappa}\left(\left(\frac{E_y}{2E^2}\right)_x + \frac{E_x E_y}{4E^3} + \frac{a\,b}{E^2}\right)\right)\frac{l_y}{\sqrt{E}}$$

$$- \left(\frac{1}{\varkappa}\right)' \frac{l}{\sqrt{E}} + \left(\left(\frac{1}{\varkappa}\right)' \frac{a}{E} + \frac{1}{\varkappa}\left(-b\,\frac{E_y}{2E^2} + \left(\frac{a}{E}\right)_x\right)\right)\frac{n}{\sqrt{E}}.$$

From (2-22) it follows that c is a circle if and only if

$$\varkappa = \text{const}, \qquad \left(\frac{E_y}{2E^2}\right)_x + \frac{E_x E_y}{4E^3} + \frac{a\,b}{E^2} = 0,$$

$$\left(\frac{a}{E}\right)_x - b\,\frac{E_y}{2E^2} = 0,$$

which is equivalent to

$$(2\text{-}23) \qquad \begin{aligned} 2a\,E_x + b\,E_y &= 0, \\ 2E E_{xy} - 3E_x E_y + 4abE &= 0. \end{aligned}$$

Analogously, the y-lines are circles if and only if

$$(2\text{-}24) \qquad \begin{aligned} b\,E_x - 2a\,E_y &= 0, \\ 2E E_{xy} - 3E_x E_y - 4abE &= 0. \end{aligned}$$

From (2-20), we calculate

$$(2\text{-}25) \qquad E_{xy} = -\sin\theta\cos\theta\, E_{uu} + \cos 2\theta\, E_{uv} + \sin\theta\cos\theta\, E_{vv}.$$

From the first equality of (2-23), using (2-20), we get $\cos^3\theta\, E_u - \sin^3\theta\, E_v = 0$. All solutions of this equation are given by

$$(2\text{-}26) \qquad E = \varphi(\sin^3\theta\, u + \cos^3\theta\, v)$$

for an arbitrary smooth function φ. Hence, $E_u = \sin^3\theta\, \varphi'$, $E_v = \cos^3\theta\, \varphi'$, $E_{uu} = \sin^6\theta\, \varphi''$, $E_{vv} = \cos^6\theta\, \varphi''$, and $E_{uv} = \sin^3\theta\cos^3\theta\, \varphi''$. From the second equality of (2-23), we obtain $\sin 2\theta\cos 2\theta\,(\varphi\varphi'' - \frac{3}{2}\varphi'^2 - 4\varphi) = 0$ using (2-25) and (2-26). Consequently, the x-lines are circles if and only if

$$(2\text{-}27) \qquad \begin{aligned} E &= \varphi(\sin^3\theta\, u + \cos^3\theta\, v), \\ 0 &= \sin 2\theta\cos 2\theta\,(\varphi\varphi'' - \tfrac{3}{2}\varphi'^2 - 4\varphi). \end{aligned}$$

Analogously, using (2-24) we obtain that the y-lines are circles if and only if

$$(2\text{-}28) \qquad \begin{aligned} E &= \varphi(\cos^3\theta\, u - \sin^3\theta\, v), \\ 0 &= \sin 2\theta\cos 2\theta\,(\varphi\varphi'' - \tfrac{3}{2}\varphi'^2 - 4\varphi). \end{aligned}$$

Thus the condition that the x-lines (or y-lines) be circles leads to the cases

(I) $\sin 2\theta = 0$, that is, $\theta = 0$ or $\theta = \pi/2$.

This case corresponds to $x = u$, $y = v$ or $x = v$, $y = -u$. From (2-27) and (2-28) we obtain that the u-lines are circles if and only if $E = \varphi(v)$, and the v-lines are circles if and only if $E = \varphi(u)$. In this case, one of the families of principal lines is a family of circles.

(II) $\cos 2\theta = 0$, that is, $\theta = \pi/4$ or $\theta = 3\pi/4$.

This case corresponds to $x = (u+v)/\sqrt{2}$, $y = (-u+v)/\sqrt{2}$ or $x = (-u+v)/\sqrt{2}$, $y = -(u+v)/\sqrt{2}$. From (2-27) and (2-28) we obtain that the x-lines are circles if and only if $E = \varphi(x)$, and the y-lines are circles if and only if $E = \varphi(y)$. In this case one of the families of bisectrices of the principal lines is a family of circles.

(III) $\varphi\varphi'' - \frac{3}{2}\varphi'^2 - 4\varphi = 0$ and $\sin 2\theta \cos 2\theta \neq 0$.

We shall prove that this case is not possible. Since $E > 0$, we have $E = \varphi(\tau) = e^{z(\tau)}$ for some function $z = z(\tau)$ with $\tau = \sin^3 \theta \, u + \cos^3 \theta \, v$ (or $\tau = \cos^3 \theta \, u - \sin^3 \theta \, v$). Moreover, $E \neq$ const, that is, $z'(\tau) \neq 0$. The equality $\varphi\varphi'' - \frac{3}{2}\varphi'^2 - 4\varphi = 0$ implies

$$(2\text{-}29) \qquad\qquad z'' - \tfrac{1}{2}z'^2 - 4e^{-z} = 0.$$

On the other hand, using identity (2-6), we obtain

$$(2\text{-}30) \qquad\qquad (\cos^6 \theta + \sin^6 \theta)z'' + 4\sinh z = 0.$$

Let us write $\lambda = \cos^6 \theta + \sin^6 \theta = $ const. Multiplying (2-30) by z' and integrating, we get

$$\tfrac{1}{2}\lambda z'^2(\tau) + 4\cosh z(\tau) = \tfrac{1}{2}\lambda z'^2(0) + 4\cosh z(0).$$

Equalities (2-29) and (2-30) imply

$$4\sinh z + \tfrac{1}{2}\lambda z'^2 + 4\lambda e^{-z} = 0.$$

Using the last two equalities, we obtain

$$4(1-\lambda)e^{-z} = \tfrac{1}{2}\lambda z'^2(0) + 4\cosh z(0) = \text{const}.$$

Since $1 - \lambda = 3\sin^2 \theta \cos^2 \theta \neq 0$, we get $e^{-z(\tau)} = $ const, that is, $z(\tau) = $ const, which contradicts the condition $z'(\tau) \neq 0$, that is, $E \neq$ const. $\qquad\square$

From Theorem 2.2 it follows that there are only two types of generalized tori in S^3: the first are those such that one of the families of bisectrices of the principal lines is a family of circles (such surfaces are *generalized tori of the first type*); the second are those such that one of the families of principal lines is a family of circles (these are the *generalized tori of the second type*).

Theorem 2.3. *Let M^2 be a generalized torus of the first type with nonconstant Gauss curvature. Then M^2 is a Lawson torus $\mathcal{M}_\alpha$ for some positive α with $\alpha \neq 1$.*

Proof. Let M^2 be a generalized torus of the first type with nonconstant Gauss curvature. In this case the derivative formulas (2-4) hold with $a = 0$, $b = 1$, and $E = E(u)$ (or $E = E(v)$). We consider only the first case, in which $E = E(u)$ and $E_u \neq 0$. The second one can be investigated analogously. The derivative formulas in this case look like

$$(2\text{-}31) \qquad \begin{aligned} l_{uu} &= \frac{E_u}{2E} l_u - E\, l, & n_u &= -\frac{1}{E} l_v, \\ l_{uv} &= \frac{E_u}{2E} l_v + n, & n_v &= -\frac{1}{E} l_u, \\ l_{vv} &= -\frac{E_u}{2E} l_u - E\, l. \end{aligned}$$

We shall prove that the parametric u-lines are great circles. Consider for constant v_0 the u-line $c : c(u) = l(u, v_0)$. From (2-31) it follows that $\dot{c} = l_u$, and the tangent vector t_c of c is $t_c = c' = \dot{c}/\dot{s} = l_u/\sqrt{E}$. We calculate

$$t_c' = \frac{\dot{t}_c}{\dot{s}} = \frac{1}{\sqrt{E}} \left(\frac{l_{uu}}{\sqrt{E}} - \frac{E_u}{2E\sqrt{E}} l_u \right) = -l.$$

Hence the curvature $\varkappa$ of c is $\varkappa = 1$, so c is a great circle. Consequently, M^2 is a one-parameter family of great circles. According to [Lawson 1970, Proposition 7.2], M^2 is an open submanifold of $\mathcal{M}_\alpha$ for some positive α with $\alpha \neq 1$. $\qquad \square$

Now we consider a generalized torus of the second type with nonconstant Gauss curvature. In this case the derivative formulas (2-4) hold with $a = 1$, $b = 0$, and $E = E(u)$ (or $E = E(v)$). We consider the case $E = E(u)$ and $E_u \neq 0$, so that the parametric v-lines are circles. We prove that the different v-lines are circles with different radii.

Now the derivative formulas are

$$(2\text{-}32) \qquad \begin{aligned} l_{uu} &= \frac{E_u}{2E} l_u - E\, l + n, & n_u &= -\frac{1}{E} l_u, \\ l_{uv} &= \frac{E_u}{2E} l_v, & n_v &= \frac{1}{E} l_v, \\ l_{vv} &= -\frac{E_u}{2E} l_u - E\, l - n. \end{aligned}$$

For $u_0 = \text{const}$, we consider the v-line $c : c(v) = l(u_0, v)$. As in the proof of Theorem 2.3, from (2-32) we calculate the curvature $\varkappa(u_0)$ of c to be

$$\varkappa(u_0) = \sqrt{1 + \frac{E_u^2(u_0)}{4E^3(u_0)} + \frac{1}{E^2(u_0)}}.$$

For different values of the constant u_0, the curvatures $\varkappa(u_0)$ are different. We note that $\varkappa(u_0) > 1$, that is, the circles are not great ones. Therefore, the generalized tori of the second type differ from the Lawson tori.

Since $E(u) > 0$, we write $E(u)$ in the form $E(u) = e^{z(u)}$ for $z(u) \neq$ const. Then the system (2-32) is rewritten in the form

$$
\begin{aligned}
l_{uu} - \tfrac{1}{2}z'(u)\,l_u + e^{z(u)}\,l - n &= 0, & n_u + e^{-z(u)}\,l_u &= 0, \\
l_{uv} - \tfrac{1}{2}z'(u)\,l_v &= 0, & n_v - e^{-z(u)}\,l_v &= 0, \\
l_{vv} + \tfrac{1}{2}z'(u)\,l_u + e^{z(u)}\,l + n &= 0.
\end{aligned}
$$
(2-33)

We look for classical solutions of the system (2-33) for the vector-valued functions $l(u, v)$, $l_u(u, v)$, $l_v(u, v)$ and $n(u, v)$ in a neighborhood of the origin under the initial conditions

$$
\begin{aligned}
l(0, 0) &= e_1, & l_v(0, 0) &= e^{s/2}e_3 = \sqrt{E(0)}\,e_3, \\
l_u(0, 0) &= e^{s/2}e_2 = \sqrt{E(0)}\,e_2, & n(0, 0) &= e_4,
\end{aligned}
$$
(2-34)

where $\{e_1, e_2, e_3, e_4\}$ is the standard orthonormal basis in $\mathbb{R}^4$, and $s = $ const $= z(0) = \ln E(0)$.

Since the function $E(u)$ satisfies identity (2-6) with $a = 1$ and $b = 0$, it follows that $z(u)$ is the solution of the ordinary differential equation

$$
z''(u) + 4\sinh z(u) = 0, \quad z(0) = s, \quad z'(0) = 2t,
$$
(2-35)

where s and t are arbitrary constants.

To find $z(u)$ explicitly, we note that the identity (2-6) holds for an arbitrary minimal surface in S^3 parametrized by isothermal parameters. So let us consider again the Lawson torus $\mathcal{M}_\alpha$ defined by (2-9). We change the parameters (x, y) by new parameters (u, v) via

$$
u = h(x) = \sqrt{\alpha}\int_0^x \frac{1}{\sqrt{\alpha^2\cos^2\tau + \sin^2\tau}}\,d\tau, \quad v = \sqrt{\alpha}\,y,
$$
(2-36)

and denote by h^{-1} the inverse function of h. Then we obtain an isothermal parametrization of $\mathcal{M}_\alpha$, and the function

$$
E(u) = e^{z(u)} = \frac{\alpha^2\cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha}
$$

satisfies (2-6) with $a = 0$ and $b = 1$. Hence the function

$$
z(u) = \ln\frac{\alpha^2\cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha}
$$
(2-37)

is a solution of (2-35) with $t = 0$ and $s = \ln \alpha$ for $\alpha > 0$, $\alpha \neq 1$. (It can be calculated directly that the function $z(u)$ defined by (2-37) satisfies (2-35) with $z(0) = \ln \alpha$ for $\alpha > 0$, $\alpha \neq 1$ and $z'(0) = 0$.)

We will prove that every solution $\tilde{z}(u)$ of (2-35) with arbitrary t and s can be obtained from (2-37) by the formula $\tilde{z}(u) = z(u + u_0)$ for a suitable choice of constants u_0 and α. Since (2-35) is an autonomous equation and $z(u)$ is its solution, $\tilde{z}(u) = z(u + u_0)$ is also a solution of this equation. Therefore we have to check only the initial conditions. Let $x_0 = h^{-1}(u_0)$. Simple calculations give us the equalities

$$\tilde{z}(0) = z(u_0) = \ln \frac{\alpha^2 \cos^2 x_0 + \sin^2 x_0}{\alpha} = s,$$

$$\tilde{z}'(0) = z'(u_0) = \frac{(1 - \alpha^2) \sin 2x_0}{\sqrt{\alpha}\sqrt{\alpha^2 \cos^2 x_0 + \sin^2 x_0}} = 2t,$$

which imply

$$\sin 2x_0 = \frac{2\alpha t e^{s/2}}{1 - \alpha^2},$$

$$\cos 2x_0 = \frac{1 + \alpha^2 - 2\alpha e^s}{1 - \alpha^2}.$$

Using $\sin^2 2x_0 + \cos^2 2x_0 = 1$, we see that α satisfies

$$e^s \alpha^2 - (1 + e^{2s} + t^2 e^s)\alpha + e^s = 0,$$

whose positive solutions are

$$\alpha = \frac{1 + e^{2s} + t^2 e^s \pm \sqrt{(1 + e^{2s} + t^2 e^s)^2 - 4e^{2s}}}{2e^s}.$$

For this choice of α and

$$u_0 = h(x_0) = \tfrac{1}{2} \arctan \frac{2\alpha t e^{s/2}}{1 + \alpha^2 - 2\alpha e^s},$$

the initial conditions are satisfied.

Note that all solutions of (2-35) are periodic with period $\omega = h(\pi)$.

To simplify system (2-33), we change the vector function $l(u, v)$ with vector function $L(u, v)$ determined by

(2-38) $$l(u, v) = e^{z(u)/2} L(u, v).$$

We get the system

$$0 = L_{uu}(u, v) + \tfrac{1}{2}z'(u)L_u(u, v) + e^{-z(u)}L(u, v) - e^{-z(u)/2}n,$$

$$0 = L_{uv}(u, v),$$

(2-39) $$0 = L_{vv}(u, v) + \tfrac{1}{2}z'(u)L_u(u, v) + ((\tfrac{1}{2}z'(u))^2 + e^{z(u)})L(u, v) + e^{-z(u)/2}n,$$

$$0 = n_u + e^{-z(u)/2}(L_u + \tfrac{1}{2}z'(u)L),$$

$$0 = n_v - e^{-z(u)/2}L_v.$$

The initial conditions (2-34) for $l(u, v)$ imply the following initial conditions for $L(u, v)$:

$$L(0, 0) = e^{-s/2}e_1, \qquad L_v(0, 0) = e_3,$$

$$L_u(0, 0) = -te^{-s/2}e_1 + e_2, \qquad n(0, 0) = e_4.$$

From the second equality in (2-39) it follows that $L(u, v) = f(u) + g(v)$, where $f(u)$ and $g(v)$ are vector functions satisfying the system

$$0 = f''(u) + \tfrac{1}{2}z'(u)f'(u) + e^{-z(u)}(f(u) + g(v)) - e^{-z(u)/2}n,$$

$$0 = g''(v) + \tfrac{1}{2}z'(u)f'(u) + ((\tfrac{1}{2}z'(u))^2 + e^{z(u)})(f(u) + g(v)) + e^{-z(u)/2}n,$$

(2-40) $$0 = n_u + e^{-z(u)/2}(f'(u) + \tfrac{1}{2}z'(u)(f(u) + g(v))),$$

$$0 = n_v - e^{-z(u)/2}g'(v) = 0,$$

$$f'(0) = -te^{-s/2}e_1 + e_2, \quad g'(0) = e_3, \quad f(0) + g(0) = e^{-s/2}e_1.$$

Without loss of generality we assume that $g(0) = 0$ and $f(0) = e^{-s/2}e_1$.

Let us fix $u = 0$ in the fourth equality of (2-40). Then after integration we get $n(0, v) = e^{-s/2}g(v) + e_4$. Now using the second equality of (2-40), we see that $g(v)$ satisfies the initial value problem

$$g''(v) + (t^2 + 2\cosh s)g(v) = -e^{s/2}e_1 - te_2 - e^{-s/2}e_4,$$

$$g(0) = 0, \quad g'(0) = e_3.$$

Simple computations give us

(2-41) $$g(v) = \frac{1}{t^2 + 2\cosh s}(\cos\sqrt{t^2 + 2\cosh s}\, v - 1)(e^{s/2}e_1 + te_2 + e^{-s/2}e_4)$$

$$+ \frac{1}{\sqrt{t^2 + 2\cosh s}}\sin\sqrt{t^2 + 2\cosh s}\, ve_3.$$

Now, multiplying (2-35) with $z'(u)$ and integrating from 0 to u, we get that $z'(u)$ satisfies the equality

(2-42) $$(z'(u))^2 + 8\cosh z(u) = 4t^2 + 8\cosh s.$$

Using the first and second equalities of (2-40), (2-41), and (2-42), and setting $v = 0$, we see that $f(u)$ satisfies the initial value problem

$$f''(u) + z'(u) f'(u) + (t^2 + 2\cosh s) f(u) = e^{s/2}e_1 + te_2 + e^{-s/2}e_4,$$
$$f(0) = e^{-s/2}e_1, \quad f'(0) = -te^{-s/2}e_1 + e_2.$$

Therefore the solution $f(u)$ of the system above can be written in the form

$$f(u) = p(u) + \frac{1}{t^2 + 2\cosh s}(e^{s/2}e_1 + te_2 + e^{-s/2}e_4),$$

where $p(u)$ is the unique solution of the linear homogeneous system

$$p''(u) + z'(u)p'(u) + (t^2 + 2\cosh s)p(u) = 0,$$
$$p(0) = \frac{1}{t^2 + 2\cosh s}[e^{-s/2}(t^2 + e^{-s})e_1 - te_2 - e^{-s/2}e_4], \quad p'(0) = -te^{-s/2}e_1 + e_2.$$

By (2-38), the solution $l(u, v)$ of problem (2-33) under condition (2-34) is

$$l(u, v) = e^{z(u)/2}p(u) + \frac{e^{z(u)/2}}{t^2 + 2\cosh s} \cos\sqrt{t^2 + 2\cosh s} \, v(e^{s/2}e_1 + te_2 + e^{-s/2}e_4)$$
$$+ \frac{e^{z(u)/2}}{\sqrt{t^2 + 2\cosh s}} \sin\sqrt{t^2 + 2\cosh s} \, ve_3.$$

If we denote $\beta = \sqrt{t^2 + 2\cosh s}$ (here β is constant), then $l(u, v)$ is rewritten as

$$l(u, v) = e^{z(u)/2}p(u) + \frac{e^{z(u)/2}}{\beta^2}(\cos\beta \, v(e^{s/2}e_1 + te_2 + e^{-s/2}e_4) + \beta\sin\beta \, ve_3).$$

The function $z(u)$ is given explicitly by (2-37). Thus we have proved this:

Theorem 2.4. *Let $M^2 : l = l(u, v)$ be a generalized torus of the second type with nonconstant Gauss curvature. Then*

$$l(u, v) = e^{z(u)/2}p(u) + \frac{e^{z(u)/2}}{\beta^2}(\cos\beta \, v(e^{s/2}e_1 + te_2 + e^{-s/2}e_4) + \beta\sin\beta \, ve_3),$$

where $\beta = \sqrt{t^2 + 2\cosh s}$ for arbitrary constants s and t. The scalar function $z(u)$ is the solution of the initial value problem

$$z'' + 4\sinh z(u) = 0, \quad z(0) = s, \quad z'(0) = 2t,$$

and is given explicitly by (2-37). The vector function $p(u)$ is a solution of the system

(2-43)
$$p''(u) + z'(u)p'(u) + \beta^2 p(u) = 0,$$
$$p(0) = \frac{1}{\beta^2}(e^{-s/2}(t^2 + e^{-s})e_1 - te_2 - e^{-s/2}e_4), \quad p'(0) = -te^{-s/2}e_1 + e_2,$$

and $\{e_1, e_2, e_3, e_4\}$ is the standard orthonormal basis in $\mathbb{R}^4$.

3. Application to the theory of minimal foliated semisymmetric hypersurfaces

We now relate the theory of minimal surfaces in S^3 to the theory of minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$.

For an n-dimensional Riemannian manifold (M^n, g), we denote by $T_p M^n$ the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X} M^n$ the algebra of all vector fields on M^n. The associated Levi-Civita connection of the metric g is denoted by ∇ and the Riemannian curvature tensor R is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \quad \text{for } X, Y \in \mathfrak{X} M^n.$$

A *semisymmetric space* is a Riemannian manifold (M^n, g) such that R satisfies the identity $R(X, Y) \cdot R = 0$ for all vector fields $X, Y \in \mathfrak{X} M^n$. (Here $R(X, Y)$ acts as a derivation on R).

According to the classification of Z. Szabó [1982], the main class of semisymmetric spaces is the class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

The foliated semisymmetric hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$ are the hypersurfaces of type number two, that is, those hypersurfaces whose second fundamental form has rank two everywhere. They are characterized by a second fundamental form $h = v_1 \, \eta_1 \otimes \eta_1 + v_2 \, \eta_2 \otimes \eta_2$ for $v_1 v_2 \neq 0$, where η_1 and η_2 are unit one-forms, and v_1 and v_2 are functions on the hypersurface M^n. The Euclidean leaves of the foliation are the integral submanifolds of the distribution Δ_0, determined by the one-forms η_1 and η_2, that is, $\Delta_0(p) = \{X \in T_p M^n \mid \eta_1(X) = 0, \ \eta_2(X) = 0\}$ for $p \in M^n$. A special class of foliated semisymmetric hypersurfaces is the class of ruled hypersurfaces.

A hypersurface M^n of type number two is minimal if $v_1 + v_2 = 0$.

Ganchev and Milousheva [2007b] have characterized the foliated semisymmetric hypersurfaces in $\mathbb{E}^{n+1}$ as follows.

Theorem 3.1. *A hypersurface M^n in Euclidean space $\mathbb{E}^{n+1}$ is locally a foliated semisymmetric hypersurface if and only if it is the envelope of a two-parameter family of hyperplanes in $\mathbb{E}^{n+1}$.*

If we consider a foliated semisymmetric hypersurface as the envelope of a two-parameter family of hyperplanes, each such hypersurface is determined by a pair consisting of a unit vector-valued function $l(u, v)$ and a scalar function $r(u, v)$, both defined in a domain $\mathscr{D} \subset \mathbb{R}^2$.

Since the vector fields l_u and l_v are linearly independent, $l(u, v)$ determines a two-dimensional surface $M^2 : l = l(u, v)$ for $(u, v) \in \mathscr{D}$ in $\mathbb{E}^{n+1}$. Without loss of generality, it can be assumed that the surface M^2 is parametrized locally by isothermal parameters, namely, $E = G$ and $F = 0$. Then the generated foliated

semisymmetric hypersurface M^n is given in [Ganchev and Milousheva 2007a] by

$$(3-1) \qquad X(u, v, w^\alpha) = r\, l + \frac{r_u}{E}\, l_u + \frac{r_v}{E}\, l_v + w^\alpha b_\alpha \quad \text{for } \alpha = 1, \ldots, n-2,$$

where $(u, v) \in \mathcal{D}$, $w^\alpha \in \mathbb{R}$ for $\alpha = 1, \ldots, n-2$ and $b_1(u, v), \ldots, b_{n-2}(u, v)$, where $(u, v) \in \mathcal{D}$ are $n-2$ mutually orthogonal unit vectors, orthogonal to $\mathrm{span}\{l, l_u, l_v\}$.

The minimal foliated semisymmetric hypersurfaces in $\mathbb{E}^{n+1}$ are characterized analytically in [Ganchev and Milousheva 2007a] as follows.

Theorem 3.2. *Let M^n be a hypersurface in $\mathbb{E}^{n+1}$ that is the envelope of a two-parameter family of hyperplanes and is determined by a unit vector-valued function $l(u, v)$, represented by isothermal parameters, and a scalar function $r(u, v)$. Then M^n is minimal if and only if $l(u, v)$ and $r(u, v)$ satisfy*

$$\Delta l(u, v) + 2E(u, v)\, l(u, v) = 0,$$
$$\Delta r(u, v) + 2E(u, v)\, r(u, v) = 0.$$

Hence, the minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$ are generated by the solutions of system (2-3), that is, by the minimal surfaces in S^3.

Now we construct examples of minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$ that are generated by some minimal surfaces in S^3.

The simplest example of a minimal surface in S^3 is the sphere $S^2 = S^3 \cap \mathbb{R}^3$. We assume that $\mathbb{R}^3$ is the subspace of $\mathbb{R}^4$ orthogonal to e_4, that is, $\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}$. An isothermal parametrization of S^2 is given by

$$S^2 : l(u, v) = \frac{1}{\cosh u} (\cos v;\ \sin v;\ \sinh u;\ 0).$$

A direct check shows that $E = \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = 1/\cosh^2 u$ and $F = \langle l_u, l_v \rangle = 0$, and obviously $l(u, v)$ satisfies the equality

$$\Delta l(u, v) + \frac{2}{\cosh^2 u}\, l(u, v) = 0.$$

The normal vector field $n(u, v)$ of S^2 is $n = e_4 = (0;\ 0;\ 0;\ 1)$. By Theorem 3.2, the corresponding differential equation for the scalar function $r(u, v)$ is

$$(3-2) \qquad \Delta r(u, v) + \frac{2}{\cosh^2 u}\, r(u, v) = 0.$$

Every solution $r(u, v)$ of (3-2) together with the sphere $S^2 : l = l(u, v)$ generates a minimal foliated semisymmetric hypersurface in $\mathbb{R}^4$ according to formula (3-1).

One solution of (3-2) is

$$r(u, v) = (v + \tfrac{1}{2}\pi) \tanh u.$$

Now consider the minimal foliated semisymmetric hypersurface M^3 generated by $l(u, v)$ and this solution $r(u, v)$. Calculating r_u, r_v, l_u, l_v, and applying formula (3-1), we obtain $M^3 : X(u, v, w) = (-\sinh u \sin v; \sinh u \cos v; v + \frac{1}{2}\pi; w)$. After changing to the parameters $u^1 = \sinh u$ and $t = v + \frac{1}{2}\pi$, we obtain the hypersurface

$$(3\text{-}3) \qquad M^3 : X(u^1, t, w) = u^1(\cos t\, e_1 + \sin t\, e_2) + t e_3 + w e_4.$$

The hypersurface M^3 whose radius vector $X = X(u^1, t, w)$ is determined by (3-3) is the *generalized helicoidal ruled hypersurface* obtained by G. Aumann [1981, Theorem 4]. It is also called a *first type helicoid* in $\mathbb{R}^4$ and is a generalization of the *right helicoid* in $\mathbb{R}^3$.

The next well-known example of a minimal surface in S^3 is the Clifford torus

$$\mathcal{M} : l(u, v) = (\cos u \cos v; \cos u \sin v; \sin u \cos v; \sin u \sin v).$$

The normal vector field $n(u, v)$ of $\mathcal{M}$ is

$$n(u, v) = (\sin u \sin v; -\sin u \cos v; -\cos u \sin v; \cos u \cos v).$$

Since $E = \langle l_u, l_u \rangle = \langle l_v, l_v \rangle = 1$, Theorem 3.2 says that the corresponding equation for the scalar function $r(u, v)$ is $\Delta r(u, v) + 2 r(u, v) = 0$. If we take the trivial solution $r(u, v) = 0$, we obtain the minimal foliated semisymmetric hypersurface M^3: $X(u, v, w) = w n(u, v) = w(\sin u \sin v; -\sin u \cos v; -\cos u \sin v; \cos u \cos v)$.

After the change of parameters $u^1 = -w \sin u$, $u^2 = w \cos u$, and $t = v + \frac{1}{2}\pi$, we obtain the hypersurface

$$(3\text{-}4) \qquad M^3 : X(u^1, u^2, t) = u^1(\cos t\, e_1 + \sin t\, e_2) + u^2(\cos t\, e_3 + \sin t\, e_4).$$

The hypersurface M^3 whose radius vector $X = X(u^1, u^2, t)$ is determined by (3-4) is the minimal ruled hypersurface obtained by G. Aumann [1981, Theorem 1], and is known as a *second type helicoid*. The helicoids of first and second type are the only minimal ruled hypersurfaces in $\mathbb{R}^4$ [Aumann 1981].

Thus we have shown that the first type helicoid is generated by the sphere S^2 in S^3, while the second type helicoid is generated by the Clifford torus.

Our scheme of constructing minimal foliated semisymmetric hypersurfaces in $\mathbb{R}^4$ can be applied to each minimal surface $M^2 : l = l(u, v)$ in S^3 and each solution of the corresponding differential equation for the scalar function $r(u, v)$.

We illustrate how this construction can be applied to the generalized torus of the second type, given in Theorem 2.4, in the special case when $t = 0$ and $s = \ln \alpha$ for positive $\alpha \neq 1$. Since the calculations are too long and complicated, we give only a short sketch. In this case the solution $l(u, v)$ is defined by

$$l(u, v) = f(u)\left(p(u) + \frac{\alpha}{\alpha^2 + 1} \cos \beta v \left(\sqrt{\alpha} e_1 + \frac{1}{\sqrt{\alpha}} e_4\right) + \sqrt{\frac{\alpha}{\alpha^2 + 1}} \sin \beta v\, e_3 \right),$$

where $f(u) = \sqrt{(\alpha^2 \cos^2 h^{-1}(u) + \sin^2 h^{-1}(u))/\alpha}$, $h^{-1}(u)$ is the inverse function of h given in (2-36), $p(u)$ is the solution of system (2-43), and $\beta = \sqrt{(\alpha^2 + 1)/\alpha}$. As a solution of the corresponding differential equation for $r(u, v)$, we choose $r(u, v) = f(u)\sqrt{\alpha/(\alpha^2 + 1)} \sin \beta v$. We calculate $l_u(u, v)$ and $l_v(u, v)$, and using (2-33) and (2-41), we find the vector-valued function

$$n(u, v) = \frac{1 - \alpha^2}{\alpha(\alpha^2 + 1)}(e_1 - \alpha e_4) + \frac{\alpha}{(\alpha^2 + 1)f(u)} \cos \beta v \left(\sqrt{\alpha} e_1 + \frac{1}{\sqrt{\alpha}} e_4\right)$$
$$+ \sqrt{\frac{\alpha}{\alpha^2 + 1}} \frac{1}{f(u)} \sin \beta v \, e_3 - \frac{p(u)}{f(u)} - \frac{1 - \alpha^2}{\alpha} \int_0^u \frac{\sin 2h^{-1}(s)}{f^2(s)} p(s) \, ds.$$

Applying formula (3-1), we obtain the following minimal foliated semisymmetric hypersurface M^3:

$$X(u, v, w) = \frac{1 - \alpha^2}{2\sqrt{\alpha(\alpha^2 + 1)}} \frac{\sin 2h^{-1}(u)}{f(u)} \sin \beta v \, p'(u)$$
$$+ \sqrt{\frac{\alpha}{\alpha^2 + 1}} \frac{\alpha^4 \cos^2 h^{-1}(u) + \sin^2 h^{-1}(u)}{\alpha^2 f^2(u)} \sin \beta v \, p(u)$$
$$- \left(\sqrt{\frac{\alpha}{\alpha^2 + 1}}\right)^3 \frac{1}{2f^2(u)} \sin 2\beta v \left(\sqrt{\alpha} e_1 + \frac{1}{\sqrt{\alpha}} e_4\right) + \left(1 - \frac{\alpha}{(\alpha^2 + 1)f^2(u)} \sin^2 \beta v\right) e_3$$
$$+ wn(u, v).$$

Unfortunately, in this case we cannot write the hypersurface M^3 in terms of elementary functions as in the previous examples, because we cannot find an explicit solution $p(u)$ of linear system (2-43).

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