**ORBIT CORRESPONDENCES FOR
REAL REDUCTIVE DUAL PAIRS**

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Suppose that one of the real vector spaces V and W is symplectic and the other is quadratic. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denote the Lie algebras of the groups of isometries of the two spaces, and let $\tau_i : V \otimes_{\mathbb{R}} W \to \mathfrak{g}_i$ be their respective moment maps for $i = 1, 2$. Suppose that $\mathbb{O}$ and $\mathbb{D}$ are nilpotent orbits in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. We prove that $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}))$ and $\tau_1(\tau_2^{-1}(\overline{\mathbb{D}}))$ are each the union of at most two closures of nilpotent orbits in $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively (where $\overline{\mathcal{P}}$ denotes the closure of a nilpotent orbit $\mathcal{P}$).

1. Introduction

Let F denote either complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$. Let $(V, \langle\,\cdot\,,\cdot\,\rangle)$ be a (nondegenerate) symplectic space, and let $(W, (\,\cdot\,,\cdot\,))$ be a quadratic space over F. The tensor product $V \otimes W$ is a symplectic space, with the form $\langle\!\langle\,\cdot\,,\cdot\,\rangle\!\rangle$ defined by

$$\langle\!\langle e \otimes v, f \otimes w \rangle\!\rangle = \langle e, f \rangle (v, w) \quad \text{for } e, f \in V \text{ and } v, w \in W.$$

The pair of groups $(\mathrm{Sp}(V), \mathrm{O}(W))$ is called a *reductive dual pair* [Howe 1989].

Let $\varphi_1 : V \otimes W \to \mathrm{Hom}(V, W)$ be the vector space isomorphism given by $\varphi_1(v \otimes w)(x) := \langle v, x \rangle w$, and let $\varphi_2 : V \otimes W \to \mathrm{Hom}(W, V)$ be the isomorphism given by $\varphi_2(v \otimes w)(y) := (w, y) v$. Define the maps $\tau_1 : V \otimes W \to \mathrm{End}(V)$ by $\tau_1(\gamma) := \varphi_2(\gamma) \circ \varphi_1(\gamma)$, and $\tau_2 : V \otimes W \to \mathrm{End}(W)$ by $\tau_2(\gamma) := \varphi_1(\gamma) \circ \varphi_2(\gamma)$. It turns out that the image of τ_1 is in $\mathfrak{sp}(V)$ and the image of τ_2 is in $\mathfrak{so}(W)$. The maps τ_1 and τ_2 are called the *moment maps* [Kazhdan et al. 1978].

An element $X \in \mathfrak{sp}(V)$ is said to correspond to an element $X' \in \mathfrak{so}(W)$ if $\tau_1^{-1}(X) \cap \tau_2^{-1}(X')$ is not empty. An orbit $\mathbb{O}$ in $\mathfrak{sp}(V)$ is said to correspond to an orbit $\mathbb{O}'$ in $\mathfrak{so}(W)$ if there is an element in $\mathbb{O}$ that corresponds to an element in $\mathbb{O}'$. Suppose that $\mathbb{O}$ is an orbit in $\mathfrak{sp}(V)$ corresponding to an orbit $\mathbb{O}'$ in $\mathfrak{so}(W)$. It is known that $\mathbb{O}$ is nilpotent if and only if $\mathbb{O}'$ is nilpotent. On the correspondence of nilpotent orbits, A. Daszkiewicz, W. Kraśkiewicz and T. Przebinda [1997] proved the next theorem:

Theorem A. *Let $F = \mathbb{C}$. Let $\tau_1 : V \otimes_F W \to \mathfrak{sp}(V)$ and $\tau_2 : V \otimes_F W \to \mathfrak{so}(W)$ be the moment maps. Let $\mathcal{O}$ and $\mathcal{Q}$ be nilpotent orbits in $\mathfrak{sp}(V)$ and $\mathfrak{so}(W)$ contained in the images of τ_1 and τ_2, respectively. There are unique nilpotent orbits $\mathcal{O}' \subseteq \mathfrak{so}(W)$ and $\mathcal{Q}' \subseteq \mathfrak{sp}(V)$ such that $\tau_2(\tau_1^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}}'$ and $\tau_1(\tau_2^{-1}(\overline{\mathcal{Q}})) = \overline{\mathcal{Q}}'$, where $\overline{\mathcal{P}}$ denotes the closure of a nilpotent orbit $\mathcal{P}$.*

In the theorem of 4.1, we give a slightly more precise form of this theorem, which coincides with [Daszkiewicz et al. 1997, Theorem 5.2]. Our main theorem is the analogous result for real orbit correspondences:

Theorem B. *Suppose $F = \mathbb{R}$. Let $\tau_1 : V \otimes_F W \to \mathfrak{sp}(V)$ and $\tau_2 : V \otimes_F W \to \mathfrak{so}(W)$ be the moment maps. Let $\mathcal{O}$ and $\mathcal{Q}$ be nilpotent orbits in $\mathfrak{sp}(V)$ and $\mathfrak{so}(W)$ contained in the images of τ_1 and τ_2, respectively. Then, $\tau_2(\tau_1^{-1}(\overline{\mathcal{O}}))$ and $\tau_1(\tau_2^{-1}(\overline{\mathcal{Q}})))$ are each the union of at most two closures of nilpotent orbits in $\mathfrak{so}(W)$ and $\mathfrak{sp}(V))$, respectively.*

The motivation for the result is this: Suppose G is a real reductive group with Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Lie algebra of the complexified group $G_{\mathbb{C}}$ of G. Let K be a maximal compact subgroup of G. Given an irreducible admissible representation (π, E) of G, we can attach to it a union of nilpotent coadjoint orbits of $G_{\mathbb{C}}$ in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, as follows: If E^K is the subspace of K-finite vectors of E, then E^K is an irreducible $(\mathfrak{g}, K)$-module. Denote the action of $\mathfrak{g}$ on E^K by π still. Let I_π be the annihilator of E^K in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The latter has a natural grading, and the associated graded algebra $\mathrm{gr}(U(\mathfrak{g}))$ is isomorphic to the algebra of polynomials on $\mathfrak{g}^*$. Let $\mathcal{V}(I_\pi)$ denote the variety in $\mathfrak{g}^*$ defined by the ideal $\mathrm{gr}(I_\pi)$ of $\mathrm{gr}(U(\mathfrak{g}))$. It is known that $\mathcal{V}(I_\pi)$ is a finite union of nilpotent orbits in $\mathfrak{g}^*$ and by [Collingwood and McGovern 1993; Joseph 1985] is the closure of a single nilpotent orbit. The variety $\mathcal{V}(I_\pi)$ is called the associated variety of the annihilator, in the universal enveloping algebra, of the Harish-Chandra module of the representation (π, E). By a result of R. Howe [1989], there is a one-to-one correspondence between some irreducible admissible representations of the metaplectic cover $\widetilde{\mathrm{Sp}}(V)$ and some irreducible admissible representations of $\widetilde{\mathrm{O}}(W)$. Therefore, one expects a relation between the correspondence of coadjoint orbits and the correspondence of irreducible representations. Some results have been established in [Adams 1983; 1987; Przebinda 1993].

Theorem B might be interesting for studying representations of real groups by using finer orbit structures, such as wave front sets or characteristic varieties. In fact, this result is used in [He 2003; He 2005].

The content of this paper is as follows: In Sections 2 and 3, we summarize some well-known results on nilpotent orbits in $\mathfrak{sp}(V)$ and $\mathfrak{so}(W)$. We describe some special realization of the Lie algebras $\mathfrak{sp}(V)$ and $\mathfrak{so}(W)$, and the parametrization of

their nilpotent orbits by some special kinds of partitions. In Section 4, we describe
the results of complex orbit correspondences; the proofs are omitted, since the re-
sults in this section are already known from [Daszkiewicz et al. 1997]. In Section 5,
we describe the basic rules for determining the real orbit correspondence. We state
our main theorem in Section 6. The construction is somewhat subtle, although the
idea is in fact simple. The proof of the main theorem is in the final Section 7.

2. Signed partitions and signed Young diagrams

2.1. *Partitions.* For nonnegative integers r_i, let $\boldsymbol{d} := \bigoplus_{i=1}^{k} [r_i]$ denote the partition
$[r_1, r_2, \ldots, r_k]$, where $r_1 \geq r_2 \geq \cdots \geq r_k$, and let $|\boldsymbol{d}| := \sum_{i=1}^{k} r_i$ denote the *size*
of $\boldsymbol{d}$. We may use $k[r]$ to denote $[r] \oplus [r] \oplus \cdots \oplus [r]$ (with k terms). Suppose that
$\boldsymbol{d}^1 := \bigoplus_{i=1}^{k_1} [r_i^1]$ and $\boldsymbol{d}^2 := \bigoplus_{i=1}^{k_2} [r_i^2]$ are two partitions of the same size. We write

$$\boldsymbol{d}^1 \leq \boldsymbol{d}^2 \quad \text{if} \quad \sum_{i=1}^{j} r_i^1 \leq \sum_{i=1}^{j} r_i^2 \quad \text{for each } j \geq 1.$$

This defines a partial ordering on the set of partitions of same size.

2.2. *Symplectic and orthogonal partitions.* A partition $\bigoplus_{i=1}^{k} \alpha_i [r_i]$ of pairwise
distinct r_i is *symplectic* if α_i is even whenever r_i is odd. A symplectic partition is
said to be *simple* if it is of the form $[2r]$ or $2[2r-1]$ for some positive integer r.
It is clear that a symplectic partition is a sum of simple symplectic partitions.

A partition $\bigoplus_{i=1}^{k} \alpha_i [r_i]$ is *orthogonal* if α_i is even whenever r_i is even. An
orthogonal partition is *simple* if it is of the form $[2r-1]$ or $2[2r]$ for some positive
integer r. An orthogonal partition is a sum of simple orthogonal partitions.

2.3. *Signed partitions and signed Young diagrams.* A *signed partition* is a sum
$\bigoplus_{i=1}^{k} [r_i]^{\bullet}$ of symbols $[r_i]^{\bullet}$, where r_i is a nonnegative integer and $\bullet$ is either plus
or minus.

The signed partitions $[r]^-$ and $[r]^+$ can be presented as rows of r with boxes
labeled alternatingly by plus and minus signs, ending with ⊟ and ⊞, respectively.
For example, $[2r]^-$ has the presentation

$$\underbrace{⊞\ ⊟\ ⊞\ \cdots\ ⊟\ ⊞\ ⊟}_{2r}.$$

By convention, we set $[0]^+ = [0]^- := \varnothing$. The signed partitions $[r]^+$ and $[r]^-$ are
respectively called positive and negative rows.

A presentation of a signed partition is called a *signed Young diagram* if the sizes
of the rows are nonincreasing. Two signed Young diagrams of a signed partition
are not regarded as the same if they differ by exchanging two rows of the same

length but opposite parity. For example, the signed Young diagrams

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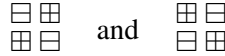
are regarded as different, although they are associated to the same signed partition $[2]^+ \oplus [2]^-$. Giving a signed Young diagram of a signed partition is equivalent to defining a total order on the set of rows of the signed partition. For example, the first signed Young diagram above gives the order $[2]^+ > [2]^-$, while the second Young diagram gives the order $[2]^- > [2]^+$. Therefore, we will use the notation (d, Q) to denote a signed Young diagram of a signed partition d, where Q is an order on the set of rows of d.

2.4. *Signature of a signed partition.* If $d := \bigoplus_{i=1}^k [r_i]^*$ is a signed partition, the *signature* of d is the ordered pair defined by

$$\mathrm{sgn}(d) := \sum_{i=1}^k \mathrm{sgn}([r_i]^*)$$

with $\mathrm{sgn}([r]^+) := (\lfloor (r+1)/2 \rfloor, \lfloor r/2 \rfloor)$ and $\mathrm{sgn}([r]^-) := (\lfloor r/2 \rfloor, \lfloor (r+1)/2 \rfloor)$, where $\lfloor a \rfloor$ stands for the largest integer not greater than a, and $(p, q) + (r, s)$ is defined to be $(p+r, q+s)$. That is, the first (respectively, second) component of the signature is the number of plus (respectively, minus) boxes in any presentation of a signed partition.

2.5. *Derivatives of signed partitions.* Define the *derivatives* of a partition or of a signed partition as follows: Let

$$[r]' := [r-1], \quad ([r]^+)' := [r-1]^+, \quad ([r]^-)' := [r-1]^-.$$

Define $d' := \bigoplus_{i=1}^k ([r_i]^*)'$ if $d = \bigoplus_{i=1}^k [r_i]^*$. Let $d^{(0)} := d$, and $d^{(k)} := (d^{(k-1)})'$ for $k \geq 1$.

For two pairs of positive integers, the order $(p, q) \leq (r, s)$ stands for $p \leq r$ and $q \leq s$. Suppose that d_1 and d_2 are two signed partitions of the same signature. We say that $d_1 \leq d_2$ if $\mathrm{sgn}(d_1^{(k)}) \leq \mathrm{sgn}(d_2^{(k)})$ for all $k \geq 0$.

2.6. *Symplectic and orthogonal signed partitions.* A signed partition d is *simple symplectic* if d is equal to $[2r]^+$, $[2r]^-$, or $[2r-1]^+ \oplus [2r-1]^-$ for some positive integer r. The signed partition $[2r-1]^+ \oplus [2r-1]^-$ is called a *double odd block*, with $[2r-1]^+$ called the *partner* of $[2r-1]^-$ and vice versa. A sum of simple symplectic signed partitions is a *symplectic signed partition*.

We say a signed partition d is *simple orthogonal* if d is $[2r-1]^+$, $[2r-1]^-$ or $[2r]^+ \oplus [2r]^-$ for some positive integer r. A sum of simple orthogonal signed partitions is an *orthogonal signed partition*.

3. Nilpotent orbits and moment maps

3.1. *Nilpotent orbits in a symplectic Lie algebra.* Let F be the field $\mathbb{C}$ or $\mathbb{R}$. Let $(V, \langle \cdot, \cdot \rangle)$ be a $2n$-dimensional symplectic space over F. The Lie algebra $\mathfrak{sp}(V)$ consists of elements $X \in \operatorname{End}_F(V)$ such that $\langle X.v, w \rangle + \langle v, X.w \rangle = 0$ for all $v, w \in V$. The Lie algebra $\mathfrak{sp}(V)$ can be realized as the space $S^2(V)$ of degree two elements in the symmetric algebra of V, with the Lie bracket given by

$$[ef, e'f'] := \langle e', e \rangle ff' + \langle f', e \rangle fe' + \langle e', f \rangle ef' + \langle f', f \rangle ee'$$

for $e, e', f, f' \in V$. The action of $S^2(V)$ on V is given by

$$(ef)(x) := \langle x, e \rangle f + \langle x, f \rangle e$$

for $e, f, x \in V$. Details can be found in [Howe 1989].

It is well known that, when $F = \mathbb{R}$, the nilpotent orbits of the symplectic group $\operatorname{Sp}(V)$ in $\mathfrak{sp}(V)$ are parametrized by the set of symplectic signed partitions of size $2n$ [Collingwood and McGovern 1993]. We can take

$$e_1 f_2 + \cdots + e_{r-1} f_r + e_r e_r$$

as representative of the nilpotent orbit corresponding to the simple signed partition $[2r]^-$, where $\{e_1, \ldots, e_r, f_1, \ldots, f_r\}$ is a self-dual basis (that is, $\langle e_i, f_j \rangle = \delta_{ij}$) of a $2r$-dimensional symplectic subspace of V. Similarly, we can take

$$e_1 f_2 + \cdots + e_{r-1} f_r - e_r e_r$$

as representative of the nilpotent orbit corresponding to $[2r]^+$, and

$$e_1 f_2 + \cdots + e_{2r-2} f_{2r-1}$$

as representative of the nilpotent orbit corresponding to $[2r-1]^+ \oplus [2r-1]^-$.

3.2. *Nilpotent orbits in an orthogonal Lie algebra.* Suppose $(W, (\cdot, \cdot))$ is an m-dimensional quadratic space over F. The Lie algebra $\mathfrak{so}(W)$ can be realized as the space $\bigwedge^2(W)$ of degree two elements in the exterior algebra of W, with the Lie bracket given by

$$[v \wedge w, v' \wedge w'] := (v, v')gw \wedge w' + (v, w')w \wedge v' + (w, v')gv \wedge w' + (w, w')v \wedge v'$$

for $v, v', w, w' \in W$. The Lie algebra $\bigwedge^2(W)$ acts on W by

$$(v \wedge w)(y) := (y, w)v - (y, v)w \quad \text{for } v, w, y \in W.$$

It is well known that, when $F = \mathbb{R}$, the nilpotent orbits of the orthogonal group $\operatorname{O}(W)$ in $\mathfrak{so}(W)$ are parametrized by the set of orthogonal signed partitions with

signature equal to that of the form $(\,\cdot\,,\cdot\,)$ [Collingwood and McGovern 1993]. We can take

$$v_1 \wedge w_2 + v_2 \wedge w_3 + \cdots + v_{r-2} \wedge w_{r-1} + v_{r-1} \wedge z$$

as representative of the nilpotent orbit corresponding to the simple orthogonal signed partition $[2r-1]^+$, where $\{z, v_1, \ldots, v_{r-1}, w_1, \ldots, w_{r-1}\}$ is a basis of a $(2r-1)$-dimensional orthogonal subspace of signature $(r, r-1)$ such that

$$(v_i, w_j) = \delta_{ij}, \quad (v_i, v_j) = (w_i, w_j) = 0, \quad (v_i, z) = (w_i, z) = 0, \quad (z, z) = 1.$$

Similarly, we can take $v_1 \wedge w_2 + v_2 \wedge w_3 + \cdots + v_{r-2} \wedge w_{r-1} + v_{r-1} \wedge z'$ as representative of the nilpotent orbit corresponding to $[2r-1]^-$, where $(z', z') = -1$, and take $v_1 \wedge w_2 + v_2 \wedge w_3 + \cdots + v_{2r-1} \wedge w_{2r}$ as representative of the nilpotent orbit corresponding to $[2r]^+ \oplus [2r]^-$.

3.3. *Derivatives and the ordering of orbits.* Let $\mathbb{O}_1$ and $\mathbb{O}_2$ be two nilpotent orbits in $\mathfrak{sp}(V)$ (or in $\mathfrak{so}(W)$), with corresponding signed partitions $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, respectively. We define $\mathbb{O}_1 \leq \mathbb{O}_2$ if $\mathbb{O}_1 \subset \overline{\mathbb{O}}_2$, the closure of $\mathbb{O}_2$.

Proposition [Djoković 1981]. *We have $\mathbb{O}_1 \leq \mathbb{O}_2$ if and only if $\mathrm{sgn}(\boldsymbol{d}_1^{(k)}) \leq \mathrm{sgn}(\boldsymbol{d}_2^{(k)})$ for every $k \geq 0$.*

3.4. *Moment maps.* Consider the vector space isomorphisms

$$\varphi_1 : V \otimes W \to \mathrm{Hom}(V, W), \quad \varphi_1(v \otimes w)(x) := \langle v, x \rangle w,$$
$$\varphi_2 : V \otimes W \to \mathrm{Hom}(W, V), \quad \varphi_2(v \otimes w)(y) := (w, y) v$$

for $v, x \in V$ and $w, y \in W$, and define the *moment maps*

$$\tau_1 : V \otimes W \to \mathrm{End}(V), \quad \tau_1(\gamma) := \varphi_2(\gamma) \circ \varphi_1(\gamma),$$
$$\tau_2 : V \otimes W \to \mathrm{End}(W), \quad \tau_2(\gamma) := \varphi_1(\gamma) \circ \varphi_2(\gamma).$$

It is known that $\tau_1(V \otimes W) \subseteq \mathfrak{sp}(V)$ and $\tau_2(V \otimes W) \subseteq \mathfrak{so}(W)$. It is not difficult to check that

$$(3\text{-}1) \qquad \tau_1(\gamma) = \sum_{i,j} (\eta_i, \eta_j) \varepsilon_i \varepsilon_j \quad \text{and} \quad \tau_2(\gamma) = \sum_{i,j} \langle \varepsilon_i, \varepsilon_j \rangle \eta_i \wedge \eta_j$$

when $\gamma := \sum_i \varepsilon_i \otimes \eta_i \in V \otimes W$ with $\varepsilon_i \in V$ and $\eta_i \in W$, under the realization $\mathfrak{sp}(V) = \mathrm{S}^2(V)$ and $\mathfrak{so}(W) = \bigwedge^2(W)$.

4. The orbit correspondences for complex reductive dual pairs

In this section, we assume that $F = \mathbb{C}$. Let V be a $2n$-dimensional complex symplectic space and W be an m-dimensional complex quadratic space. Let τ_1 and τ_2 be the moment maps from 3.4.

4.1. The purpose of Section 4 is, for a nilpotent orbit $\mathcal{O}$ in $\mathfrak{sp}(V)$, to define a nilpotent orbit $\Phi(\lambda(\mathcal{O}))$ in $\mathfrak{so}(W)$ such that the following theorem holds.

Theorem. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sp}(V)$ that is contained in the image of τ_1. Then $\tau_2(\tau_1^{-1}(\overline{\mathcal{O}})) = \overline{\Phi(\lambda(\mathcal{O}))}$.*

4.2. Let $d := \bigoplus_{i=1}^{k} \alpha_i[r_i]$ be a symplectic partition of size $2n$, with $r_1 > r_2 > \cdots > r_k > 0$. Define the numbers $r_i^1 = r_i^2 = \cdots = r_i^{\alpha_i} := r_i$ for each $i = 1, 2, \ldots, k$, so we may write

$$(4\text{-}1) \qquad\qquad d = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [r_i^j].$$

Let $\alpha := \sum_{i=1}^{k} \alpha_i$ be the number of rows of d. We define

$$(4\text{-}2) \qquad \varphi(d) := \begin{cases} \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [r_i^j - 1] \oplus (\alpha + m - 2n)[1] & \text{if } -\alpha \leq m - 2n, \\ [0] & \text{otherwise.} \end{cases}$$

It is not difficult to see that the partition $\varphi(d)$ is either $[0]$ or an orthogonal partition of size m. Define

$$\varphi(\mathcal{O}) := \begin{cases} \mathcal{O}_{\varphi(d)} & \text{if } \varphi(d) \neq [0], \\ \varnothing & \text{if } \varphi(d) = [0]; \end{cases}$$

that is, if it is nonempty, $\varphi(\mathcal{O})$ is a nilpotent orbit in $\mathfrak{so}(W)$.

4.3. We define two partitions $\psi_1(d)$ and $\psi_2(d)$ according to the following cases:

(i) Suppose that $\alpha < m - 2n$; define

$$\psi_1(d) := \bigoplus_{i=1}^{k} \alpha_i[r_i] \oplus \alpha[1] \quad \text{and} \quad \psi_2(d) := (m - 2n - \alpha)[1].$$

(ii) Suppose that $0 \leq m - 2n \leq \alpha$; define

$$\psi_1(d) := \bigoplus_{i=1}^{k} \alpha_i[r_i] \oplus (m - 2n)[1] \quad \text{and} \quad \psi_2(d) := [0].$$

(iii) Suppose that $-\alpha \leq m - 2n < 0$. Let x_0 and y_0 be the indices such that $r_{x_0}^{y_0}$ be the $(2n - m)$-th number from the smallest in the ordered sequence

$$(4\text{-}3) \quad r_1^1 \geq \cdots \geq r_1^{\alpha_1} > r_2^1 \geq \cdots \geq r_2^{\alpha_2} > \cdots > r_{k-1}^1 \geq \cdots \geq r_{k-1}^{\alpha_{k-1}} > r_k^1 \geq \cdots \geq r_k^{\alpha_k},$$

that is, choose x_0 and y_0 so that $\alpha_{x_0} - y_0 + 1 + \sum_{i=x_0+1}^{k} \alpha_i = 2n - m$. Define

$$\psi_1(d) := \bigoplus_{i=1}^{x_0-1} \alpha_i[r_i] \oplus (y_0 - 1)[r_{x_0}] \quad \text{and}$$

$$\psi_2(d) := (\alpha_{x_0} - y_0 + 1)[r_{x_0} - 1] \oplus \bigoplus_{i=x_0+1}^{k} \alpha_i[r_i - 1].$$

(iv) Suppose that $m - 2n < -\alpha$; define $\psi_1(d) = \psi_2(d) := [0]$.

Note that the definitions of $\psi_1(d)$ and $\psi_2(d)$ depend on the dimensions m and $2n$.

4.4. For $\psi_1(d)$ as given in 4.3, we define the element $\Phi^*(\psi_1(d))$. If $\psi_1(d) = [0]$, we define $\Phi^*(\psi_1(d)) := [0]$. Assume next that $\psi_1(d) \neq [0]$. Suppose that

$$\psi_1(d) = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{\beta_i} [f_i^j],$$

where the numbers f_i^j are ordered as

(4-4) $\quad f_1^1 = \cdots = f_1^{\beta_1} > f_2^1 = \cdots = f_2^{\beta_2} > \cdots$

$$> f_{l-1}^1 = \cdots = f_{l-1}^{\beta_{l-1}} > f_l^1 = \cdots = f_l^{\beta_l} > 0.$$

Let $\beta := \sum_{i=1}^{l} \beta_i$ be the number of rows of $\psi_1(d)$. We define the number s_x^y according to the following cases:

(i) Suppose that β is even, and let $f_{x_2}^{y_2} \geq f_{x_1}^{y_1}$ be the two median numbers in (4-4).

  (a) If $f_{x_1}^{y_1} = f_{x_2}^{y_2}$ are odd and y_1 is even, we define s_x^y by

$$s_x^y := \begin{cases} f_x^y + 1 & \text{if either } x = x_2 \text{ and } y < y_2, \text{ or } x < x_2; \\ f_x^y & \text{if either } x = x_1 \text{ and } y = y_1, \text{ or } x = x_2 \text{ and } y = y_2; \\ f_x^y - 1 & \text{if either } x = x_1 \text{ and } y > y_1, \text{ or } x > x_1. \end{cases}$$

  (b) Otherwise, we define

$$s_x^y := \begin{cases} f_x^y + 1 & \text{if either } x = x_2 \text{ and } y \leq y_2, \text{ or } x < x_2; \\ f_x^y - 1 & \text{if either } x = x_1 \text{ and } y \geq y_1, \text{ or } x > x_1. \end{cases}$$

(ii) Suppose that β is odd, and let $f_{x_3}^{y_3} \geq f_{x_2}^{y_2} \geq f_{x_1}^{y_1}$ be the three median numbers in (4-4).

  (a) If $f_{x_2}^{y_2} = f_{x_3}^{y_3}$ are odd and y_2 is even, we define

$$s_x^y := \begin{cases} f_x^y + 1 & \text{if either } x = x_3 \text{ and } y < y_3, \text{ or } x < x_3; \\ f_x^y & \text{if either } x = x_2 \text{ and } y = y_2, \text{ or } x = x_3 \text{ and } y = y_3; \\ f_x^y - 1 & \text{if either } x = x_1 \text{ and } y \geq y_1, \text{ or } x > x_1. \end{cases}$$

(b) Otherwise, we define

$$s_x^y := \begin{cases} f_x^y + 1 & \text{if either } x = x_3 \text{ and } y \le y_3, \text{ or } x < x_3; \\ f_x^y - 1 & \text{if either } x = x_2 \text{ and } y \ge y_2, \text{ or } x > x_2. \end{cases}$$

Then, we define

(4-5)
$$\Phi^*(\psi_1(\boldsymbol{d})) := \begin{cases} \bigoplus_{i=1}^l \bigoplus_{j=1}^{\beta_i} [s_i^j] & \text{for case (i.a) and (i.b)}, \\ \bigoplus_{i=1}^l \bigoplus_{j=1}^{\beta_i} [s_i^j] \oplus [1] & \text{for case (ii.a) and (ii.b)}; \end{cases}$$
$$\Phi(\boldsymbol{d}) := \Phi^*(\psi_1(\boldsymbol{d})) \oplus \psi_2(\boldsymbol{d}).$$

The partition $\Phi(\boldsymbol{d})$ is either $[0]$ or an orthogonal partition of size m. Define

$$\Phi(\mathbb{O}) := \begin{cases} \mathbb{O}_{\Phi(\boldsymbol{d})} & \text{if } \Phi(\boldsymbol{d}) \ne [0], \\ \varnothing & \text{if } \Phi(\boldsymbol{d}) = [0], \end{cases}$$

that is, $\Phi(\mathbb{O})$ is a nilpotent orbit in $\mathfrak{so}(W)$ if it is nonempty.

4.5. Example. Suppose that $2n = 24$, $m = 25$, and $\boldsymbol{d} = [6] \oplus 2[5] \oplus [4] \oplus 2[2]$; that is, $r_1^1 = 6$, $r_2^1 = 5$, $r_2^2 = 5$, $r_3^1 = 4$, $r_4^1 = 2$, and $r_4^2 = 2$. We have $\alpha = 1 + 2 + 1 + 2 = 6$ and $m - 2n = 25 - 24 = 1$, so we are in case (ii) of 4.3. Thus, $\psi_2(\boldsymbol{d}) = [0]$ and $\psi_1(\boldsymbol{d}) = [6] \oplus 2[5] \oplus [4] \oplus 2[2] \oplus [1]$. Now, $\alpha + (m - 2n) = 7$ is odd, and the three median numbers of sequence Equation (4-4) are $5 > 4 > 2$. We are in case (ii.b) of 4.4 and, hence, $s_1^1 = 7$, $s_2^1 = 6$, $s_2^2 = 6$, $s_3^1 = 3$, $s_4^1 = 1$, $s_4^2 = 1$, and $s_5^1 = 0$. Thus, $\Phi(\boldsymbol{d}) = \Phi^*(\psi_1(\boldsymbol{d})) = [7] \oplus 2[6] \oplus [3] \oplus 3[1]$ and $\varphi(\boldsymbol{d}) = [5] \oplus 2[4] \oplus [3] \oplus 9[1]$.

4.6. Proposition. *If $\mathbb{O}$ is a nilpotent orbit contained in the image of τ_1, then $\Phi(\mathbb{O})$ is the unique maximal element and $\varphi(\mathbb{O})$ is the unique minimal element in the set of nilpotent orbits in $\tau_2(\tau_1^{-1}(\mathbb{O}))$.*

4.7. Suppose that we are not in the case (iv) of 4.3. We want to define a new partition $\lambda(\boldsymbol{d})$ of size $2n$. If $m - 2n \ge \alpha$, we define $\lambda(\boldsymbol{d}) := \boldsymbol{d}$. Next, we suppose that $m - 2n < \alpha$. Define a row $[r_{x_0}^{y_0}]$ as follows: Let $[r_{x_0}^{y_0}]$ be the smallest element in the ordered sequence (4-3) satisfying the condition

(4-6)
$$\sum_{i=x_0+1}^k \alpha_i r_i + (\alpha_{x_0} - y_0 + 1) r_{x_0} + (m - 2n) \ge y_0 - 1 + \sum_{i=1}^{x_0-1} \alpha_i,$$

where x_0 is some number between 1 and k, and y_0 is between 1 and α_{x_0}. Define

$$\sigma := \sum_{i=x_0+1}^k \alpha_i r_i + (\alpha_{x_0} - y_0 + 1) r_{x_0} + (m - 2n) - \left(y_0 - 1 + \sum_{i=1}^{x_0-1} \alpha_i \right),$$

$$\delta := y_0 - 1 + \sum_{i=1}^{x_0-1} \alpha_i - (m - 2n).$$

It is clear that $0 \leq \sigma < r_{x_0}^{y_0}$ and $0 \leq \delta$. We define $\lambda(\boldsymbol{d})$ according to the following cases:

(i) If $r_{x_0}^{y_0}$ is even or y_0 is odd, we define

$$\lambda(\boldsymbol{d}) := \boldsymbol{e} \oplus \begin{cases} [\sigma - 1] \oplus (\delta + 1)[1], & \text{if } \sigma \text{ is odd,} \\ \delta[1] & \text{if } \sigma = 0, \\ [\sigma - 2] \oplus (\delta + 2)[1] & \text{if } \sigma \text{ is positive and even,} \end{cases}$$

where

$$\boldsymbol{e} := \bigoplus_{i=1}^{x_0-1} \bigoplus_{j=1}^{\alpha_i} [r_i^j] \oplus \bigoplus_{j=1}^{y_0-1} [r_{x_0}^j].$$

(ii) If $r_{x_0}^{y_0}$ is odd and y_0 is even, we define

$$\lambda(\boldsymbol{d}) := \boldsymbol{e}' \oplus \begin{cases} [\sigma - 1] \oplus (\delta + 2)[1], & \text{if } \sigma \text{ is odd,} \\ [\sigma] \oplus (\delta + 1)[1], & \text{if } \sigma \text{ is even,} \end{cases}$$

where

$$\boldsymbol{e}' := \bigoplus_{i=1}^{x_0-1} \bigoplus_{j=1}^{\alpha_i} [r_i^j] \oplus \bigoplus_{j=1}^{y_0-2} [r_{x_0}^j] \oplus [r_{x_0}^{y_0-1} - 1].$$

It is not difficult to see that $\lambda(\boldsymbol{d})$ is a symplectic partition of size $2n$ and $\lambda(\boldsymbol{d}) \leq \boldsymbol{d}$. Finally, we define

$$\lambda(\mathbb{O}_{\boldsymbol{d}}) := \mathbb{O}_{\lambda(\boldsymbol{d})}.$$

4.8. *Example.* Suppose that $2n = m = 60$ and $\boldsymbol{d} = 6[8] \oplus 2[6]$. Then, $m - 2n = 0$, $\alpha = 6 + 2 = 8$, $\psi_2(\boldsymbol{d}) = [0]$ and $\psi_1(\boldsymbol{d}) = \boldsymbol{d}$. Hence,

$$\Phi(\boldsymbol{d}) = \Phi^*(\psi_1(\boldsymbol{d})) = 4[9] \oplus 2[7] \oplus 2[5].$$

Now, $r_1^1 = r_1^2 = \cdots = r_1^6 = 8$, $r_2^1 = r_2^2 = 6$, $\alpha_1 = 6$, $\alpha_2 = 2$ and $k = 2$. Therefore, $r_{x_0}^{y_0} = r_2^1$, that is, $x_0 = 2$ and $y_0 = 1$. Hence, $\sigma = 0 + (2-1+1) \cdot 6 + 0 - (1-1+6) = 6$, which is even, and $\delta = (1-1+6) - 0 = 6$. Then, $\lambda(\boldsymbol{d}) = 6[8] \oplus [4] \oplus 8[1]$. The number of rows for $\lambda(\boldsymbol{d})$ is $6 + 1 + 8 = 15$. Hence, $\psi_1(\lambda(\boldsymbol{d})) = \lambda(\boldsymbol{d})$ and $\psi_2(\lambda(\boldsymbol{d})) = [0]$. Write $\psi_1(\lambda(\boldsymbol{d})) = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{\beta_i} [f_i^j]$, where $f_1^1 = f_1^2 = \cdots = f_1^6 = 8$, $f_2^1 = 4$, and $f_3^1 = f_3^2 = \cdots = f_3^8 = 1$. The number of rows of $\psi_1(\lambda(\boldsymbol{d}))$ is odd. So now we are in the case (ii.b) of 4.4 and, therefore, $\Phi(\lambda(\boldsymbol{d})) = \Phi^*(\psi_1(\lambda(\boldsymbol{d}))) = 6[9] \oplus [5] \oplus [1]$. It is clear that $\lambda(\boldsymbol{d}) < \boldsymbol{d}$, but $\Phi(\lambda(\boldsymbol{d})) > \Phi(\boldsymbol{d})$.

4.9. Proposition. *Let $\mathbb{O}$ be a nilpotent orbit contained in the image of τ_1. Then, $\Phi(\lambda(\mathbb{O}))$ is the unique maximal element in the set $\{\Phi(\mathbb{Q}) \mid \mathbb{Q} \leq \mathbb{O}\}$.*

5. The rules for simple signed partitions

From now on to the end of this paper we assume that $F = \mathbb{R}$. Let τ_1 and τ_2 be the maps defined in 3.4.

5.1. Suppose $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{sp}(V)$ that is in the image of τ_1, that $\boldsymbol{d}$ is the signed partition corresponding to $\mathcal{O}$, and that γ is an element in $V \otimes W$ such that $\tau_1(\gamma)$ is in $\mathcal{O}$. Assume $V = V^1 \oplus \cdots \oplus V^k$ is an orthogonal decomposition corresponding to the Jordan blocks of the decomposition $\boldsymbol{d} = \bigoplus_{i=1}^k \boldsymbol{d}^i$, such that each $\boldsymbol{d}^i$ is a simple symplectic signed partition, as introduced in 2.6. Now, γ can be written as $(\gamma^1, \ldots, \gamma^k)$, where $\gamma^i \in V^i \otimes W$, and $\tau_1(\gamma) \in \bigoplus_{i=1}^k \mathfrak{sp}(V^i) \subseteq \mathfrak{sp}(V)$. Write $\tau_1(\gamma) = (\pi^1(\tau_1(\gamma)), \ldots, \pi^k(\tau_1(\gamma)))$ with $\pi^i(\tau_1(\gamma)) \in \mathfrak{sp}(V^i)$ for each i, and assume that

$$\pi^i(\tau_1(\gamma)) = e_1^i f_2^i + \cdots + e_{r_i-1}^i f_{r_i}^i + e_{r_i}^i e_{r_i}^i;$$

that is, $\pi^i(\tau_1(\gamma))$ is an element of the nilpotent orbit in $\mathfrak{sp}(V^i)$ corresponding to $[2r_i]^-$, as in 3.1. Then, there is a self-dual set $S = \{v_l^i, w_l^i, z^i \mid l = 1, \ldots, r_i - 1\}$ and an isotropic vector ω^i (possibly trivial) orthogonal to S, such that γ^i is of the form

$$e_1^i \otimes v_1^i + e_2^i \otimes v_2^i + \cdots + e_{r_i-1}^i \otimes v_{r_i-1}^i + e_{r_i}^i \otimes z^i$$
$$+ f_1^i \otimes \omega^i + f_2^i \otimes w_1^i + \cdots + f_{r_i-1}^i \otimes w_{r_i-2}^i + f_{r_i}^i \otimes w_{r_i-1}^i.$$

Therefore, by Equation (3-1), $\tau_2(\gamma)$ has a block of the form

$$v_1^i \wedge \omega^i + v_2^i \wedge w_1^i + \cdots + v_{r_i-1}^i \wedge w_{r_i-2}^i + z^i \wedge w_{r_i-1}^i.$$

This block corresponds to $[2r_i + 1]^+$ or $[2r_i - 1]^+$, depending on whether the vector ω^i can be nonzero or not.

5.2. Assume that $\pi^i(\tau_1(\gamma)) = e_1^i f_2^i + \cdots + e_{r_i-1}^i f_{r_i}^i - e_{r_i}^i e_{r_i}^i$; that is, $\pi^i(\tau_1(\gamma))$ is an element in the nilpotent orbit in $\mathfrak{sp}(V^i)$ corresponding to $[2r_i]^+$. Then, there is a set $S = \{v_l^i, w_l^i, z'^i \mid l = 1, \ldots, r_i - 1\}$ such that

$$(v_r^i, w_s^i) = \delta_{rs}, \quad (v_r^i, v_s^i) = (w_r^i, w_s^i) = (v_r^i, z'^i) = (w_r^i, z'^i) = 0, \quad (z'^i, z'^i) = -1$$

for any $r, s = 1, \ldots, l$ and an isotropic vector ω^i orthogonal to S, such that γ^i is of the form

$$e_1^i \otimes v_1^i + e_2^i \otimes v_2^i + \cdots + e_{r_i-1}^i \otimes v_{r_i-1}^i + e_{r_i}^i \otimes z'^i$$
$$+ f_1^i \otimes \omega^i + f_2^i \otimes w_1^i + \cdots + f_{r_i-1}^i \otimes w_{r_i-2}^i + f_{r_i}^i \otimes w_{r_i-1}^i.$$

Therefore, $\tau_2(\gamma)$ has a block of the form

$$v_1^i \wedge \omega^i + v_2^i \wedge w_1^i + \cdots + v_{r_i-1}^i \wedge w_{r_i-2}^i + z'^i \wedge w_{r_i-1}^i.$$

This block corresponds to $[2r_i + 1]^-$ or $[2r_i - 1]^-$, depending on whether the vector ω^i can be nonzero or not.

5.3. Assume now that

$$\pi^i(\tau_1(\gamma)) = e_1^i f_2^i + \cdots + e_{2r_i-2}^i f_{2r_i-1}^i;$$

that is, $\pi^i(\tau_1(\gamma))$ is an element in the nilpotent orbit in $\mathfrak{sp}(V^i)$ corresponding to $[2r_i - 1]^+ \oplus [2r_i - 1]^-$. There is a self-dual set $S = \{v_l^i, w_l^i \mid l = 1, \ldots, 2r_i - 2\}$ and two isotropic vectors ω_0^i and $\omega_{2r_i-1}^i$, orthogonal to each other and orthogonal to S, such that γ^i is of the form

$$e_1^i \otimes v_1^i + e_2^i \otimes v_2^i + \cdots + e_{2r^i-2}^i \otimes v_{2r_i-2}^i + e_{2r_i-1}^i \otimes \omega_{2r_i-1}^i$$
$$+ f_1^i \otimes \omega_0^i + f_2^i \otimes w_1^i + \cdots + f_{2r_i-2}^i \otimes w_{2r_i-3}^i + f_{2r_i-1}^i \otimes w_{2r_i-2}^i.$$

Therefore, $\tau_2(\gamma)$ has a block of the form

$$v_1^i \wedge \omega_0^i + v_2^i \wedge w_1^i + \cdots + v_{2r_i-2}^i \wedge w_{2r_i-3}^i + \omega_{2r_i-1}^i \wedge w_{2r_i-2}^i.$$

This block corresponds to

$$\begin{cases} [2r_i]^+ \oplus [2r_i]^- & \text{if both } \omega_0^i \text{ and } \omega_{2r_i-1}^i \text{ are allowed to be nonzero;} \\ [2r_i - 1]^+ \oplus [2r_i - 1]^- & \text{if only one of } \omega_0^i \text{ and } \omega_{2r_i-1}^i \text{ is allowed to be nonzero;} \\ [2r_i - 2]^+ \oplus [2r_i - 2]^- & \text{if both } \omega_0^i \text{ and } \omega_{2r_i-1}^i \text{ have to be zero.} \end{cases}$$

5.4. *Example.* Suppose that $2n = 8$, $p = 4$, $q = 4$ and $d = [4]^+ \oplus [2]^- \oplus [1]^+ \oplus [1]^-$. Let γ be an element in $V \otimes W$ such that $\tau_1(\gamma)$ is in the nilpotent orbit corresponding to d. Now, $d = d^1 \oplus d^2 \oplus d^3$, where $d^1 := [4]^+$, $d^2 := [2]^-$ and $d^3 := [1]^+ \oplus [1]^-$. We have the corresponding decomposition $V^1 \oplus V^2 \oplus V^3$ of V, where V^1 is 4-dimensional, and V^2 and V^3 are 2-dimensional. Therefore, a nilpotent orbit $\mathbb{O}_{d'}$ in $\mathfrak{so}(W)$ is in $\tau_2(\tau_1^{-1}(\mathbb{O}_d))$ if and only if

$$d' = d_1' \oplus d_2' \oplus d_3' \oplus \tfrac{1}{2}(8 - |d_1'| - |d_2'| - |d_2'|)([1]^+ \oplus [1]^-),$$

such that d_1' is $[5]^-$ or $[3]^-$, d_2' is $[3]^+$ or $[1]^+$, d_3' is $[2]^+ \oplus [2]^-$, $[1]^+ \oplus [1]^-$ or $\varnothing$, and $\sum_{i=1}^3 |d_i'| \leq 8$. The only possible combinations for d' are

$$e_1 := [5]^- \oplus [3]^+, \qquad\qquad e_4 := [3]^- \oplus [1]^+ \oplus [2]^+ \oplus [2]^-,$$
$$e_2 := [5]^- \oplus [1]^+ \oplus [1]^+ \oplus [1]^-, \quad e_5 := [3]^- \oplus [1]^+ \oplus [1]^+ \oplus [1]^- \oplus [1]^+ \oplus [1]^-,$$
$$e_3 := [3]^- \oplus [3]^+ \oplus [1]^+ \oplus [1]^-.$$

Hence,

$$\tau_2(\tau_1^{-1}(\mathbb{O}_d)) = \mathbb{O}_{e_1} \cup \mathbb{O}_{e_2} \cup \mathbb{O}_{e_3} \cup \mathbb{O}_{e_4} \cup \mathbb{O}_{e_5}.$$

6. The construction for real cases

In this section, we assume that $F = \mathbb{R}$, that V is a $2n$-dimensional real symplectic space and that W is an m-dimensional real quadratic space with signature (p, q). Let τ_1 and τ_2 be the moment maps given in 3.4.

6.1. Our purpose is to define an orthogonal signed partition $\Phi(\lambda(d, Q))$ for an admissible signed Young diagram (d, Q) (see 2.3), such that the following main theorem holds:

Theorem. *Let $\mathbb{O}_d$ be a nilpotent orbit in $\mathfrak{sp}(V)$ contained in the image of τ_1. Then*

$$\tau_2(\tau_1^{-1}(\overline{\mathbb{O}_d})) = \bigcup_{(d,Q)} \overline{\mathbb{O}_{\Phi(\lambda(d,Q))}},$$

where the union runs over all possible admissible signed Young diagrams of the signed partition d. The union consists of at most two closures of nilpotent orbits.

The theorem will be proved in 7.8.

6.2. Let

$$(6\text{-}1) \qquad d := \bigoplus_{i=1}^{k} \alpha_i [a_i]^+ \oplus \bigoplus_{j=1}^{l} \beta_j [b_j]^-$$

be a symplectic signed partition of sized $2n$, where $a_1 > a_2 > \cdots > a_k > 0$ and $b_1 > b_2 > \cdots > b_l > 0$. We can rewrite d as

$$(6\text{-}2) \qquad d = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [a_i^j]^+ \oplus \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{\beta_i} [b_i^j]^-$$

such that the numbers a_i^j and b_i^j satisfy the conditions

$$a_1^1 = \cdots = a_1^{\alpha_1} > a_2^1 = \cdots = a_2^{\alpha_2} > \cdots > a_k^1 = \cdots = a_k^{\alpha_k},$$
$$b_1^1 = \cdots = b_1^{\beta_1} > b_2^1 = \cdots = b_2^{\beta_2} > \cdots > b_l^1 = \cdots = b_l^{\beta_l},$$

where $a_i^1 = \cdots = a_i^{\alpha_i} = a_i$ and $b_j^1 = \cdots = b_j^{\beta_j} = b_j$ for each i and j. To simplify the notation, we define an order on the set $\{[a_x^y]^+ \mid x, y\}$, as follows. We say that $[a_{x_1}^{y_1}]^+ > [a_{x_2}^{y_2}]^+$ if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. It is a total order on the set $\{[a_x^y]^+ \mid x, y\}$; that is, we have the sequence

$$(6\text{-}3) \quad [a_1^1]^+ > \cdots > [a_1^{\alpha_1}]^+ > [a_2^1]^+ > \cdots > [a_2^{\alpha_2}]^+ > \cdots$$
$$> [a_{k-1}^1]^+ > \cdots > [a_{k-1}^{\alpha_{k-1}}]^+ > [a_k^1]^+ > \cdots > [a_k^{\alpha_k}]^+.$$

We define a total order on the set $\{[b_x^y]^- \mid x, y\}$ analogously.

6.3. *The ten cases.* We want to define two signed partitions $\psi_1(\boldsymbol{d})$ and $\psi_2(\boldsymbol{d})$ from the symplectic signed partition $\boldsymbol{d}$. The definition depends on the signature (p, q) of W. Without loss of generality we assume that $p \geq q$. Let $\alpha := \sum_{i=1}^{k} \alpha_i$, $\beta := \sum_{j=1}^{l} \beta_j$, $d_1 := \min\{p - n, \beta\}$ and $d_2 := \min\{q - n, \alpha\}$. We define $\psi_1(\boldsymbol{d})$ and $\psi_2(\boldsymbol{d})$ according to the following cases:

(i) Suppose $q \geq n$ and $\beta - \alpha \geq p - n$. We have $p \geq n$ and $\beta > \beta - \alpha - (p - n) \geq 0$. Therefore, there exists a unique number ν between 0 and l such that $\sum_{i=\nu}^{l} \beta_i > \beta - \alpha - (p - n) \geq \sum_{i=\nu+1}^{l} \beta_i$. Let $\beta' := \sum_{i=\nu}^{l} \beta_i - (\beta - \alpha - (p - n))$, and define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [a_i^j]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- \geq [b_\nu^{\beta'}]^-} [b_x^y]^- \right) \oplus (p - n)[1]^+,$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[b_x^y]^- < [b_\nu^{\beta'}]^-} [b_x^y - 1]^+ \right) \oplus (\beta - \alpha + q - p)[1]^-.$$

(ii) Suppose that $q \geq n$ and $p - n > \beta - \alpha \geq p - q$. It is clear that both d_1 and $p - n - d_1$ are nonnegative. Now, $\alpha - \beta + d_1$ is equal to α or $\alpha - \beta + p - n$, and hence $\alpha - \beta + d_1$ is also nonnegative. Now, $\beta - \alpha + q - n - d_1 \geq p - n - d_1$. So, $\beta - \alpha + q - n - d_1$ is nonnegative. Define

$$\psi_1(\boldsymbol{d}) := \boldsymbol{d} \oplus d_1[1]^+ \oplus (\alpha - \beta + d_1)[1]^-,$$

$$\psi_2(\boldsymbol{d}) := (p - n - d_1)[1]^+ \oplus (\beta - \alpha + q - n - d_1)[1]^-.$$

(iii) Suppose that $q \geq n$ and $p - q > \beta - \alpha \geq -(q - n)$. Clearly both d_2 and $q - n - d_2$ are nonnegative. Now, $\beta - \alpha + d_2$ is equal to β or $\beta - \alpha + (q - n)$. Hence, $\beta - \alpha + d_2$ is nonnegative. Next, $\alpha - \beta + p - n - d_2 > q - n - d_2$. So, $\alpha - \beta + p - n - d_2$ is also nonnegative. Define

$$\psi_1(\boldsymbol{d}) := \boldsymbol{d} \oplus (\beta - \alpha + d_2)[1]^+ \oplus d_2[1]^-,$$

$$\psi_2(\boldsymbol{d}) := (\alpha - \beta + p - n - d_2)[1]^+ \oplus (q - n - d_2)[1]^-.$$

(iv) Suppose that $q \geq n$ and $-(q - n) > \beta - \alpha$. Since $\alpha \geq \alpha - \beta - (q - n) > 0$, there is a unique number μ between 0 and k such that $\sum_{i=\mu}^{k} \alpha_i > \alpha - \beta - (q - n) \geq \sum_{i=\mu+1}^{k} \alpha_i$. Let $\alpha' := \sum_{i=\mu}^{k} \alpha_i - (\alpha - \beta - (q - n))$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ \geq [a_\mu^{\alpha'}]^+} [a_x^y]^+ \right) \oplus \left(\bigoplus_{i=1}^{l} \bigoplus_{j=1}^{\beta_i} [b_i^j]^- \right) \oplus (q - n)[1]^-,$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ < [a_\mu^{\alpha'}]^+} [a_x^y - 1]^- \right) \oplus (\alpha - \beta + p - q)[1]^+.$$

(v) Suppose that $p \geq n > q \geq n - \beta$ and $\beta - \alpha \geq p - q$. We have $\beta \geq \beta - \alpha - (p-n) \geq n - q > 0$. Let ν be the number such that

$$\sum_{i=\nu}^{l} \beta_i > \beta - \alpha - (p-n) \geq \sum_{i=\nu+1}^{l} \beta_i,$$

and let $\beta' := \sum_{i=\nu}^{l} \beta_i - (\beta - \alpha - (p-n))$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [a_i^j]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- \geq [b_\nu^{\beta'}]^-} [b_x^y]^- \right) \oplus (p-n)[1]^+,$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[b_x^y]^- < [b_\nu^{\beta'}]^-} [b_x^y - 1]^+ \right) \oplus (\beta - \alpha + q - p)[1]^-.$$

(vi) Suppose that $p \geq n > q \geq n - \beta$ and $p - q > \beta - \alpha \geq n - q$. Now, both $\beta - \alpha + q - n$ and $p - q - \beta + \alpha$ are nonnegative. Since $\beta \geq n - q > 0$, there is a unique number ν' such that $\sum_{i=\nu'}^{l} \beta_i > n - q \geq \sum_{i=\nu'+1}^{l} \beta_i$. Let $\beta'' := \sum_{i=\nu'}^{l} \beta_i - (n - q)$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\alpha_i} [a_i^j]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- \geq [b_{\nu'}^{\beta''}]^-} [b_x^y]^- \right) \oplus (\beta - \alpha + q - n)[1]^+,$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[b_x^y]^- < [b_{\nu'}^{\beta''}]^-} [b_x^y - 1]^+ \right) \oplus (\alpha - \beta + p - q)[1]^+.$$

(vii) Suppose that $p \geq n > q \geq n - \beta$ and $n - q > \beta - \alpha$. Hence, $\alpha \geq \alpha - \beta - q + n > 0$ and $\beta \geq n - q > 0$. Let μ and ν' be the numbers such that

$$\sum_{i=\mu}^{k} \alpha_i > \alpha - \beta - (q - n) \geq \sum_{i=\mu+1}^{k} \alpha_i \quad \text{and} \quad \sum_{i=\nu'}^{l} \beta_i > n - q \geq \sum_{i=\nu'+1}^{l} \beta_i.$$

Let $\alpha' := \sum_{i=1}^{\mu} \alpha_i - (\alpha - \beta - (q-n))$ and $\beta'' := \sum_{i=1}^{\nu'} \beta_i - (n-q)$. Now, $\alpha - \beta + p - q \geq \alpha - \beta + n - q \geq 0$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ \geq [a_\mu^{\alpha'}]^+} [a_x^y]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- \geq [b_{\nu'}^{\beta''}]^-} [b_x^y]^- \right),$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ < [a_\mu^{\alpha'}]^+} [a_x^y - 1]^- \right) \oplus \left(\bigoplus_{[b_x^y]^- < [b_{\nu'}^{\beta''}]^-} [b_x^y - 1]^+ \right) \oplus (\alpha - \beta + p - q)[1]^+.$$

(viii) Suppose that $n > p \geq n - \alpha$, $n > q \geq n - \beta$ and $\beta - \alpha \geq p - q$. Hence, $\alpha \geq n - p > 0$ and $\beta \geq \beta - \alpha + n - p > 0$. Let μ' and ν be the numbers such that

$$\sum_{i=\mu'}^{k} \alpha_i > n - p \geq \sum_{i=\mu'+1}^{k} \alpha_i \quad \text{and} \quad \sum_{i=\nu}^{l} \beta_i > \beta - \alpha - (p-n) \geq \sum_{i=\nu+1}^{l} \beta_i.$$

Let $\alpha'' := \sum_{i=1}^{\mu} \alpha_i - (n - p)$ and $\beta' := \sum_{i=v}^{l} \beta_i - (\beta - \alpha - (p - n))$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ > [a_{\mu}^{\alpha''}]^+} [a_x^y]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- > [b_v^{\beta'}]^-} [b_x^y]^- \right),$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ \leq [a_{\mu'}^{\alpha''}]^+} [a_x^y - 1]^- \right) \oplus \left(\bigoplus_{[b_x^y]^- \leq [b_v^{\beta'}]^-} [b_x^y - 1]^+ \right) \oplus (\beta - \alpha + q - p)[1]^-.$$

(ix) Suppose that $n > p \geq n - \alpha$, $n > q \geq n - \beta$ and $p - q > \beta - \alpha$. Hence, $\alpha \geq \alpha - \beta + n - q > 0$ and $\beta \geq n - q > 0$. Let μ and v' be the numbers such that $\sum_{i=\mu}^{k} \alpha_i > \alpha - \beta - (q - n) \geq \sum_{i=\mu+1}^{k} \alpha_i$ and $\sum_{i=v'}^{l} \beta_i > n - q \geq \sum_{i=v'+1}^{l} \beta_i$. Let $\alpha' := \sum_{i=\mu}^{k} \alpha_i - (\alpha - \beta - (q - n))$ and $\beta'' := \sum_{i=v}^{l} \beta_i - (n - q)$. Define

$$\psi_1(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ \geq [a_{\mu}^{\alpha'}]^+} [a_x^y]^+ \right) \oplus \left(\bigoplus_{[b_x^y]^- \geq [b_{v'}^{\beta''}]^-} [b_x^y]^- \right),$$

$$\psi_2(\boldsymbol{d}) := \left(\bigoplus_{[a_x^y]^+ < [a_{\mu}^{\alpha'}]^+} [a_x^y - 1]^- \right) \oplus \left(\bigoplus_{[b_x^y]^- < [b_{v'}^{\beta''}]^-} [b_x^y - 1]^+ \right) \oplus (\alpha - \beta + p - q)[1]^+.$$

(x) Suppose that $p < n - \alpha$, or $q < n - \beta$. In this case we define

$$\psi_1(\boldsymbol{d}) = \psi_2(\boldsymbol{d}) := \varnothing.$$

6.4. Lemma. *Let $\psi_1(\boldsymbol{d})$ be defined as in 6.3. If we write*

$$\psi_1(\boldsymbol{d}) := \bigoplus_{i=1}^{k'} \bigoplus_{j=1}^{\alpha_i'} [a_i'^j]^+ \oplus \bigoplus_{i=1}^{l'} \bigoplus_{j=1}^{\beta_i'} [b_i'^j]^-,$$

then $\sum_{i=1}^{k'} \alpha_i' = \sum_{i=1}^{l'} \beta_i'$; that is, the number of positive rows of $\psi_1(\boldsymbol{d})$ is equal to the number of negative rows.

Proof. For case (i) in 6.3, we have

$$\sum_{i=1}^{k'} \alpha_i' = \sum_{i=1}^{k} \alpha_i + p - n = \alpha + p - n,$$
$$\sum_{i=1}^{l'} \beta_i' = \sum_{i=1}^{v} \beta_i + \beta' = \sum_{i=1}^{v} \beta_i + \sum_{i=1}^{v} \beta_i - (\beta - \alpha - (p - n)) = \alpha - (p - n).$$

Hence, $\sum_{i=1}^{k'} \alpha_i' = \sum_{i=1}^{l'} \beta_i'$ for this case.

Next, we consider case (ii). We have

$$\sum_{i=1}^{k'} \alpha_i' = \sum_{i=1}^{k} \alpha_i + d_1 = \alpha + d_1$$
$$\sum_{i=1}^{l'} \beta_i' = \sum_{i=1}^{l} \beta_i + d_1 - (\beta - \alpha) = \alpha + d_1.$$

Hence, $\sum_{i=1}^{k'} \alpha_i' = \sum_{i=1}^{l'} \beta_i'$. The proofs for the other cases are similar, and we skip them. □

From this lemma we know that the number of rows of $\psi_1(\boldsymbol{d})$ is even.

6.5. Lemma. *Let $[r_{x_1}^{y_1}]^* < [r_{x_2}^{y_2}]^*$ be the two median elements with respect to an order on the set of rows of $\psi_1(\boldsymbol{d})$. If $[r_x^{y-1}]^* \oplus [r_x^y]^*$ is a double odd block of $\boldsymbol{d}$ such that $[r_x^y]^*$ is in $\psi_1(\boldsymbol{d})$ but $[r_x^{y-1}]^*$ is not in $\psi_1(\boldsymbol{d})$, then $[r_x^y]^* \leq [r_{x_1}^{y_1}]^*$ with respect to the same order.*

Proof. For case (i) in 6.3, $[r_x^y]^*$ is $[r_x^y]^+$; that is, it ends with ⊞. If $[r_z^w]^* < [r_x^y]^+$, then $[r_z^w]^*$ also ends with ⊞. The lemma follows from 6.4, because no more than half the number of rows end with ⊞. For cases (ii) and (iii), the lemma is true because no such $[r_x^y]^*$ exist. For case (iv), $[r_x^y]^*$ is $[r_x^y]^-$; that is, it ends with ⊟. If $[r_z^w]^* < [r_x^y]^-$, then $[r_z^w]^*$ also ends with ⊟. This lemma holds again, by 6.4. The proofs for the other cases are similar, and we omit them. □

6.6. *Minimal and maximal elements.* We set $\varphi(\boldsymbol{d}) := [0]$ for case (x) in 6.3, and

$$(6\text{-}4) \quad \varphi(\boldsymbol{d}) := \left(\bigoplus_{i=1}^k \alpha_i [a_i - 1]^- \right) \oplus \left(\bigoplus_{j=1}^l \beta_j [b_j - 1]^+ \right)$$
$$\oplus (\alpha + p - n)[1]^+ \oplus (\beta + q - n)[1]^-$$

for the other cases.

Suppose that $\psi_1(\boldsymbol{d}) \neq [0]$. Let $(\psi_1(\boldsymbol{d}), Q)$ be a signed Young diagram of $\psi_1(\boldsymbol{d})$. Then, as in Equation (4-5), we can define the element $\Phi^*(\psi_1(\boldsymbol{d}), Q)$ as follows: Assume that $\psi_1(\boldsymbol{d})$ is written as $\bigoplus_{i=1}^{\delta} \bigoplus_{j=1}^{\rho_i} [f_i^j]^*$. Let s_x^y be defined as in 4.4. If $s_x^y = f_x^y$, let the sign of $[s_x^y]^*$ be the same as that of $[f_x^y]^*$; if $s_x^y \neq f_x^y$, let the sign of $[s_x^y]^*$ be opposite to the sign of $[f_x^y]^*$. Finally, define

$$\Phi^*(\psi_1(\boldsymbol{d}), Q) := \bigoplus_{i=1}^{\delta} \bigoplus_{j=1}^{\rho_i} [s_i^j]^*$$

$$\Phi(\boldsymbol{d}, Q) := \Phi^*(\psi_1(\boldsymbol{d}), Q) \oplus \psi_2(\boldsymbol{d}).$$

Proposition. *If $\Phi(\boldsymbol{d}, Q)$ and $\varphi(\boldsymbol{d})$ are not $[0]$, then they are signed orthogonal partitions of signature (p, q).*

Proof. From Equation (6-4), it is clear that the signature of $\varphi(\boldsymbol{d})$ is (p, q). We know that $\boldsymbol{d}$ is a finite sum of simple even blocks or double odd blocks. Hence, it is clear that $\varphi(\boldsymbol{d})$ is a finite sum of simple odd blocks or double even blocks. Therefore, $\varphi(\boldsymbol{d})$ is an orthogonal signed partition.

Next, we consider $\Phi(\boldsymbol{d}, Q)$. The meaning of 6.5 is that the image of two rows of a double block still have the same length after applying Φ^*. Hence, from its definition, it is not difficult to see that $\Phi(\boldsymbol{d}, Q)$ is an orthogonal signed partition. If we check the definitions of $\psi_1(\boldsymbol{d})$ and $\psi_2(\boldsymbol{d})$, we see that the signature of $\psi_1(\boldsymbol{d}) \oplus \psi_2(\boldsymbol{d})$ is (p, q). Clearly, the signature of $\psi_1(\boldsymbol{d})$ is the same as the signature of $\Phi^*(\psi_1(\boldsymbol{d}), Q)$ for any order Q. Hence, $\Phi(\boldsymbol{d}, Q)$ has signature (p, q). □

Let $\mathbb{O}_d$ be the nilpotent orbit in $\mathfrak{sp}(V)$ corresponding to d. From the previous proposition, we know that each of $\varphi(d)$ and $\Phi(d, Q)$ correspond to a nilpotent orbit in $\mathfrak{so}(W)$. We define

$$\varphi(\mathbb{O}_d) := \mathbb{O}_{\varphi(d)} \quad \text{and} \quad \Phi(\mathbb{O}_d) := \bigcup_Q \mathbb{O}_{\Phi(d, Q)},$$

where the union $\bigcup_Q \mathbb{O}_{\Phi(d, Q)}$ is running over all possible signed Young diagrams $(\psi_1(d), Q)$ of the signed partition $\psi_1(d)$. Of course, it is a finite union.

6.7. *Example.* Suppose that $2n = 36$, $p = 21$, $q = 19$ and $d = 2[4]^+ \oplus [3]^+ \oplus 5[4]^- \oplus [3]^- \oplus [2]^-$. Now, $p - q = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha = 2 + 1 = 3$, $\beta_1 = 5$, $\beta_2 = 1$, $\beta_3 = 1$, $\beta = 5 + 1 + 1 = 7$, $d_1 = \min\{21 - 18, 7\} = 3$, $d_2 = \min\{19 - 18, 3\} = 1$ and $\beta - \alpha = 4$. So, we are in case (i) of 6.3. Now, $\beta_2 + \beta_3 > \beta - \alpha - (p - n) \geq \beta_3$. Then, $\nu = 2$ and $\beta' = (1 + 1) - (7 - 3 - (21 - 18)) = 1$. Therefore, $\psi_1(d) = 2[4]^+ \oplus [3]^+ \oplus 5[4]^- \oplus [3]^- \oplus 3[1]^+$ and $\psi_2(d) = [1]^+ \oplus 2[1]^-$. Note that the number of positive blocks in $\psi_1(d)$ is equal to the number of negative blocks. It is not difficult to see that there are 42 different signed Young diagrams $(\psi_1(d), Q)$ of the signed partition $\psi_1(d)$. However, they only produce two different $\Phi^*(\psi_1(d), Q)$, which are

$$\Phi^*(\psi_1(d), Q_1) = 5[5]^+ \oplus [2]^+ \oplus [5]^- \oplus [3]^- \oplus [2]^-,$$
$$\Phi^*(\psi_1(d), Q_2) = 4[5]^+ \oplus [3]^+ \oplus [2]^+ \oplus 2[5]^- \oplus [2]^-.$$

Therefore, $\Phi(\mathbb{O}_d) = \mathbb{O}_{e_1} \cup \mathbb{O}_{e_2}$ and $\varphi(\mathbb{O}_d) = \mathbb{O}_f$, where

$$e_1 := 5[5]^+ \oplus [2]^+ \oplus [1]^+ \oplus [5]^- \oplus [3]^- \oplus [2]^- \oplus 2[1]^-,$$
$$e_2 := 4[5]^+ \oplus [3]^+ \oplus [2]^+ \oplus [1]^+ \oplus 2[5]^- \oplus [2]^- \oplus 2[1]^-,$$
$$f := 5[3]^+ \oplus [2]^+ \oplus 7[1]^+ \oplus 2[3]^- \oplus [2]^- \oplus 8[1]^-.$$

6.8. *Admissible signed Young diagrams.* Let d be a symplectic signed partition, as in 6.2. Rewrite d as

$$d = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{\gamma_i} [r_i^j]^\bullet,$$

where each $\bullet$ is plus or minus, as usual, and these r_i^j satisfy the condition $r_1^1 = \cdots = r_1^{\gamma_1} > r_2^1 = \cdots = r_2^{\gamma_2} > \cdots > r_s^1 = \cdots = r_s^{\gamma_s}$. Let r_x be the size of a row $[r_x^y]$. We define a special kind of signed Young diagrams of d called *admissible*, as follows: The definition depends on the signature (p, q) of W. Recall that we assume that $p \geq q$ and, in a Young diagram, a row of longer length is always above a shorter row.

(i) Among the rows of same length, if the length r_x is even, we put a row $[r_x^y]^*$ with sign such that the difference of the negative rows and the positive rows that are no lower than the row $[r_x^y]^*$ is as close to $q - p$ as possible.

(ii) If r_x is odd, then the number of positive rows of length r_x is equal to the number of negative rows of the same length. Let the signs of these $[r_x^y]^*$ appear alternatively, in the order starting from the sign such that the difference of negative rows and positive rows that are no lower than the row $[r_x^y]^*$ is as close to $p - q$ as possible.

If (d, Q) is an admissible signed Young diagram of d, then Q is called an *admissible order* on the set of rows in the signed partition d.

6.9. Example. Suppose that $2n = 26$, $p = 14$, $q = 13$ and $d = 2[4]^+ \oplus 3[4]^- \oplus [3]^+ \oplus [3]^-$. The rows of d have two lengths, namely 4 and 3. So, the top five rows are of size 4, and the next two rows are of size 3. Since now $p - q = 1$, the first row of an admissible signed Young diagram of d must be negative. The second row is allowed to be positive or negative. If the second row is positive, then the third row has to be negative. If the second row is negative, then the third row is forced to be positive, and so on. Therefore, we have the following eight possible admissible signed Young diagrams associated to the signed partition d:
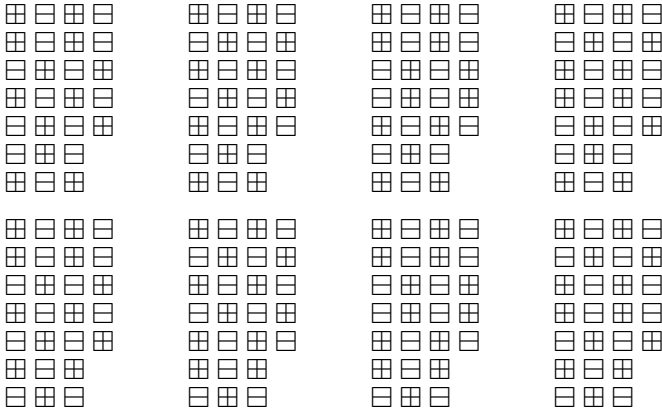
6.10. Let (d, Q) be an admissible signed Young diagram of a symplectic signed partition d of size $2n$. Recall that Q is an order on the set $\{[r_x^y]^*\}$. Let $[r_{x_0}^{y_0}]^*$ be the smallest element under the order Q, subject to the condition that $v \leq p' + p - n$ and $u \leq q' + q - n$, where

$$(p', q') := \mathrm{sgn}\left(\bigoplus_{[r_x^y]^* \leq [r_{x_0}^{y_0}]^*} [r_x^y]^* \right),$$

u is the cardinality of the set $\{[r_x^y]^+ \mid [r_x^y]^+ > [r_{x_0}^{y_0}]^*\}$ and v is the cardinality of the set $\{[r_x^y]^- \mid [r_x^y]^- > [r_{x_0}^{y_0}]^*\}$. Set

$$\sigma := \min\{p' + p - n - v, \ q' + q - n - u\}.$$

Let $[2\sigma]^*$ be a simple signed partition, with the same sign as the sign of the row next to $[r_{x_0}^{y_0}]^*$. We define $\lambda(\boldsymbol{d}, Q)$ according to the following three cases:

(i) If $r_{x_0}^{y_0}$ is odd, y_0 is even, and $[r_{x_0}^{y_0}]^*$ is a positive row, then we define

$$\lambda(\boldsymbol{d}, Q) := \bigoplus_{[r_x^y]^* > [r_{x_0}^{y_0-1}]^*} [r_x^y]^* \oplus [r_{x_0}^{y_0-1} - 1]^- \oplus [2\sigma]^* \oplus (p' - \sigma)[1]^+ \oplus (q' - \sigma + 1)[1]^-.$$

(ii) If $r_{x_0}^{y_0}$ is odd, y_0 is even, and $[r_{x_0}^{y_0}]^*$ is a negative row, then we define

$$\lambda(\boldsymbol{d}, Q) := \bigoplus_{[r_x^y]^* > [r_{x_0}^{y_0-1}]^*} [r_x^y]^* \oplus [r_{x_0}^{y_0-1} - 1]^+ \oplus [2\sigma]^* \oplus (p' - \sigma + 1)[1]^+ \oplus (q' - \sigma)[1]^-.$$

(iii) In the remaining cases, set

$$\lambda(\boldsymbol{d}, Q) := \bigoplus_{[r_x^y]^* > [r_{x_0}^{y_0}]^*} [r_x^y]^* \oplus [2\sigma]^* \oplus (p' - \sigma)[1]^+ \oplus (q' - \sigma)[1]^-.$$

Clearly, $\lambda(\boldsymbol{d}, Q)$ is a signed partition of signature (n, n). Moreover, we can check that $p' + 1 = q'$ for case (i), $p' = q' + 1$ for case (ii), and $p' = q'$ for case (iii). Hence, $\lambda(\boldsymbol{d}, Q)$ is a symplectic signed partition.

6.11. Example. Suppose $2n = 22$, $\boldsymbol{d} = [6]^+ \oplus [6]^- \oplus [5]^+ \oplus [5]^-$, and $p = q = 11$. There are four admissible orders on the set of rows of $\boldsymbol{d}$, namely

$$\begin{aligned}
Q_1: & \quad [6]^+ > [6]^- > [5]^+ > [5]^-, \\
Q_2: & \quad [6]^- > [6]^+ > [5]^+ > [5]^-, \\
Q_3: & \quad [6]^+ > [6]^- > [5]^- > [5]^+, \\
Q_4: & \quad [6]^- > [6]^+ > [5]^- > [5]^+.
\end{aligned}$$

First, we consider Q_1. Under the admissible order Q_1, it is easy to check that $[r_{x_0}^{y_0}]^* = [5]^-$. Then, $(p', q') = (2, 3)$, $u = 2$ and $v = 1$. Hence,

$$\sigma = \min\{2 + 11 - 11 - 1,\ 3 + 11 - 11 - 2\} = 1.$$

Now, $r_{x_0}^{y_0} = 5$, y_0 is 2, and $[r_{x_0}^{y_0}]^*$ is a negative row. So, we are in case (ii) of 6.10, and

$$\lambda(\boldsymbol{d}, Q_1) = [6]^+ \oplus [6]^- \oplus [4]^+ \oplus [2]^- \oplus 2[1]^+ \oplus 2[1]^-.$$

For Q_2, we have $[r_{x_0}^{y_0}]^* = [5]^-$. By the same argument, we have

$$\lambda(\boldsymbol{d}, Q_2) = [6]^+ \oplus [6]^- \oplus [4]^+ \oplus [2]^- \oplus 2[1]^+ \oplus 2[1]^-.$$

For Q_3, we have $[r_{x_0}^{y_0}]^* = [5]^+$. Then, $(p', q') = (3, 2)$, $u = 1$ and $v = 2$. Hence, $\sigma = \min\{3 + 11 - 11 - 2, 2 + 11 - 11 - 1\} = 1$. Now, $r_{x_0}^{y_0} = 5$, y_0 is 2 and $[r_{x_0}^{y_0}]^*$ is a positive row. So, we are in case (i) of 6.10, and

$$\lambda(\boldsymbol{d}, Q_3) = [6]^+ \oplus [6]^- \oplus [4]^- \oplus [2]^+ \oplus 2[1]^+ \oplus 2[1]^-.$$

For Q_4, we have $[r_{x_0}^{y_0}]^* = [5]^+$. By the same argument as for Q_3, we have

$$\lambda(d, Q_4) = [6]^+ \oplus [6]^- \oplus [4]^- \oplus [2]^+ \oplus 2[1]^+ \oplus 2[1]^-.$$

Note that $\lambda(d, Q_1) = \lambda(d, Q_2) \neq \lambda(d, Q_3) = \lambda(d, Q_4)$.

6.12. Lemma. *For all possible admissible signed Young diagrams (d, Q) of a signed partition d, there are at most two different signed partitions $\lambda(d, Q)$.*

Proof. Let $[a]^*$ be a row of the signed partition d. Suppose that $[a]^*$ is $[r_{x_0}^{y_0}]^*$, under some admissible order Q. If $[a]^*$ is a row in a double odd block with partner $[b]^*$, then it might be that there is another admissible order Q' such that $[r_{x_0}^{y_0}]^*$ is $[b]^*$. If $[a]^*$ is a row of even length, it is also possible that there is another admissible order Q' such that $[r_{x_0}^{y_0}]^*$ is a row of the same length, but opposite sign. From the construction, we know that these two situations are the only chances to obtain different signed partitions $\lambda(d, Q)$, $\lambda(d, Q')$. $\qquad\square$

6.13. Lemma. *If (d, Q) is an admissible signed Young diagram of a signed partition d, then $\lambda(d, Q) \leq d$ and $\Phi(\mathbb{O}_{\lambda(d,Q)})$ is a single nilpotent orbit.*

Proof. The first part is obvious from the definition of $\lambda(d, Q)$ in 6.10.

We prove the second part. Let Q_1 and Q_2 be two different admissible orders of the signed partition $\psi_1(\lambda(d, Q))$. From the construction of $\lambda(d, Q)$ and definition of Φ, it is not difficult to see that $\Phi^*(\psi_1(\lambda(d), Q), Q_1) = \Phi^*(\psi_1(\lambda(d), Q), Q_2)$. Hence,

$$\Phi(\lambda(d, Q), Q_1) = \Phi(\lambda(d, Q), Q_2).$$

Therefore, $\Phi(\mathbb{O}_{\lambda(d,Q)})$ is a single nilpotent orbit. $\qquad\square$

Because the signed partition $\Phi(\lambda(d, Q), Q')$ is independent of the admissible order Q' of $\psi_1(\lambda(d, Q))$, we will write $\Phi(\lambda(d, Q))$ instead of $\Phi(\lambda(d, Q), Q')$.

6.14. *Example.* Suppose that $2n = 26$, $p = 14$, $q = 13$ and $d = 2[4]^+ \oplus 3[4]^- \oplus [3]^+ \oplus [3]^-$. We see that $[r_{x_0}^{y_0}]^* = [3]^+$ for any admissible order Q on the set of rows of d. Hence, $(p', q') = \text{sgn}([3]^+ [3]^-) = (3, 3)$, $(p' + p - n, q' + q - n) = (4, 3)$, $u = 2$, $v = 3$ and $\sigma = \min\{4 - 3, 3 - 2\} = 1$. Now, $r_{x_0}^{y_0} = 3$ is odd, $y_0 = 2$ is even, $[r_{x_0}^{y_0}]^*$ is a positive row, and $\sigma \neq 0$. So we are in case (iii) of 6.10. Therefore, $\lambda(d, Q) = 2[4]^+ \oplus 3[4]^- \oplus 3[1]^+ \oplus 3[1]^-$ for all four different admissible orders Q. It is obvious that $\lambda(d, Q) \leq d$. Now, $\psi_1(\lambda(d, Q)) = 2[4]^+ \oplus 3[4]^- \oplus 3[1]^+ \oplus 2[1]^-$ and $\psi_2(\lambda(d, Q)) = [1]^+ \oplus [1]^-$. Therefore, $\Phi(\lambda(d, Q)) = 3[5]^+ \oplus 2[5]^- \oplus [1]^+ \oplus [1]^-$. It is not difficult to check that there are two signed Young diagrams (d, Q_1) and (d, Q_2) of d such that

$$\Phi(d, Q_1) = 2[5]^+ \oplus 2[5]^- \oplus [3]^+ \oplus [2]^+ \oplus [2]^-,$$
$$\Phi(d, Q_2) = 3[5]^+ \oplus [5]^- \oplus [3]^- \oplus [2]^+ \oplus [2]^-.$$

Clearly, $\Phi(\lambda(d, Q)) > \Phi(d, Q_1)$ and $\Phi(\lambda(d, Q)) > \Phi(d, Q_2)$.

7. The orbit correspondences for real reductive dual pairs

7.1. Keep the assumption that $F := \mathbb{R}$, and let φ, Φ be the maps defined in 6.6.

Proposition. *If $\mathbb{O}$ is a nilpotent orbit in $\mathfrak{sp}(V)$ such that $\mathbb{O}$ is in the image of τ_1, then $\Phi(\mathbb{O})$ is the union of maximal elements in the set of nilpotent orbits contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$.*

The proof of this proposition is in 7.4.

7.2. Complexification. Let $W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$. The real Lie algebra $\mathfrak{so}(W)$ is naturally contained in the complex Lie algebra $\mathfrak{so}(W^{\mathbb{C}})$. Hence, a real orbit $\mathcal{Q}$ in $\mathfrak{so}(W)$ is contained in a unique complex orbit in $\mathfrak{so}(W^{\mathbb{C}})$, denoted by $\mathcal{Q}^{\mathbb{C}}$. The complex orbit $\mathcal{Q}^{\mathbb{C}}$ is called the *complexification* of $\mathcal{Q}$. If $\boldsymbol{d}$ is the signed partition corresponding to the real orbit $\mathcal{Q}$, then we can obtain the partition corresponding to the complex orbit $\mathcal{Q}^{\mathbb{C}}$ by forgetting the signs in $\boldsymbol{d}$.

Lemma. *Let $\mathbb{O}_{\boldsymbol{d}}$ be a nilpotent orbit contained in the image of τ_1. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two nilpotent orbits contained in $\Phi(\mathbb{O}_{\boldsymbol{d}})$, then $\mathcal{Q}_1^{\mathbb{C}} = \mathcal{Q}_2^{\mathbb{C}}$.*

Proof. Let Q_1 and Q_2 be the orders of rows of $\psi_1(\boldsymbol{d})$ such that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are nilpotent orbits corresponding to

$$\Phi^*(\psi_1(\boldsymbol{d}), Q_1) \oplus \psi_2(\boldsymbol{d}) \quad \text{and} \quad \Phi^*(\psi_1(\boldsymbol{d}), Q_2) \oplus \psi_2(\boldsymbol{d}),$$

respectively. From the construction, it is clear that the left summands have the same complexification. Hence, the sums have the same complexification. $\square$

7.3. Lemma. *Let $\mathbb{O}$ be a nilpotent orbit in the image of τ_1. Suppose that $\mathcal{P}$ is a nilpotent orbit contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$. There exists a nilpotent orbit $\mathcal{Q}$ contained in $\Phi(\mathbb{O})$ such that $\mathcal{P} \leq \mathcal{Q}$.*

Proof. Let $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$. Let $\tau_1^{\mathbb{C}} : V^{\mathbb{C}} \otimes_{\mathbb{C}} W^{\mathbb{C}} \to \mathfrak{sp}(V^{\mathbb{C}})$ and $\tau_2^{\mathbb{C}} : V^{\mathbb{C}} \otimes_{\mathbb{C}} W^{\mathbb{C}} \to \mathfrak{so}(W^{\mathbb{C}})$ be the complex moment maps. Clearly, the complex nilpotent orbit $\mathcal{P}^{\mathbb{C}}$ is contained in $\tau_2^{\mathbb{C}}(\tau_1^{\mathbb{C}-1}(\mathbb{O}^{\mathbb{C}}))$. Moreover, if $\mathcal{Q}'$ is a nilpotent orbit contained in $\Phi(\mathbb{O})$, then clearly $\mathcal{Q}'^{\mathbb{C}} = \Phi(\mathbb{O}^{\mathbb{C}})$, where $\Phi(\mathbb{O}^{\mathbb{C}})$ is as defined in 4.4. Hence, we have $\mathcal{P}^{\mathbb{C}} \leq \mathcal{Q}'^{\mathbb{C}}$ by 4.6. Clearly, we need only consider the case when there is no nilpotent orbit $\mathcal{P}' \subseteq \tau_2(\tau_1^{-1}(\mathbb{O}))$ such that $\mathcal{P} < \mathcal{P}'$ and $\mathcal{P}'^{\mathbb{C}} \leq \mathcal{Q}^{\mathbb{C}}$.

Let $\boldsymbol{d} = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{\gamma_i} [r_i^j]^*$ be the symplectic signed partition corresponding to the nilpotent orbit $\mathbb{O}$. From 5.1, 5.2 and 5.3, we know that a nilpotent orbit contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$ should correspond to an orthogonal signed partition $\boldsymbol{e}$, constructed from $\boldsymbol{d}$ according the following rules:

(i) If $[r_i^j]^+$ (respectively, $[r_i^j]^-$) is an even single row of $\boldsymbol{d}$, then $\boldsymbol{e}$ has an odd single row $[r_i^j + 1]^-$ or $[r_i^j + 1]^-$ (respectively, $[r_i^j + 1]^+$ or $[r_i^j + 1]^+$); that is, we can remove or add a signed box to an even single row of $\boldsymbol{d}$. We shall say that the row $[r_i^j + 1]^*$ or $[r_i^j - 1]^*$ is the *image* in $\boldsymbol{e}$ of the row $[r_i^j]^*$.

(ii) If $[r_i^{j-1}]^+ \oplus [r_i^j]^-$ is an odd double row, then $\boldsymbol{e}$ has a double row

$$[r_i^{j-1} - 1]^- \oplus [r_i^j - 1]^+, \quad [r_i^{j-1}]^+ \oplus [r_i^j]^-, \quad \text{or} \quad [r_i^{j-1} + 1]^- \oplus [r_i^j + 1]^+.$$

(iii) The signature of $\boldsymbol{e}$ is (p, q).

Suppose that $\mathscr{P} \not\leq \mathscr{Q}'$, for otherwise the lemma is proved. Give an order on the set of rows of $\boldsymbol{d}$ as in Equation (6-3). Let $[r_{x_1}^{y_1}]^*$ be the largest row in $\boldsymbol{d}$ whose image in the signed partition corresponding to $\mathscr{P}$ has longer length than its image in the signed partition corresponding to $\mathscr{Q}'$. From the construction of $\Phi(\mathbb{O})$, we see that $\boldsymbol{d}$ has a row $[r_{x_2}^{y_2}]^*$ such that its image in the signed partition corresponding to $\mathscr{Q}'$ has longer length than that in the signed partition corresponding to $\mathscr{P}$, and $r_{x_2}^{y_2} \geq r_{x_1}^{y_1}$. However, $\mathscr{P} \not\leq \mathscr{Q}'$, and there is no nilpotent orbit $\mathscr{P}'$ contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$ such that $\mathscr{P} < \mathscr{P}'$ and $\mathscr{P}'^{\mathbb{C}} \leq \mathscr{Q}'^{\mathbb{C}}$, so $r_{x_2}^{y_2}$ is equal to $r_{x_1}^{y_1}$, while $[r_{x_1}^{y_1}]^*$ and $[r_{x_2}^{y_2}]^*$ have opposite parity. Then, there is a nilpotent orbit $\mathscr{Q}'' \in \Phi(\mathbb{O})$ such that the largest row in $\boldsymbol{d}$, with its image in the signed partition corresponding to $\mathscr{P}$ having longer length than its image in the signed partition corresponding to $\mathscr{Q}''$, is strictly smaller than $[r_{x_1}^{y_1}]^*$. Hence, the result can be obtained by induction. $\square$

7.4. Proof of the proposition in 7.1. It is clear from the construction and the rules in 5.1, 5.2 and 5.3 that every nilpotent $\mathscr{Q}$ in $\Phi(\mathbb{O})$ is contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$. Suppose that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are two different nilpotent orbits contained in $\Phi(\mathbb{O})$. From the construction, we know that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ have the same complexification. Therefore, they are clearly not comparable. Suppose that $\mathscr{P}$ is a nilpotent orbit contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$. Then, by 7.3, $\mathscr{P} \leq \mathscr{Q}$ for some nilpotent orbit $\mathscr{Q}$ contained in $\Phi(\mathbb{O})$. Hence, every nilpotent orbit contained in $\Phi(\mathbb{O})$ is a maximal element in the ordered set of nilpotent orbits contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$. Moreover, every maximal element in the set of nilpotent orbits contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$ is contained in $\Phi(\mathbb{O})$. $\square$

7.5. Proposition. *If $\mathbb{O}_1$ and $\mathbb{O}_2$ are nilpotent orbits in $\mathfrak{sp}(V)$ such that $\mathbb{O}_1 \geq \mathbb{O}_2$, then $\varphi(\mathbb{O}_1) \geq \varphi(\mathbb{O}_2)$. Moreover, $\varphi(\mathbb{O})$ is the unique minimal element among the nilpotent orbits contained in $\tau_2(\tau_1^{-1}(\mathbb{O}))$.*

Proof. Use the definition of φ in 6.6 and the rules in 5.1, 5.2 and 5.3. $\square$

7.6. Lemma. *If $\mathbb{O}_d$ is a nilpotent orbit contained in the image of τ_1, then $\mathbb{O}_{\Phi(\lambda(d, Q))}$ is a maximal element in the set of nilpotent orbits in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$. Every maximal element in the set of nilpotent orbits in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$ is of this form for some Q.*

Proof. Suppose that $\mathbb{O}_{\Phi(\lambda(d, Q))}$ is not a maximal element. Then, $\mathbb{O}_{\Phi(\lambda(d, Q))} < \mathscr{P}$ for some nilpotent orbit $\mathscr{P}$ in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$. Hence, $\mathbb{O}_{\Phi(\lambda(d, Q))}^{\mathbb{C}} < \mathscr{P}^{\mathbb{C}}$, contradicting 4.9. It is also clear from 4.9 that every maximal element in the set of nilpotent orbits in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$ is of the form $\mathbb{O}_{\Phi(\lambda(d, Q))}$ for some Q. $\square$

7.7. Lemma. *If* $\mathbb{O}_{d_1}$, $\mathbb{O}_{d_2}$ *are two nilpotent orbits contained in the image of* τ_1 *such that* $\mathbb{O}_{d_1} \leq \mathbb{O}_{d_2}$, *then, for every nilpotent orbit* $\mathfrak{Q}_1$ *contained in* $\bigcup_{(d_1, Q_1)} \mathbb{O}_{\Phi(\lambda(d_1, Q_1))}$, *there is a nilpotent orbit* $\mathfrak{Q}_2$ *contained in* $\bigcup_{(d_2, Q_2)} \mathbb{O}_{\Phi(\lambda(d_2, Q_2))}$ *such that* $\mathfrak{Q}_1 \leq \mathfrak{Q}_2$, *where the union* $\bigcup_{(d_i, Q_i)}$ *runs over the admissible signed Young diagrams of the signed partition* d_i.

Proof. Suppose (d_1, Q_1) is an admissible signed Young diagram of d_1 such that $\mathfrak{Q}_1 = \mathbb{O}_{\Phi(\lambda(d_1, Q_1))}$. We have $\lambda(d_1, Q_1) \leq d_1 \leq d_2$. So, $\mathbb{O}_{\Phi(\lambda(d_1, Q_1))} \subset \tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_{d_2}))$. By 6.12, we know that a maximal element in the set of nilpotent orbits contained in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_{d_2}))$ is equal to $\mathbb{O}_{\Phi(\lambda(d_2, Q_2))}$ for some admissible signed Young diagram (d_2, Q_2) of d_2. Hence, $\mathfrak{Q}_1 \leq \mathbb{O}_{\Phi(\lambda(d_2, Q_2))}$. □

7.8. *Proof of 6.1.* First, we show that

$$\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d)) \subseteq \bigcup_{(d, Q)} \overline{\mathbb{O}_{\Phi(\lambda(d, Q))}}.$$

Suppose $\mathfrak{Q}$ is a nilpotent orbit in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$. Then, $\mathfrak{Q}$ is in $\tau_2(\tau_1^{-1}(\mathbb{O}_{d_1}))$ for some nilpotent orbit $\mathbb{O}_{d_1}$ in $\mathfrak{sp}(V)$ such that $\mathbb{O}_{d_1} \leq \mathbb{O}_d$. From 7.1, we know that $\mathfrak{Q} \leq \mathfrak{Q}_1$ for some nilpotent orbit $\mathfrak{Q}_1$ contained in $\Phi(\mathbb{O}_{d_1})$. By 7.6, there exists an admissible signed Young diagram (d_1, Q_1) such that $\mathfrak{Q}_1 \leq \mathbb{O}_{\Phi(\lambda(d_1, Q_1))}$. Since $\mathbb{O}_{d_1} \leq \mathbb{O}_d$, by 7.7, there is an admissible signed Young diagram (d, Q) such that $\mathbb{O}_{\Phi(\lambda(d_1, Q_1))} \leq \mathbb{O}_{\Phi(\lambda(d, Q))}$. Hence, $\mathfrak{Q}_1 \leq \mathbb{O}_{\Phi(\lambda(d, Q))}$ and therefore $\mathfrak{Q}_1 \subseteq \bigcup_{(d, Q)} \overline{\mathbb{O}_{\Phi(\lambda(d, Q))}}$.

Next, we prove the opposite inclusion

$$\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d)) \supseteq \bigcup_{(d, Q)} \overline{\mathbb{O}_{\Phi(\lambda(d, Q))}}.$$

Let $\mathfrak{Q}'$ be a nilpotent orbit contained in $\bigcup_{(d, Q)} \overline{\mathbb{O}_{\Phi(\lambda(d, Q))}}$. Then, $\mathfrak{Q}' \leq \mathbb{O}_{\Phi(\lambda(d, Q))}$ for some admissible signed Young diagram (d, Q). By a construction similar to 6.6, we can define a map φ', from the set of nilpotent orbits in $\mathfrak{so}(W)$ to the set of nilpotent orbits in $\mathfrak{sp}(V)$, such that $\varphi'(\mathbb{O}')$ is the unique minimal element among the nilpotent orbits contained in $\tau_1(\tau_2^{-1}(\mathbb{O}'))$ for any nilpotent orbit $\mathbb{O}'$ in $\mathfrak{so}(W)$. Moreover,

$$(7\text{-}1) \qquad\qquad \varphi'(\mathfrak{Q}') \leq \varphi'(\mathbb{O}_{\Phi(\lambda(d, Q))}).$$

We know that $\mathbb{O}_{\Phi(\lambda(d, Q))}$ is contained in $\tau_2(\tau_1^{-1}(\mathbb{O}_{\lambda(d, Q)}))$. Hence, the nilpotent orbit $\mathbb{O}_{\lambda(d, Q)}$ is contained in $\tau_1(\tau_2^{-1}(\mathbb{O}_{\Phi(\lambda(d, Q))}))$. Thus, $\varphi'(\mathbb{O}_{\Phi(\lambda(d, Q))}) \leq \mathbb{O}_{\lambda(d, Q)}$ because $\varphi'(\mathbb{O}_{\Phi(\lambda(d, Q))})$ is the unique minimal element among the nilpotent orbits contained in $\tau_1(\tau_2^{-1}(\mathbb{O}_{\Phi(\lambda(d, Q))}))$. Hence, we have $\varphi'(\mathfrak{Q}') \leq \mathbb{O}_{\lambda(d, Q)}$ from (7-1). Then, by 6.13, we have

$$(7\text{-}2) \qquad\qquad \varphi'(\mathfrak{Q}') \leq \mathbb{O}_d.$$

From the construction, we know that $\varphi'(\mathfrak{Q}')$ is contained in $\tau_1(\tau_2^{-1}(\mathfrak{Q}'))$. Hence, $\mathfrak{Q}'$ is contained in $\tau_2(\tau_1^{-1}(\varphi'(\mathfrak{Q}')))$. By (7-2), we see that $\mathfrak{Q}'$ is contained in $\tau_2(\tau_1^{-1}(\overline{\mathbb{O}}_d))$.

From 6.12, we know that $\bigcup_{(d,Q)} \overline{\mathbb{O}_{\Phi(\lambda(d,Q))}}$ is a union of at most two closures of nilpotent orbits. $\square$

Acknowledgments

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