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SOME ELLIPTIC PDES ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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We study some rigidity properties of stable solutions of elliptic equations set on manifolds with boundary. Our results are classified by the dimension of the manifold and the sign of its Ricci curvature. As a consequence of our results on boundary reactions, we obtain several symmetry and Liouville results for nonlocal equations.

1. Introduction

Let $(\mathcal{M}, \bar{g})$ be a complete, connected, smooth, $(n+1)$-dimensional manifold with boundary $\partial\mathcal{M}$, endowed with a smooth Riemannian metric $\bar{g} = \{\bar{g}_{ij}\}_{i,j=1,\dots,n}$.

The volume element is written in local coordinates as

$$dV_{\bar{g}} = \sqrt{|\bar{g}|}\, dx^1 \wedge \cdots \wedge dx^{n+1}, \tag{1-1}$$

where $\{dx^1, \dots, dx^{n+1}\}$ is the basis of 1-forms dual to the basis $\{\partial_i, \dots, \partial_{n+1}\}$ of vectors and we use the standard notation $|\bar{g}| = \det(\bar{g}_{ij}) \geqslant 0$.

We denote by $\operatorname{div}_{\bar{g}} X$ the divergence of a smooth vector field X on $\mathcal{M}$, that is, in local coordinates,

$$\operatorname{div}_{\bar{g}} X = \frac{1}{\sqrt{|\bar{g}|}} \partial_i \big(\sqrt{|\bar{g}|}\, X^i \big),$$

where we use the Einstein summation convention.

We also denote by $\nabla_{\bar{g}}$ the Riemannian gradient and by $\Delta_{\bar{g}}$ the Laplace–Beltrami operator, that is, in local coordinates,

$$(\nabla_{\bar{g}}\phi)^i = \bar{g}^{ij} \partial_j \phi \tag{1-2}$$

and

$$\Delta_{\bar{g}}\phi = \operatorname{div}_{\bar{g}}(\nabla_{\bar{g}}\phi) = \frac{1}{\sqrt{|\bar{g}|}} \partial_i \big(\sqrt{|\bar{g}|}\, \bar{g}^{ij} \partial_j \phi \big),$$

for any smooth function $\phi : \mathcal{M} \to \mathbb{R}$.

We let $\langle \cdot , \cdot \rangle$ be the scalar product induced by $\bar{g}$. Given a vector field X, we also write $|X| = \sqrt{\langle X, X \rangle}$.

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Also (see, for instance [Jost 1998, Definition 3.3.5]), it is customary to define the Hessian of a smooth function ϕ as the symmetric 2-tensor given in a local patch by

$$(H_{\bar{g}}\phi)_{ij} = \partial^2_{ij}\phi - \Gamma^k_{ij}\partial_k\phi,$$

where Γ^k_{ij} are the Christoffel symbols, namely

$$\Gamma^k_{ij} = \tfrac{1}{2}\bar{g}^{hk}(\partial_i \bar{g}_{hj} + \partial_j \bar{g}_{ih} - \partial_h \bar{g}_{ij}).$$

Given a tensor A, we define its norm by $|A| = \sqrt{AA^*}$, where A^* is the adjoint.

This paper studies special solutions of elliptic equations on manifolds with boundary and is, in some sense, a follow up to the paper by the authors and Farina [Farina et al. 2008b], which studied the case without boundary. In a Euclidean context, that is, $\mathcal{M} = \mathbb{R}^{n+1}_+$ with the flat metric, the rigidity features of the stable solutions has been investigated in [Sire and Valdinoci 2009; Cabré and Sire 2010; Cabré and Solà-Morales 2005].

More precisely, we study rigidity properties of stable solutions of nonlinear problems for which the nonlinearity is prescribed on the boundary of $\mathcal{M}$. Via a theorem of Caffarelli and Silvestre [2007], boundary problems are related to nonlocal equations involving fractional powers of the Laplacian. An analogue of their results has been obtained in a more geometric context by means of scattering theory [Fefferman and Graham 2002; Graham et al. 1992; Graham and Zworski 2003; Chang and del Mar González 2010].

In this paper, we will mainly focus on the two specific models:

- *Product manifolds* of the type

$$(\mathcal{M} = M \times \mathbb{R}^+, \bar{g} = g + |dx|^2),$$

  where (M, g) is a complete, smooth Riemannian manifold without boundary. The boundary of $\mathcal{M}$ is precisely the manifold M.

- *The hyperbolic halfspace*, that is,

$$(\mathcal{M} = \mathbb{H}^{n+1}, \bar{g} = (|dy|^2 + |dx|^2)/x^2),$$

  where $x > 0$ and $y \in \mathbb{R}^n$.

These models comprise both the positive and the negative curvature cases. Also, we will use here that the manifold $\mathbb{H}^{n+1}$ with metric $\bar{g} = (|dy|^2 + |dx|^2)/x^2$ is conformal to $\mathbb{R}^{n+1}_+$ with the flat metric, and, in fact, $(\mathbb{H}^{n+1}, \bar{g})$ is the main example of a conformally compact Einstein manifold, as we discuss in Section 5.1.

We also want to deal with nonlocal equations. More precisely, let (M, g) be a smooth connected Riemannian manifold without boundary, and consider $(-\Delta_g)^\gamma$

for $\gamma \in (0, 1)$, the pseudodifferential operator with symbol $|\xi|^{2\gamma}$ (a practical construction is provided starting on 482). We investigate the problem

$$(-\Delta_g)^\gamma u = f(u) \quad \text{on } M, \tag{1-3}$$

where f is a $C^1(\mathbb{R})$ nonlinearity (in fact, up to minor modifications, the proofs we present also work for locally Lipschitz nonlinearities). Problem (1-3) is clearly nonlocal, which makes its analysis challenging. To study it special solutions, we will realize the nonlocal operators as boundary operators of a suitable extension on $\mathcal{M}$. More precisely, we will make explicit the link between (1-3) and the boundary problem

$$\begin{cases} \nabla_{\bar g} \cdot (x^{1-2\gamma} \nabla_{\bar g} u) = 0 & \text{in } \mathcal{M} = M \times \mathbb{R}^+, \\ -x^{1-2\gamma} \partial_x u = f(u) & \text{on } M \times \{0\}. \end{cases} \tag{1-4}$$

Remark. It would be interesting, and we leave it as an open problem, to investigate problem (1-4) in a more general context than product manifolds. Indeed, it has been shown in [Chang and del Mar González 2010] that one can relate fractional-order conformally covariant operators on manifolds M to extension operators on manifolds $\mathcal{M}$ when M is the conformal infinity of $\mathcal{M}$. These extension operators have a form similar to (1-4) except that they involve lower-order terms. This makes their study with our method more involved.

2. A weighted Poincaré inequality for stable solutions of (1-4) for $\gamma = 1/2$

Definition 2.1. We call u a *weak solution* of (1-4) if, for every $\xi \in C_0^\infty(M \times \mathbb{R})$, we have

$$\int_{\mathcal{M}} \langle \nabla_{\bar g} u, \nabla_{\bar g} \xi \rangle \, dV_{\bar g} = \int_{\partial \mathcal{M}} f(u) \xi \, dV_g. \tag{2-1}$$

We focus on an important class of solutions of (1-4), namely the so-called stable solutions. These solutions play an important role in the calculus of variations and are characterized by the fact that the second variation of the energy functional is nonnegative definite. This condition may be explicitly written in the case of problem (1-4) by saying that a weak solution u of (1-4) is stable if

$$\int_{\mathcal{M}} |\nabla_{\bar g} \xi|^2 dV_{\bar g} - \int_{\partial \mathcal{M}} f'(u) \xi^2 \, dV_g \geqslant 0 \tag{2-2}$$

for every $\xi \in C_0^\infty(M \times \mathbb{R})$.

To simplify notation, we write ∇ instead of $\nabla_{\bar g}$ for the gradient on $M \times \mathbb{R}^+$, but we will keep the notation ∇_g for the Riemannian gradient on M.

Recalling (1-2), we have

$$\nabla = (\nabla_g, \partial_x). \tag{2-3}$$

In Theorem 2.2, we obtain a formula involving the geometry, in a quite implicit way, of the level sets of stable solutions of (1-4).

Such a formula may be considered a geometric version of the Poincaré inequality, since the L^2 norm of the gradient of any test function bounds the L^2 norm of the test function itself. Remarkably, these L^2 norms are weighted and the weights have a neat geometric interpretation.

These type of geometric Poincaré inequalities were first obtained by Sternberg and Zunbrun [1998a; 1998b] in the Euclidean setting, and similar estimates have been recently widely used for rigidity results in PDEs; see, for instance, [Farina et al. 2008a; Sire and Valdinoci 2009; Ferrari and Valdinoci 2009].

Theorem 2.2. *Let u be a stable weak solution of (1-4) such that $\nabla_g u$ is bounded. Then, for every $\varphi \in C_0^\infty(M \times \mathbb{R})$,*

$$\int_{M\times\mathbb{R}^+} (\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_g u|^2 - |\nabla_g |\nabla_g u||^2)\varphi^2 \leqslant \int_{M\times\mathbb{R}^+} |\nabla_g u|^2 |\nabla\varphi|^2. \tag{2-4}$$

Proof. Recall the classical Bochner–Weitzenböck formula for a smooth function $\phi : \mathcal{M} \to \mathbb{R}$ (see for instance [Berger et al. 1971] and references therein):

$$\tfrac{1}{2}\Delta_{\bar g}|\nabla_{\bar g}\phi|^2 = |H_{\bar g}\phi|^2 + \langle \nabla_{\bar g}\Delta_{\bar g}\phi, \nabla_{\bar g}\phi\rangle + \mathrm{Ric}_{\bar g}(\nabla_{\bar g}\phi, \nabla_{\bar g}\phi). \tag{2-5}$$

The proof of Theorem 2.2 consists in plugging the test function $\xi = |\nabla_g u|\varphi$ into the stability condition (2-2): After a simple computation, this gives

$$\int_{\mathcal{M}} \varphi^2 |\nabla|\nabla_g u||^2 + \tfrac{1}{2}\langle\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle + |\nabla_g u|^2|\nabla\varphi|^2 - \int_M f'(u)|\nabla_g u|^2\varphi^2 \geqslant 0. \tag{2-6}$$

Also, by recalling (2-3), we have

$$\langle\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle = \langle\nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle + \partial_x|\nabla_g u|^2\partial_x\varphi^2. \tag{2-7}$$

Moreover, since M is boundaryless, we can use on M the Green formula — see, for example, [Gallot et al. 1990, page 184] — and obtain

$$\begin{aligned}\int_{\mathcal{M}} \langle\nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle &= \int_{\mathbb{R}^+}\int_M \langle\nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle \\ &= -\int_{\mathbb{R}^+}\int_M \Delta_g|\nabla_g u|^2\varphi^2 = -\int_{\mathcal{M}} \Delta_g|\nabla_g u|^2\varphi^2.\end{aligned} \tag{2-8}$$

Hence, using (2-5), (2-7) and (2-8), we conclude that

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}} \langle\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= \frac{1}{2}\int_{\mathcal{M}} \partial_x|\nabla_g u|^2\partial_x\varphi^2 \\ &\quad - \int_{\mathcal{M}} \varphi^2(|H_g u|^2 + \langle\nabla_g\Delta_g u, \nabla_g u\rangle + \mathrm{Ric}_g(\nabla_g u, \nabla_g u)).\end{aligned} \tag{2-9}$$

Using the first equation in (1-4), we obtain $\Delta_g u = -\partial_{xx} u$, so (2-9) becomes

$$\frac{1}{2}\int_{\mathcal{M}} \langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle = \frac{1}{2}\int_{\mathcal{M}} \partial_x|\nabla_g u|^2 \partial_x \varphi^2 - \int_{\mathcal{M}} \varphi^2 |H_g u|^2 + \int_{\mathcal{M}} \varphi^2 \langle \nabla_g \partial_{xx} u, \nabla_g u\rangle - \int_{\mathcal{M}} \varphi^2 \operatorname{Ric}_g(\nabla_g u, \nabla_g u). \tag{2-10}$$

Furthermore, integrating by parts, we see that

$$\begin{aligned}\int_{\mathcal{M}} \partial_x|\nabla_g u|\partial_x\varphi^2 &= \int_M \int_0^{+\infty} \partial_x|\nabla_g u|\partial_x\varphi^2 \\ &= -\int_M (\partial_x|\nabla_g u|\varphi^2)|_{x=0} - \int_M\int_0^{+\infty} \partial_{xx}|\nabla_g u|\varphi^2 \\ &= -\int_M (\partial_x|\nabla_g u|\varphi^2)|_{x=0} - \int_{\mathcal{M}} \partial_{xx}|\nabla_g u|\varphi^2.\end{aligned}$$

Consequently, (2-10) becomes

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}} \langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= -\int_{\mathcal{M}} \varphi^2\big(\tfrac{1}{2}\partial_{xx}|\nabla_g u|^2 + |H_g u|^2 + \operatorname{Ric}_g(\nabla_g u, \nabla_g u)\big) \\ &\quad + \int_{\mathcal{M}} \varphi^2\langle \nabla_g \partial_{xx} u, \nabla_g u\rangle - \tfrac{1}{2}(\partial_x|\nabla_g u|^2\varphi^2)|_{x=0}.\end{aligned} \tag{2-11}$$

Now, we use the boundary condition in (1-4) to obtain that, on M,

$$f'(u)\nabla_g u = \nabla_g(f(u)) = \nabla_g \partial_\nu u = -\nabla_g\partial_x u.$$

Therefore,

$$-\frac{1}{2}\int_M (\partial_x|\nabla_g u|^2\varphi^2)|_{x=0} - \int_M \langle \nabla_g u_x, \nabla_g u\rangle\varphi^2 = \int_M f'(u)|\nabla_g u|^2\varphi^2. \tag{2-12}$$

All in all, by collecting the results in (2-6), (2-11), and (2-12), we obtain

$$\begin{aligned}\int_{\mathcal{M}} \varphi^2|\nabla|\nabla_g u||^2 &- \int_{\mathcal{M}} \varphi^2\big(\tfrac{1}{2}\partial_{xx}|\nabla_g u|^2 + |H_g u|^2 + \operatorname{Ric}_g(\nabla_g u, \nabla_g u)\big) \\ &+ \int_{\mathcal{M}} \varphi^2\langle \nabla_g\partial_{xx}u, \nabla_g u\rangle + \int_{\mathcal{M}} |\nabla_g u|^2|\nabla\varphi|^2 \geqslant 0.\end{aligned} \tag{2-13}$$

Also, we observe that

$$|\partial_x|\nabla_g u||^2 + \langle \nabla_g\partial_{xx}u, \nabla_g u\rangle - \tfrac{1}{2}\partial_{xx}|\nabla_g u|^2 = |\partial_x|\nabla_g u||^2 - |\partial_x\nabla_g u|^2 \leqslant 0$$

by the Cauchy–Schwarz inequality.

Accordingly, we get (2-4) using (2-13) and

$$|\nabla|\nabla_g u||^2 = |\nabla_g|\nabla_g u||^2 + |\partial_x|\nabla_g u||^2 \leqslant \tfrac{1}{2}\partial_{xx}|\nabla_g u|^2 - \langle \nabla_g\partial_{xx}u, \nabla_g u\rangle. \qquad \square$$

3. The case of product manifolds

Now we present our results in the case of product manifolds $\mathcal{M} = M \times \mathbb{R}^+$.

Theorem 3.1. *Let $\gamma = 1/2$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$. Assume furthermore that M is compact and satisfies $\mathrm{Ric}_g \geqslant 0$ with Ric_g not vanishing identically. Then every bounded stable weak solution u of (1-4) is constant.*

The assumption on the boundedness of u is needed. For example, the function $u(x, y) = x$ is a stable solution of

$$\begin{cases} \Delta_{\bar{g}} u = 0 & \text{in } M \times \mathbb{R}^+, \\ \partial_\nu u = -1 & \text{on } M \times \{0\}\,. \end{cases}$$

From Theorem 3.1, one also obtains the following Liouville-type theorem for the half-Laplacian on compact manifolds; for the definition and basic functional properties of fractional operators, see for example [Kato 1995].

Theorem 3.2. *Let (M, g) be a compact manifold and let $u : M \to \mathbb{R}$ be a smooth bounded solution of*

$$(-\Delta_g)^{1/2} u = f(u), \tag{3-1}$$

with

$$\int_{\mathcal{M}} (|\nabla_g \xi|^2 + |\nabla_x \xi|^2) - \int_{\partial\mathcal{M}} f'(u)\xi^2 \geqslant 0, \tag{3-2}$$

for every $\xi \in C_0^\infty(\mathcal{M})$. Assume furthermore that $\mathrm{Ric}_g \geqslant 0$ and Ric_g does not vanish identically. Then u is constant.

Remark. Results for $(-\Delta_g)^\gamma$ with $\gamma \in (0, 1)$ may be obtained similarly. See Section 4.

Theorem 3.3. *Let $\gamma = 1/2$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$, that M is complete, and $\mathrm{Ric}_g \geqslant 0$, with Ric_g not vanishing identically. Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)^2$ for some $C > 0$. Then every bounded stable weak solution u of (1-4) is constant.*

Next theorem is a flatness result when the Ricci tensor of M vanishes identically.

Theorem 3.4. *Let $\gamma = 1/2$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is $\bar{g} = g + |dx|^2$ and Ric_g vanishes identically. Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)^2$ for some $C > 0$. Then for every $x > 0$ and $c \in \mathbb{R}$, every connected component of the submanifold $\mathcal{S}_x = \{y \in M : u(x, y) = c\}$ is a geodesic, where u is a bounded stable solution of (1-4).*

With (2-4) in hand, one can prove Theorems 3.1–3.4. First, we recall a lemma.

Lemma 3.5 [Farina et al. 2008b, Lemma 9, Section 2]. *For any smooth $\phi : \mathcal{M} \to \mathbb{R}$,*

$$|H_{\bar g}\phi|^2 \geqslant |\nabla_{\bar g}|\nabla_{\bar g}\phi||^2 \quad \textit{almost everywhere.} \tag{3-3}$$

Lemma 3.6. *Let u be a bounded solution of (1-4). Assume that $\mathrm{Ric}_g \geqslant 0$ and that Ric_g does not vanish identically on M. Suppose that*

$$\mathrm{Ric}_g(\nabla_g u, \nabla_g u) \textit{ vanishes identically on } \mathcal{M}. \tag{3-4}$$

Then, u is constant on $\mathcal{M}$.

Proof. By assumption, Ric_g is strictly positive definite in a suitable nonempty open set $U \subseteq M$. Then, (3-4) gives that $\nabla_g u$ vanishes identically in $U \times \mathbb{R}^+$.

This means that, for any fixed $x \in \mathbb{R}^+$, the map $U \ni y \mapsto u(x, y)$ does not depend on y. Accordingly, there exists a function $\tilde u : \mathbb{R}^+ \to \mathbb{R}$ such that $u(x, y) = \tilde u(x)$, for any $y \in U$. Thus, from (1-4),

$$0 = \Delta_{\bar g} u = \tilde u_{xx} \quad \text{in } U \times \mathbb{R}^+$$

and so there exist $a, b \in \mathbb{R}$ for which

$$u(x, y) = \tilde u(x) = a + bx \quad \text{for any } x \in \mathbb{R}^+ \text{ and any } y \in U.$$

Since u is bounded, we have that $b = 0$, so u is constant in $U \times \mathbb{R}^+$.

By the unique continuation principle (see [Kazdan 1988, Theorem 1.8]), the solution u is constant on $M \times \mathbb{R}^+$. □

Proof of Theorem 3.1. Points in $\mathcal{M}$ will be denoted here as (x, y), with $x \in \mathbb{R}^+$ and $y \in M$.

Take φ in (2-4) to be the function $\varphi(x, y) = \phi(x/R)$, where $R > 0$ and ϕ is a smooth cut-off, that is $\phi = 0$ on $|x| \geqslant 2$ and $\phi = 1$ on $|x| \leqslant 1$. We stress that this is an admissible test function, since M is assumed to be compact in Theorem 3.1. Moreover,

$$|\nabla\varphi(x, y)| \leqslant \|\phi\|_{C^1(\mathbb{R})}\, \chi_{(0,2R)}(x)/R. \tag{3-5}$$

Also, since u is bounded, elliptic regularity gives that ∇u is bounded in $M \times \mathbb{R}^+$.

Therefore, using (2-4), Lemma 3.5 and (3-5), we obtain

$$\int_{M\times\mathbb{R}^+} (\mathrm{Ric}_g(\nabla_g u, \nabla_g u))\varphi^2 \leqslant \frac{C}{R^2}\int_{M\times(0,2R)} dV_{\bar g} \leqslant \frac{C}{R} \tag{3-6}$$

for some constant $C > 0$. Sending $R \to +\infty$ and using the fact that $\mathrm{Ric}_g \geqslant 0$, we conclude that $\mathrm{Ric}_g(\nabla_g u, \nabla_g u)$ vanishes identically.

Thus, by Lemma 3.6, we deduce that u is constant. □

Proof of Theorem 3.2. We put coordinates $(y, x) \in \mathcal{M} = M \times \mathbb{R}^+$.

Given a smooth and bounded $u_o : M \to \mathbb{R}$, we can define the harmonic extension $\mathscr{E}u_o : M \times \mathbb{R}^+ \to \mathbb{R}$ as the unique bounded function solving

$$\begin{cases} \Delta_{\bar{g}}(\mathscr{E}u_o) = 0 & \text{in } M \times \mathbb{R}^+, \\ \mathscr{E}u_o = u_o & \text{on } M \times \{0\}. \end{cases} \tag{3-7}$$

See [Cabré and Solà-Morales 2005, Section 2.4] for further details.

Then, we define

$$\mathscr{L}u_o := \partial_\nu(\mathscr{E}u_o)|_{x=0}. \tag{3-8}$$

We claim that, for any point in $M \to \mathbb{R}$,

$$-\partial_x(\mathscr{E}u_o) = \mathscr{E}(\mathscr{L}u_o). \tag{3-9}$$

Indeed, by differentiating the PDE in (3-7), we get $\Delta_{\bar{g}}\partial_x(\mathscr{E}u_o) = 0$. On the other hand, $-\partial_x(\mathscr{E}u_o)(0, y) = \partial_\nu(\mathscr{E}u_o)(0, y) = \mathscr{L}u_o$, thanks to (3-8). Moreover, $\partial_x(\mathscr{E}u_o)$ is bounded by elliptic estimates, since so is u_o. Consequently, $-\partial_x(\mathscr{E}u_o)$ is a bounded solution of (3-7) with u_o replaced by $\mathscr{L}u_o$. Thus, by the uniqueness of bounded solutions of (3-7), we obtain (3-9).

By exploiting (3-8) and (3-9), we see that

$$\begin{aligned} \mathscr{L}^2 u_o &= \partial_\nu(\mathscr{E}(\mathscr{L}u_o))|_{x=0} = -\partial_x(\mathscr{E}(\mathscr{L}u_o))|_{x=0} \\ &= -\partial_x(-\partial_x(\mathscr{E}u_o))|_{x=0} = \partial_{xx}(\mathscr{E}u_o)|_{x=0}. \end{aligned} \tag{3-10}$$

On the other hand, using the PDE in (3-7), we get

$$0 = \Delta_{\bar{g}}(\mathscr{E}u_o) = \Delta_g(\mathscr{E}u_o) + \partial_{xx}(\mathscr{E}u_o),$$

so (3-10) becomes

$$\mathscr{L}^2 u_o(y) = \partial_{xx}(\mathscr{E}u_o)(0, y) = -\Delta_g(\mathscr{E}u_o)(0, y) = -\Delta_g u_o(y),$$

for any $y \in M$, that is

$$\mathscr{L} = (-\Delta_g)^{1/2}. \tag{3-11}$$

With these observations in hand, we now take u as in the statement of Theorem 3.2 and define $v := \mathscr{E}u$.

From (3-8) and (3-11), we have $\partial_\nu v|_{x=0} = \partial_\nu(\mathscr{E}u)|_{x=0} = \mathscr{L}u = (-\Delta_g)^{1/2}u$. Consequently, recalling (3-1), we obtain that v is a bounded solution of (1-4). Furthermore, the function v is stable, thanks to (4-12). Hence v is constant by Theorem 3.1, and so we obtain the desired result for $u = v|_{x=0}$. □

Proof of Theorem 3.3. Given $p=(m,x)\in M\times\mathbb{R}^+$, we define $d_g(m)$ to be the geodesic distance of m in M (with respect to a fixed point) and

$$d(p):=\sqrt{d_g(m)^2+x^2}.$$

Let also $\hat{B}_R:=\{p\in M\times\mathbb{R}^+ : d(p)<R\}$, for any $R>0$. Notice that $|\nabla_g u|\in L^\infty(M\times\mathbb{R}^+)$ by elliptic estimates, and that $\hat{B}_R\subseteq B_R\times[0,R]$, where B_R is the corresponding geodesic ball in M.

As a consequence, by our assumption on the volume of B_R, we obtain

$$\int_{\hat{B}_R}|\nabla_g u|^2\,dV_{\bar{g}}\leqslant\int_{\hat{B}_R}|\nabla_{\bar{g}} u|^2\,dV_{\bar{g}}=\int_{\partial\hat{B}_R}uu_\nu\leqslant C(R+1)^2\,\|\nabla_g u\|_{L^\infty(M\times\mathbb{R}^+)}\|u\|_\infty.$$

That is,

$$\int_{\hat{B}_R}|\nabla_g u|^2\,dV_{\bar{g}}\leqslant CR^2\quad\text{for any } R\geqslant 1. \tag{3-12}$$

Also, since d_g is a distance function on M (see [Petersen 1998, pages 34 and 123]), we have

$$|\nabla d(p)|=|(d_g(m)\nabla_g d_g(m),x)|/d(p)\leqslant 1. \tag{3-13}$$

Also, given $R\geqslant 1$, we define

$$\phi_R(p):=\begin{cases}1 & \text{if } d(p)\leqslant\sqrt{R},\\ (\log\sqrt{R})^{-1}\big(\log R-\log(d(p))\big) & \text{if } d(p)\in(\sqrt{R},R),\\ 0 & \text{if } d(p)\geqslant R.\end{cases}$$

Notice that by (3-13), up to a set of zero $V_{\bar{g}}$-measure,

$$|\nabla\phi_R(p)|\leqslant\frac{\chi_{\hat{B}_R\setminus\hat{B}_{\sqrt{R}}}(p)}{\log\sqrt{R}\,d(p)}.$$

As a consequence,

$$\begin{aligned}(\log\sqrt{R})^2\int_{M\times\mathbb{R}^+}|\nabla_g u|^2|\nabla\phi_R|^2\,dV_{\bar{g}}&\leqslant\int_{\hat{B}_R\setminus\hat{B}_{\sqrt{R}}}\frac{|\nabla_g u(p)|^2}{d(p)^2}\,dV_{\bar{g}}(p)\\ &=\int_{\hat{B}_R\setminus\hat{B}_{\sqrt{R}}}|\nabla_g u(p)|^2\Big(\frac{1}{R^2}+\int_{d(p)}^R\frac{2\,dt}{t^3}\Big)\,dV_{\bar{g}}(p)\\ &\leqslant\frac{1}{R^2}\int_{\hat{B}_R}|\nabla_g u(p)|^2\,dV_{\bar{g}}(p)+\int_{\sqrt{R}}^R\int_{\hat{B}_t}\frac{2|\nabla_g u(p)|^2}{t^3}\,dV_{\bar{g}}(p)\,dt.\end{aligned}$$

Therefore, by (3-12),

$$(\log\sqrt{R})^2\int_{M\times\mathbb{R}^+}|\nabla_g u|^2|\nabla\phi_R|^2\,dV_{\bar{g}}\leqslant C\Big(1+\int_{\sqrt{R}}^R\frac{2\,dt}{t}\Big)\leqslant 3C\log R.$$

Consequently, from (2-4),

$$\int_{M\times\mathbb{R}^+} (\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_g u|^2 - |\nabla_g|\nabla_g u||^2)\phi_R^2 \leqslant \frac{12C}{\log R}. \tag{3-14}$$

From this and (3-3), we conclude that

$$\int_{M\times\mathbb{R}^+} \mathrm{Ric}_g(\nabla_g u, \nabla_g u)\phi_R^2 \leqslant \frac{12C}{\log R}.$$

By sending $R \to +\infty$, we obtain that $\mathrm{Ric}_g(\nabla_g u, \nabla_g u)$ vanishes identically.

Hence, u is constant, thanks to Lemma 3.6, proving Theorem 3.3. □

Proof of Theorem 3.4. The proof of Theorem 3.3 can be carried out in this case too, up to formula (3-14).

Then, (3-14) in this case gives that

$$\int_{M\times\mathbb{R}^+} (|H_g u|^2 - |\nabla_g|\nabla_g u||^2)\phi_R^2 \leqslant \frac{12C}{\log R}.$$

By sending $R \to +\infty$, and by recalling (3-3), we conclude that $|H_g u|$ is identically equal to $|\nabla_g|\nabla_g u||$ on $(M \times \{x\}) \cap \{\nabla_g u \neq 0\}$ for any fixed $x > 0$.

Consequently, by [Farina et al. 2008b, Lemma 5], for any $k = 1, \dots, n$ there exist $\kappa^k : M \to \mathbb{R}$ such that

$$\nabla_g(\nabla_g u)^k(p) = \kappa^k(p)\nabla_g u(p) \quad \text{for any } p \in (M \times \{x\}) \cap \{\nabla_g u \neq 0\}.$$

From this and [Farina et al. 2008b, the computation starting on formula (23)], we see that every connected component of $\{y \in M : u(x, y) = c\}$ is a geodesic. □

4. The case $\gamma \neq 1/2$

In this section, we provide the suitable adaptations of the previous arguments to deal with the case $\gamma \neq 1/2$. We also construct the nonlocal operators. Recall here that M is boundaryless, so that we do need to take care of the traces.

Given $\gamma \in (0, 1)$, let $\alpha = 1-2\gamma \in (-1, 1)$. Using variables $(x, y) \in (0, +\infty) \times M$, the space $H^\gamma(M)$ coincides with the trace on $\partial\mathcal{M}$ of

$$H^1(x^\alpha) := \left\{ u \in H^1_{\mathrm{loc}}(\mathcal{M}) : \int_{\mathcal{M}} x^\alpha (u^2 + |\nabla u|^2_{\bar{g}})\, dx\, dy < +\infty \right\}.$$

In other words, $v := u|_{\partial\mathcal{M}} \in H^\gamma(M)$ for any function $u \in H^1(x^\alpha) \cap C(\bar{\mathcal{M}})$, and there exists a constant $C > 0$ such that $\|v\|_{H^\gamma(M)} \leqslant C\|u\|_{H^1(x^\alpha)}$. So, by a standard density argument (see [Chiadò Piat and Serra Cassano 1994] in the case of $M = \mathbb{R}^n$), every $u \in H^1(x^\alpha)$ has a well-defined trace $v \in H^\gamma(M)$. Conversely, any $v \in H^\gamma(M)$ is the trace of a function $u \in H^1(x^\alpha)$. In addition, the function $u \in H^1(x^\alpha)$ defined by

$$u := \arg\min\left\{\int_{\mathcal{M}} x^\alpha |\nabla w|_{\bar g}^2 \, dx : w|_{\partial\mathcal{M}} = v\right\} \tag{4-1}$$

solves the PDE

$$\begin{cases} \operatorname{div}_{\bar g}(x^\alpha \nabla_{\bar g} u) = 0 & \text{in } \mathcal{M}, \\ u = v & \text{on } \partial\mathcal{M}. \end{cases} \tag{4-2}$$

By standard elliptic regularity, u is smooth in $\mathcal{M}$. It turns out that $x^\alpha u_x(x,\cdot)$ converges in $H^{-\gamma}(M)$ to a distribution $f \in H^{-\gamma}(M)$, as $x \to 0^+$, that is, u solves

$$\begin{cases} \operatorname{div}_{\bar g}(x^\alpha \nabla_{\bar g} u) = 0 & \text{in } \mathcal{M}, \\ -x^\alpha u_x = f & \text{on } \partial\mathcal{M}. \end{cases} \tag{4-3}$$

Consider the Dirichlet-to-Neumann operator

$$\Gamma_\alpha : H^\gamma(M) \to H^{-\gamma}(M), \qquad v \mapsto \Gamma_\alpha(v) = f := -x^\alpha u_x|_{\partial\mathcal{M}},$$

where u is the solution of (4-1)–(4-3).

Definition 4.1. There exists a constant $d_{n,\gamma} > 0$ defined by the condition that $(-\Delta_g)^\gamma v = d_{n,\gamma}\Gamma_\alpha(v)$ for every $v \in H^\gamma(M)$, where $\alpha = 1 - 2\gamma$.

In other words, given $f \in H^{-\gamma}(M)$, a function $v \in H^\gamma(M)$ solves the equation

$$\frac{1}{d_{n,\gamma}}(-\Delta_g)^\gamma v = f \quad \text{in } \mathbb{M} \tag{4-4}$$

if and only if its lifting $u \in H^1(x^\alpha)$ solves $u = v$ on $\partial\mathcal{M}$ and (4-3). For a proof of the claims that lead us to Definition 4.1, see [Caffarelli and Silvestre 2007] where such a construction is provided for $M = \mathbb{R}^n$.

Observe that Definition 4.1 does not give a proper way of defining $(-\Delta)^s v$ for arbitrary $v \in C^2(M)$. However, Definition 4.1 can be extended to the class of *bounded* functions $v \in C^2(M)$ and they coincide, using the construction in Section 3. Several works have been devoted to equations of the type (4-3), starting with the pioneering work of Cabré and Sola-Morales [2005] in the case $\alpha = 0$. Sire and Cabré [2010] have extended their techniques to any power $\alpha \in (-1, 1)$.

The previous discussion, in addition to the techniques in [Cabré and Sire 2010; Cabré and Solà-Morales 2005] and Section 3, allows us to prove the following results.

We first provide the weighted Poincaré inequality.

Theorem 4.2. *Let u be a stable solution of*

$$\begin{cases} \operatorname{div}_{\bar g}(x^\alpha \nabla_{\bar g})u = 0 & \textit{in } \mathcal{M} = M \times \mathbb{R}^+, \\ -x^\alpha \partial_x u = f(u) & \textit{on } M \times \{0\} \end{cases} \tag{4-5}$$

such that $x^\alpha \nabla_g u$ is bounded. Then, for every $\varphi \in C_0^\infty(M \times \mathbb{R})$, we have

$$\int_{M\times\mathbb{R}^+} x^\alpha(\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + |H_g u|^2 - |\nabla_g|\nabla_g u||^2)\varphi^2 \leqslant \int_{M\times\mathbb{R}^+} x^\alpha |\nabla_g u|^2 |\nabla\varphi|^2. \tag{4-6}$$

Proof. We use the technique used to prove Theorem 2.2, just making sure that we are able to control all the terms because of the weight x^α. We plug the test function test function $\xi = |\nabla_g u|\varphi$ into the stability condition, giving

$$\int_{\mathcal{M}} x^\alpha(\varphi^2|\nabla|\nabla_g u||^2 + \tfrac{1}{2}\langle\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle + |\nabla_g u|^2|\nabla\varphi|^2) - \int_M f'(u)|\nabla_g u|^2\varphi^2 \geqslant 0. \tag{4-7}$$

We have also

$$\begin{aligned}\int_{\mathcal{M}} x^\alpha \langle\nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle &= \int_{\mathbb{R}^+} x^\alpha \int_M \langle\nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle \\ &= -\int_{\mathbb{R}^+} x^\alpha \int_M \Delta_g|\nabla_g u|^2\varphi^2 = -\int_{\mathcal{M}} x^\alpha \Delta_g|\nabla_g u|^2\varphi^2.\end{aligned}$$

Hence, using the Bochner formula (2-5), we have

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}} \langle x^\alpha\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= \frac{1}{2}\int_{\mathcal{M}} x^\alpha \partial_x|\nabla_g u|^2\partial_x\varphi^2 \\ &\quad - \int_{\mathcal{M}} x^\alpha\varphi^2(|H_g u|^2 + \langle\nabla_g\Delta_g u, \nabla_g u\rangle + \mathrm{Ric}_g(\nabla_g u, \nabla_g u)).\end{aligned} \tag{4-8}$$

Using the equation for u, the first term on the right becomes, by just integrating by parts,

$$\begin{aligned}\int_{\mathcal{M}} x^\alpha\partial_x|\nabla_g u|^2\partial_x\varphi^2 &= \int_M\int_0^{+\infty} x^\alpha\partial_x|\nabla_g u|^2\partial_x\varphi^2 \\ &= -\int_M (x^\alpha\partial_x|\nabla_g u|^2\varphi^2)|_{x=0} - \int_M\int_0^{+\infty}\partial_x(x^\alpha\partial_x|\nabla_g u|^2)\varphi^2\end{aligned}$$

Consequently, (4-8) becomes

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}} x^\alpha\langle\nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= -\int_{\mathcal{M}} x^\alpha\varphi^2(\langle\nabla_g\Delta_g u, \nabla_g u\rangle + |H_g u|^2 + \mathrm{Ric}_g(\nabla_g u, \nabla_g u)) \\ &\quad - \frac{1}{2}\int_{\mathcal{M}}\varphi^2\partial_x(x^\alpha\partial_x|\nabla_g u|^2) - \frac{1}{2}\int_M (x^\alpha\partial_x|\nabla_g u|^2\varphi^2)|_{x=0}.\end{aligned} \tag{4-9}$$

We use the equation, noticing that $\Delta_g u = -(\alpha/x)\partial_x u$ on $\mathcal{M}$. This gives

$$\frac{1}{2}\int_{\mathcal{M}} x^\alpha \langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle = -\int_{\mathcal{M}} x^\alpha \varphi^2(|H_g u|^2 + \operatorname{Ric}_g(\nabla_g u, \nabla_g u)) - \frac{1}{2}\int_{\mathcal{M}} \varphi^2 x^\alpha \partial_{xx}|\nabla_g u|^2 - \frac{1}{2}\int_M (x^\alpha \partial_x |\nabla_g u|^2 \varphi^2)|_{x=0}. \tag{4-10}$$

Finally, we use the boundary condition to obtain that

$$f'(u)\nabla_g u = \nabla_g(f(u)) = \nabla_g(x^\alpha \partial_x u) = -\nabla_g(x^\alpha\partial_x)u \quad \text{on } M. \qquad \square$$

As a consequence of the previous theorem and adapting the proof of the case $\gamma = 1/2$, one has the following series of results.

Theorem 4.3. *Let $\alpha \in (-1, 1)$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$. Assume furthermore that M is compact and satisfies $\operatorname{Ric}_g \geqslant 0$, with Ric_g not vanishing identically. Then every bounded stable weak solution u of (4-5) is constant.*

Theorem 4.4. *Let $\gamma \in (0, 1)$. Let (M, g) be a compact manifold and let $u : M \to \mathbb{R}$ be a smooth bounded solution of*

$$(-\Delta_g)^\gamma u = f(u), \tag{4-11}$$

with

$$\int_{\mathcal{M}} (|\nabla_g \xi|^2 + |\nabla_x \xi|^2) - \int_{\partial\mathcal{M}} f'(u)\xi^2 \geqslant 0 \quad \textit{for every } \xi \in C_0^\infty(\mathcal{M}). \tag{4-12}$$

Assume that $\operatorname{Ric}_g \geqslant 0$ and Ric_g does not vanish identically. Then u is constant.

Theorem 4.5. *Let $\alpha \in (-1, 1)$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$, that M is complete, and $\operatorname{Ric}_g \geqslant 0$, with Ric_g not vanishing identically. Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)^2$ for some $C > 0$. Then every bounded stable weak solution u of (4-5) is constant.*

Theorem 4.6. *Let $\alpha \in (-1, 1)$. Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$ and Ric_g vanishes identically. Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)^2$ for some $C > 0$. Then for every $x > 0$ and $c \in \mathbb{R}$, every connected component of the submanifold*

$$\mathcal{S}_x = \{y \in M : u(x, y) = c\}$$

is a geodesic, where u is a bounded stable solution of (4-5).

5. The case of hyperbolic space

We now turn to a boundary problem in hyperbolic space. To do so we start with the construction of the nonlocal operators.

5.1. ***Scattering theory and construction of the nonlocal operators.*** Let M be a compact manifold of dimension n. Given a metric g on M, the conformal class $[g]$ of g is defined as the set of metrics $\hat{g}$ that can be written as $\hat{g} = \varphi g$ for a positive conformal factor φ.

Let $\mathcal{M}$ be a smooth manifold of dimension $n+1$ with boundary $\partial\mathcal{M} = M$. A function x is a *defining function* of $\partial\mathcal{M}$ in $\mathcal{M}$ if

$$x > 0 \quad \text{in } \mathcal{M}, \qquad x = 0 \quad \text{on } \partial\mathcal{M}, \qquad dx \neq 0 \quad \text{on } \partial\mathcal{M}.$$

We say that h is a *conformally compact* metric on $\mathcal{M}$ with conformal infinity $(M, [g])$ if there exists a defining function x such that the manifold $(\bar{\mathcal{M}}, \bar{h})$ is compact for $\bar{h} = x^2 g$, and $\bar{h}|_M \in [g]$.

Given a conformally compact, asymptotically hyperbolic manifold $(\mathcal{M}^{n+1}, h)$ and a representative $\hat{g}$ in $[g]$ on the conformal infinity M, there is a uniquely defined function x such that h has the normal form $h = x^{-2}(dx^2 + g_x)$ on $M \times (0, \epsilon)$ in $\mathcal{M}$, where g_x is a one-parameter family of metrics on M such that $g_x|_M = \hat{g}$; see [Graham and Zworski 2003] for precise statements and further details.

In this setting, the scattering matrix of M is defined as follows. Consider the following eigenvalue problem in $(\mathcal{M}, h)$:

$$-\Delta_h u - s(n-s)u = 0 \quad \text{in } \mathcal{M}, \tag{5-1}$$

where $s \in \mathbb{C}$. Problem (5-1) is solvable unless $s(n-s)$ belongs to the spectrum of $-\Delta_h$.

However, we have $\sigma(-\Delta_h) = [(n/2)^2, \infty) \cup \sigma_{\mathrm{pp}}(\Delta_h)$, where the pure point spectrum $\sigma_{\mathrm{pp}}(\Delta_h)$, that is, the set of L^2 eigenvalues, is finite and is contained in $(0, (n/2)^2)$.

Moreover, given any $f \in C^\infty(M)$, Graham and Zworski [2003] obtained a meromorphic family of solutions $u = \mathcal{P}(s)f$ such that $\mathcal{P}(s)f = Fx^{n-s} + Hx^s$, where $F, H \in \mathscr{C}^\infty(\mathcal{M})$ and $F|_M = f$. The scattering operator is defined as $S(s)f = H|_M$, which is a meromorphic family of pseudodifferential operators in $\mathrm{Re}(s) > n/2$ with poles at $s = n/2 + \mathbb{N}$ of finite rank residues. The relation between f and $S(s)f$ is like that of the Dirichlet to Neumann operator in standard harmonic analysis. Note that the principal symbol is

$$\sigma(S(s)) = 2^{n-2s} \frac{\Gamma(n/2 - s)}{\Gamma(s - n/2)} \sigma((-\Delta_g)^{s-n/2}).$$

The operators obtained when $s = n/2+\gamma$ and $\gamma \in (0, n/2) \setminus \mathbb{N}$ have been well studied; one defines $S(n/2+\gamma) = c_\gamma P_\gamma[h, g]$, and $P_\gamma[h, g]$ are the conformally invariant powers of the Laplacian constructed by [Fefferman and Graham 2002; Graham et al. 1992]. For a change of metric $g_u = u^{4/(n-2\gamma)} g$, we have

$$P_\gamma[h, g_u] f = u^{-(n+2\gamma)/(n-2\gamma)} P_\gamma[h, g](uf).$$

In particular, when $\gamma = 1$ we have the conformal Laplacian

$$P_1 = -\Delta_g + \frac{n-2}{4(n-1)} R_g$$

and when $\gamma = 2$, the Paneitz operator

$$P_2 = (-\Delta_g)^2 + \delta(a_n R_g g + b_n \operatorname{Ric}_g) d + \tfrac{1}{2}(n-4) Q_2.$$

See [del Mar González et al. 2010], where some geometric properties associated to these operators are investigated.

Remark. An important feature of these operators is their dependence on the metric on M and the metric on $\mathcal{M}$.

The following result, found in [Chang and del Mar González 2010], establishes a link between scattering theory on $\mathcal{M}$ and a local problem in the half-space.

Theorem 5.1. *Fix $0 < \gamma < 1$ and let $s = n/2+\gamma$. Assume that u is a smooth solution of*

$$\begin{cases} -\Delta_h u - s(n-s)u = 0 & \text{in } \mathbb{H}^{n+1}, \\ P_\gamma[h, |dy|^2] u = v & \text{on } \partial\mathbb{H}^{n+1} \end{cases} \tag{5-2}$$

for some smooth function v defined on $\partial\mathbb{H}^{n+1}$. Then the function $U = x^{s-n} u$ solves

$$\begin{cases} \operatorname{div}(x^{1-2\gamma} \nabla U) = 0 & \text{for } y \in \mathbb{R}^n,\ x \in (0, +\infty), \\ U(0, \cdot) = u & \text{in } \mathbb{R}^n, \\ -\lim_{x\to 0} x^{1-2\gamma} \partial_x U = Cv \end{cases} \tag{5-3}$$

for some constant C.

We consider the problem $(-\Delta_{|dy|^2})^\gamma u = f(u)$ on $\partial\mathbb{H}^{n+1}$. Chang and del Mar González [2010] proved that

$$(-\Delta_g)^\gamma = P_\gamma\left[\frac{dx^2 + |dy|^2}{x^2}, |dy|^2\right].$$

We consider then the nonlinear problem

$$\begin{cases} -\Delta_h u - s(n-s)u = 0 & \text{in } \mathcal{M} = \mathbb{H}^{n+1}, \\ P_\gamma[h, |dy|^2] u = f(u) & \text{on } \partial\mathbb{H}^{n+1}. \end{cases} \tag{5-4}$$

where f is $C^1(\mathbb{R})$ and the real parameter s in (5-4) is chosen to be $s = n/2 + \gamma$, where $\gamma \in (0, 1)$ and the metric h is given by $h = (dx^2 + |dy|^2)/x^2$.

5.2. *Results for the hyperbolic space.* The next theorem provides a flatness result when the manifold $\mathcal{M}$ is $\mathbb{H}^3$.

Theorem 5.2. *Let $n = 2$. Let u be a smooth solution of (5-4) and let $s = n/2 + \gamma$, where $\gamma \in (0, 1)$. We assume furthermore that the function $x^{n-s}u$ is bounded. Suppose that u is a monotone function, that is,*

$$\partial_{y_2} u > 0. \tag{5-5}$$

Then, for every $x > 0$ and $c \in \mathbb{R}$, each of the submanifolds

$$\mathcal{S}_x = \{y \in \mathbb{R}^n \mid u(x, y) = cx^{n-s}\}$$

is a Euclidean straight line.

The proof of Theorem 5.2 contains two main ingredients:

(1) We first notice that the metric on $\mathbb{H}^{n+1}$ is conformal to the flat metric on $\mathbb{R}^{n+1}_+$.

(2) We then use some results from [Sire and Valdinoci 2009] (see also [Cabré and Sire 2010; Cabré and Solà-Morales 2005] for related problems) to get the desired result.

Remark. The results do not depend on the nonlinearity f. This feature was already known in the case of standard interior reactions; see [Alberti et al. 2001].

Remark. The assumption on the Ricci curvature for the case of product manifolds is important. Indeed, already for Ricci flat manifolds such as $\mathbb{R}^n$, very little is known, as far as stable solutions are concerned, for dimensions $n \geqslant 3$.

Proof of Theorem 5.2. Let u be a solution as stated. By Theorem 5.1, the function $U = x^{s-n}u$ satisfies in a weak sense

$$\begin{cases} \operatorname{div}(x^{1-2\gamma} \nabla U) = 0 & \text{for } y \in \mathbb{R}^2,\ x \in (0, +\infty), \\ U(0, \cdot) = u, \\ -\lim_{x \to 0} x^{1-2\gamma} \partial_x U = f(U). \end{cases} \tag{5-6}$$

Notice that $\partial_{y_2} U > 0$, thanks to (5-5). Furthermore, U is bounded.

Therefore, we can apply the following theorem in [Sire and Valdinoci 2009]:

Theorem 5.3. *Let $v \in C^2_{\mathrm{loc}}(\mathbb{R}^n)$ be a bounded solution of*

$$\begin{cases} \operatorname{div}(x^{1-2\gamma} \nabla v) = 0 & \text{for } y \in \mathbb{R}^2,\ x \in (0, +\infty), \\ -\lim_{x \to 0} x^{1-2\gamma} \partial_x v = f(v), \end{cases} \tag{5-7}$$

with f locally Lipschitz. Suppose that

$$\partial_{y_2} v > 0. \tag{5-8}$$

Then, there exist $\omega \in \mathrm{S}^1$ *and* $v_o : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *such that* $v(x, y) = v_o(x, \omega \cdot y)$ *for any* $y \in \mathbb{R}^2$.

Therefore, $U(x, y) = U_o(x, \omega \cdot y)$ for suitable $U_o : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\omega \in S^1$. This gives directly the desired result. □

Remark. This result on the hyperbolic space cannot be obtained directly by the methods of the previous section. Indeed, it is an open problem to use weighted Poincaré inequalities for manifolds with negative curvature. As mentioned in the introduction, the general case of conformally compact manifolds is still open.

References

[Alberti et al. 2001] G. Alberti, L. Ambrosio, and X. Cabré, "On a long-standing conjecture of E. De Giorgi: Symmetry in 3D for general nonlinearities and a local minimality property", *Acta Appl. Math.* **65**:1-3 (2001), 9–33. MR 2002f:35080 Zbl 1121.35312

[Berger et al. 1971] M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Mathematics **194**, Springer, Berlin, 1971. MR 43 #8025 Zbl 0223.53034

[Cabré and Sire 2010] X. Cabré and Y. Sire, "Nonlinear equations involving fractional Laplacians: I and II", manuscript, 2010.

[Cabré and Solà-Morales 2005] X. Cabré and J. Solà-Morales, "Layer solutions in a half-space for boundary reactions", *Comm. Pure Appl. Math.* **58**:12 (2005), 1678–1732. MR 2006i:35116 Zbl 1102.35034

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7-9 (2007), 1245–1260. MR 2009k:35096 Zbl 1143.26002

[Chang and del Mar González 2010] A. Chang and M. del Mar González, "Fractional Laplacian in conformal geometry", manuscript, 2010.

[Chiadò Piat and Serra Cassano 1994] V. Chiadò Piat and F. Serra Cassano, "Relaxation of degenerate variational integrals", *Nonlinear Anal.* **22**:4 (1994), 409–424. MR 95a:49005 Zbl 0799.49012

[Farina et al. 2008a] A. Farina, B. Sciunzi, and E. Valdinoci, "Bernstein and De Giorgi type problems: New results via a geometric approach", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **7**:4 (2008), 741–791. MR 2009j:58020 Zbl 1180.35251

[Farina et al. 2008b] A. Farina, Y. Sire, and E. Valdinoci, "Stable solutions on Riemannian manifolds", preprint, 2008, available at http://www.mat.uniroma2.it/~valdinoc/manifold.pdf.

[Fefferman and Graham 2002] C. Fefferman and C. R. Graham, "Q-curvature and Poincaré metrics", *Math. Res. Lett.* **9**:2-3 (2002), 139–151. MR 2003f:53053 Zbl 1016.53031

[Ferrari and Valdinoci 2009] F. Ferrari and E. Valdinoci, "A geometric inequality in the Heisenberg group and its applications to stable solutions of semilinear problems", *Math. Ann.* **343**:2 (2009), 351–370. MR 2009j:35044 Zbl 1173.35057

[Gallot et al. 1990] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 2nd ed., Springer, Berlin, 1990. MR 91j:53001 Zbl 0716.53001

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR 2004c:58064 Zbl 1030.58022

[Graham et al. 1992] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, "Conformally invariant powers of the Laplacian, I: Existence", *J. London Math. Soc.* (2) **46**:3 (1992), 557–565. MR 94c:58226 Zbl 0726.53010

[Jost 1998] J. Jost, *Riemannian geometry and geometric analysis*, 2nd ed., Springer, Berlin, 1998. MR 99g:53025 Zbl 0997.53500

[Kato 1995] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin, 1995. MR 96a:47025 Zbl 0836.47009

[Kazdan 1988] J. L. Kazdan, "Unique continuation in geometry", *Comm. Pure Appl. Math.* **41**:5 (1988), 667–681. MR 89k:35039 Zbl 0632.35015

[del Mar González et al. 2010] M. del Mar González, R. Mazzeo, and Y. Sire, "Singular solutions of fractional order conformal Laplacians", manuscript, 2010.

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics **171**, Springer, New York, 1998. MR 98m:53001 Zbl 0914.53001

[Sire and Valdinoci 2009] Y. Sire and E. Valdinoci, "Fractional Laplacian phase transitions and boundary reactions: A geometric inequality and a symmetry result", *J. Funct. Anal.* **256**:6 (2009), 1842–1864. MR 2010c:35201 Zbl 1163.35019

[Sternberg and Zumbrun 1998a] P. Sternberg and K. Zumbrun, "Connectivity of phase boundaries in strictly convex domains", *Arch. Rational Mech. Anal.* **141**:4 (1998), 375–400. MR 99c:49045 Zbl 0911.49025

[Sternberg and Zumbrun 1998b] P. Sternberg and K. Zumbrun, "A Poincaré inequality with applications to volume-constrained area-minimizing surfaces", *J. Reine Angew. Math.* **503** (1998), 63–85. MR 99g:58028 Zbl 0967.53006

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