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For an almost contact metric manifold N, we find conditions under which either the total space of an S^1-bundle over N or the Riemannian cone over N admit a strong Kähler with torsion (SKT) structure. In so doing, we construct new 6-dimensional SKT manifolds. Moreover, we study the geometric structure induced on a hypersurface of an SKT manifold and use it to construct new SKT manifolds via appropriate evolution equations. We also study hyper-Kähler with torsion (HKT) structures on the total space of an S^1-bundle over manifolds with three almost contact structures.

1. Introduction

On any Hermitian manifold (M^{2n}, J, h) there exists a unique Hermitian connection ∇^B with totally skew-symmetric torsion, which is called the Bismut connection after [Bismut 1989]. The torsion 3-form $h(X, T^B(Y, Z))$ of ∇^B can be identified with the 3-form

$$-J\,dF(\cdot\,,\cdot\,,\cdot\,) = -dF(J\cdot\,, J\cdot\,, J\cdot\,),$$

where $F(\cdot\,,\cdot\,) = h(\cdot\,, J\cdot\,)$ is the fundamental 2-form associated to the Hermitian structure (J, h).

Hermitian structures with closed $J\,dF$ are called *strong Kähler with torsion* (in short, SKT) or *pluriclosed* [Egidi 2001]. Since $\partial\bar{\partial}$ acts as $\frac{1}{2}dJd$ on forms of bidegree $(1, 1)$, the latter condition is equivalent to $\partial\bar{\partial}F = 0$. SKT structures have been recently studied by many authors, and they also have applications in type II string theory and in 2-dimensional supersymmetric σ-models [Gates et al. 1984; Strominger 1986; Ivanov and Papadopoulos 2001].

The class of SKT metrics includes of course the Kähler metrics, but as in [Fino et al. 2004], we are interested on non-Kähler geometry, so by an *SKT metric* we

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will mean a Hermitian metric h whose fundamental 2-form F is $\partial\bar{\partial}$-closed but not d-closed.

Gauduchon [1984] showed that on a compact complex surface, an SKT metric can be found in the conformal class of any given Hermitian metric, but in higher dimensions the situation is more complicated.

SKT structures on 6-dimensional nilmanifolds, that is, on compact quotients of nilpotent Lie groups by discrete subgroups, were classified in [Fino et al. 2004; Ugarte 2007]. Simply connected examples of 6-dimensional SKT manifolds have been found in [Grantcharov et al. 2008] by using torus bundles, and recently Swann [2010] has reproduced them via the twist construction, by extending them to higher dimensions and finding new compact simply connected SKT manifolds. Moreover, Fino and Tomassini [2009] showed that the SKT condition is preserved by the blow-up construction.

The odd-dimensional analogue of a Hermitian structure is given by a normal almost contact metric structure. Indeed, on the product $N^{2n+1}\times\mathbb{R}$ of a $(2n+1)$-dimensional almost contact metric manifold N^{2n+1} by the real line $\mathbb{R}$, it is possible to define a natural almost complex structure, which is integrable if and only if the almost contact metric structure on N^{2n+1} is normal [Sasaki and Hatakeyama 1961]. More generally, it is possible to construct Hermitian manifolds starting from an almost contact metric manifold N^{2n+1} by considering a principal fiber bundle P with base space N^{2n+1} and structural group S^1, that is, an S^1-bundle over N^{2n+1}; see [Ogawa 1963]. Indeed, by using the almost contact metric structure on N^{2n+1} and the connection 1-form θ, Ogawa constructed an almost Hermitian structure (J, h) on P and found conditions under which J is integrable and (J, h) is Kähler.

In Section 2, we determine in Theorem 2.3 general conditions under which an S^1-bundle over an almost contact metric $(2n+1)$-dimensional manifold N^{2n+1} is SKT. We study the particular case when N^{2n+1} is quasi-Sasakian, that is, when it has an almost contact metric structure for which the fundamental form is closed (Corollary 2.4). In this way, we are able to construct some new 6-dimensional SKT examples, starting from 5-dimensional quasi-Sasakian Lie algebras, and also from Sasakian ones.

A Sasakian structure can be also seen as the analogue, in odd dimensions, of a Kähler structure. Indeed, by [Boyer and Galicki 1999], a Riemannian manifold (N^{2n+1}, g) of odd dimension $2n+1$ admits a compatible Sasakian structure if and only if the Riemannian cone $N^{2n+1}\times\mathbb{R}^+$ is Kähler. In Section 3, Theorem 3.1 gives the conditions that must be satisfied by the compatible almost contact metric structure on a Riemannian manifold (N^{2n+1}, g) in order that the Riemannian cone $N^{2n+1}\times\mathbb{R}^+$ be SKT. We provide an example of an SKT manifold constructed as a Riemannian cone, and in Section 4 we consider the case when the Riemannian cone is 6-dimensional. This case is interesting since one can impose that the SKT structure is in addition an SKT SU(3)-structure, and one can find relations with the SU(2)-structures studied by Conti and Salamon [2007].

In Section 5, we study the geometric structure induced naturally on any oriented hypersurface N^{2n+1} of a $(2n+2)$-dimensional manifold M^{2n+2} carrying an SKT structure, and in Section 6, we use such structures in Theorem 6.4 to construct new SKT manifolds via appropriate evolution equations [Hitchin 2001; Conti and Salamon 2007] starting from a 5-dimensional manifold endowed with an SU(2)-structure.

A good quaternionic analogue of Kähler geometry is given by *hyper-Kähler with torsion* (in short, HKT) geometry. An HKT manifold is a hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ admitting a hyper-Hermitian connection with totally skew-symmetric torsion, that is, one in which the three Bismut connections associated with the three Hermitian structures (J_r, h) coincide for $r = 1, 2, 3$. This geometry was introduced by Howe and Papadopoulos [1996] and later studied in [Grantcharov and Poon 2000; Fino and Grantcharov 2004; Barberis et al. 2009; Barberis and Fino 2008; Swann 2010].

In the interesting special case in which the torsion 3-form of such a hyper-Hermitian connection is closed, the HKT manifold is called *strong*.

In Section 7, Theorem 7.1 gives conditions under which an S^1-bundle over a $(4n+3)$-dimensional manifold endowed with three almost contact metric structures is HKT and in particular when it is strong HKT.

2. SKT structures arising from S^1-bundles

Consider a $(2n+1)$-manifold N^{2n+1} endowed with an almost contact metric structure (I, ξ, η, g); that is, I is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form, and g is a Riemannian metric on N^{2n+1}, satisfying together the conditions

$$I^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(IU, IV) = g(U, V) - \eta(U)\eta(V)$$

for any vector fields U and V on N^{2n+1}. Denote by ω the fundamental 2-form of (I, ξ, η, g); that is, ω is the 2-form on N^{2n+1} given by

$$\omega(\cdot\,,\cdot) = g(\cdot\,, I\,\cdot).$$

Given the tensor field I, consider its Nijenhuis torsion $[I, I]$, defined by

$$[I, I](X, Y) = I^2[X, Y] + [IX, IY] - I[IX, Y] - I[X, IY]. \tag{1}$$

On the product $N^{2n+1} \times \mathbb{R}$, one can define a natural almost complex structure

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(IX + f\xi,\ -\eta(X)\frac{d}{dt}\Big),$$

where f is a $\mathscr{C}^\infty$-function on $N^{2n+1} \times \mathbb{R}$ and t is the coordinate on $\mathbb{R}$.

Definition 2.1 [Sasaki and Hatakeyama 1961]. *We call an almost contact metric structure* (I, ξ, η, g) *on* N^{2n+1} normal *if the almost complex structure* J *on* $N^{2n+1} \times \mathbb{R}$ *is integrable, or equivalently if*

$$[I, I](X, Y) + 2d\eta(X, Y)\xi = 0$$

for any vector fields X, Y *on* N^{2n+1}.

By [Blair 1967, Lemma 2.1], one has $i_\xi d\eta = 0$ for a normal almost contact metric structure (I, ξ, η, g).

Remark 2.2. The normality of the almost contact structure implies as well that $I d\eta = d\eta$. Indeed, $d(\eta - i\,dt) = d\eta$ has no $(0, 2)$-part and therefore has no $(2, 0)$-part since $d\eta$ is real. Thus, $J\,d\eta = d\eta$, but $J\,d\eta = I\,d\eta$ as well since $i_\xi d\eta = 0$.

We recall that a Hermitian manifold (M, J, h) is SKT if and only if the 3-form $J\,dF$ is closed, where F is the fundamental 2-form of (J, h). We will use the convention that J acts on r-forms β by

$$(J\beta)(X_1, \dots, X_r) = \beta(JX_1, \dots, JX_r) \quad \text{for any vector fields } X_1, \dots, X_r.$$

We now show general conditions under which an S^1-bundle over an almost contact metric $(2n+1)$-dimensional manifold is SKT.

Let (N^{2n+1}, I, ξ, η) be a $(2n+1)$-dimensional almost contact manifold, and let Ω be a closed 2-form on N^{2n+1} that represents an integral cohomology class on N^{2n+1}. From the well-known result of Kobayashi [1956], we can consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$ and the connection 1-form θ on P whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N^{2n+1}$ is the projection.

By using the almost contact structure (I, ξ, η) and the connection 1-form θ, one can define an almost complex structure J on P as follows [Ogawa 1963]. For any right-invariant vector field X on P, the vector field JX is given by

$$\theta(JX) = -\pi^*(\eta(\pi_* X)) \quad \text{and} \quad \pi_*(JX) = I(\pi_* X) + \tilde{\theta}(X)\xi, \tag{2}$$

where $\tilde{\theta}(X)$ is the unique function on N^{2n+1} such that

$$\pi^*\tilde{\theta}(X) = \theta(X). \tag{3}$$

This definition can be extended to an arbitrary vector field X on P since X can be written in the form $X = \sum_j f_j X_j$, with f_j smooth functions on P, and X_j right-invariant vector fields. Then $JX = \sum_j f_j\, JX_j$.

Ogawa [1963] showed that when (N^{2n+1}, I, ξ, η) is normal, the almost complex structure J on P defined by (2) is integrable if and only if $d\theta$ is J-invariant, that is, if $J(d\theta) = d\theta$ or equivalently $d\theta(JX, Y) + d\theta(X, JY) = 0$ for any vector fields X and Y on P. That is, $d\theta$ is a complex 2-form on P having bidegree $(1, 1)$ with respect to J.

In terms of the 2-form Ω, whose lift to P is the curvature of the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$, the previous condition means that Ω is I-invariant, that is, $I(\Omega) = \Omega$. Therefore $i_\xi \Omega = 0$.

If $\{e^1, \dots, e^{2n}, \eta\}$ is an adapted coframe on a neighborhood U on N^{2n+1}, that is,

$$Ie^{2j-1} = -e^{2j} \quad \text{and} \quad Ie^{2j} = e^{2j-1} \quad \text{for } 1 \le j \le n,$$

then we can take $\{\pi^* e^1, \dots, \pi^* e^{2n}, \pi^*\eta, \theta\}$ as a coframe in $\pi^{-1}(U)$. By using the coframe $\{\pi^* e^1, \dots, \pi^* e^{2n}\}$, we may write

$$d\theta = \pi^*\alpha + \pi^*\beta \wedge \pi^*\eta,$$

where α is a 2-form in $\bigwedge^2 \langle e^1, \dots, e^{2n}\rangle$, and $\beta \in \bigwedge^1 \langle e^1, \dots, e^{2n}\rangle$.

Suppose that N^{2n+1} has a normal almost contact metric structure (I, ξ, η, g). We consider a principal S^1-bundle P with base space N^{2n+1} and connection 1-form θ, and endow P with the almost complex structure J (associated to θ) defined by (2). Since N^{2n+1} has a Riemannian metric g, a Riemannian metric h on P compatible with J (see [Ogawa 1963]) is given by

$$h(X, Y) = \pi^* g(\pi_* X, \pi_* Y) + \theta(X)\,\theta(Y) \tag{4}$$

for any right-invariant vector fields X and Y. This definition can be extended to any vector field on P.

Theorem 2.3. *Consider a $(2n+1)$-dimensional almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$, and let Ω be a closed 2-form on N^{2n+1} that represents an integral cohomology class. Consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$ with connection 1-form θ, whose curvature form is $d\theta = \pi^*(\Omega)$ for the projection $\pi : P \to N^{2n+1}$.*

The almost Hermitian structure (J, h) on P defined by (2) and (4) is SKT if and only if (I, ξ, η, g) is normal, $d\theta$ is J-invariant, and

$$\begin{aligned} d(\pi^*(I(i_\xi d\omega))) &= 0, \\ d(\pi^*(I(d\omega) - d\eta \wedge \eta)) &= (-\pi^*(I(i_\xi d\omega)) + \pi^*\Omega) \wedge \pi^*\Omega, \end{aligned} \tag{5}$$

where ω is the fundamental form of the almost contact metric structure (I, ξ, η, g).

Proof. As we mentioned, a result of Ogawa [1963] asserts that the almost complex structure J is integrable if and only if (I, ξ, η, g) is normal and $J(d\theta) = d\theta$. Thus, (J, h) is SKT if and only if the 3-form $J\,dF$ is closed. Using the first equality in (2), we find that the fundamental 2-form F on P is

$$\begin{aligned} F(X, Y) &= h(X, JY) \\ &= \pi^* g(\pi_* X, \pi_* JY) + \theta(X)\,\theta(JY) \\ &= \pi^* g(\pi_* X, \pi_* JY) - \theta(X)\,\pi^*\eta(\pi_* Y). \end{aligned}$$

Therefore, taking into account that we are working with a circle bundle (whose fiber is thus 1-dimensional), we have

$$
\begin{aligned}
F &= \pi^*\omega + \pi^*\eta \wedge \theta, \\
dF &= \pi^*(d\omega) + \pi^*(d\eta)\wedge\theta - \pi^*\eta \wedge d\theta, \\
J\,dF &= J(\pi^*(d\omega)) - J(\pi^*(d\eta)) \wedge \pi^*\eta - \theta \wedge d\theta
\end{aligned} \tag{6}
$$

since $J(\pi^*\eta) = \theta$ and J is integrable, and so $J(d\theta) = d\theta$. Moreover,

$$J(\pi^*(d\omega)) = \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta. \tag{7}$$

Indeed, locally and in terms of the adapted basis $\{e^1, \dots, e^{2n+1}\}$ with

$$Ie^{2j-1} = -e^{2j} \quad \text{for } 1 \le j \le n, \qquad Ie^{2n+1} = 0, \qquad \text{and} \qquad \eta = e^{2n+1},$$

we can write $d\omega = \alpha + \beta \wedge \eta$, where the local forms $\alpha \in \bigwedge^3\langle e^1, \dots, e^{2n}\rangle$ and $\beta \in \bigwedge^2\langle e^1, \dots, e^{2n}\rangle$ are generated only by $e^1, \dots, e^{2n}$. Furthermore, we have $I\alpha = I(d\omega)$ and $\beta = i_\xi d\omega$. Thus

$$J(\pi^*(d\omega)) = J(\pi^*(\alpha)) + J(\pi^*(i_\xi d\omega)) \wedge \theta.$$

Now, by using (2) and (3), we see that $J(\pi^*(\alpha)) = \pi^*(I\alpha)$ and $J(\pi^*(i_\xi d\omega)) = \pi^*(I(i_\xi d\omega))$, which proves (7). As a consequence of Remark 2.2,

$$J(\pi^*(d\eta)) = \pi^*(I(d\eta)) - \pi^*(I(i_\xi d\eta)) \wedge \theta = \pi^*(d\eta) \tag{8}$$

since $i_\xi d\eta = 0$ and $I\,d\eta = d\eta$.

Using (7) and (8), we get

$$J\,dF = \pi^*(I(d\omega)) + \pi^*(I(i_\xi d\omega)) \wedge \theta - \pi^*(d\eta) \wedge \pi^*\eta - \theta \wedge d\theta. \tag{9}$$

Therefore,

$$
\begin{aligned}
d(J\,dF) = d(\pi^*(I(d\omega))) &+ d(\pi^*(I(i_\xi d\omega))) \wedge \theta + \pi^*(I(i_\xi d\omega)) \wedge d\theta \\
&- d(\pi^*(d\eta)) \wedge \pi^*\eta - \pi^*(d\eta) \wedge d\pi^*\eta - d\theta \wedge d\theta.
\end{aligned}
$$

Consequently, $d(J\,dF) = 0$ if and only if

$$
\begin{aligned}
d(\pi^*(I(i_\xi d\omega))) &= 0, \\
d(\pi^*(I(d\omega) - d\eta \wedge \eta)) &= (\pi^*(-I(i_\xi d\omega)) + d\theta) \wedge d\theta. \qquad \square
\end{aligned}
$$

An almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$ is *quasi-Sasakian* if it is normal and its fundamental form ω is closed. In particular, if $d\eta = \alpha\,\omega$, then the almost contact metric structure is called *α-Sasakian*. When $\alpha = -2$, the structure is said to be *Sasakian*.

By [Friedrich and Ivanov 2002, Theorem 8.2], an almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$ admits a connection ∇^c that preserves the almost contact metric structure and has totally skew-symmetric torsion tensor if and only if the Nijenhuis tensor of I, given by (1), is skew-symmetric and ξ is a Killing vector field. This connection is unique.

In particular, on any quasi-Sasakian manifold $(N^{2n+1}, I, \xi, \eta, g)$ there exists a unique connection ∇^c with totally skew-symmetric torsion, such that

$$\nabla^c I = 0, \quad \nabla^c g = 0, \quad \nabla^c \eta = 0.$$

Such a connection ∇^c is uniquely determined by

$$g(\nabla^c_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2}(d\eta \wedge \eta)(X, Y, Z), \tag{10}$$

where ∇^g is the Levi-Civita connection and $\frac{1}{2}(d\eta \wedge \eta)$ is the torsion 3-form of ∇^c.

Corollary 2.4. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a quasi-Sasakian $(2n+1)$-manifold, and let Ω be a closed 2-form on N^{2n+1} that represents an integral cohomology class. Consider the circle bundle $S^1 \hookrightarrow P \to N^{2n+1}$ with connection 1-form θ whose curvature form is $d\theta = \pi^*(\Omega)$ for the projection $\pi : P \to N^{2n+1}$. The almost Hermitian structure (J, h) on P defined by (2) and (4) is SKT if and only if Ω is I-invariant, $i_\xi \Omega = 0$, and*

$$d\eta \wedge d\eta = -\Omega \wedge \Omega. \tag{11}$$

The Bismut connection ∇^B of (J, h) on P and the connection ∇^c on N given by (10) are related by

$$h(\nabla^B_X Y, Z) = \pi^* g(\nabla^c_{\pi_* X} \pi_* Y,\ \pi_* Z) \tag{12}$$

for any vector fields $X, Y, Z \in \operatorname{Ker}\theta$.

Proof. Since $d\omega = 0$, if we impose the SKT condition, then we get by using the previous theorem the equation (11).

The Bismut connection ∇^B associated to the Hermitian structure (J, h) on P is

$$h(\nabla^B_X Y, Z) = h(\nabla^h_X Y, Z) - \tfrac{1}{2} dF(JX, JY, JZ) \tag{13}$$

for any vector fields X, Y, Z on P, where ∇^h is the Levi-Civita connection associated to h. Then, for any X, Y, Z in the kernel of θ, we have

$$h(\nabla^B_X Y, Z) = \pi^* g(\nabla^h_X Y, Z) + \tfrac{1}{2}(\pi^*(d\eta) \wedge \pi^*\eta)(X, Y, Z).$$

By [Ogawa 1963, Lemma 3] and the definition of ∇^c, we get

$$\begin{aligned} h(\nabla^B_X Y, Z) &= \pi^* g(\nabla^g_{\pi_* X} \pi_* Y, \pi_* Z) + \tfrac{1}{2}(\pi^*(d\eta) \wedge \pi^*\eta)(X, Y, Z) \\ &= \pi^* g(\nabla^c_{\pi_* X} \pi_* Y, \pi_* Z) \end{aligned}$$

for any X, Y, Z in the kernel of θ. $\square$

Remark 2.5. If the structure (I, ξ, η, g) is α-Sasakian, equation (11) reads

$$\Omega \wedge \Omega = -\alpha^2 \omega \wedge \omega.$$

In the case of a trivial S^1-bundle, that is, if we consider the natural almost Hermitian structure on the product $N^{2n+1} \times \mathbb{R}$, we get this:

Corollary 2.6. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold. Impose on the product $N^{2n+1} \times \mathbb{R}$ the almost complex structure J given by*

$$JX = IX \quad \textit{for } X \in \operatorname{Ker} \eta \quad \textit{and} \quad J\xi = -\frac{d}{dt},$$

and the metric h given by $h = g + (dt)^2$. The Hermitian structure (J, h) is SKT if and only if (I, ξ, η, g) is normal, $d(I(d\omega)) = d(d\eta \wedge \eta)$ and $d(I(i_\xi d\omega)) = 0$, where ω denotes the fundamental 2-form of the almost contact metric structure (g, I, ξ, η).

Corollary 2.7. *Let $(N^{2n+1}, I, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi-Sasakian manifold with $d\eta \wedge d\eta = 0$. The Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}$ is SKT. Moreover, its Bismut connection ∇^B coincides with the unique connection ∇^c on N^{2n+1} given by (10).*

Proof. In this case, since $d\omega = 0$ we get $d(J\,dF) = -d(d\eta \wedge \eta)$. By using (12), we get $h(\nabla^B_X Y, Z) = g(\nabla^c_X Y, Z)$ for any vector fields X, Y, Z on N^{2n+1}. $\square$

2.1. *Examples.* We will present three examples of quasi-Sasakian Lie algebras satisfying the condition $d\eta \wedge d\eta = 0$. By applying Corollary 2.7, one gets an SKT structure on the product of the corresponding simply connected Lie group by $\mathbb{R}$.

Example 2.8. Let $\mathfrak{s}$ be the 5-dimensional Lie algebra with structure equations

$$\begin{aligned}
de^1 &= e^{13} + e^{23} + e^{25} - e^{34} + e^{35},\\
de^2 &= 2e^{12} - 2e^{13} + e^{14} - e^{15} - e^{24} + e^{34} + e^{45},\\
de^3 &= -e^{12} + e^{13} + e^{14} - e^{15} + 2e^{24} - 2e^{34} + e^{45},\\
de^4 &= -e^{12} - e^{23} + e^{24} - e^{25} - e^{35},\\
de^5 &= e^{12} - e^{13} - e^{24} + e^{34},
\end{aligned}$$

where $e^{ij} = e^i \wedge e^j$. On $\mathfrak{s}$, take the quasi-Sasakian structure (I, ξ, η, g) given by

$$\eta = e^5, \quad Ie^1 = -e^2, \quad Ie^3 = -e^4, \quad \omega = -e^{12} - e^{34}, \quad g = \textstyle\sum_{j=1}^5 (e^j)^2. \tag{14}$$

This quasi-Sasakian structure satisfies the condition $d(d\eta \wedge \eta) = 0$. The Lie algebra $\mathfrak{s}$ is 2-step solvable since the commutator

$$\mathfrak{s}^1 = [\mathfrak{s}, \mathfrak{s}] = \mathbb{R}\langle e_1 - e_4,\ e_2 + e_3,\ e_1 - e_2 + 2e_3 - e_5\rangle$$

is abelian, where $\{e_1, \dots, e_5\}$ denotes the dual basis of $\{e^1, \dots, e^5\}$. Moreover, $\mathfrak{s}$ has trivial center, is irreducible and nonunimodular, since the trace of ad_{e_1} is -3.

Example 2.9. Consider the family of 2-step solvable Lie algebras $\mathfrak{s}_a$ for $a \in \mathbb{R} - \{0\}$, given by

$$\begin{aligned} de^1 &= a\,e^{23} + 3e^{25}, & de^3 &= a\,e^{34}, \\ de^2 &= -a\,e^{13} - 3e^{15}, & de^4 &= 0, \\ & & de^5 &= -\tfrac{1}{3}a^2 e^{34}. \end{aligned}$$

The almost contact metric structure (I, ξ, η, g) defined in (14) is quasi-Sasakian and satisfies the condition $d\eta \wedge d\eta = 0$. The second cohomology group of $\mathfrak{s}_a$ is generated by e^{12} and e^{45}.

Example 2.10. Another family of quasi-Sasakian Lie algebras that satisfies the condition $d\eta \wedge d\eta = 0$ is $\mathfrak{g}_b$ for $b \in \mathbb{R} - \{0\}$, with structure equations

$$\begin{aligned} de^1 &= b(e^{13} + e^{14} - e^{23} + e^{24}) + e^{25}, & de^3 &= 2e^{45}, \\ de^2 &= b(-e^{13} + e^{14} - e^{23} - e^{24}) - e^{15}, & de^4 &= -2e^{35}, \\ & & de^5 &= -4b^2 e^{34}, \end{aligned}$$

and endowed with the quasi-Sasakian structure given by (14). The second cohomology group of $\mathfrak{g}_b$ is generated by e^{12}. The Lie algebras $\mathfrak{g}_b$ are not solvable since the commutators are $[\mathfrak{g}_b, \mathfrak{g}_b] = \mathfrak{g}_b$.

The Lie groups underlying Examples 2.9 and 2.10 also satisfy the conditions of Corollary 2.4 with $\Omega \wedge \Omega = 0$, by just taking as connection 1-form the 1-form e^6 such that $de^6 = \lambda e^{12}$. Then, $\Omega = \lambda e^{12}$. With this expression of de^6, we have

$$d^2 e^6 = 0, \quad J(de^6) = de^6, \quad \text{and} \quad de^6 \wedge de^6 = 0.$$

Therefore, equation (11) is satisfied. Observe that $\lambda = 0$ provides examples of trivial S^1-bundles.

The next example allows us to recover one of the 6-dimensional nilmanifolds found in [Fino et al. 2004]:

Example 2.11. Consider the 5-dimensional nilpotent Lie algebra with structure equations

$$\begin{aligned} de^j &= 0 \quad \text{for } j = 1, \dots, 4, \\ de^5 &= e^{12} + e^{34}, \end{aligned}$$

and endowed with the quasi-Sasakian structure given by (14). If we consider the closed 2-form $\Omega = e^{13} + e^{24}$ and apply Corollary 2.4, we see that there exists a nontrivial S^1-bundle over the corresponding 5-dimensional nilmanifold. Since $de^5 \wedge de^5 = -\Omega \wedge \Omega \neq 0$, the total space of this S^1-bundle is an SKT nilmanifold. More precisely, according to the classification given in [Fino et al. 2004] (see also

[Ugarte 2007]), the nilmanifold is the one with underlying Lie algebra isomorphic to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, where by $\mathfrak{h}_3$ we denote the real 3-dimensional Heisenberg Lie algebra.

Since the starting Lie algebra from Example 2.11 is Sasakian, it is natural to start with other 5-dimensional Sasakian Lie algebras to construct new SKT structures in dimension 6. A classification of 5-dimensional Sasakian Lie algebras was obtained in [Andrada et al. 2009].

Example 2.12. Consider the 5-dimensional Lie algebra $\mathfrak{k}_3$ with structure equations

$$\begin{aligned} de^1 &= 0, & de^4 &= 0,\\ de^2 &= -e^{13}, & de^5 &= \lambda e^{14} + \mu e^{23},\\ de^3 &= e^{12}, \end{aligned}$$

where $\lambda, \mu < 0$. By [Andrada et al. 2009], this algebra admits the Sasakian structure given by

$$Ie^1 = e^4, \quad Ie^2 = e^3, \quad \eta = e^5,$$
$$g = -\tfrac{1}{2}\lambda(e_1)^2 - \tfrac{1}{2}\lambda(e_2)^2 - \tfrac{1}{2}\mu(e_3)^2 - \tfrac{1}{2}\mu(e_4)^2 + (e_5)^2,$$

and is isomorphic to $\mathbb{R} \ltimes (\mathfrak{h}_3 \times \mathbb{R})$; moreover, the corresponding solvable simply connected Lie group admits a compact quotient by a discrete subgroup.

Consider on $\mathfrak{k}_3$ the closed 2-form $\Omega = \lambda e^{14} - \mu e^{23}$. The form Ω is I-invariant and satisfies $\Omega \wedge \Omega = -2\lambda\mu e^{1234}$. Since e^5 is the contact form and $de^5 \wedge de^5 = 2\lambda\mu e^{1234}$, we get again by Corollary 2.4 an SKT structure on a nontrivial S^1-bundle over the 5-dimensional solvmanifold. We denote by e^6 the connection 1-form.

The orthonormal basis $\{\alpha^1 = e^1,\ \alpha^2 = e^4,\ \alpha^3 = e^2,\ \alpha^4 = e^3,\ \alpha^5 = e^5,\ \alpha^6 = \theta\}$ for the SKT metric satisfies the equations

$$\begin{aligned} d\alpha^1 &= d\alpha^2 = 0, & d\alpha^3 &= -\alpha^{14}, \quad d\alpha^4 = \alpha^{13},\\ d\alpha^5 &= \lambda\alpha^{12} + \mu\alpha^{34}, & d\alpha^6 &= \lambda\alpha^{12} - \mu\alpha^{34}, \end{aligned}$$

and the complex structure is given by $J(X_1) = X_2$, $J(X_3) = X_4$ and $J(X_5) = X_6$, where $\{X_i\}_{i=1}^6$ denotes the basis dual to $\{\alpha^i\}_{i=1}^6$. Since the fundamental 2-form is $F = \alpha^{12} + \alpha^{34} + \alpha^{56}$, the 3-form torsion T of the SKT structure is

$$T = \lambda\alpha^{12}(\alpha^5 + \alpha^6) + \mu\alpha^{34}(\alpha^5 - \alpha^6).$$

Moreover, $*T = \lambda\alpha^{12}(\alpha^5 + \alpha^6) - \mu\alpha^{34}(\alpha^5 - \alpha^6)$, where $*$ denotes the metric's Hodge operator; this implies that the torsion form is also coclosed.

The only nonzero curvature forms $(\Omega^B)^i_j$ of the Bismut connection ∇^B are

$$(\Omega^B)^1_2 = -2\lambda^2\alpha^{12} \quad \text{and} \quad (\Omega^B)^3_4 = -2\mu^2\alpha^{34}.$$

A direct calculation shows that the 1-forms α^5 and α^6 and the 2-forms α^{12} and α^{34} are parallel with respect to the Bismut connection, which implies that $\nabla^B T = 0$.

Finally, $\mathrm{Hol}(\nabla^B) = U(1) \times U(1) \subset U(3)$ since $\nabla^B \alpha^i \neq 0$ for $i = 1, 2, 3, 4$.

3. SKT structures arising from Riemannian cones

Let N^{2n+1} be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (I, ξ, η, g), and denote by ω its fundamental 2-form.

The Riemannian cone of N^{2n+1} is defined as the manifold $N^{2n+1} \times \mathbb{R}^+$ equipped with the cone metric

$$h = t^2 g + (dt)^2. \tag{15}$$

The cone $N^{2n+1} \times \mathbb{R}^+$ has a natural almost Hermitian structure defined by

$$F = t^2 \omega + t\eta \wedge dt. \tag{16}$$

The almost complex structure J on $N^{2n+1} \times \mathbb{R}^+$ defined by (F, h) is given by

$$JX = IX \quad \text{for } X \in \operatorname{Ker} \eta \quad \text{and} \quad J\xi = -t\frac{d}{dt}.$$

In terms of a local orthonormal adapted coframe $\{e^1, \dots, e^{2n}\}$ for g with

$$\omega = -\sum_{j=1}^{n} e^{2j-1} \wedge e^{2j}, \tag{17}$$

we have

$$\begin{aligned} Je^{2j-1} &= -e^{2j}, \qquad Je^{2j} = e^{2j-1} \quad \text{for } j = 1, \dots, n, \\ J(te^{2n+1}) &= dt, \qquad J(dt) = -te^{2n+1}. \end{aligned} \tag{18}$$

The almost Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}^+$ is Kähler if and only if the almost contact metric structure (I, ξ, η, g) on N^{2n+1} is Sasakian, that is, a normal contact metric structure.

If we impose that the almost Hermitian structure (J, h) on $N^{2n+1} \times \mathbb{R}^+$ is SKT, we can prove the following:

Theorem 3.1. *Consider a $(2n+1)$-dimensional almost contact metric manifold $(N^{2n+1}, I, \xi, \eta, g)$. The almost Hermitian structure (J, h) given by (15) and (16) on the Riemannian cone $(N^{2n+1} \times \mathbb{R}^+, h)$ is SKT if and only if (I, ξ, η, g) is normal and*

$$-4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta = d(I(i_\xi d\omega)), \tag{19}$$

where ω denotes the fundamental 2-form of the almost contact metric structure (I, ξ, η, g).

Proof. J is integrable if and only if the almost contact metric structure is normal. We compute $J\,dF$. We have

$$dF = 2t dt \wedge \omega + t^2 d\omega + t\, d\eta \wedge dt, \quad \text{and so}$$
$$J dF = -2t^2 \eta \wedge \omega + t^2 J(d\omega) - t^2 d\eta \wedge \eta$$

since $J\omega = \omega$, $J(dt) = -t\eta$ and $J\,d\eta = d\eta$. Moreover, with respect to an adapted basis $\{e^1, \dots, e^{2n+1}\}$ we can get, in a way similar to the proof of Theorem 2.3, that

$$J\,d\omega = I(d\omega) + I(i_\xi d\omega) \wedge J\eta. \tag{20}$$

As a consequence, we get $J\,dF = -2t^2\eta\wedge\omega + t^2 I(d\omega) + t\,dt \wedge I(i_\xi d\omega) - t^2 d\eta \wedge \eta$. Therefore, by imposing that $d(J\,dF) = 0$, we obtain

$$-4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta - d(I(i_\xi d\omega)) = 0,$$
$$-2d(\eta \wedge \omega) + d(I(d\omega)) - d(d\eta \wedge \eta) = 0.$$

Since the second equation is a consequence of the first, the Hermitian structure (F, h) on the Riemannian cone $N^{2n+1} \times \mathbb{R}^+$ is SKT if and only if the almost contact metric structure $(I, \eta, \xi, g, \omega)$ on N^{2n+1} satisfies equation (19). □

Remark 3.2. As a consequence of previous theorem, when $n = 1$, equation (19) is satisfied if and only if the 3-dimensional manifold N is Sasakian. On the other hand, if $n > 1$ and the almost contact metric structure on N^{2n+1} is quasi-Sasakian (that is, $d\omega = 0$), then the structure has to be Sasakian, that is, $d\eta = -2\omega$.

Example 3.3. Consider the 5-dimensional Lie algebras $\mathfrak{g}_{a,b,c}$ with structure equations

$$\begin{aligned}
de^1 &= a e^{23} + 2e^{25} + \Big(-\tfrac{1}{2}ab + \frac{b^3}{2a} + 2\frac{b}{a}\Big) e^{34} + b e^{45},\\
de^2 &= -a e^{13} - 2e^{15} - \tfrac{1}{2} b c e^{34} - b e^{35},\\
de^3 &= \Big(-\frac{4}{a} - \frac{b^2}{a}\Big) e^{34},\\
de^4 &= c e^{34},\\
de^5 &= 2e^{12} + b e^{14} - b e^{23} + (2 + b^2) e^{34},
\end{aligned}$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. They are endowed with the normal almost contact metric structure $(I, \xi, \eta, g, \omega)$ with

$$I e^1 = -e^2, \quad I e^3 = -e^4, \quad \eta = e^5, \quad \omega = -e^{12} - e^{34}.$$

This structure satisfies (19), and therefore the Riemannian cones over the corresponding simply connected Lie groups are SKT.

4. SKT SU(3)-structures

Let (M^6, J, h) be a 6-dimensional almost Hermitian manifold. An SU(3)-structure on M^6 is determined by the choice of a (3, 0)-form $\Psi = \Psi_+ + i\Psi_-$ of unit norm. If Ψ is closed, then the underlying almost complex structure J is integrable and the manifold is Hermitian. We will denote the SU(3)-structure (J, h, Ψ) simply by (F, Ψ), where F is the fundamental 2-form, since from F and Ψ we can reconstruct the almost Hermitian structure.

Definition 4.1. *We say that an* SU(3)*-structure* (F, Ψ) *on* M^6 *is* SKT *if*

$$d\Psi = 0 \quad \text{and} \quad d(J\,dF) = 0, \tag{21}$$

where J is the associated complex structure.

We will see the relation between SKT SU(3)-structures in dimension 6 and SU(2)-structures in dimension 5.

First, we recall some facts about SU(2)-structures on a 5-dimensional manifold. An SU(2)-*structure* on a 5-dimensional manifold N^5 is an SU(2)-reduction of the principal bundle of linear frames on N^5. By [Conti and Salamon 2007, Proposition 1], these structures are in one-to-one correspondence with quadruplets $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form and ω_i are 2-forms on N^5 satisfying $\omega_i \wedge \omega_j = \delta_{ij} v$ and $v \wedge \eta \neq 0$ for some 4-form v, and $\omega_2(X, Y) \geq 0$ if $i_X \omega_3 = i_Y \omega_1$, where i_X denotes the contraction by X. Equivalently, an SU(2)-structure on N^5 can be viewed as the datum of (η, ω_1, Φ), where η is a 1-form, ω_1 is a 2-form, and $\Phi = \omega_2 + i\,\omega_3$ is a complex 2-form such that

$$\eta \wedge \omega_1 \wedge \omega_1 \neq 0, \quad \Phi \wedge \Phi = 0, \quad \omega_1 \wedge \Phi = 0, \quad \Phi \wedge \overline{\Phi} = 2\omega_1 \wedge \omega_1,$$

and Φ is of type (2, 0) with respect to ω_1.

Conti and Salamon [2007] locally characterize an SU(2)-structure as follows. If $(\eta, \omega_1, \omega_2, \omega_3)$ is an SU(2)-structure on a 5-manifold N^5, then locally there exists an orthonormal basis of 1-forms $\{e^1, \dots, e^5\}$ such that

$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}, \quad \eta = e^5\,.$$

We can also consider the local tensor field I given by

$$Ie^1 = -e^2, \quad Ie^2 = e^1, \quad Ie^3 = -e^4, \quad Ie^4 = e^3, \quad Ie^5 = 0.$$

This tensor gives rise to a global tensor field of type (1, 1) on the manifold N^5, defined by $\omega_1(X, Y) = g(X, IY)$ for any vector fields X and Y on N^5, where g is the Riemannian metric on N^5 underlying the SU(2)-structure. The tensor field I satisfies $I^2 = -\operatorname{Id} + \eta \otimes \xi$, where ξ is the vector field on N^5 dual to the 1-form η.

Therefore, given an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ we also have an almost contact metric structure (I, ξ, η, g) on the manifold, where ω_1 is its fundamental form.

Remark 4.2. Notice that we have two more almost contact metric structures when we consider ω_2 and ω_3 as fundamental forms.

If N^5 has an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$, the product $N^5 \times \mathbb{R}$ has a natural SU(3)-structure given by

$$F = \omega_1 + \eta \wedge dt \quad \text{and} \quad \Psi = (\omega_2 + i\omega_3) \wedge (\eta - i\,dt). \tag{22}$$

By Corollary 2.6, the previous SU(3)-structure is SKT if and only if

$$\begin{aligned} d(I(d\omega_1)) &= d(d\eta \wedge \eta), & d\omega_2 &= -3\omega_3 \wedge \eta, \\ d(I(i_\xi d\omega_1)) &= 0, & d\omega_3 &= 3\omega_2 \wedge \eta, \end{aligned} \tag{23}$$

proving this:

Theorem 4.3. *Suppose N^5 is a 5-dimensional manifold endowed with an* SU(2)-*structure $(\eta, \omega_1, \omega_2, \omega_3)$. The* SU(3)-*structure (F, Ψ) given by (22) on the product $N^5 \times \mathbb{R}$ is SKT if and only if the equations (23) are satisfied.*

Example 4.4. On the 5-dimensional Lie algebras introduced in Examples 2.8, 2.9 and 2.10, consider the SU(2)-structure given by

$$\omega = \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}.$$

For Example 2.8, we have

$$d\omega_2 = -2\omega_3 \wedge \eta - 4(e^{124} - e^{134}) \quad \text{and} \quad d\omega_3 = 2\omega_2 \wedge \eta + 4(e^{123} + e^{234}).$$

For Examples 2.9 and 2.10, we get $d\omega_2 = -3\omega_3 \wedge \eta$ and $d\omega_3 = 3\,\omega_2 \wedge \eta$. Therefore one gets an SKT SU(3)-structure on the product of the corresponding simply connected Lie groups by $\mathbb{R}$.

We will study the existence of SKT SU(3)-structures on a Riemannian cone over a 5-dimensional manifold N^5 endowed with an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$. Then N^5 has an induced almost contact metric structure (I, ξ, η, g), and ω_1 is its fundamental form.

The Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ of (N^5, g) has a natural SU(3)-structure defined by

$$F = t^2\omega_1 + t\eta \wedge dt \quad \text{and} \quad \Psi = t^2(\omega_2 + i\omega_3) \wedge (t\eta - i\,dt).$$

In terms of a local orthonormal coframe $\{e^1, \dots, e^5\}$ for g such that

$$\omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{13} + e^{24}, \quad \omega_3 = -e^{14} - e^{23}, \quad \eta = e^5,$$

we have

$$\begin{aligned} Je^1 &= -e^2, & Je^2 &= e^1, & Je^3 &= -e^4, \\ Je^4 &= e^3, & J(te^5) &= dt, & J(dt) &= -te^5. \end{aligned}$$

We recall that the SU(3)-structure (F, Ψ) on $N^5 \times \mathbb{R}^+$ is integrable if and only if the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 is Sasaki–Einstein, or equivalently if and only if

$$d\eta = -2\omega_1, \quad d\omega_2 = -3\omega_3 \wedge \eta, \quad d\omega_3 = 3\omega_2 \wedge \eta.$$

For the Riemannian cones, we can prove the following

Corollary 4.5. *Let N^5 be a 5-dimensional manifold endowed with an* SU(2)-*structure $(\eta, \omega_1, \omega_2, \omega_3)$. The* SU(3)-*structure (F, Ψ) on the Riemannian cone $(N^5 \times \mathbb{R}^+, h)$ is SKT if and only if*

$$\begin{aligned} -4\eta \wedge \omega_1 + 2I(d\omega_1) - 2d\eta \wedge \eta &= d(I(i_\xi d\omega_1)),\\ d\omega_2 &= 3\omega_3 \wedge \eta,\\ d\omega_3 &= -3\omega_2 \wedge \eta. \end{aligned} \tag{24}$$

Proof. By imposing that $d\Psi = 0$, we get the conditions $d\omega_2 = -3\,\omega_3 \wedge \eta$ and $d\omega_3 = 3\,\omega_2 \wedge \eta$. By imposing $d(J\,dF) = 0$, we get, as in the proof of Theorem 3.1, equation (19) for $\omega = \omega_1$. □

5. Almost contact metric structure induced on a hypersurface

We study the almost contact metric structure induced naturally on any oriented hypersurface N^{2n+1} of a $(2n+2)$-manifold M^{2n+2} equipped with an SKT structure.

Let $f : N^{2n+1} \to M^{2n+2}$ be an oriented hypersurface of a $(2n+2)$-dimensional manifold M^{2n+2} endowed with an SKT structure (J, h, F), and denote by $\mathbb{U}$ the unitary normal vector field. It is well known that N^{2n+1} inherits an almost contact metric structure (I, ξ, η, g) such that η and the fundamental 2-form ω are given by

$$\eta = -f^*(i_{\mathbb{U}} F) \quad \text{and} \quad \omega = f^* F, \tag{25}$$

where F is the fundamental 2-form of the almost Hermitian structure; see, for instance, [Blair 2002].

Proposition 5.1. *Suppose $f : N^{2n+1} \to M^{2n+2}$ is an immersion of an oriented $(2n+1)$-dimensional manifold into a $(2n+2)$-dimensional Hermitian manifold. If the Hermitian structure (J, h) on M^{2n+2} is SKT, then the induced almost contact metric structure (I, ξ, η, g) on N^{2n+1}, with η and ω given by (25), satisfies*

$$d(I d\omega - I(f^*(i_{\mathbb{U}} dF)) \wedge \eta) = 0. \tag{26}$$

Proof. Locally we can choose an adapted coframe $\{e^1, \dots, e^{2n+2}\}$ for the Hermitian structure so that the unitary normal vector field $\mathbb{U}$ is dual to e^{2n+2}. Since the almost complex structure J is given in this adapted basis by

$$\begin{aligned} &Je^{2j-1} = -e^{2j}, \qquad Je^{2j} = e^{2j-1} \quad \text{for } j=1,\dots,n,\\ &Je^{2n+1} = e^{2n+2}, \quad Je^{2n+2} = -e^{2n+1}, \end{aligned}$$

it follows that the tensor field I on N^{2n+1} has $If^*e^i = f^*Je^i$ for $i=1,\dots,2n+1$. That is,

$$If^*e^{2j-1} = -f^*e^{2j}, \quad If^*e^{2j} = f^*e^{2j-1} \text{ for } j=1,\dots,n, \quad If^*e^{2n+1}=0.$$

However, $If^*e^{2n+2} = 0 \neq f^*e^{2n+1} = -f^*Je^{2n+2}$.

We compute $f^*J\,dF$. First, we decompose (locally and in terms of the adapted basis) the differential of F as

$$dF = \alpha + \beta\wedge e^{2n+1} + \gamma\wedge e^{2n+2} + \mu\wedge e^{2n+1}\wedge e^{2n+2},$$

where the local forms

$$\alpha \in \bigwedge^3\langle e^1,\dots,e^{2n}\rangle, \quad \beta,\gamma \in \bigwedge^2\langle e^1,\dots,e^{2n}\rangle, \quad \mu \in \bigwedge^1\langle e^1,\dots,e^{2n}\rangle$$

are generated only by $e^1,\dots,e^{2n}$. Then,

$$J\,dF = J\alpha + J\beta\wedge e^{2n+2} - J\gamma\wedge e^{2n+1} + J\mu\wedge e^{2n+1}\wedge e^{2n+2}.$$

Since $f^*e^{2n+2} = 0$ and $f^*e^{2n+1} = \eta$, we have $f^*J\,dF = f^*J\alpha - (f^*J\gamma)\wedge\eta$. However, $f^*(i_{\mathbb{U}}dF) = f^*\gamma + f^*\mu\wedge\eta$, which implies that

$$I(f^*(i_{\mathbb{U}}dF)) = If^*\gamma = f^*J\gamma.$$

On the other hand, $Id\omega = Idf^*F = If^*dF = If^*\alpha = f^*J\alpha$. We conclude that

$$f^*J\,dF = f^*J\alpha - (f^*J\gamma)\wedge\eta = Id\omega - I(f^*(i_{\mathbb{U}}dF))\wedge\eta.$$

Now, if the Hermitian structure is SKT, then $J\,dF$ is closed and the induced structure satisfies (26). □

Remark 5.2. Notice that, using $i_{\mathbb{U}}dF = \mathcal{L}_{\mathbb{U}}F - di_{\mathbb{U}}F$, we can write (26) as

$$d(Id\omega - I(f^*(\mathcal{L}_{\mathbb{U}}F) + d\eta)\wedge\eta) = 0.$$

Therefore, if $f^*(\mathcal{L}_{\mathbb{U}}F)=0$, then the induced almost contact metric structure has to satisfy the equation $d(Id\omega - I(d\eta)\wedge\eta)=0$. In the case of the product $N^{2n+1}\times\mathbb{R}$, the condition $f^*(\mathcal{L}_{\mathbb{U}}F)=0$ is satisfied. In the case of the Riemannian cone, we have $\mathcal{L}_{d/dt}F = 2t\omega + dt\wedge\eta$ and therefore $f^*(\mathcal{L}_{d/dt}F) = 2\omega$. In this way, we recover some of the equations obtained in Corollary 2.6 and Theorem 3.1.

Now we study the structure that is induced naturally on any oriented hypersurface N^5 of a 6-manifold M^6 equipped with an SKT SU(3)-structure.

Let $f\colon N^5 \to M^6$ be an oriented hypersurface of a 6-manifold M^6 endowed with an SU(3)-structure $(F,\ \Psi = \Psi_+ + i\ \Psi_-)$, and denote by $\mathbb{U}$ the unitary normal vector field. Then N^5 inherits an SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ given by

$$\eta = -f^*(i_{\mathbb{U}} F), \quad \omega_1 = f^* F, \quad \omega_2 = -f^*(i_{\mathbb{U}} \Psi_-), \quad \omega_3 = f^*(i_{\mathbb{U}} \Psi_+). \tag{27}$$

Corollary 5.3. *Let $f\colon N^5 \to M^6$ be an immersion of an oriented 5-dimensional manifold into a 6-dimensional manifold with an* SU(3)*-structure. If the* SU(3)*-structure is SKT, then the induced* SU(2)*-structure on N^5 given by (27) satisfies*

$$d(I d\omega_1 - I f^*(i_{\mathbb{U}} dF) \wedge \eta) = 0, \tag{28}$$

$$d(\omega_2 \wedge \eta) = 0 \quad \textit{and} \quad d(\omega_3 \wedge \eta) = 0. \tag{29}$$

Proof. Equation (28) follows from Proposition 5.1 by taking $\omega = \omega_1$. Locally, we can choose an adapted coframe $\{e^1, \dots, e^5, e^6\}$ for the SU(3)-structure such that the unitary normal vector field $\mathbb{U}$ is dual to e^6. From (27), it follows that $\omega_2 \wedge \eta = f^* \Psi_+$ and $\omega_3 \wedge \eta = f^* \Psi_-$. If $\Psi = \Psi_+ + i\ \Psi_-$ is closed, then the induced structure satisfies (29). □

5.1. ***A simple example.*** Consider the 6-dimensional nilmanifold M^6 whose underlying nilpotent Lie algebra has structure equations

$$de^j = 0 \quad \text{for } j = 1, 2, 3, 6, \quad de^4 = e^{12}, \quad de^5 = e^{14},$$

and is endowed with the SU(3)-structure given by

$$F = -e^{14} - e^{26} - e^{53} \quad \text{and} \quad \Psi = (e^1 - i e^4) \wedge (e^2 - i e^6) \wedge (e^5 - i e^3).$$

The oriented hypersurface with normal vector field dual to e^2 is a 5-dimensional nilmanifold N^5 that by [Conti and Salamon 2007] has no invariant hypo structures. However, the SU(2)-structure on N^5, namely,

$$\eta = e^2, \quad \omega_1 = -e^{14} - e^{53}, \quad \omega_2 = -e^{15} - e^{34}, \quad \omega_3 = -e^{13} - e^{45}, \tag{30}$$

satisfies (28) and (29). In Section 6, we will show that by using this SU(2)-structure and appropriate evolution equations, we can construct an SKT SU(3)-structure on the product of N^5 with an open interval.

6. SKT evolution equations

The goal here is to construct SKT SU(3)-structures by using appropriate evolution equations, starting from a suitable SU(2)-structure on a 5-dimensional manifold. We follow ideas of [Hitchin 2001] and [Conti and Salamon 2007].

Lemma 6.1. *Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of* SU(2)*-structures on a 5-dimensional manifold N^5 for $t \in (a, b)$. The* SU(3)*-structure on $M^6 = N^5 \times (a, b)$*

given by $F=\omega_1(t)+\eta(t)\wedge dt$ *and* $\Psi=(\omega_2(t)+i\omega_3(t))\wedge\big(\eta(t)-i\,dt\big)$ *satisfies the condition* $d\Psi=0$ *if and only if* $(\eta(t),\omega_1(t),\omega_2(t),\omega_3(t))$ *is an* SU(2)*-structure such that, for any t in the open interval* (a,b),

$$\begin{aligned}\hat d(\omega_2(t)\wedge\eta(t))=0,\quad \partial_t(\omega_2(t)\wedge\eta(t))=-\hat d\omega_3(t),\\ \hat d(\omega_3(t)\wedge\eta(t))=0,\quad \partial_t(\omega_3(t)\wedge\eta(t))=\hat d\omega_2(t).\end{aligned}\tag{31}$$

Here, $\hat d$ denotes the exterior differential on N^5 and d is the exterior differential on M^6. We now present the additional evolution equations to be added to the last two of (31) in order to ensure that $dJdF=0$.

Proposition 6.2. *Let* $(\eta(t),\omega_1(t),\omega_2(t),\omega_3(t))$ *be a family of* SU(2)*-structures on* N^5 *for* $t\in(a,b)$. *The* SU(3)*-structure on* $M^6=N^5\times(a,b)$ *given by*

$$F=\omega_1(t)+\eta(t)\wedge dt\quad\text{and}\quad\Psi=(\omega_2(t)+i\omega_3(t))\wedge(\eta(t)-i\,dt),\tag{32}$$

has JdF *closed if and only if* $(\eta(t),\omega_1(t),\omega_2(t),\omega_3(t))$ *satisfies the evolution equations*

$$\begin{aligned}&\hat d\big(I_t\hat d\omega_1(t)-I_t(\partial_t\omega_1(t)+\hat d\eta(t))\wedge\eta(t)\big)=0,\\ &\partial_t\big(I_t\hat d\omega_1(t)-I_t(\partial_t\omega_1(t)+\hat d\eta(t))\wedge\eta(t)\big)\\ &\qquad=-\hat d\big(I_t(i_\xi\hat d\omega_1(t))-I_t(i_\xi(\partial_t\omega_1(t)+\hat d\eta(t)))\wedge\eta(t)\big),\end{aligned}\tag{33}$$

where $\xi(t)$ *denotes the vector field on* N^5 *dual to* $\eta(t)$ *for each* $t\in(a,b)$.

Proof. Since $F=\omega_1(t)+\eta(t)\wedge dt$, we have $dF=\hat d\omega_1+(\partial_t\omega_1+\hat d\eta)\wedge dt$. Define $\{e^1(t),\dots,e^4(t),\eta(t)\}$ to be a local adapted basis for the SU(2)-structure $(\eta(t),\omega_1(t),\omega_2(t),\omega_3(t))$. Then, $\{e^1(t),\dots,e^4(t),\eta(t),dt\}$ is an adapted basis for the SU(3)-structure (32), and J is given by

$$\begin{aligned}&Je^1(t)=-e^2(t),\quad Je^2(t)=e^1(t),\quad J\eta(t)=dt,\\ &Je^3(t)=-e^4(t),\quad Je^4(t)=e^3(t),\quad Jdt=-\eta(t).\end{aligned}$$

For each t, the structures I_t induced on N^5 are given by

$$\begin{aligned}&I_te^1(t)=-e^2(t),\quad I_te^2(t)=e^1(t),\\ &I_te^3(t)=-e^4(t),\quad I_te^4(t)=e^3(t),\quad I_t\eta(t)=0.\end{aligned}$$

Now, we can locally decompose a given $\tau(t)\in\Omega^k(N^5)$ for $t\in(a,b)$ as

$$\tau(t)=\alpha(t)+\beta(t)\wedge\eta(t),$$

where $\alpha(t)\in\bigwedge^k\langle e^1(t),\dots,e^4(t)\rangle$ and $\beta(t)\in\bigwedge^{k-1}\langle e^1(t),\dots,e^4(t)\rangle$. Therefore,

$$\begin{aligned}J\tau(t)&=J\alpha(t)+J\beta(t)\wedge J\eta(t)=I_t\alpha(t)+I_t\beta(t)\wedge dt\\ &=I_t\tau(t)-(-1)^kI_t(i_{\xi(t)}\tau(t))\wedge dt.\end{aligned}$$

Applying this to $J\,dF$, we get

$$\begin{aligned} J\,dF &= J\hat{d}\omega_1 - J(\partial_t\omega_1 + \hat{d}\eta)\wedge\eta(t) \\ &= I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta)\wedge\eta(t) + I_t(i_{\xi(t)}\hat{d}\omega_1)\wedge dt \\ &\qquad\qquad - I_t(i_\xi(\partial_t\omega_1 + \hat{d}\eta))\wedge\eta(t)\wedge dt. \end{aligned}$$

Finally, taking the differential of $J\,dF$, we get

$$\begin{aligned} dJ\,dF = {}& \hat{d}\big(I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta)\wedge\eta(t)\big) \\ & + \partial_t\big(I_t\hat{d}\omega_1 - I_t(\partial_t\omega_1 + \hat{d}\eta)\wedge\eta(t)\big)\wedge dt \\ & + \hat{d}\big(I_t(i_{\xi(t)}\hat{d}\omega_1) - I_t(i_\xi(\partial_t\omega_1 + \hat{d}\eta))\wedge\eta(t)\big)\wedge dt. \quad \square \end{aligned}$$

Remark 6.3. Observe that the first equation in (33) is exactly condition (28) for $F = \omega_1(t) + \eta(t)\wedge dt$. See Remark 5.2.

As a consequence of Lemma 6.1 and Proposition 6.2, we get the following:

Theorem 6.4. *For $t \in (a, b)$, let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of* SU(2)-*structures on a* 5-*dimensional manifold N^5 such that*

$$\hat{d}(\omega_2(t)\wedge\eta(t)) = 0 \quad \textit{and} \quad \hat{d}(\omega_3(t)\wedge\eta(t)) = 0 \tag{34}$$

for any t. If the evolution equations

$$\begin{aligned} &\hat{d}(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t))\wedge\eta(t)) = 0, \\ &\partial_t(I_t\hat{d}\omega_1(t) - I_t(\partial_t\omega_1(t) + \hat{d}\eta(t))\wedge\eta(t)) \\ &\qquad\qquad = -\hat{d}(I_t(i_\xi\hat{d}\omega_1(t)) - I_t(i_\xi(\partial_t\omega_1(t) + \hat{d}\eta(t)))\wedge\eta(t)), \\ &\partial_t(\omega_2(t)\wedge\eta(t)) = -\hat{d}\omega_3(t), \\ &\partial_t(\omega_3(t)\wedge\eta(t)) = \hat{d}\omega_2(t), \end{aligned} \tag{35}$$

are satisfied, then the SU(3)-*structure on $M = N\times(a, b)$ given by*

$$F = \omega_1(t) + \eta(t)\wedge dt \quad \textit{and} \quad \Psi = (\omega_2(t) + i\omega_3(t))\wedge(\eta(t) - i\,dt) \tag{36}$$

is SKT.

Example 6.5. Consider the Lie algebra with structure equations

$$de^j = 0 \quad \text{for } = 1, 2, 3, \quad de^4 = e^{12}, \quad de^5 = e^{14},$$

which underlies the 5-dimensional nilmanifold N^5 considered in Section 5.1. We endow it with the SU(2)-structure given by (30). It is easy to verify that

$$d(\omega_2\wedge\eta) = d(\omega_3\wedge\eta) = d(\omega_1\wedge\omega_1) = 0.$$

We evolve this SU(2)-structure by

$$\begin{aligned} \omega_1(t) &= -e^{14} - e^{53}, & \omega_2(t) &= -(1+\tfrac{3}{2}t)^{1/3}e^{15} - (1+\tfrac{3}{2}t)^{-1/3}e^{34}, \\ \eta(t) &= (1+\tfrac{3}{2}t)^{1/3}e^{2}, & \omega_3(t) &= -(1+\tfrac{3}{2}t)^{1/3}e^{13} - (1+\tfrac{3}{2}t)^{-1/3}e^{45}, \end{aligned}$$

where $t \in (-2/3, \infty)$.

For any $t \in (-2/3, \infty)$, the family $(\omega_1(t), \omega_2(t), \omega_3(t), \eta(t))$ satisfies equations (34) and the last two equations of (35). Moreover, it satisfies the conditions

$$\partial_t \omega_1(t) = 0, \quad \hat{d}(\eta(t)) = 0, \quad i_\xi(\hat{d}(\omega_1(t))) = 0, \quad \partial_t(I_t(\hat{d}\omega_1(t))) = 0,$$

which implies that the evolution equations (33) are also satisfied.

On the product $N^5 \times \mathbb{R}$, we consider the local basis of 1-forms

$$\begin{aligned} &\beta^1 = (1+\tfrac{3}{2}t)^{1/3}e^1, & &\beta^2 = (1+\tfrac{3}{2}t)^{-1/3}e^4, & &\beta^3 = e^5, \\ &\beta^4 = e^3, & &\beta^5 = (1+\tfrac{3}{2}t)^{1/3}e^2, & &\beta^6 = dt. \end{aligned}$$

The structure equations are

$$\begin{aligned} d\beta^1 &= -\tfrac{1}{2}(1+\tfrac{3}{2}t)^{-1}\beta^{16}, & d\beta^4 &= 0, \\ d\beta^2 &= (1+\tfrac{3}{2}t)^{-1}(\beta^{15} + \tfrac{1}{2}\beta^{26}), & d\beta^5 &= -\tfrac{1}{2}(1+\tfrac{3}{2}t)^{-1}\beta^{56}, \\ d\beta^3 &= \beta^{12}, & d\beta^6 &= 0. \end{aligned}$$

Locally, J is given by $J\beta^1 = -\beta^2$, $J\beta^3 = -\beta^4$ and $J\beta^5 = \beta^6$. The fundamental form $F = -\beta^{12} - \beta^{34} + \beta^{56}$ satisfies $d(J\,dF) = 0$, and the $(3,0)$-form $\Psi = (\beta^1 + i\beta^2) \wedge (\beta^3 + i\beta^4) \wedge (\beta^5 - i\beta^6)$ is closed. Therefore, (F, Ψ) is a local SKT SU(3)-structure on $N^5 \times \mathbb{R}$.

Remark 6.6. A Hermitian structure (J, h) on a 6-dimensional manifold M^6 is called *balanced* if $F \wedge F$ is closed, F being the associated fundamental 2-form. The paper [Fernández et al. 2009] introduced the notion of balanced SU(2)-structures on 5-dimensional manifolds, together with appropriate evolution equations whose solution gives rise to a balanced SU(3)-structure in six dimensions.

If M^6 is compact, then a balanced structure cannot be SKT; see, for instance, [Fino et al. 2004].

The SU(2)-structure (30) from the previous example is also balanced, and it gives rise to a balanced metric on the product of N^5 with a open interval; see [Fernández et al. 2009, (11)]. However, one can check directly that this solution is not SKT.

If G is the nilpotent Lie group underlying N^5, the product $G \times \mathbb{R}$ has no left-invariant SKT structures and does not admit any left-invariant complex structures; however, we can find a local SKT SU(3)-structure on it.

7. HKT structures

We will now find conditions under which an S^1-bundle over a $(4n+3)$-dimensional manifold endowed with three almost contact metric structures is hyper-Kähler with torsion (HKT, for short). Recall that a $4n$-dimensional hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is a hypercomplex manifold (M^{4n}, J_1, J_2, J_3) endowed with a Riemannian metric h compatible with the complex structures J_r for $r = 1, 2, 3$; that is, h satisfies

$$h(J_r X, J_r Y) = h(X, Y)$$

for any $r = 1, 2, 3$ and any vector fields X and Y on M^{4n}.

A hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, h)$ is called HKT if and only if

$$J_1 dF_1 = J_2 dF_2 = J_3 dF_3, \tag{37}$$

where F_r denotes the fundamental 2-form associated to the Hermitian structure (J_r, h); see [Grantcharov and Poon 2000].

We consider a $(4n+3)$-dimensional manifold N^{4n+3} endowed with three almost contact metric structures (I_r, ξ_r, η_r, g) for $r = 1, 2, 3$, and satisfying

$$\begin{aligned} I_k &= I_i I_j - \eta_j \otimes \xi_i = -I_j I_i + \eta_i \otimes \xi_j, \\ \xi_k &= I_i \xi_j = -I_j \xi_i, \quad \eta_k = \eta_i I_j = -\eta_j I_i. \end{aligned} \tag{38}$$

By applying Theorem 2.3, we can construct hyper-Hermitian structures on S^1-bundles over N^{4n+3} and study when they are strong HKT.

Theorem 7.1. *Let N^{4n+3} be a $(4n+3)$-dimensional manifold with three normal almost contact metric structures (I_r, ξ_r, η_r, g) for $r = 1, 2, 3$, and satisfying (38). Let Ω be a closed 2-form on N^{4n+3} that represents an integral cohomology class, and that is I_r-invariant for every $r = 1, 2, 3$. Consider the circle bundle $S^1 \hookrightarrow P \to N^{4n+3}$ with a connection 1-form θ whose curvature form is $d\theta = \pi^*(\Omega)$, where $\pi : P \to N$ is the projection. The hyper-Hermitian structure (J_1, J_2, J_3, h) on P defined by (2) and (4) is HKT if and only if*

$$\begin{aligned} \pi^*(I_1(d\omega_1)) - \pi^*(d\eta_1) \wedge \pi^*\eta_1 &= \pi^*(I_2(d\omega_2)) - \pi^*(d\eta_2) \wedge \pi^*\eta_2 \\ &= \pi^*(I_3(d\omega_3)) - \pi^*(d\eta_3) \wedge \pi^*\eta_3, \\ \pi^*(I_1(i_{\xi_1} d\omega_1)) = \pi^*(I_2(i_{\xi_2} d\omega_2)) &= \pi^*(I_3(i_{\xi_3} d\omega_3)), \end{aligned} \tag{39}$$

where ω_r is the fundamental form of the almost contact structure (I_r, ξ_r, η_r, g). Moreover, the HKT structure is strong if and only if

$$\begin{aligned} & d(\pi^*(I_r(i_{\xi_r} d\omega_r))) = 0, \\ & d(\pi^*(I_r(d\omega_r) - d\eta_r \wedge \eta_r)) = (\pi^*(-I_r(i_{\xi_r} d\omega_r)) + \pi^*\Omega) \wedge \pi^*\Omega \end{aligned} \tag{40}$$

for every $r = 1, 2, 3$.

Proof. The almost hyper-Hermitian structure (J_1, J_2, J_3, h) on P defined by (2) and (4) is hyper-Hermitian if and only (I_r, ξ_r, η_r, g) is normal and $d\theta$ is J_r-invariant for every $r = 1, 2, 3$. The HKT condition is equivalent to (37). By (9), we have

$$J_r dF_r = \pi^*(I_r(d\omega_r)) + \pi^*(I_r(i_{\xi_r} d\omega_r)) \wedge \theta - \pi^*(d\eta_r) \wedge \pi^*\eta_r - \theta \wedge d\theta,$$

where F_r is the fundamental 2-form of (J_r, h). Therefore, condition (37) is satisfied if and only if (39) holds. Finally, the $J_r dF_r$ are closed if and only if (40) holds. □

On $N^{4n+3} \times \mathbb{R}$, consider the almost Hermitian structures (J_r, F_r, h) defined by

$$\begin{aligned} h &= g + (dt)^2, \qquad & F_r &= \omega_r + \eta_r \wedge dt, \\ J_r(\eta_r) &= dt, \qquad & J_r(X) &= I_r(X) \quad \text{for } X \in \operatorname{Ker} \eta_r. \end{aligned} \tag{41}$$

By (38), we have

$$J_1 J_2 = J_3 = -J_2 J_1,$$
$$J_1 \eta_2 = I_1 \eta_2 = -\eta_3, \quad J_2 \eta_3 = I_2 \eta_3 = -\eta_1, \quad J_3 \eta_1 = I_3 \eta_1 = -\eta_2.$$

Therefore, (J_r, F_r, h) for $r = 1, 2, 3$ is a hyper-Hermitian structure on $N^{4n+3} \times \mathbb{R}$ if and only if the structures (I_r, ξ_r, η_r, g) are normal.

Corollary 7.2. *Let N^{4n+3} be a $(4n+3)$-dimensional manifold endowed with three normal almost contact metric structures (I_r, ξ_r, η_r, g) for $r = 1, 2, 3$. On the product $N^{4n+3} \times \mathbb{R}$, consider the hyper-Hermitian structure (J_1, J_2, J_3, h) defined by (41). Then, (J_1, J_2, J_3, h) is HKT if and only if*

$$I_1(d\omega_1) - d\eta_1 \wedge \eta_1 = I_2(d\omega_2) - d\eta_2 \wedge \eta_2 = I_3(d\omega_3) - d\eta_3 \wedge \eta_3,$$
$$I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3).$$

The HKT structure is strong if and only if

$$d(I_r(i_{\xi_r} d\omega_r)) = 0 \quad \text{and} \quad d(I_r(d\omega_r) - d\eta_r \wedge \eta_r) = 0 \quad \text{for every } r = 1, 2, 3.$$

Moreover, if (J_1, J_2, J_3, h) is such that $d\eta_1 \wedge \eta_1 = d\eta_2 \wedge \eta_2 = d\eta_3 \wedge \eta_3$ and one of the conditions

(a) *$d\omega_r = 0$ for any $r = 1, 2, 3$, that is, (I_r, ξ_r, η_r, g) is quasi-Sasakian for any $r = 1, 2, 3$; or*

(b) *$d\omega_i \wedge \eta_j \wedge \eta_k \neq 0$, where (i, j, k) is a permutation of $(1, 2, 3)$, as well as*

$$I_1(d\omega_1) = I_2(d\omega_2) = I_3(d\omega_3) \quad \text{and} \quad I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3),$$

is satisfied, then (J_1, J_2, J_3, h) is HKT. In case (a), the HKT structure is strong. In case (b), the HKT structure is strong if and only if $d(I_1(d\omega_1)) = d(I_1(i_{\xi_1} d\omega_1)) = 0$.

Proof. By Theorem 7.1, the hyper-Hermitian structure (J_r, F_r, h) for $r = 1, 2, 3$ is HKT if and only if

$$\begin{aligned} I_1(d\omega_1) - d\eta_1 \wedge \eta_1 = I_2(d\omega_2) - d\eta_2 \wedge \eta_2 = I_3(d\omega_3) - d\eta_3 \wedge \eta_3, \\ I_1(i_{\xi_1} d\omega_1) = I_2(i_{\xi_2} d\omega_2) = I_3(i_{\xi_3} d\omega_3). \end{aligned} \tag{42}$$

Locally, we write

$$d\omega_r = \alpha_r + \sum_{i=1}^{3} \beta_i^r \wedge \eta_i + \sum_{i<j=1}^{3} \gamma_{ij}^r \wedge \eta_i \wedge \eta_j + \rho_r \eta_1 \wedge \eta_2 \wedge \eta_3, \tag{43}$$

where ρ_r are smooth functions, while α_r, β_i^r, and γ_{ij}^r are respectively 3-forms, 2-forms, and 1-forms in $\bigcap_{i=1}^3 \operatorname{Ker} \eta_i$.

By first using the normality of the three almost contact metric structures, and then that $i_{\xi_r} d\eta_r = 0$ and $I_r(d\eta_r) = d\eta_r$, locally we can write

$$\begin{aligned} d\eta_1 &= A_1 + B_1 \wedge \eta_2 - I_1 B_1 \wedge \eta_3 + C_1 \eta_2 \wedge \eta_3, \\ d\eta_2 &= A_2 + B_2 \wedge \eta_1 + I_2 B_2 \wedge \eta_3 + C_2 \eta_1 \wedge \eta_3, \\ d\eta_3 &= A_3 + B_3 \wedge \eta_1 - I_3 B_3 \wedge \eta_2 + C_3 \eta_1 \wedge \eta_2, \end{aligned} \tag{44}$$

where $I_r A_r = A_r$. Here, the A_r and B_r are respectively 2-forms and 1-forms in $\bigcap_{i=1}^3 \operatorname{Ker} \eta_i$, while the C_r are smooth functions. We have

$$J_r(dF_r) = J_r(d\omega_r) + J_r(d\eta_r \wedge dt) = J_r(d\omega_r) - d\eta_r \wedge \eta_r.$$

Therefore, by using (43) and (44), we obtain

$$\begin{aligned} J_1(dF_1) = {} & I_1\alpha_1 + I_1\beta_1^1 \wedge dt - A_1 \wedge \eta_1 - I_1\beta_3^1 \wedge \eta_2 - I_1\beta_2^1 \wedge \eta_3 \\ & - I_1\gamma_{13}^1 \wedge \eta_2 \wedge dt + I_1\gamma_{12}^1 \wedge \eta_3 \wedge dt + B_1 \wedge \eta_1 \wedge \eta_2 - I_1 B_1 \wedge \eta_1 \wedge \eta_3 \\ & + I_1\gamma_{23}^1 \wedge \eta_2 \wedge \eta_3 + \rho_1 \eta_2 \wedge \eta_3 \wedge dt - C_1 \eta_1 \wedge \eta_2 \wedge \eta_3, \\ J_2(dF_2) = {} & I_2\alpha_2 + I_2\beta_2^2 \wedge dt - I_2\beta_3^2 \wedge \eta_1 - A_2 \wedge \eta_2 + I_2\beta_1^2 \wedge \eta_3 \\ & + I_2\gamma_{23}^2 \wedge \eta_1 \wedge dt + I_2\gamma_{12}^2 \wedge \eta_3 \wedge dt - B_2 \wedge \eta_1 \wedge \eta_2 + I_2\gamma_{13}^2 \wedge \eta_1 \wedge \eta_3 \\ & + I_2 B_2 \wedge \eta_2 \wedge \eta_3 - \rho_2 \eta_1 \wedge \eta_3 \wedge dt + C_2 \eta_1 \wedge \eta_2 \wedge \eta_3, \\ J_3(dF_3) = {} & I_3\alpha_3 + I_3\beta_3^3 \wedge dt + I_3\beta_2^3 \wedge \eta_1 - I_3\beta_1^3 \wedge \eta_2 - A_3 \wedge \eta_3 \\ & + I_3\gamma_{23}^3 \wedge \eta_1 \wedge dt - I_3\gamma_{13}^3 \wedge \eta_2 \wedge dt + I_3\gamma_{12}^3 \wedge \eta_1 \wedge \eta_2 - B_3 \wedge \eta_1 \wedge \eta_3 \\ & + I_3 B_3 \wedge \eta_2 \wedge \eta_3 + \rho_3 \eta_1 \wedge \eta_2 \wedge dt - C_3 \eta_1 \wedge \eta_2 \wedge \eta_3. \end{aligned}$$

The conditions (42) are satisfied if and only if

$$
\begin{aligned}
&\gamma_{12}^1 = \gamma_{13}^1 = \gamma_{12}^2 = \gamma_{23}^2 = \gamma_{13}^3 = \gamma_{23}^3 = 0, \\
&\rho_r = 0, \qquad\qquad\qquad C_1 = -C_2 = C_3, \\
&I_1\alpha_1 = I_2\alpha_2 = I_3\alpha_3, \qquad I_1\beta_1^1 = I_2\beta_2^2 = I_3\beta_3^3, \\
&A_1 = I_2\beta_3^2 = -I_3\beta_2^3, \qquad A_2 = -I_1\beta_3^1 = I_3\beta_1^3, \qquad A_3 = I_1\beta_2^1 = -I_2\beta_1^2, \\
&B_1 = -B_2 = I_3\gamma_{12}^3, \quad -I_1B_1 = -B_3 = I_2\gamma_{13}^2, \quad I_2B_2 = I_3B_3 = I_1\gamma_{23}^1.
\end{aligned}
\tag{45}
$$

Since $I_rA_r = A_r$ the coefficients β_i^r for $r \neq i = 1, 2, 3$ must satisfy the conditions

$$I_i(\beta_j^i - I_k\beta_j^i) = 0 \quad \text{for all } i, j, k = 1, 2, 3 \text{ with } i \neq j,\ j \neq k \text{ and } k \neq i.$$

The last three equations in (45) are satisfied if and only if $\gamma_{23}^1 = \gamma_{13}^2 = \gamma_{12}^3 = 0$.

Thus, finally, we obtain

$$
\begin{aligned}
&d\omega_r = \alpha_r + \textstyle\sum_{i=1}^3 \beta_i^r \wedge \eta_i, \quad d\eta_i = A_i + \lambda\,\eta_j \wedge \eta_k, \\
&\quad 0 = I_i(\beta_j^i - I_k\beta_j^i) \quad \text{for all } i, j, k = 1, 2, 3 \text{ with } i \neq j,\ j \neq k \text{ and } k \neq i, \\
&I_1\alpha_1 = I_2\alpha_2 = I_3\alpha_3, \\
&A_1 = I_2\beta_3^2 = -I_3\beta_2^3, \quad A_2 = -I_1\beta_3^1 = I_3\beta_1^3, \quad A_3 = I_1\beta_2^1 = -I_2\beta_1^2
\end{aligned}
\tag{46}
$$

for any even permutation of $(1, 2, 3)$.

The expression for $d(J_1dF_1)$ is

$$
\begin{aligned}
d(J_1dF_1) &= d(I_1(d\omega_1) + I_1(i_{\xi_1}d\omega_1) \wedge dt) - d((d\eta_1) \wedge \eta_1) \\
&= d(I_1(d\omega_1)) + d(I_1(i_{\xi_1}d\omega_1)) \wedge dt - d\eta_1 \wedge d\eta_1 \\
&= d(I_1(d\omega_1) - d\eta_1 \wedge \eta_1) + d(I_1(i_{\xi_1}d\omega_1)) \wedge dt,
\end{aligned}
$$

and thus the HKT structure is strong if and only if

$$d(I_1(d\omega_1) - d\eta_1 \wedge \eta_1) = 0 \quad \text{and} \quad d(I_1(i_{\xi_1}d\omega_1)) = 0.$$

To prove the last part of the corollary it suffices to consider coefficients $\beta_r^i = 0$ if $r \neq i$ in expression (43). $\square$

Example 7.3. Consider the 7-dimensional Lie group $G = \mathrm{SU}(2) \ltimes \mathbb{R}^4$, with structure equations

$$
\begin{aligned}
de^1 &= -\tfrac{1}{2}e^{25} - \tfrac{1}{2}e^{36} - \tfrac{1}{2}e^{47}, & de^5 &= e^{67}, \\
de^2 &= \tfrac{1}{2}e^{15} + \tfrac{1}{2}e^{37} - \tfrac{1}{2}e^{46}, & de^6 &= -e^{57}, \\
de^3 &= \tfrac{1}{2}e^{16} - \tfrac{1}{2}e^{27} + \tfrac{1}{2}e^{45}, & de^7 &= e^{56}. \\
de^4 &= \tfrac{1}{2}e^{17} + \tfrac{1}{2}e^{26} - \tfrac{1}{2}e^{35}, & &
\end{aligned}
$$

By [Fino and Tomassini 2008], G admits a compact quotient $M^7 = \Gamma \backslash G$ by a uniform discrete subgroup Γ, and is endowed with a weakly generalized G_2-structure. By [Barberis and Fino 2008], $M^7 \times S^1$ admits a strong HKT structure. We can show that M^7 has three normal almost contact metric structures (I_r, ξ_r, η_r, g) for $r = 1, 2, 3$ that are given by

$$\begin{aligned}
&I_1 e^1 = e^2, && I_1 e^3 = e^4, && I_1 e^5 = e^6, && \eta_1 = e^7,\\
&I_2 e^1 = e^3, && I_2 e^2 = -e^4, && I_2 e^5 = -e^7, && \eta_2 = e^6,\\
&I_3 e^1 = e^4, && I_3 e^2 = e^3, && I_3 e^6 = e^7, && \eta_3 = e^5
\end{aligned}$$

and that satisfy the conditions of Corollary 7.2(a).

References

[Andrada et al. 2009] A. Andrada, A. Fino, and L. Vezzoni, "A class of Sasakian 5-manifolds", *Transformation Groups* **14**:3 (2009), 493–512. MR 2534796 Zbl 1185.53046 arXiv 0807.1800

[Barberis and Fino 2008] M. L. Barberis and A. Fino, "New HKT manifolds arising from quaternionic representations", preprint, 2008. To appear in *Math. Z.* arXiv 0805.2335

[Barberis et al. 2009] M. L. Barberis, I. G. Dotti, and M. Verbitsky, "Canonical bundles of complex nilmanifolds, with applications to hypercomplex geometry", *Math. Res. Lett.* **16**:2 (2009), 331–347. MR 2010g:53083 Zbl 1178.32014

[Bismut 1989] J.-M. Bismut, "A local index theorem for non-Kähler manifolds", *Math. Ann.* **284**:4 (1989), 681–699. MR 91i:58140

[Blair 1967] D. E. Blair, "The theory of quasi-Sasakian structures", *J. Differential Geometry* **1** (1967), 331–345. MR 37 #2127 Zbl 0163.43903

[Blair 2002] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser, Boston, MA, 2002. MR 2002m:53120 Zbl 1011.53001

[Boyer and Galicki 1999] C. Boyer and K. Galicki, "3-Sasakian manifolds", pp. 123–184 in *Surveys in differential geometry: Essays on Einstein manifolds*, edited by C. LeBrun and M. Wang, Surv. Differ. Geom. **6**, Int. Press, Boston, 1999. MR 2001m:53076 Zbl 1008.53047

[Conti and Salamon 2007] D. Conti and S. Salamon, "Generalized Killing spinors in dimension 5", *Trans. Amer. Math. Soc.* **359**:11 (2007), 5319–5343. MR 2008h:53077 Zbl 1130.53033

[Egidi 2001] N. Egidi, "Special metrics on compact complex manifolds", *Differential Geom. Appl.* **14**:3 (2001), 217–234. MR 2002b:32038 Zbl 0981.32014

[Fernández et al. 2009] M. Fernández, A. Tomassini, L. Ugarte, and R. Villacampa, "Balanced Hermitian metrics from SU(2)-structures", *J. Math. Phys.* **50**:3 (2009), 033507, 15. MR 2010f:53124 Zbl 05646541

[Fino and Grantcharov 2004] A. Fino and G. Grantcharov, "Properties of manifolds with skew-symmetric torsion and special holonomy", *Adv. Math.* **189**:2 (2004), 439–450. MR 2005k:53062 Zbl 1114.53043

[Fino and Tomassini 2008] A. Fino and A. Tomassini, "Generalized G_2-manifolds and SU(3)-structures", *Internat. J. Math.* **19**:10 (2008), 1147–1165. MR 2009m:53054 Zbl 1168.53012

[Fino and Tomassini 2009] A. Fino and A. Tomassini, "Blow-ups and resolutions of strong Kähler with torsion metrics", *Adv. Math.* **221**:3 (2009), 914–935. MR 2010d:32012

[Fino et al. 2004] A. Fino, M. Parton, and S. Salamon, "Families of strong KT structures in six dimensions", *Comment. Math. Helv.* **79**:2 (2004), 317–340. MR 2005f:53061 Zbl 1062.53062

[Friedrich and Ivanov 2002] T. Friedrich and S. Ivanov, "Parallel spinors and connections with skew-symmetric torsion in string theory", *Asian J. Math.* **6**:2 (2002), 303–335. MR 2003m:53070 Zbl 1127.53304

[Gates et al. 1984] S. J. Gates, Jr., C. M. Hull, and M. Roček, "Twisted multiplets and new supersymmetric nonlinear σ-models", *Nuclear Phys. B* **248**:1 (1984), 157–186. MR 87b:81108

[Gauduchon 1984] P. Gauduchon, "La 1-forme de torsion d'une variété hermitienne compacte", *Math. Ann.* **267**:4 (1984), 495–518. MR 87a:53101 Zbl 0523.53059

[Grantcharov and Poon 2000] G. Grantcharov and Y. S. Poon, "Geometry of hyper-Kähler connections with torsion", *Comm. Math. Phys.* **213**:1 (2000), 19–37. MR 2002a:53059

[Grantcharov et al. 2008] D. Grantcharov, G. Grantcharov, and Y. S. Poon, "Calabi–Yau connections with torsion on toric bundles", *J. Differential Geom.* **78**:1 (2008), 13–32. MR 2009g:32052 Zbl 1171.53044

[Hitchin 2001] N. Hitchin, "Stable forms and special metrics", pp. 70–89 in *Global differential geometry: The mathematical legacy of Alfred Gray* (Bilbao, 2000), edited by M. Fernández and J. A. Wolf, Contemp. Math. **288**, Amer. Math. Soc., Providence, RI, 2001. MR 2003f:53065 Zbl 1004.53034

[Howe and Papadopoulos 1996] P. S. Howe and G. Papadopoulos, "Twistor spaces for hyper-Kähler manifolds with torsion", *Phys. Lett. B* **379**:1-4 (1996), 80–86. MR 97h:53073

[Ivanov and Papadopoulos 2001] S. Ivanov and G. Papadopoulos, "Vanishing theorems and string backgrounds", *Classical Quantum Gravity* **18**:6 (2001), 1089–1110. MR 2002h:53076 Zbl 0990.53078

[Kobayashi 1956] S. Kobayashi, "Principal fibre bundles with the 1-dimensional toroidal group", *Tôhoku Math. J.* (2) **8** (1956), 29–45. MR 18,328a Zbl 0075.32103

[Ogawa 1963] Y. Ogawa, "Some properties on manifolds with almost contact structures", *Tôhoku Math. J.* (2) **15** (1963), 148–161. MR 27 #704 Zbl 0147.41002

[Sasaki and Hatakeyama 1961] S. Sasaki and Y. Hatakeyama, "On differentiable manifolds with certain structures which are closely related to almost contact structure, II", *Tôhoku Math. J.* (2) **13** (1961), 281–294. MR 25 #1513 Zbl 0112.14002

[Strominger 1986] A. Strominger, "Superstrings with torsion", *Nuclear Phys. B* **274**:2 (1986), 253–284. MR 87m:81177

[Swann 2010] A. Swann, "Twisting Hermitian and hypercomplex geometries", *Duke Math. J.* **155** (2010), 403–431.

[Ugarte 2007] L. Ugarte, "Hermitian structures on six-dimensional nilmanifolds", *Transformation Groups* **12**:1 (2007), 175–202. MR 2008e:53139 Zbl 1129.53052

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