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MICHAEL B. HENRY

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CONNECTIONS BETWEEN FLOER-TYPE INVARIANTS AND MORSE-TYPE INVARIANTS OF LEGENDRIAN KNOTS

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We define an algebraic/combinatorial object on the front projection Σ of a Legendrian knot, called a Morse complex sequence, abbreviated MCS. This object is motivated by the theory of generating families and provides new connections between generating families, normal rulings, and augmentations of the Chekanov–Eliashberg DGA. In particular, we place an equivalence relation on the set of MCSs on Σ and construct a surjective map from the equivalence classes to the set of chain homotopy classes of augmentations of L_Σ, where L_Σ is the Ng resolution of Σ. In the case of Legendrian knot classes admitting representatives with two-bridge front projections, this map is bijective. We also exhibit two standard forms for MCSs and give explicit algorithms for finding these forms. The definition of an MCS, the equivalence relation, and the statements of some of the results originate from unpublished work of Petya Pushkar.

1. Introduction

Legendrian knot theory is a rich refinement of smooth knot theory with deep connections to low-dimensional topology, symplectic and contact geometry, and singularity theory. In this article, we investigate connections between Legendrian knot invariants derived from symplectic field theory and from the theory of generating families. Specifically, we relate augmentations, derived from symplectic field theory, and Morse complex sequences, derived from generating families. A Legendrian knot K in $\mathbb{R}^3$ is a smooth knot whose tangent space sits in the standard contact structure ξ on $\mathbb{R}^3$, where ξ is the kernel of the 1-form $dz - y\,dx$. Legendrian knot theory is the study of Legendrian knots up to isotopy through Legendrian knots.

We begin by recalling existing connections between Legendrian knot invariants. The Chekanov–Eliashberg differential graded algebra (abbreviated CE-DGA) of a Legendrian knot K is a differential graded algebra $(\mathcal{A}(L), \partial)$ associated to the

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xy-projection L of K. The CE-DGA is derived from the symplectic field theory of [Eliashberg 1998; Eliashberg et al. 2000] and is developed in the latter and [Chekanov 2002a]. The homology of $(\mathcal{A}(L), \partial)$ is a Legendrian invariant and, if we consider $(\mathcal{A}(L), \partial)$ up to a certain algebraic equivalence, the resulting DGA class is also a Legendrian invariant. Geometrically, the CE-DGA is Floer theoretic in nature. See [Chekanov 2002b; Etnyre et al. 2002] for a more detailed introduction.

An *augmentation* is a type of algebra homomorphism from the CE-DGA of a Legendrian knot to the base field $\mathbb{Z}_2$. We denote the set of augmentations of $(\mathcal{A}(L), \partial)$ by $\mathrm{Aug}(L)$. There is a natural algebraic equivalence relation on $\mathrm{Aug}(L)$ and we denote the set of equivalence classes of augmentations of L by $\mathrm{Aug}^{ch}(L)$. The cardinality of $\mathrm{Aug}^{ch}(L)$ is a Legendrian isotopy invariant.

A second source of Legendrian invariants is the theory of generating families; see any of [Pushkar and Chekanov 2005; Jordan and Traynor 2006; Ng and Traynor 2004; Traynor 1997; Traynor 2001]. A *generating family* for a Legendrian knot K encodes the xz-projection of K as the Cerf diagram of a one-parameter family of functions F_x. In particular, let W be a smooth manifold and $F : \mathbb{R} \times W \to \mathbb{R}$ be a smooth function. Let $\mathscr{C}_x \subset \{x\} \times W$ denote the set of critical points of $F_x = F(x, \cdot)$ and let $\mathscr{C}_F = \bigcup_{x \in \mathbb{R}} \mathscr{C}_x$. If the rank of the matrix of second derivatives of F is maximal at all points in $\mathscr{C}_F$, then $\Sigma_F = \{(x, F(x, w)) \mid (x, w) \in \mathscr{C}_F\}$ is the xz-projection of an immersed Legendrian submanifold in $(\mathbb{R}^3, \xi)$ and we say F is a generating family for this submanifold; see Figure 1. If we restrict our attention to generating families that are sufficiently nice outside a compact set of the domain, then it can be shown that the existence of a generating family for a Legendrian knot is a Legendrian isotopy invariant; see [Jordan and Traynor 2006].

Throughout this article, we will let Σ denote the xz-projection of a Legendrian knot and call it the *front projection* of K. Every Legendrian knot can be Legendrian isotoped in an arbitrarily small neighborhood of itself so that the singularities of Σ are left cusps, right cusps, and transverse double points and so that the x-coordinates of these singularities are all distinct. We say a front is *σ-generic* if its singularities are arranged in this manner.

From a generating family, we may derive a combinatorial object called a graded normal ruling. Suppose a Legendrian knot K with σ-generic front Σ admits a generating family $F : \mathbb{R} \times W \to \mathbb{R}$. The generating family may be chosen so that F_x is a Morse function whose critical points have distinct critical values for all but finitely many values of x. By placing an appropriate metric g on $\mathbb{R} \times W$, we may construct the Morse–Smale chain complex (C_x, ∂_x, g_x) of the pair (F_x, g_x). A *graded normal ruling* is a combinatorial object on Σ that encodes a certain pairing of the generators of (C_x, ∂_x, g_x) as x varies. Chekanov and Pushkar [2005] work with a more general object called a pseudoinvolution; their Section 12 explains in detail the connection between generating families and pseudoinvolutions, including

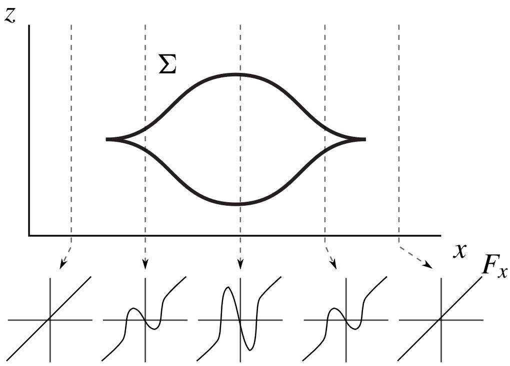


Figure 1. A generating family for a Legendrian unknot.

the restrictions placed on the types of generating families considered and on the metric g.

Many connections exist between augmentations, generating families, and graded normal rulings; see any of [Pushkar and Chekanov 2005; Fuchs 2003; Fuchs and Ishkhanov 2004; Fuchs and Rutherford 2008; Kálmán 2006; Ng and Sabloff 2006; Sabloff 2005]. For a fixed Legendrian knot K with Lagrangian projection L and front projection Σ, the following results are known.

Theorem 1.1. *The CE-DGA* $(\mathcal{A}(L), \partial)$ *admits a graded augmentation if and only if* Σ *admits a graded normal ruling. In particular, there exists a many-to-one map from the set of graded augmentations of* $(\mathcal{A}(L), \partial)$ *to the set of graded normal rulings of* Σ.

The reverse implication of the first statement was proved by Fuchs [2003]. Fuchs and Ishkhanov [2004], and independently Sabloff [2005], proved the forward implication. Ng and Sabloff [2006] proved the second statement.

Theorem 1.2. *A Legendrian knot K admits a generating family if and only if* Σ *admits a graded normal ruling.*

Chekanov and Pushkar [2005] proved the forward direction and state the converse without proof. Fuchs and Rutherford [2008] then proved the converse.

By encoding the Morse theory data inherent in a generating family, we hope to refine the many-to-one map in Theorem 1.1. We encode this data in a finite sequence of chain complexes. The resulting algebraic object is called a *Morse complex sequence*. Geometrically, it should be thought of as a sequence from the 1-parameter family of chain complexes (C_x, ∂_x, g_x) from a generating family. In this article we will not work explicitly with generating families, though they provide important geometric intuition. We let $\mathrm{MCS}(\Sigma)$ denote the set of MCSs of Σ and let $\widehat{\mathrm{MCS}}(\Sigma)$ denote the set of MCSs of Σ up to a natural equivalence.

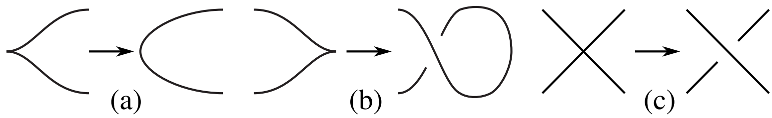


Figure 2. Cusps and crossings in the Ng resolution procedure.

In 1999, Petya Pushkar began a program to combinatorialize the Morse theory data coming from a generating family. The work with pseudoinvolutions in [Pushkar and Chekanov 2005] may be considered the first step in this program. In emails to D. Fuchs in 2001 and 2008, Pushkar outlines his "Spring Morse theory," which encodes the sequence (C_x, ∂_x, g_x) coming from a generating family and provides an equivalence relation on the resulting objects. The equivalence relation is the result of understanding the evolution of one-parameter families of functions and metrics. The ideas behind Morse complex sequences and the equivalence relation we define in this article originate with Petya Pushkar.

1a. *Results.* Given a Legendrian knot K with σ-generic front projection Σ, we form the Ng L_Σ resolution of Σ by resolving the cusps and crossings as indicated in Figure 2. The CE-DGA of L_Σ is equal to the CE-DGA of an xy-projection for the Legendrian knot class of K. Therefore we may use L_Σ to compare objects defined on Σ with objects derived from the CE-DGA of K. Given Σ and L_Σ, we have the following results.

Theorem 1.3. *For a fixed Legendrian knot K with σ-generic front projection Σ and Ng resolution L_Σ, there exists a surjective map*

$$\widehat{\Psi} : \widehat{\mathrm{MCS}}(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma).$$

Theorem 1.4. *If Σ has exactly two left cusps, then the map $\widehat{\Psi}$ above is a bijection.*

In his 2008 email, Pushkar announced, without proof, results very similar to Theorem 1.3.

In the case of a front projection with exactly two left cusps, we can explicitly calculate $|\mathrm{Aug}^{ch}(L_\Sigma)| = |\widehat{\mathrm{MCS}}(\Sigma)|$. The language in this corollary is defined in Section 7.

Corollary 1.5. *Suppose Σ has exactly two left cusps, and let $N(\Sigma)$ denote the set of graded normal rulings on Σ. For each $N \in N(\Sigma)$, define $\nu(N)$ to be the number of graded departure-return pairs in N. Then*

$$|\mathrm{Aug}^{ch}(L_\Sigma)| = |\widehat{\mathrm{MCS}}(\Sigma)| = \sum_{N \in N(\Sigma)} 2^{\nu(N)}.$$

In Section 6e, we describe two standard forms for MCSs on Σ. The algorithms used to find these forms allow us to understand $\widehat{\Psi}$, while avoiding the involved algebraic arguments required to prove Theorem 1.3. We will use the $S\bar{R}$-form to calculate bounds on the number of MCS classes associated to a fixed graded normal ruling. The A-form of an MCS $\mathscr{C}$ allows us to easily compute the augmentation class $\widehat{\Psi}([\mathscr{C}])$.

Theorem 1.6. *Every MCS is equivalent to an MCS in $S\bar{R}$-form and an MCS in A-form.*

1b. ***Outline of the rest of the article.*** In Section 2, we provide the necessary background material in Legendrian knot theory. The front and Lagrangian projections of a Legendrian knot are used to develop combinatorial descriptions of the Chekanov–Eliashberg DGA, augmentations, and graded normal rulings. The definition of a Morse complex sequence (MCS) is given in Section 3, along with an equivalence relation on MCSs. Section 4 reviews properties of differential graded algebras, DGA morphisms and DGA chain homotopies and applies them to the case of the CE-DGA and augmentations. We also sketch the proof that $\text{Aug}^{ch}(L)$ is a Legendrian knot invariant. In Section 5 we use a variation of the splash construction first developed in [Fuchs 2003] to write down the boundary map of the CE-DGA of a "dipped" version of L_Σ as a system of local matrix equations. This gives us local control over augmentations and chain homotopies of augmentations. A number of lemmas are proved involving extending augmentations to dipped diagrams. In Section 6 we develop the connections between MCSs and augmentations. We use the lemmas on dipped diagrams from Section 5 and the lemmas concerning chain homotopies from Section 4 to explicitly construct $\widehat{\Psi}$ and prove Theorem 1.3. In Section 6e, we describe two standard forms for MCSs on Σ and prove Theorem 1.6 using explicit algorithms. In Section 7 we prove Theorem 1.4 and Corollary 1.5.

2. Background

We assume the reader is familiar with the basic concepts in Legendrian knot theory, including front and Lagrangian projections, and the classical invariants. Throughout this article, Σ and L denote the front and Lagrangian projections of a knot K, respectively. Legendrian knots with nonzero rotation number do not admit augmentations, generating families, and graded normal rulings. Thus we will always assume the rotation number of K is 0. An in-depth survey of Legendrian knot theory can be found in [Etnyre 2005].

2a1. *The Ng resolution.* In [2003], Ng algorithmically constructs a Legendrian isotopy of K such that the Lagrangian projection L' of the resulting Legendrian knot K' is topologically similar to Σ. Definition 2.1 gives a combinatorial description of the Lagrangian projection produced by Ng's resolution algorithm.

Definition 2.1. Given a Legendrian knot K with front projection Σ, we form the *Ng resolution* L_Σ by smoothing the left cusps as in Figure 2(a), smoothing and twisting the right cusps as in Figure 2(b), and resolving the double points as in Figure 2(c).

The projection L_Σ is regularly homotopic to L', and the CE-DGA of L_Σ is equal to the CE-DGA of L'. Given that Σ and L_Σ are combinatorially very similar, the Ng resolution algorithm provides a natural first step towards finding connections between generating families and the CE-DGA.

2b. *The Chekanov–Eliashberg DGA.* Chekanov [2002a] and Eliashberg [2000] develop a differential graded algebra, henceforth referred to as the CE-DGA, that has led to the discovery of several new Legendrian isotopy invariants. The definitions and statements in this section originated in [Chekanov 2002a]; see also [Etnyre et al. 2002].

2b1. *The algebra.* Label the crossings of a Lagrangian projection L by $q_1, \dots, q_n$. Let $A(L)$ denote the $\mathbb{Z}_2$ vector space freely generated by the elements of $Q = \{q_1, \dots, q_n\}$. The algebra $\mathcal{A}(L)$ is the unital tensor algebra $TA(L)$. We consider $\mathcal{A}(L)$ to be a *based* algebra since the algebra basis Q is part of the data of $\mathcal{A}(L)$. An element of $\mathcal{A}(L)$ looks like the sum of noncommutative words in the letters q_i.

2b2. *The grading.* We define a $\mathbb{Z}$-grading $|q_i|$ on the generators q_i and extend it to all monomials in $\mathcal{A}(L)$ by requiring $\left|\prod_{j=1}^{l} q_{i_j}\right| = \sum_{j=1}^{l} |q_{i_j}|$. We isotope K slightly so that the two strands meeting at each crossing of L are orthogonal. Let γ_i be a path in K that begins on the overstrand of L at q_i and follows L until it first returns to q_i. We let $r(\gamma_i)$ denote the fractional winding number of the tangent space of γ_i with respect to the trivialization $\{\partial_x, \partial_y\}$ of the tangent space of $\mathbb{R}^2$. The *grading of a crossing* q_i is defined to be $|q_i| = 2r(\gamma_i) - 1/2$. The grading is well-defined since we have assumed K has rotation number 0.

If L_Σ is the Ng resolution of a front projection Σ, then we can calculate the grading of the crossings of L_Σ from Σ using a Maslov potential.

Definition 2.2. A *strand* in Σ is a smooth path in Σ from a left cusp to a right cusp. A *Maslov potential* on Σ is a map μ from the strands of Σ to $\mathbb{Z}$ satisfying the relation shown in Figure 3(a).

Given a crossing q in Σ, the grading of the corresponding resolved crossing q in L_Σ is computed by $|q| = \mu(T) - \mu(B)$, where T and B are the strands crossing at q and T has smaller slope. The crossings created by resolving right cusps have grading 1.

2b3. *The differential.* The differential of the CE-DGA counts certain disks in the xy-plane with polygonal corners at the crossings of L. We begin by decorating each corner of q_i with a $+$ or $-$ sign as in Figure 3(b).

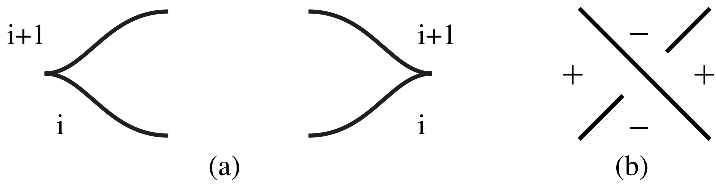


Figure 3. Left: a Maslov potential near left and right cusps. Right: the Reeb sign of a crossing.

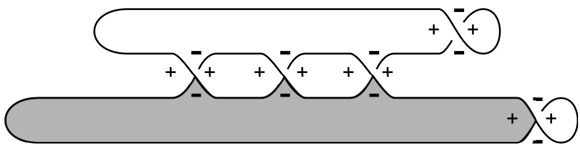


Figure 4. A convex immersed polygon contributing $q_3q_2q_1$ to ∂q_5. Crossings are labeled from left to right.

Definition 2.3. Let D be the unit disk in $\mathbb{R}^2$ and let $\mathscr{X} = \{x_0, \ldots, x_n\}$ be a set of distinct points along ∂D in counterclockwise order. A *convex immersed polygon* is a continuous map $f : D \to \mathbb{R}^2$ such that

(1) f is an orientation-preserving immersion on the interior of D;

(2) the restriction of f to $\partial D \setminus \mathscr{X}$ is an immersion whose image lies in L; and

(3) the image of each $x_i \in \mathscr{X}$ under f is a crossing of L, and the image of a neighborhood of x_i covers a convex corner at the crossing. Call a corner of an immersed polygon *positive* if it covers a $+$ sign and *negative* otherwise.

Remark 2.4. Given a crossing q_i, let γ_i be the path in $\mathbb{R}^3$ that begins and ends on K, has constant x and y coordinates, and projects to the crossing q_i. Such paths are called *Reeb chords*. We define a height function $h : Q \to \mathbb{R}^+$ on the crossings of L by defining $h(q_i)$ to be length of the path γ_i.

In the Ng resolution algorithm, we may arrange the Legendrian isotopy so that the height function on L' is strictly increasing as we move from left to right along the x-axis. Thus the height function provides an ordering on the crossings of L_Σ corresponding to the ordering coming from the x-axis.

Lemma 2.5. *Let f be a convex immersed polygon and let γ_i be the Reeb chord that lies over the corner $f(x_i)$. Then*

$$\sum_{x_i \text{ positive}} h(\gamma_i) - \sum_{x_j \text{ negative}} h(\gamma_j) = \mathrm{Area}(f(D)).$$

As a consequence, every convex immersed polygon has at least one positive corner and in the case of an immersed polygon with a single positive corner, the height of the positive corner is greater than the height of each of the negative corners. The differential ∂q_i of the generator q_i is a mod 2 count of convex immersed polygons with a single positive corner at q_i.

Definition 2.6. Let $\tilde{\Delta}(q_i; q_{j_1}, \ldots, q_{j_k})$ be the set of convex immersed polygons with a positive corner at q_i and negative corners at $q_{j_1}, \ldots, q_{j_k}$. The negative corners are ordered by the counterclockwise order of the marked points along ∂D. Let $\Delta(q_i; q_{j_1}, \ldots, q_{j_k})$ be $\tilde{\Delta}(q_i; q_{j_1}, \ldots, q_{j_k})$ modulo smooth reparameterization.

Definition 2.7. The differential ∂ on the algebra $\mathcal{A}(L)$ is defined on a generator $q_i \in Q$ by the formula

$$\partial q_i = \sum_{\Delta(q_i; q_{j_1}, \ldots, q_{j_k})} \#(\Delta(q_i; q_{j_1}, \ldots, q_{j_k})) q_{j_1} \cdots q_{j_k},$$

where $\#(\Delta(\cdots))$ is the mod 2 count of the elements in $\Delta(\cdots)$. We extend ∂ to all of $\mathcal{A}(L)$ by linearity and the Leibniz rule.

The convex immersed polygon contributing the monomial $q_3 q_2 q_1$ to ∂q_5 is given in Figure 4. Given the differential and grading defined above, $(\mathcal{A}(L), \partial)$ is a differential graded algebra.

Theorem 2.8 [Chekanov 2002a]. *The differential ∂ satisfies*

(1) $|\partial q| = |q| - 1$ *modulo* $2r(K)$, *and*

(2) $\partial \circ \partial = 0$.

Remark 2.9. In the case of an Ng resolution L_Σ, we noted in Remark 2.4 that the heights of the crossings increase as we move from left to right along the x-axis. Thus the negative corners of a convex immersed polygon contributing to ∂ in $(\mathcal{A}(L_\Sigma), \partial)$ always appear to the left of the positive corner. In Section 5, this fact will allow us to find all convex immersed polygons contributing to ∂.

Chekanov [2002a] defines an algebraic equivalence on DGAs, called *stable tame isomorphism*, and proves that, up to this equivalence, the CE-DGA is a Legendrian isotopy invariant. In general, it is difficult to determine if two CE-DGAs are stable tame isomorphic.

Definition 2.10. Given two algebras $\mathcal{A}$ and $\mathcal{A}'$, a grading-preserving identification of their generating sets Q and Q', and a generator q_j for $\mathcal{A}$, an *elementary isomorphism* ϕ is an graded algebra map defined by

$$\phi(q_i) = \begin{cases} q_i' & \text{if } i \neq j, \\ q_j' + u & \text{if } i = j \text{ and } u \text{ is a term in } \mathcal{A}' \text{ not containing } q_j'. \end{cases}$$

A composition of elementary isomorphisms is called a *tame isomorphism*. A *tame isomorphism of DGAs* between $(\mathcal{A}, \partial)$ and $(\mathcal{A}', \partial')$ is a tame isomorphisms Φ that is also a chain map, that is, $\Phi \circ \partial = \partial' \circ \Phi$.

Definition 2.11. Given a DGA $(\mathscr{A}, \partial)$ with generating set Q, we define the *degree i stabilization* $S_i(\mathscr{A}, \partial)$ to be the differential graded algebra generated by the set $Q \cup \{e_1, e_2\}$, where $|e_1| = i$ and $|e_2| = i - 1$ and with the differential extended by $\partial e_1 = e_2$ and $\partial e_2 = 0$.

Definition 2.12. Two DGAs $(\mathscr{A}, \partial)$ and $(\mathscr{A}', \partial')$ are *stable tame isomorphic* if there exist stabilizations $S_{i_1}, \dots, S_{i_m}$ and $S'_{j_1}, \dots, S'_{j_n}$ and a tame isomorphism

$$\psi : S_{i_1}(\dots S_{i_m}(\mathscr{A}) \dots) \to S'_{j_1}(\dots S'_{j_n}(\mathscr{A}') \dots)$$

of DGAs such that the composition of maps is a chain map.

A stable tame isomorphism preserves the homology of the DGA and so the homology of $(\mathscr{A}(L), \partial)$ is also a Legendrian isotopy invariant, which is called the *Legendrian contact homology* of K.

2c. *Augmentations.* Chekanov [2002a] implicitly defines a class of DGA chain maps called augmentations.

Definition 2.13. An *augmentation* is an algebra map $\epsilon : (\mathscr{A}(L), \partial) \to \mathbb{Z}_2$ with the properties that $\epsilon(1) = 1$, that $\epsilon \circ \partial = 0$, and that $|q_i| = 0$ if $\epsilon(q_i) = 1$. We let $\mathrm{Aug}(L)$ denote the set of augmentations of $(\mathscr{A}(L), \partial)$.

A tame isomorphism between DGAs induces a bijection on the corresponding sets of augmentations. Stabilizing a DGA may double the number of augmentations, depending on the grading of the new generators. It is possible to normalize the number of augmentations by an appropriate power of two and obtain an integer Legendrian isotopy invariant; see [Mishachev 2003; Ng and Sabloff 2006].

2d. *Graded normal rulings.* The geometric motivation for a graded normal ruling on Σ comes from examining the one-parameter family of Morse–Smale chain complexes that comes from a suitably generic generating family F for Σ. Pushkar and Chekanov [2005, Section 12] provide a detailed explanation of the connection between generating families and graded normal rulings, including the restrictions placed on the types of generating families considered. In their language, a graded normal ruling is a positive Maslov pseudoinvolution. As combinatorial objects, rulings originated in the work of Chekanov [2002b] and Fuchs [2003].

Definition 2.14. A *ruling* on the front diagram Σ is a one-to-one correspondence between the set of left cusps and the set of right cusp and, for each corresponding pair of cusps, two paths in Σ that join them. The paths are such that

(1) any two paths in the ruling meet only at crossings or at cusps; and

(2) the two paths joining corresponding cusps meet only at the cusps, and hence their interiors are disjoint.

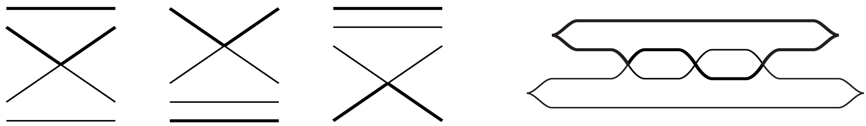

(a) (b)

Figure 5. Left: the three possible configurations of a normal switch. Right: a graded normal ruling on the standard Legendrian trefoil.

The two paths joining corresponding cusps are called *companions* of each other. Together the two paths bound a disk in the plane called the *ruling disk*. At a crossing, two paths either pass through each other or one path lies entirely above the other. In the latter case, we call the crossing a *switch*.

Definition 2.15. We say a ruling is *graded* if each switched crossing has grading 0, where the grading comes from a Maslov potential as defined in Section 2b2. A ruling is *normal* if at each switch the two paths at the crossing and their companion strands are arranged as in Figure 5(a). Let $N(\Sigma)$ denote the set of graded normal rulings of a Legendrian knot with front projection Σ.

Figure 5(b) gives a graded normal ruling of the standard Legendrian trefoil.

Theorem 2.16 [Pushkar and Chekanov 2005]. *If K and K' are Legendrian isotopic and Σ and Σ' are σ-generic, then Σ and Σ' admit the same number of graded normal rulings.*

In a normal ruling, there are two types of unswitched crossings. A *departure* is an unswitched crossing in which, to the left of the crossing, the two ruling disks are either disjoint or one is nested inside the other. A *return* is an unswitched crossing in which the two ruling disks partially overlap to the left of the crossing.

3. Defining Morse complex sequences

As indicated in the introduction, the ideas behind *Morse complex sequences* and the equivalence relation we define in this section originate with Petya Pushkar. Equivalence classes of Morse complex sequences correspond in the language of his 2008 email to "combinatorial generating families".

A *Morse complex sequence* is a finite sequence of chain complexes and a set of chain maps relating consecutive chain complexes. Its definition is geometrically motivated by the sequence of Morse–Smale chain complexes (C_x, ∂_x, g_x) coming from a suitably generic generating family F and metric g. The local moves used to define an equivalence relation are found by considering two-parameter families of function/metric pairs (F_x^t, g_x^t). Hatcher and Wagoner [1973] describe possible relationships between the sequences (C_x, ∂_x, g_x^0) and (C_x, ∂_x, g_x^1). In his 2001 email,

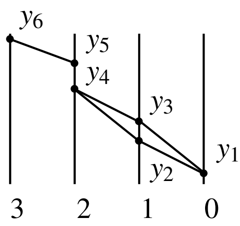


Figure 6. The graphical presentation of an ordered chain complex. The sloped lines from y_4 to y_3 and y_2 indicate that $\partial y_4 = y_2 + y_3$.

Pushkar identifies the relationships that are necessary for working with Legendrian front projections.

3a. ***Ordered chain complexes.*** We begin by defining the chain complexes that comprise a Morse complex sequence.

Definition 3.1. An *ordered chain complex* is a $\mathbb{Z}_2$ vector space C with ordered basis $y_1 < y_2 < \cdots < y_m$, a $\mathbb{Z}$ grading $|y_j|$ on $y_1, \ldots, y_m$, and a linear map $\partial : C \to C$, such that

(1) $\partial \circ \partial = 0$,

(2) $|\partial y_j| = |y_j| - 1$, and

(3) $\partial y_j = \sum_{i<j} a_{j,i} y_i$, where $a_{j,i} \in \mathbb{Z}_2$.

We denote an ordered chain complex by (C, ∂) when the ordered basis and grading are understood. We let $\langle \partial y_j \,|\, y_i \rangle$ denote the contribution of the generator y_i to ∂y_j, that is, $\langle \partial y_j \,|\, y_i \rangle = a_{j,i}$.

Remark 3.2. The $m \times m$ lower triangular matrix $\mathcal{D}$ defined by $(\mathcal{D})_{j,i} = a_{j,i}$ for $j > i$ is a matrix representative of the map ∂. Indeed, $\partial^2 = 0$ implies $\mathcal{D}^2 = 0$ and with respect to the basis $\{y_m\}$ the $\mathbb{Z}_2$ coefficients of ∂y_j are given by $e_j \mathcal{D}$, where e_j is the j-th standard basis row vector. In Section 6, we use the matrix representatives of a sequence of ordered chain complexes to associate an augmentation to an MCS.

In Figure 6 we give an example of the graphical encoding of an ordered chain complex (C, ∂) used in [Barannikov 1994]. The vertical lines indicate the gradings of the generators $y_1, \ldots, y_6$, the height of the vertices on the vertical lines indicates the ordering of the generators, and the sloped lines connecting vertices represent the boundary map ∂.

3a1. *Matrix notation.* All of the matrices in this article have entries in $\mathbb{Z}_2$ and matrix operations are done mod 2. For $k > l$, we let $\mathcal{H}_{k,l}$ denote a square matrix with 1 in the (k, l) position and zeros everywhere else. We let $E_{k,l} = I + \mathcal{H}_{k,l}$, where I denotes the identity matrix. We let $P_{i+1,i}$ denote the square permutation matrix obtained by interchanging rows i and $i+1$ of the identity matrix. Finally, we let J_{i-1} denote the matrix obtained by inserting two columns of zeros after column $i-1$ in the identity matrix. The matrix J_{i-1}^T is the transpose of J_{i-1}.

3b. *Morse complex sequences.* Consecutive ordered chain complexes in a Morse complex sequence are related by one of the following four chain maps.

Definition 3.3. Suppose (C, ∂) and (C', ∂') are ordered chain complexes with ordered generating sets $y_1 < \cdots < y_n$ and $y'_1 < \cdots < y'_m$, respectively. We define four types of chain isomorphisms.

(1) Suppose $n = m$, $1 \le l < k \le n$, and $|y_k| = |y_l|$. We say (C, ∂) and (C', ∂') are *related by a handleslide between k and l* if and only if the linear extension of the map on generators defined by

$$\phi_1(y_i) = \begin{cases} y'_i & \text{if } i \neq k, \\ y'_k + y'_l & \text{if } i = k \end{cases}$$

is a chain isomorphism from (C, ∂) to (C', ∂'). As matrices, this means $\mathfrak{D} = E_{k,l}\mathfrak{D}' E_{k,l}^{-1}$.

(2) Suppose $n = m$ and $1 \le k < n$. We say (C, ∂) and (C', ∂') are *related by interchanging critical values at k* if and only if the linear extension of the map on generators defined by

$$\phi_2(y_i) = \begin{cases} y'_i & \text{if } i \notin \{k, k+1\}, \\ y'_{k+1} & \text{if } i = k, \\ y'_k & \text{if } i = k+1 \end{cases}$$

is a chain isomorphism from (C, ∂) to (C', ∂'). As matrices, this means $\mathfrak{D} = P_{k+1}\mathfrak{D}' P_{k+1}^{-1}$.

(3) Suppose $n = m-2$, $1 \le k < m$ and $\langle \partial' y'_{k+1} \,|\, y'_k \rangle = 1$. We say (C, ∂) and (C', ∂') are *related by the birth of two generators at k* if and only if (C, ∂) is chain isomorphic to the quotient of (C', ∂') by the acyclic subcomplex generated by $\{y'_{k+1}, \partial' y'_{k+1}\}$ by the map

$$\phi_3(y_i) = \begin{cases} [y'_i] & \text{if } i < k, \\ [y'_{i+2}] & \text{if } i > k+1. \end{cases}$$

The matrix $\mathfrak{D}$ is computed explicitly as follows.

Denote by $y'_{k+1} < y'_{u_1} < y'_{u_2} < \cdots < y'_{u_s}$ the generators of C' satisfying $\langle \partial' y' \,|\, y'_k \rangle = 1$. Let $y'_{v_r} < \cdots < y'_{v_1} < y'_k$ denote the generators of C' such that $\langle \partial' y'_{k+1} \,|\, y' \rangle = 1$. Let $E = E_{k,v_r} \ldots E_{k,v_1} E_{u_1,k+1} \ldots E_{u_s,k+1} I$. Then, as matrices, $\mathfrak{D} = J_{i-1} E \mathfrak{D}' E^{-1} J_{i-1}^T$.

We say (C, ∂) and (C', ∂') are *related by the death of two generators at k* if the roles of (C, ∂) and (C', ∂') are exchanged in the map above. In particular, $n = m+2$, $\langle \partial y_{k+1} \,|\, y_k \rangle = 1$ for some $1 \le k < n$, and (C', ∂') is chain isomorphic to the quotient of (C, ∂) by the acyclic subcomplex generated by $\{y_{k+1}, \partial y_{k+1}\}$ by the map ϕ_4 given by $y'_i \mapsto [y_i]$ if $i < k$ and $y'_i \mapsto [y_{i+2}]$ otherwise.

In the matrix equation in (3), $E\mathcal{D}'E^{-1}$ represents a series of handleslide moves on the chain complex (C', ∂'). The generators y_{k+1} and y_k form a trivial acyclic subcomplex in $E\mathcal{D}'E^{-1}$, and $\mathcal{D} = J_{i-1}E\mathcal{D}'E^{-1}J^T_{i-1}$ is the result of quotienting out this subcomplex.

A Morse complex sequence encodes possible algebraic changes resulting from a generic deformation between two Morse functions without critical points. A generating family for a Legendrian knot is such a deformation.

Definition 3.4. A *Morse complex sequence* $\mathscr{C}$ is a finite sequence of ordered chain complexes $(C_1, \partial_1) \ldots (C_m, \partial_m)$ with ordered generating sets $y^j_1 < \cdots < y^j_{n_j}$ for each (C_j, ∂_j) for $1 \le j \le m$, and $\tau_j \in \mathbb{Z} \oplus \mathbb{Z}$ for $1 \le j < m$ satisfying the following:

(1) For (C_1, ∂_1) and (C_m, ∂_m),

$$n_1 = n_m = 2, \quad \langle \partial_1 y^1_2 \,|\, y^1_1 \rangle = 1, \quad \langle \partial_m y^m_2 \,|\, y^m_1 \rangle = 1.$$

In particular, both (C_1, ∂_1) and (C_m, ∂_m) have trivial homology.

(2) $|n_{j+1} - n_j| \in \{0, 2\}$ for each $1 \le j < m$.

(3) If $n_j = n_{j+1} - 2$, then $\tau_j = (k, 0)$ for some k, and (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by the birth of two generators at k.

(4) If $n_j = n_{j+1} + 2$, then $\tau_j = (k, 0)$ for some k, and (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by the death of two generators at k.

(5) If $n_j = n_{j+1}$, then either
   - $\tau_j = (k, 0)$ for some k and (C_j, ∂_j), and $(C_{j+1}, \partial_{j+1})$ are related by interchanging critical values at k, or
   - $\tau_j = (k, l)$ for some $1 \le l < k \le n_j$, and (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by a handleslide between k and l.

3c. ***Associating a marked front projection to an MCS.*** We may encode an MCS graphically using a front projection and certain vertical line segments. The vertical marks encode the handleslides occurring in the chain isomorphisms of type (1) and (3) in Definition 3.3.

Definition 3.5. A *handleslide mark* on a σ-generic front projection Σ with Maslov potential μ is a vertical line segment in the xz-plane with endpoints on Σ. We require that the line segment not intersect the crossings or cusps of Σ and the endpoints sit on strands of Σ with the same Maslov potential. A *marked front*

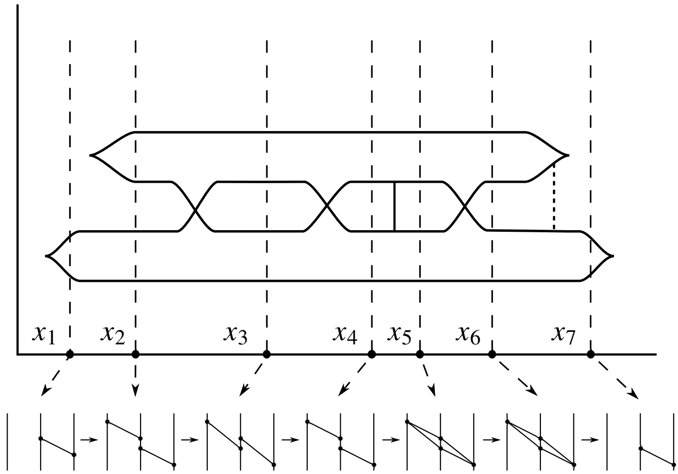


Figure 7. An example of an MCS and its associated marked front projection.

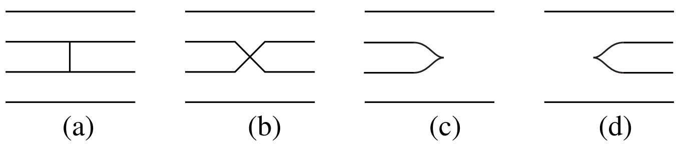


Figure 8. The four possible tangles used in Section 3c.

projection is a σ-generic front projection with a collection of handleslide marks; see Figure 7.

Let $(C_1, \partial_1) \cdots (C_m, \partial_m)$ and $\tau_1, \ldots, \tau_{m-1} \in \mathbb{Z} \times \mathbb{Z}$ be an MCS $\mathscr{C}$. We build the marked front projection for $\mathscr{C}$ inductively beginning with a single left cusp and building to the right by adjoining tangles of the types in Figure 8.

For each $1 \leq j < m$, we adjoin a tangle from Figure 8. At each step, we assume the strands of our tangle are numbered from top to bottom. If (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by a handleslide between k and l, then we adjoin a tangle of type (a) with a handleslide mark between strands k and l. If (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by interchanging critical values at k, then we adjoin a tangle of type (b) with a crossing between strands k and $k+1$. If (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by the death of two generators at k, then we adjoin a tangle of type (c) where the right cusp connects strands k and $k+1$. If (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by the birth of two generators at k, then we adjoin a tangle of type (d) where the left cusp sits above strand $k-1$.

We will refer to the marks arising from handleslides between consecutive chain complexes as *explicit handleslide marks*. At the cusps corresponding to births or deaths, we also place marks in a small neighborhood of the cusp. These marks correspond to the composition of handleslides represented by the matrix product E in Definition 3.3(3). We call these *implicit handleslide marks* and use dotted vertical lines to distinguish them from the explicit handleslide marks.

The front projection we have constructed is σ-generic and has a natural Maslov potential coming from the index of the generators in $(C_1, \partial_1) \cdots (C_m, \partial_m)$. In addition, the handleslide marks connect strands of the same Maslov potential. Hence, we have constructed a marked front projection. Figure 7 gives an MCS and its associated marked front projection.

Remark 3.6. (1) If a marked front projection is associated to $\mathscr{C}$, then by reversing the construction defined above, $(C_1, \partial_1) \cdots (C_m, \partial_m)$ and $\tau_1, \ldots, \tau_{m-1}$ are uniquely reconstructed from the cusps, crossings and handleslide marks of the marked front projection using Definition 3.4(1) and the chain isomorphisms from Definition 3.3. Thus, a marked front projection may be associated with at most one MCS.

(2) By Definition 3.3, $\langle \partial_j y^j_{i+1} \,|\, y^j_i \rangle$ is equal to 1 if (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are related by the death of two generators at i, and equal to 0 if they are related by interchanging critical values at i.

(3) In an MCS $\mathscr{C}$, the homologies of (C_j, ∂_j) and $(C_{j+1}, \partial_{j+1})$ are isomorphic. Thus, by Definition 3.4(1), all of the homologies in $\mathscr{C}$ are trivial.

Definition 3.7. Given a σ-generic front projection Σ, an MCS $\mathscr{C}$ is in the set $\text{MCS}(\Sigma)$ if and only if the marked front projection associated to $\mathscr{C}$ has a front projection that is planar isotopic to Σ by a planar isotopy through σ-generic front projections. Such an isotopy pulls back to a Legendrian isotopy in $\mathbb{R}^3$.

Pushkar (in the emails to Fuchs) defines a "spring sequence" to be a sequence of ordered chain complexes over a commutative ring $\mathbb{E}$ with consecutive complexes connected by maps similar to those defined in Definition 3.3. Originally defined in [Pushkar and Chekanov 2005, Section 12], these chain complexes are called M-complexes. Pushkar encodes a spring sequence by adding vertical marks, called "springs", to the front projection of an associated Legendrian knot.

3d. ***An equivalence relation on*** $\mathbf{MCS(\Sigma)}$**.** In this section, we describe a set of local moves used to define an equivalence relation on $\text{MCS}(\Sigma)$. Since an MCS is completely determined by its associated marked front projection, these moves are defined as graphical changes in the handleslide marks. The equivalence relation encodes local algebraic changes resulting from a generic deformation of a 1-parameter family of Morse functions. Such deformations are studied extensively in [Hatcher and Wagoner 1973].

Figures 9, 10, and 11 describe local changes in the handleslide marks of a marked front projection. We call these *MCS moves*. Other strands may appear

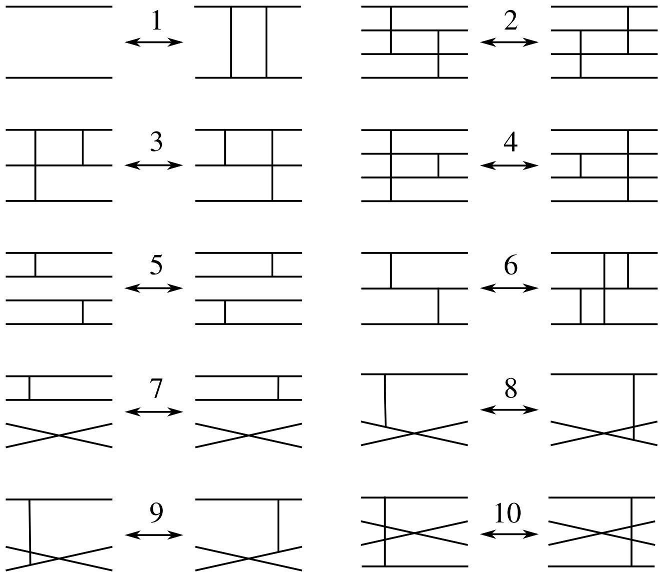


Figure 9. MCS moves 1–10. We also allow reflections about the horizontal axis.

in a local neighborhood of an MCS move, although we assume no other crossings or cusps appear. The ordering of implicit handleslide marks at a birth or death is irrelevant; thus we will not consider analogues of moves 1–6 for implicit handleslide marks. In moves 11–14, there may be other implicit marks that the indicated explicit handleslide mark commutes past without incident. Additional implicit marks may also appear at the birth or death in moves 15 and 16.

MCS move 17 requires explanation. Let $\mathscr{C} \in \mathrm{MCS}(\Sigma)$ and suppose (C, ∂) is an ordered chain complex in $\mathscr{C}$ with generators $y_1 < \cdots < y_m$ and a pair of generators $y_l < y_k$ such that $|y_l| = |y_k| + 1$. Let $y_{u_1} < y_{u_2} < \cdots < y_{u_s}$ denote the generators of C satisfying $\langle \partial y \,|\, y_k \rangle = 1$; see the left three arrows in Figure 11. Let $y_{v_r} < \cdots < y_{v_1} < y_i$ denote the generators of C appearing in ∂y_l; see the right two arrows in Figure 11. Let $E = E_{k,v_r} \cdots E_{k,v_1} E_{u_1,l} \cdots E_{u_s,l}$. Then over $\mathbb{Z}_2$, we have

$$\mathscr{D} = E \mathscr{D} E^{-1}. \tag{1}$$

The MCS move in Figure 11 says that we may either introduce or remove the handleslides represented by E and this move is local in the sense that it does not change any of the other chain complexes in $\mathscr{C}$. The next proposition shows that all of the MCS moves are local in this sense.

Proposition 3.8. *Suppose $\mathscr{C} = (C_1, \partial_1) \ldots (C_m, \partial_m)$ is an MCS on Σ and that in the interval $[a, b]$ of the x-axis we modify the marked front projection of $\mathscr{C}$ by one of*

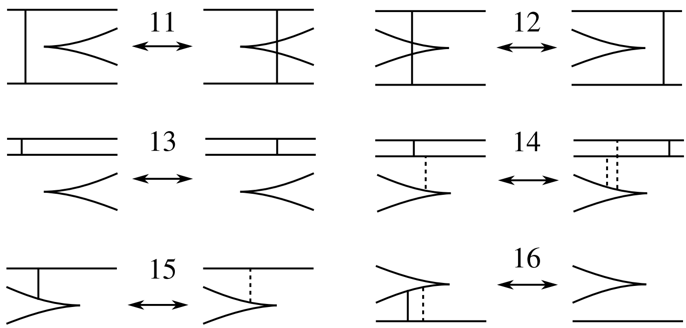


Figure 10. MCS moves 11–16. We also allow reflections about the horizontal and vertical axes.

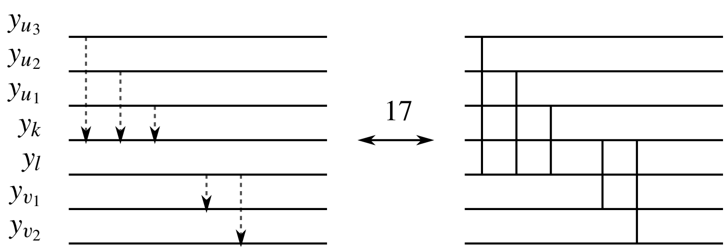


Figure 11. MCS move 17.

the MCS moves in Figures 9, 10, or 11. Then the resulting marked front projection $\mathscr{C}'$ determines an MCS. In addition, the ordered chain complexes of $\mathscr{C}$ and $\mathscr{C}'$ agree outside of the interval $[a, b]$.

Proof. As marked front projections, $\mathscr{C}$ and $\mathscr{C}'$ agree outside of $[a, b]$ and so by Remark 3.6(1) the chain complexes they determine agree to the left of $[a, b]$. The chain complexes to the right of $[a, b]$ depend on the handleslide marks of each MCS and the chain complex to the immediate right of $[a, b]$, so it suffices to show that $\mathscr{C}$ and $\mathscr{C}'$ determine the same chain complex to the immediate right of $[a, b]$. This follows from the matrix equations in Definition 3.3 and the following matrix equations:

Move 1: $E_{k,l}E_{k,l} = I$ for $k > l$.

Moves 2–5: $E_{k_1,l_1}E_{k_2,l_2} = E_{k_2,l_2}E_{k_1,l_1}$ for $k_1 > l_1$ and $k_2 > l_2$ with $k_1 \neq l_2$ and $k_2 \neq l_1$.

Move 6: $E_{a,b}E_{b,c} = E_{b,c}E_{a,c}E_{a,b}$ for $a > b > c$.

Moves 7 and 10: $E_{k,l}P_{i+1,i} = P_{i+1,i}E_{k,l}$ for $k > l$ with $k, l \notin \{i+1, i\}$.

Moves 8 and 9: $E_{k,i}P_{i+1,i} = P_{i+1,i}E_{k,i+1}$ for $k > i+1$.

Moves 15 and 16: Suppose the explicit handleslide mark occurs between strands k and $i+1$. If E (respectively F) represents the sequence of implicit handleslide moves in the formula for the differential of the chain complex of $\mathscr{C}$ (respectively $\mathscr{C}'$) occurring after the right cusp, then $F = EE_{k,i+1}$.

In moves 11–13, the explicit handleslide mark does not involve the generators of the acyclic subcomplex that is quotiented out, so the chain complexes of $\mathscr{C}$ and $\mathscr{C}'$ to the right of $[a, b]$ are equal. Move 14 is a composition of moves 6, 13, and 15. Indeed, moves 15 and 6 allow us to commute the explicit mark past the implicit mark and then move 13 pushes the explicit mark past the cusp. Finally, the localness of move 17 was detailed in the discussion surrounding equation (1). $\square$

Definition 3.9. We say $\mathscr{C}$ and $\mathscr{C}'$ in $\mathrm{MCS}(\Sigma)$ are *equivalent*, denoted $\mathscr{C} \sim \mathscr{C}'$, if there exists a finite sequence $\mathscr{C} = \mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n = \mathscr{C}' \in \mathrm{MCS}(\Sigma)$ where for each $1 \le i < n$ the marked front projections of $\mathscr{C}_i$ and $\mathscr{C}_{i+1}$ are related by an MCS move.

We let $\widehat{\mathrm{MCS}}(\Sigma)$ denote the equivalence classes of $\mathrm{MCS}(\Sigma)/\sim$, and we let $[\mathscr{C}]$ denote an equivalence class in $\widehat{\mathrm{MCS}}(\Sigma)$.

3e. *Associating a normal ruling to an MCS.*

Definition 3.10 [Barannikov 1994]. An ordered chain complex (C, ∂) with generators $y_1, \dots, y_m$ is in *simple form* if the following hold:

- For all i, either $\partial y_i = 0$ or $\partial y_i = y_j$ for some j.
- If $\partial y_i = \partial y_k = y_j$, then $k = i$.

Lemma 3.11 [Barannikov 1994, Lemma 2]. *Let (C, ∂) be an ordered chain complex with generators $y_1, \dots, y_m$. Then after a series of handleslide moves as in Definition 3.3(1), we can reduce (C, ∂) to simple form. In addition, this simple form is unique.*

Barannikov proves this lemma by using an inductive construction to find the simple forms of each of the subcomplexes (C_k, ∂) generated by $y_1, \dots, y_k$ for $k \le m$.

Remark 3.12. The following two observations follow directly from Lemma 3.11.

(1) If (C, ∂) and (C', ∂') are chain isomorphic by a handleslide move, that is, $\mathscr{D} = E_{k,l}\mathscr{D}' E_{k,l}^{-1}$, then they have the same simple form.

(2) For consecutive generators in (C, ∂), a handleslide move does not change the value of $\langle \partial y_{j+1} \,|\, y_j \rangle$. Thus, if $\langle \partial y_{j+1} \,|\, y_j \rangle = 1$, then $\partial y_{j+1} = y_j$ in the simple form for (C, ∂).

Definition 3.13. Suppose that (C, ∂) is an ordered chain complex with generators $y_1, \dots, y_m$ and trivial homology. By Lemma 3.11, there exists a fixed-point free involution $\tau : \{1, \dots, m\} \to \{1, \dots, m\}$ such that in the simple form of (C, ∂), either $\partial y_i = y_{\tau(i)}$ or $\partial y_{\tau(i)} = y_i$ for all i. We call τ the *pairing* of (C, ∂).

The next lemma assigns a normal ruling to an MCS. Pushkar and Chekanov [2005, Section 12.4] prove this lemma for a generating family using the language of pseudoinvolutions.

Lemma 3.14. *The pairings determined by the simple forms of the ordered chain complexes in an MCS* $\mathscr{C} = (C_1, \partial_1) \cdots (C_m, \partial_m)$ *determine a graded normal ruling* $N_{\mathscr{C}}$ *on* Σ.

Proof. Each (C_j, ∂_j) has trivial homology, and thus, a pairing τ_i. Given a pair of consecutive singularities x_i and x_{i+1} of Σ, we define a pairing on the strands of the tangle $\Sigma \cap (x_i, x_{i+1})$ using the pairing of a chain complex of $\mathscr{C}$ between x_i and x_{i+1}. There may be several chain complexes of $\mathscr{C}$ between x_i and x_{i+1}. However, these chain complexes differ by handleslide moves and so by Remark 3.12(1) their pairings are identical. Remark 3.12(2) justifies that two strands entering a cusp are paired.

It remains to check that the switched crossings are graded and normal. The normality follows from [Barannikov 1994, Lemma 4]. The two strands meeting at a switch exchange companion strands and the Maslov potentials of two companioned strands differ by 1. Hence, two strands meeting at a switch must have the same Maslov potential and so the grading of a switched crossing is 0. □

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are equivalent by a single MCS move, then by Proposition 3.8 the chain complexes in $\mathscr{C}_1$ and $\mathscr{C}_2$ are equal outside of a neighborhood of the MCS move. Thus, we may use chain isomorphic chain complexes in $\mathscr{C}_1$ and $\mathscr{C}_2$ to determine the graded normal rulings $N_{\mathscr{C}_1}$ and $N_{\mathscr{C}_1}$. As a consequence we have the following.

Proposition 3.15. *If* $\mathscr{C}_1 \sim \mathscr{C}_2$ *then* $N_{\mathscr{C}_1} = N_{\mathscr{C}_2}$.

We let $N_{[\mathscr{C}]} \in N(\Sigma)$ denote the graded normal ruling associated to the MCS class $[\mathscr{C}]$.

3f. ***MCSs with simple births.*** A *simple birth* in an MCS is a birth with no implicit handleslides. In the language of Definition 3.3(3)), this says $E = I$. From now on we will restrict our attention to MCSs with only simple births and MCSs equivalence moves that do not involve implicit handleslide marks at births. The connection between MCSs and augmentations is clearer under this assumption. For the sake of simplicity, the results stated in the introduction were given in terms of $\widehat{\mathrm{MCS}}(\Sigma)$. The corresponding results proved in the remainder of the article are given in terms of MCSs with simple births. In light of Proposition 3.17 below, there is no harm in this ambiguity.

Definition 3.16. Let $\mathrm{MCS}_b(\Sigma)$ be the set of MCSs in $\mathrm{MCS}(\Sigma)$ that have only simple births. Two MCSs $\mathscr{C}, \mathscr{C}' \in \mathrm{MCS}_b(\Sigma)$ are equivalent if they are equivalent in $\mathrm{MCS}(\Sigma)$ by a sequence of MCS moves $\mathscr{C} = \mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n = \mathscr{C}'$ such that $\mathscr{C}_i \in \mathrm{MCS}_b(\Sigma)$ for all i.

We let $\widehat{\mathrm{MCS}}_b(\Sigma)$ denote the equivalence classes of $\mathrm{MCS}_b(\Sigma)/\sim$, and we let $[\mathscr{C}]$ denote an equivalence class in $\widehat{\mathrm{MCS}}_b(\Sigma)$.

Proposition 3.17. *The inclusion map* $i : \mathrm{MCS}_b(\Sigma) \to \mathrm{MCS}(\Sigma)$ *determines a bijection* $\hat{i} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \widehat{\mathrm{MCS}}(\Sigma)$ *by* $[\mathscr{C}] \mapsto [i(\mathscr{C})]$.

We sketch the argument for injectivity. Suppose $[i(\mathscr{C})] = [i(\mathscr{C}')]$ in $\mathrm{MCS}(\Sigma)$ and $\mathfrak{X} = \{i(\mathscr{C}) = \mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_{n-1}, i(\mathscr{C}') = \mathscr{C}_n\}$ is a sequence of MCS moves. Since $i(\mathscr{C})$ and $i(\mathscr{C}')$ have simple births, each implicit handleslide appearing at a birth in an MCS in $\mathfrak{X}$ is introduced and eventually eliminated using one of moves 14, 15, or 16. We may modify the sequence of MCS moves in $\mathfrak{X}$ inductively beginning with $\mathscr{C}_1$ and $\mathscr{C}_2$ so that $\mathscr{C}_i \in \mathrm{MCS}_b(\Sigma)$ for all i. We do this by replacing occurrences of move 16 at lefts cusps with move 1 and removing the occurrences of move 15 at left cusps. As a result, we must also change the occurrences of move 14 to a composition of moves 6 and 13 and possibly include additional moves 2–5. The map $\hat{i}$ is surjective since MCS move 15 may be used to find a representative of $[\mathscr{C}] \in \widehat{\mathrm{MCS}}(\Sigma)$ with simple births.

Remark 3.18. If we assume all births are simple, then we do not need to indicate the implicit handleslide marks at deaths in the associated marked front projection, since they can be determined by reconstructing the ordered chain complexes as in Remark 3.6(1). Thus, from now on, marked front projections will include only explicit handleslide marks and we will no longer indicate implicit handleslide marks in the MCS moves.

4. Chain homotopy classes of augmentations

We begin by defining DGA morphisms and chain homotopy on arbitrary DGAs and then restrict to the case of CE-DGAs. Definition 4.1 and Lemma 4.2 follow directly from [Kálmán 2005, Section 2.3].

Definition 4.1. Let $(\mathscr{A}, \partial)$ and $(\mathscr{B}, \partial')$ be differential graded algebras over a commutative ring R and graded by a cyclic group Γ. A *DGA morphism* $\varphi : (\mathscr{A}, \partial) \to (\mathscr{B}, \partial')$ is a grading-preserving algebra homomorphism satisfying $\varphi \circ \partial = \partial' \circ \varphi$. Given two DGA morphisms $\varphi, \psi : (\mathscr{A}, \partial) \to (\mathscr{B}, \partial')$, a *chain homotopy between* φ *and* ψ is a linear map $H : (\mathscr{A}, \partial) \to (\mathscr{B}, \partial')$ such that

(1) $|H(a)| = |a| + 1$ for all $a \in \mathscr{A}$,

(2) $H(ab) = H(a)\psi(b) + (-1)^{|a|}\varphi(a)H(b)$ for all $a, b \in \mathscr{A}$, and

(3) $\varphi - \psi = H \circ \partial + \partial' \circ H$.

We refer to condition (2) as the *derivation product property*.

Augmentations are DGA morphisms between the CE-DGA and the DGA whose only nonzero chain group is a copy of $\mathbb{Z}_2$ in grading 0. From [Kálmán 2005, Lemma 2.18], a chain homotopy between augmentations is determined by its action on the generators of the CE-DGA.

Lemma 4.2. *Let $(\mathscr{A}(L), \partial)$ be the CE-DGA of the Lagrangian projection L with generating set Q, and let $\epsilon_1, \epsilon_2 \in \mathrm{Aug}(L)$. If a map $H : Q \to \mathbb{Z}_2$ has support on generators of grading -1, then H can be uniquely extended by linearity and the derivation product property to a map, also denoted H, on all of $(\mathscr{A}(L), \partial)$ that has support on elements of grading -1. Moreover, if the extension satisfies $\epsilon_1(q) - \epsilon_2(q) = H \circ \partial(q)$ for all $q \in Q$, then $\epsilon_1 - \epsilon_2 = H \circ \partial$ on all of $\mathscr{A}(L)$, and thus H is a chain homotopy between ϵ_1 and ϵ_2.*

We say ϵ_1 and ϵ_2 are *chain homotopic* and write $\epsilon_1 \simeq \epsilon_2$ if a chain homotopy H exists between ϵ_1 and ϵ_2.

Lemma 4.3 [Félix et al. 1995]. *The relation $\simeq$ is an equivalence relation.*

We let $\mathrm{Aug}^{ch}(L)$ denote $\mathrm{Aug}(L)/\simeq$ and $[\epsilon]$ denote an augmentation class in $\mathrm{Aug}^{ch}(L)$. The following lemma will be necessary for our later work connecting augmentation classes and MCS classes.

Lemma 4.4. *Let L and L' be Lagrangian projections with CE-DGAs $(\mathscr{A}(L), \partial)$ and $(\mathscr{A}(L'), \partial')$, respectively. Let $f : (\mathscr{A}(L), \partial) \to (\mathscr{A}(L'), \partial')$ be a DGA morphism. Then f induces a map $F : \mathrm{Aug}^{ch}(L') \to \mathrm{Aug}^{ch}(L)$ by $[\epsilon] \mapsto [\epsilon \circ f]$. If $g : (\mathscr{A}(L), \partial) \to (\mathscr{A}(L'), \partial')$ is also a DGA morphism and H is a chain homotopy between f and g, then $F = G$.*

Proof. We begin by checking F is well-defined, that is, $\epsilon \circ f$ is an augmentation and $\epsilon \simeq \epsilon'$ implies $\epsilon \circ f \simeq \epsilon' \circ f$. Since f is a degree-preserving chain map and $\epsilon \circ \partial' = 0$, the equality $\epsilon \circ f(q) = 1$ implies $|q| = 1$ and $\epsilon \circ f \circ \partial = \epsilon \circ \partial' \circ f = 0$. Therefore $\epsilon \circ f$ is an augmentation. If H is a chain homotopy between ϵ and ϵ', then $H \circ f$ is a chain homotopy between $\epsilon \circ f$ and $\epsilon' \circ f$.

Suppose $g : (\mathscr{A}(L), \partial) \to (\mathscr{A}(L'), \partial')$ is also a DGA morphism and H is a chain homotopy between f and g. The map $H' = \epsilon \circ H : (\mathscr{A}(L), \partial) \to \mathbb{Z}_2$ is a chain homotopy between $\epsilon \circ f$ and $\epsilon \circ g$. Thus, $F = G$ since

$$F([\epsilon]) = [\epsilon \circ f] = [\epsilon \circ g] = G([\epsilon]). \qquad \square$$

In the case of the CE-DGA, the number of chain homotopy classes is a Legendrian invariant.

Proposition 4.5. *If L and L' are Lagrangian projections of Legendrian isotopic knots K and K', then there is a bijection between $\mathrm{Aug}^{ch}(L)$ and $\mathrm{Aug}^{ch}(L')$. Thus, $|\mathrm{Aug}^{ch}(L)|$ is a Legendrian invariant of the Legendrian isotopy class of K.*

We sketch the proof of Proposition 4.5. Since K and K' are Legendrian isotopic, $(\mathcal{A}(L), \partial)$ and $(\mathcal{A}(L'), \partial')$ are stable tame isomorphic. Thus, we need only consider the case of a single elementary isomorphism and a single stabilization. Suppose $(S_i(\mathcal{A}), \partial)$ is an index i stabilization of a DGA $(\mathcal{A}, \partial)$. If $i \neq 0$, then $\epsilon \in \mathrm{Aug}(\mathcal{A})$ extends uniquely to an augmentation $\tilde{\epsilon} \in \mathrm{Aug}(S_i(\mathcal{A}, \partial))$ by sending both e_1 and e_2 to 0. If $i = 0$, then $|e_1| = 0$ and $\epsilon \in \mathrm{Aug}(\mathcal{A})$ extends to two different augmentations in $S_0(\mathcal{A}, \partial)$; the first, denoted $\tilde{\epsilon}$, sends both e_1 and e_2 to 0 and the second sends e_1 to 1 and e_2 to 0. However, these two augmentations are chain homotopic by the chain homotopy H that sends e_2 to 1 and all of the other generators to 0. Regardless of i, the map defined by $[\epsilon] \mapsto [\tilde{\epsilon}]$ is the desired bijection. Finally, if $\phi : (\mathcal{A}, \partial) \to (\mathcal{B}, \partial')$ is an elementary isomorphism between two DGAs, then the map

$$\Phi : \mathrm{Aug}^{ch}(\mathcal{B}) \to \mathrm{Aug}^{ch}(\mathcal{A}), \quad [\epsilon'] \mapsto [\epsilon' \circ \phi]$$

is the desired bijection.

In the next section, we will concentrate on understanding the chain homotopy classes of a fixed Lagrangian projection. By using a procedure called "adding dips", we may reformulate the augmentation condition $\epsilon \circ \partial$ and the chain homotopy condition $\epsilon_1 - \epsilon_2 = H \circ \partial$ as a system of matrix equations. Understanding the chain homotopy classes of augmentations then reduces to understanding solutions to these matrix equations.

5. Dipped resolution diagrams

Fuchs [2003] modifies the Lagrangian projection of a Legendrian knot so that the differential in the CE-DGA is easy to compute, at the cost of increasing the number of generators. This technique has proved to be very useful; see [Fuchs and Ishkhanov 2004; Fuchs and Rutherford 2008; Ng and Sabloff 2006; Sabloff 2005]. We will use the version of this philosophy implemented in [Sabloff 2005].

5a. ***Adding dips to a Ng resolution diagram.*** By acting on L_Σ by a series of Lagrangian Reidemeister type II moves, we can limit the types of convex immersed polygon appearing in the differential of the CE-DGA. A *dip*, denoted D, is the collection of crossings created by performing type II moves on the strands of L_Σ as in Figure 12. At the location of the dip, label the strands of L_Σ from bottom to top with the integers $1, \dots, n$. For all $k > l$, there is a type II move that pushes strand k over strand l. If $k < i$, then k crosses over l before i crosses over any strand. If $l < j < k$, then k crosses over l before k crosses over j. The notation $(k, l) \prec (i, j)$ denotes that k crosses over l before i crosses over j.

The *a-lattice* of D is composed of all crossings to the right of the vertical line of symmetry, and the *b-lattice* is composed of all crossings to the left. The label $a^{k,l}$ denotes the crossing in the a-lattice of strand k over strand l where $k > l$; see

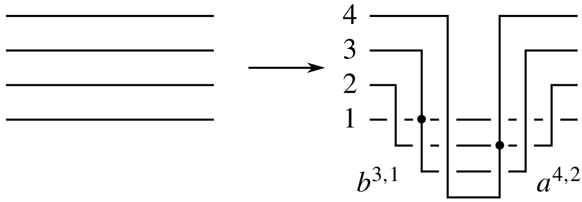


Figure 12. Adding a dip to L_Σ and labeling the resulting crossings. Crossings $b^{3,1}$ and $a^{4,2}$ are indicated.

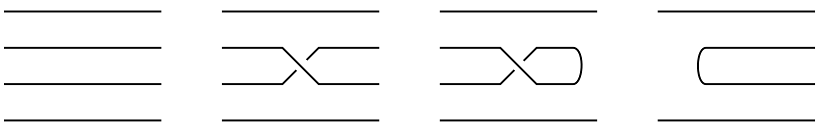


Figure 13. The four possible inserts in a sufficiently dipped diagram L_Σ^d: (1) parallel lines, (2) a single crossing, (3) a resolved right cusp, and (4) a resolved left cusp. In each case, any number of horizontal strands may exist.

Figure 12. The label $b^{k,l}$ is similarly defined. The gradings of the crossings in a dip are computed by $|b^{k,l}| = \mu(k) - \mu(l)$ and $|a^{k,l}| = |b^{k,l}| - 1$, where μ is a Maslov potential on Σ.

The type II moves may be arranged so when strand k is pushed over strand l, a crossing q has height less than $b^{k,l}$ if and only if q appears to the left of the dip, or $q = b^{r,s}$ or $q = a^{r,s}$, where $r - s \leq k - l$. Similarly, a crossing q has height less than $a^{k,l}$ if and only if q appears to the left of the dip, or $q = b^{r,s}$ or $q = a^{r,s}$, where $r - s \leq k - l$.

Definition 5.1. Given a Legendrian knot K with front projection Σ and Ng resolution L_Σ, a *dipped diagram*, denoted L_Σ^d, is the result of adding some number of dips to L_Σ. We require that during the process of adding dips, we not allow a dip to be added between the crossings of an existing dip and we not add dips in the loop of a resolved right cusp.

We let $D_1, \ldots, D_m$ denote the m dips of L_Σ^d, ordered from left to right with respect to the x-axis. For consecutive dips D_{j-1} and D_j, we let I_j denote the tangle between D_{j-1} and D_j. We define I_1 to be the tangle to the left of D_1 and I_{m+1} to be the tangle to the right of D_m. We call $I_1, \ldots, I_{m+1}$ the *inserts* of L_Σ^d.

Definition 5.2. We say a dipped diagram L_Σ^d is *sufficiently dipped* if each insert $I_1, \ldots, I_{n+1}$ is isotopic to one of those in Figure 13.

Remark 5.3. If we slide a dip of L_Σ^d left or right without passing it by a crossing, resolved cusp or another dip, then the resulting dipped diagram is topologically identical to the original. In particular, they determine the same CE-DGA. Thus, we will not distinguish between two dipped diagrams that differ by such a change.

5b. ***The CE-DGA on a sufficiently dipped diagram.*** A convex immersed polygon cannot pass completely through a dip. Therefore, we may calculate the boundary map of $(\mathcal{A}(L^d_\Sigma), \partial)$ by classifying convex immersed polygons that lie between consecutive dips or that sit entirely within a single dip. This classification is aided by our understanding of the heights of the crossings in L^d_Σ.

Let L^d_Σ be a sufficiently dipped diagram of L_Σ with dips $D_1, \dots, D_m$ and inserts $I_1, \dots, I_{m+1}$. Let $q_i, \dots, q_M$ and $z_j, \dots, z_N$ denote the crossings found in the inserts of type (2) and (3), respectively. The subscripts on q and z correspond to the subscripts of the inserts in which the crossings appear.

Remark 5.4. In Section 5a, we labeled the crossings in the a-lattice and b-lattice of a dip using a labeling of the strands at the location of the dip. When calculating ∂ on the crossings in D_j, it will make our computations clearer and cleaner if we change slightly the labeling we use for the strands in D_{j-1} and D_j.

If I_j is of type (3) and the right cusp occurs between strands $i+1$ and i, we will label the strands of D_{j-1} by $1, \dots, n$ and the strands of D_j by $1, \dots, i-1, i+2, \dots, n$. If I_j is of type (4) and the left cusp occurs between strands $i+1$ and i, we will label the strands of D_{j-1} by $1, \dots, i-1, i+2, \dots, n$ and the strands of D_j by $1, \dots, n$. If I_j is of type (1) or (2), we do not change the labeling of the strands. These changes are local, so the labeling of the strands in D_j may vary depending on whether we are calculating ∂ on the crossings in D_j or D_{j+1}.

Fix an insert I_j with $j \neq m+1$, and label the strands of D_j and D_{j-1} as indicated in Remark 5.4. For $i \in \{j, j-1\}$, A_i (respectively B_i) is the strictly lower triangular square matrices with rows and columns labeled the same as the strands of D_i and entries given by $(A_i)_{k,l} = a_i^{k,l}$ (respectively $(B_i)_{k,l} = b_i^{k,l}$) for $k > l$. The following formulas define a matrix $\tilde{A}_{j-1}$ using the a_{j-1}-lattice and possible crossings in I_j. The subscript $j-1$ has been left off the a to make the text more readable.

(a) $\tilde{A}_{j-1} = \mathcal{H}_{2,1}$ for $j = 1$, where $\dim(\mathcal{H}_{2,1}) = 2$.

(b) Suppose I_j is of type (1). Then $\tilde{A}_{j-1} = A_{j-1}$.

(c) Suppose I_j is of type (2), A_j has dimension n, and q_j involves strands $i+1$ and i. Then $\tilde{A}_{j-1}$ is the $n \times n$ matrix with (u, v) entry $\tilde{a}^{u,v}_{j-1}$ defined by

$$\tilde{a}^{u,v} = \begin{cases} a^{u,v} & \text{if } u, v \notin \{i, i+1\}, \\ 0 & \text{if } u = i+1,\ v = i, \end{cases}$$

$$\tilde{a}^{i,v} = a^{i+1,v} + q_j a^{i,v}, \qquad \tilde{a}^{i+1,v} = a^{i,v},$$

$$\tilde{a}^{u,i} = a^{u,i+1}, \qquad \tilde{a}^{u,i+1} = a^{u,i} + a^{u,i+1} q_j.$$

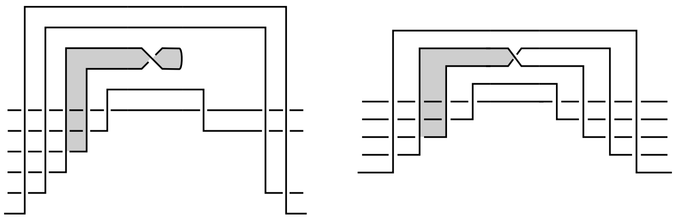


Figure 14. Disks contributing to ∂z_r and ∂q_s.

(d) Suppose I_j is of type (3), A_j has dimension n, and z_j involves strands $i+1$ and i. Then $\tilde{A}_{j-1}$ is an $n \times n$ matrix with rows and columns numbered $1, \dots, i-1, i+2, \dots, n+2$ and (u, v) entry $\tilde{a}^{u,v}_{j-1}$ defined by

$$\tilde{a}^{u,v} = \begin{cases} a^{u,v} & \text{if } u < i \text{ or } v > i+1, \\ a^{u,v} + a^{u,i}a^{i+1,v} + a^{u,i+1}z_j a^{i+1,v} & \\ \qquad + a^{u,i}z_j a^{i,v} + a^{u,i+1}z_j z_j a^{i,v} & \text{if } u > i+1 > i > v. \end{cases}$$

(e) Suppose I_j is of type (4), A_j has dimension n and the resolved left cusp strands are $i+1$ and i. Then $\tilde{A}_{j-1}$ is the $n \times n$ matrix with (u, v) entry $\tilde{a}^{u,v}_{j-1}$ defined by

$$\tilde{a}^{u,v} = \begin{cases} a^{u,v} & \text{if } u, v \notin \{i, i+1\}, \\ 1 & \text{if } u = i+1,\ v = i, \\ 0 & \text{if otherwise.} \end{cases}$$

We can now give matrix equations for the boundary map of $(\mathcal{A}(L^d_\Sigma), \partial)$.

Lemma 5.5. *The CE-DGA boundary map ∂ of a sufficiently dipped diagram of a Ng resolution L_Σ is computed as follows*:

(1) $\partial q_s = a^{i+1,i}_{s-1}$ *for a crossing q_s between strands $i+1$ and i.*

(2) $\partial z_r = 1 + a^{i+1,i}_{r-1}$ *for a crossing z_r between strands $i+1$ and i, where z_r is a crossing between strands $i+1$ and i.*

(3) $\partial A_j = A_j^2$ *for all j.*

(4) $\partial B_j = (I + B_j)A_j + \tilde{A}_{j-1}(I + B_j)$ *for all j.*

Here I is the identity matrix of appropriate dimension, and the matrices ∂A_j and ∂B_j are the result of applying ∂ to A_j and B_j entry by entry.

This result appears in [Fuchs and Rutherford 2008] in the language of "splashed diagrams". In the next three sections we justify these formulas.

5b1. *Computing ∂ on q_s and z_r.* The height ordering on the crossings of L^d_Σ ensures that q_s is the rightmost convex corner of any nontrivial disk in ∂q_s. Thus, the disk in Figure 14 is the only disk contributing to ∂q_s. The case of ∂z_r is similar.

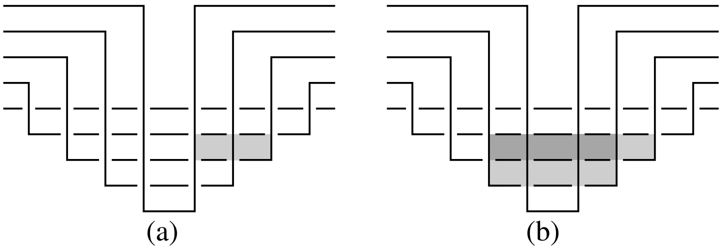


Figure 15. Left: the disk $a_j^{5,3}a_j^{3,2}$ appearing in $\partial a_j^{5,2}$. Right: the disks $b_j^{4,3}a_j^{3,2}$ and $a_j^{4,2}$ appearing in $R(\partial b_j^{4,2})$.

5b2. *Computing ∂ on the A_j lattice.* The combinatorics of the dip D_j along with the height ordering on the crossings of L_Σ^d require that disks in $\partial a_j^{k,l}$ sit within the A_j lattice. In addition, a convex immersed polygon sitting within the A_j lattice must have exactly two negative convex corners; see Figure 15(a). From this we compute $\partial a_j^{k,l} = \sum_{l<i<k} a_j^{k,i} a_j^{i,l}$, which is the (k,l)-entry in the matrix A_j^2.

5b3. *Computing ∂ on the B_j lattice.* The disks in $\partial b_j^{k,l}$ are of two types. The first type sit within D_j and include the lower right corner of $b_j^{k,l}$ as their positive convex corner. We denote these disks by $R(\partial b_j^{k,l})$. Figure 15(b) shows the two types of convex immersed polygons found in $R(\partial b_j^{k,l})$. From this, we compute

$$R(\partial b_j^{k,l}) = a_j^{k,l} + \sum_{l<i<k} b_j^{k,i} a_j^{i,l}.$$

Disks of the second type include the upper left corner of $b_j^{k,l}$ as their rightmost convex corner. Thus these disks sit between the B_j lattice and the A_{j-1} lattice. We let $L(\partial b_j^{k,l})$ denote these disks. The possible disks appear in Figures 17(a), 17(b), 17, and 18. Computing $L(\partial b_j^{k,l})$ requires understanding which of these disks appear with a positive convex corner at $b_j^{k,l}$. The entries in the matrix $\tilde{A}_{j-1}$ are defined to encode these disks. In fact, for a fixed $b_j^{k,l}$,

$$L(\partial b_j^{k,l}) = \tilde{a}_{j-1}^{k,l} + \sum_{l<i<k} \tilde{a}_{j-1}^{k,i} b_j^{i,l}.$$

Regardless of the type of I_j, $R(\partial b_j^{k,l}) + L(\partial b_j^{k,l})$ is the (k,l)-entry in the matrix $(I + B_j)A_j + \tilde{A}_{j-1}(I + B_j)$.

In Section 5c, we study the extension of augmentations across individual type II moves and then across dips, and thus, understand the connections between augmentations on L_Σ and augmentations on a sufficiently dipped diagram L_Σ^d. In Section 6, we use an argument from [Fuchs and Rutherford 2008] to associate an augmentation of L_Σ^d to an MCS of Σ.

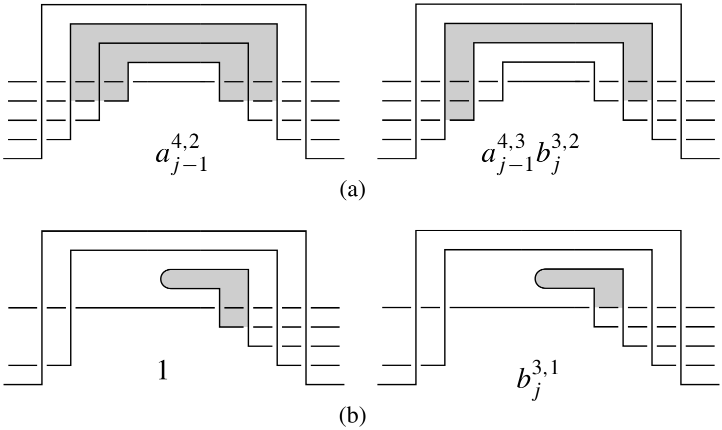


Figure 16. Disks contributing to $L(\partial b_j^{k,l})$ when I_j is of type (1) or (4). The disks in (a) may occur in an insert of any type.

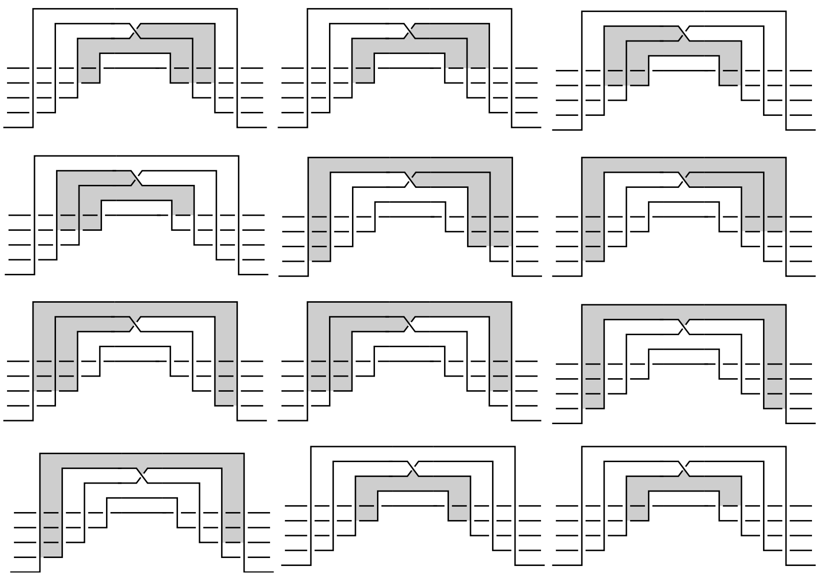


Figure 17. Disks contributing to $L(\partial B_j)$ when I_j is of type (2).

5c. ***Extending an augmentation across a dip.*** If the Lagrangian projection L_2 is obtained from L_1 by a single type II move introducing new crossings a and b with $|b| = i$ and $|a| = i-1$, then there exists a stable tame isomorphism from $(\mathcal{A}(L_2), \partial)$ to $(\mathcal{A}(L_1), \partial')$. As we demonstrated in Proposition 4.5, such a stable tame isomorphism induces a map from $\mathrm{Aug}(L_1)$ to $\mathrm{Aug}(L_2)$. By understanding this map we may keep track of augmentations as we add dips to L_Σ. This careful analysis will

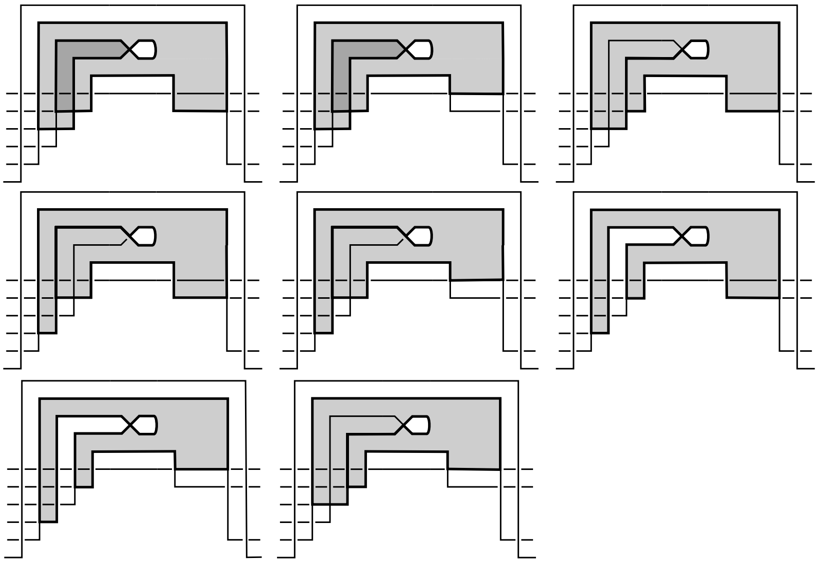


Figure 18. Disks contributing to $L(\partial B_j)$ when I_j is of type (3). The boundary of each disk is highlighted.

allow us in Section 6 to assign an MCS class of Σ to an augmentation class of L_Σ and, hence, prove the surjectivity of the map $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$.

The stable tame isomorphism from $(\mathcal{A}(L_2), \partial)$ to $(\mathcal{A}(L_1), \partial')$ was first explicitly written down in [Chekanov 2002a]. We will use the formulation found in [Sabloff 2005]. The stable tame isomorphism is a single stabilization S_i introducing generators α and β followed by a tame isomorphism ψ. The tame isomorphism $\psi : (\mathcal{A}(L_2), \partial) \to S_i(\mathcal{A}(L_1), \partial')$ is defined as follows. Let $\{x_1, \dots, x_N\}$ denote the generators with height less than $h(a)$ and labeled so that $h(x_1) < \cdots < h(x_N)$. Let $\{y_1, \dots, y_M\}$ denote the generators with height greater than $h(b)$ and labeled so that $h(y_1) < \cdots < h(y_M)$. Recall that ∂ lowers height, so we may write ∂b as $\partial b = a + v$, where v consists of words in the letters $x_1, \dots, x_N$. We begin by defining a vector space map H on $S_i(\mathcal{A}(L_1), \partial')$ by

$$H(w) = \begin{cases} 0 & \text{if } w \in \mathcal{A}(L_1), \\ 0 & \text{if } w = Q\beta R \text{ with } Q \in \mathcal{A}(L_1) \text{ and } R \in S_i(\mathcal{A}(L_1)), \\ Q\beta R & \text{if } w = Q\alpha R \text{ with } Q \in \mathcal{A}(L_1) \text{ and } R \in S_i(\mathcal{A}(L_1)). \end{cases}$$

We build ψ up from a sequence of maps $\psi_i : (\mathcal{A}(L_2), \partial) \to S_i(\mathcal{A}(L_1), \partial')$ for $0 \le i \le M$:

$$\psi_0(w) = \begin{cases} \beta & \text{if } w = b, \\ \alpha + v & \text{if } w = a, \\ w & \text{otherwise}, \end{cases} \qquad \psi_i(w) = \begin{cases} y_i + H \circ \psi_{i-1}(\partial y_i) & \text{if } w = y_i, \\ \psi_{i-1}(w) & \text{otherwise.} \end{cases}$$

If we extend the map ψ_M by linearity from a map on generators to a map on all of $(\mathcal{A}(L_2), \partial)$, then the resulting map $\psi : (\mathcal{A}(L_2), \partial) \to S_i(\mathcal{A}(L_1), \partial')$ is the desired DGA isomorphism; see [Chekanov 2002a; Sabloff 2005].

If $i \neq 0$, then an augmentation $\epsilon \in \mathrm{Aug}(L_1)$ is extended to an augmentation $\epsilon \in \mathrm{Aug}(S_i(L_1))$ by $\epsilon(\beta) = \epsilon(\alpha) = 0$. If $i = 0$, then we may choose to extend $\epsilon \in \mathrm{Aug}(L_1)$ either by $\epsilon(\beta) = \epsilon(\alpha) = 0$ or by $\epsilon(\beta) = 1$ and $\epsilon(\alpha) = 0$. Given the tame isomorphism ψ defined above, we would like to understand how our choice of an extension from $\mathrm{Aug}(L_1)$ to $\mathrm{Aug}(S_i(L_1))$ affects the induced map from $\mathrm{Aug}(L_1)$ to $\mathrm{Aug}(L_2)$. Recall $\psi : (\mathcal{A}(L_2), \partial) \to S_i(\mathcal{A}(L_1), \partial')$ induces a bijection $\Psi : \mathrm{Aug}(S_i(L_1)) \to \mathrm{Aug}(L_2)$ by $\epsilon \mapsto \epsilon \circ \psi$. Let $\epsilon' = \epsilon \circ \psi$. From the formulas for ψ we note that $\epsilon'(x_i) = \epsilon(x_i)$ for all x_i, $\epsilon'(b) = \epsilon(\beta)$, and $\epsilon'(a) = \epsilon(v)$, where $\partial b = a + v$. The next two lemmas follow from [Sabloff 2005, Lemma 3.2] and detail the behavior of ϵ' on $y_1, \ldots, y_M$.

Lemma 5.6. *If we extend an augmentation $\epsilon \in \mathrm{Aug}(L_1)$ to an $\epsilon \in \mathrm{Aug}(S_i(L_1))$ by $\epsilon(\beta) = \epsilon(\alpha) = 0$, the augmentation $\epsilon' \in \mathrm{Aug}(L_2)$ given by $\epsilon' = \epsilon \circ \psi$ satisfies*

(1) $\epsilon'(b) = 0$,

(2) $\epsilon'(a) = \epsilon(v)$, *where* $\partial b = a + v$, *and*

(3) $\epsilon' = \epsilon$ *on all other crossings.*

Proof. We are left to show $\epsilon' = \epsilon$ on $y_1, \ldots, y_M$. This follows from observing that $\epsilon(\beta) = 0$ implies $\epsilon \circ H \circ \psi(\partial y_j) = 0$ for all j. □

Lemma 5.7. *Suppose we extend an augmentation $\epsilon \in \mathrm{Aug}(L_1)$ to an augmentation $\epsilon \in \mathrm{Aug}(S_i(L_1))$ by $\epsilon(\beta) = 1$ and $\epsilon(\alpha) = 0$. Suppose in $(\mathcal{A}(L_2), \partial)$ the generator a appears in the boundary of each of the generators $\{y_{j_1}, \ldots, y_{j_l}\}$. Suppose that for $y \in \{y_{j_1}, \ldots, y_{j_l}\}$, each disk contributing a to ∂y has the form QaR, where $Q, R \in \mathcal{A}(L_2)$ and Q and R do not contain a or b. Then the augmentation $\epsilon' \in \mathrm{Aug}(L_2)$ satisfies the following*:

(1) $\epsilon'(b) = 1$;

(2) $\epsilon'(a) = \epsilon(v)$, *where* $\partial b = a + v$;

(3) *for each* $y \in \{y_{j_1}, \ldots, y_{j_l}\}$, *we have* $\epsilon'(y) = \epsilon(y)$ *if and only if the generator a appears in an even number of terms in ∂y that are of the form QaR with $\epsilon(Q) = \epsilon(R) = 1$;*

(4) $\epsilon' = \epsilon$ *on all other crossings.*

Proof. Cases (1), (2), and (4) follow as in Lemma 5.6. Let $y \in \{y_1, \ldots, y_M\}$. Note that $\epsilon'(y) = \epsilon(y) + \epsilon \circ H \circ \psi(\partial y)$. Suppose a does not appear in ∂y. Then $H \circ \psi(\partial y) = 0$, so $\epsilon'(y) = \epsilon(y)$. Suppose a does appear in ∂y. By assumption, each disk contributing a to ∂y has the form QaR, where $Q, R \in \mathcal{A}(L_2)$ and Q and R do not contain a or b. Thus $H \circ \psi(QaR) = Q\beta R$ for each disk of the form QaR in ∂y, and $H \circ \psi$ is 0 on all of the other disks in ∂y. Now $\epsilon \circ H \circ \psi(QaR) = \epsilon(Q)\epsilon(R)$, so $\epsilon \circ H \circ \psi(\partial y) = 0$ if and only if a appears in an even number of terms in ∂y that are of the form QaR with $\epsilon(Q) = \epsilon(R) = 1$. □

Now we follow an augmentation through the complete process of adding a dip to L_Σ. Suppose L^d_Σ is a dipped diagram of L_Σ with dips $D_1, \ldots, D_m$. We would like to add a new dip D to L^d_Σ away from $D_1, \ldots, D_m$. We will denote the resulting dipped diagram by $L^{d'}_\Sigma$. We want to understand how n augmentation in $\mathrm{Aug}(L^d_\Sigma)$ extends to one in $\mathrm{Aug}(L^{d'}_\Sigma)$ using the stable tame isomorphisms defined above. This work is motivated by the construction defined in [Sabloff 2005, Section 3.3].

We set the following abbreviated notation. Let $L = L^d_\Sigma$ and $L' = L^{d'}_\Sigma$. We will assume that, with respect to the ordering on dips coming from the x-axis, D is not the leftmost dip in L'. Let I denote the insert in L' bounded on the right by D. Let D_j denote the dip bounding I on the left. At the location of the dip D, label the strands of L^d_Σ from bottom to top with the integers $1, \ldots, n$. The creation of the dip D gives a sequence of Lagrangian projections $L, L_{2,1}, \ldots, L_{n,n-1} = L'$, where $L_{k,l}$ denotes the result of pushing strand k over strand l. Let $\partial_{k,l}$ denote the boundary map of the CE-DGA of $L_{k,l}$. Let $D_{k,l}$ denote the partial dip in $L_{k,l}$. In each Lagrangian projection $L_{k,l}$, the insert I and dip D_j sit to the left of the partial dip $D_{k,l}$.

Suppose (r, s) denotes the type II move immediately preceding the type II move (k, l). Then $L_{k,l}$ is the result of pushing strand k over strand l in $L_{r,s}$.

5c1. *Extending* $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$ *by* 0. Suppose we have extended $\epsilon \in \mathrm{Aug}(L_{r,s})$ so that $\epsilon'(b^{k,l}) = 0$. Lemma 5.6 implies $\epsilon'(a^{k,l}) = \epsilon(v)$ where $\partial_{k,l} b^{k,l} = a^{k,l} + v$, and $\epsilon' = \epsilon$ on all other crossings. If the insert I is of one of the four types in Figure 13, we can describe the disks in v. Note that $a^{k,l}$ is the only disk in $\partial_{k,l} b^{k,l}$ with the lower right corner of $b^{k,l}$ as its positive convex corner; thus we need only understand the disks with the upper left corner of $b^{k,l}$ as their positive convex corner. In the proof of Lemma 5.5, we let $L(\partial b^{k,l})$ denote the disks in $\partial b^{k,l}$ that include the upper left corner of $b^{k,l}$ as their positive convex corner. Here $\partial b^{k,l}$ refers to the boundary map in the CE-DGA of $L^{d'}_\Sigma$. The order in which we perform the type II moves that create D ensures that when the crossing $b^{k,l}$ is created, all of the disks in $L(\partial b^{k,l})$ appear in $\partial_{k,l} b^{k,l}$. In fact, the restrictions placed on convex immersed polygons by the height function imply that any disk appearing in $\partial_{k,l} b^{k,l}$ must also appear in $\partial b^{k,l}$. Thus, we have $v = L(\partial b^{k,l})$ and so $\epsilon'(a^{k,l}) = \epsilon(L(\partial b^{k,l}))$, which is equal to the (k, l)-entry in the matrix $\epsilon(\tilde{A}_j(I + B))$. Since ϵ is an algebra map and $\epsilon'(B) = 0$, we see that $\epsilon(\tilde{A}_j(I + B)) = \epsilon(\tilde{A}_j)$. Thus we have the following:

Lemma 5.8. *Suppose L^d_Σ is a dipped diagram of the Ng resolution L_Σ. Suppose we use the dipping procedure to add a dip D between the existing dips D_j and D_{j+1} and thus create a new dipped diagram $L^{d'}_\Sigma$. Suppose the insert I between D_j*

and D is of one of the four types in Figure 13. Let $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$. Then if at every type II move in the creation of the dip D we choose to extend ϵ so that ϵ' is 0 on the new crossing in the b-lattice, then the stable tame isomorphism from Section 5c maps ϵ to $\epsilon' \in \mathrm{Aug}(L^{d'}_\Sigma)$ satisfying

- $\epsilon'(B) = 0$,
- $\epsilon'(A) = \epsilon(\tilde{A}_j)$, *and*
- $\epsilon' = \epsilon$ *on all other crossings.*

Definition 5.9. If $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$, then we say that ϵ is *extended by 0* if after each type II move in the creation of D, we extend ϵ so that ϵ sends the new crossing of the b-lattice of D to 0.

In the next corollary, we investigate $\epsilon(\tilde{A}_j)$ by revisiting the definition of $\tilde{A}_j$ from Section 5b. The assumptions on $\epsilon(q)$ and $\epsilon(z)$ in cases (2) and (4) simplify the matrices $\epsilon(\tilde{A}_j)$ considerably, although verifying case (4) is still a slightly tedious matrix calculation.

Corollary 5.10. *Suppose we are in the setup of Lemma 5.8. Let $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$, and let $\epsilon' \in \mathrm{Aug}(L^{d'}_\Sigma)$ be the extension of ϵ described in Lemma 5.8.*

(1) *If I is of type (1), then $\epsilon'(A) = \epsilon(A_j)$.*

(2) *Suppose I is of type (2) with crossing q between strands $i+1$ and i. If $\epsilon(q) = 0$, then $\epsilon'(A) = P_{i+1,i}\epsilon(A_j)P_{i+1,i}$.*

(3) *Suppose I is of type (3) with crossing z between strands $i+1$ and i.*
   - *Let $i+1 < u_1 < u_2 < \cdots < u_s$ denote the strands at dip D_j that satisfy $\epsilon(a_j^{u_1,i}) = \cdots = \epsilon(a_j^{u_s,i}) = 1$.*
   - *let $v_r < \cdots < v_1$ denote the strands at dip D_j that satisfy $\epsilon(a_j^{i+1,v_1}) = \epsilon(a_j^{i+1,v_2}) = \cdots = \epsilon(a_j^{i+1,v_r}) = 1$; and*
   - *let $E = E_{i,v_r} \ldots E_{i,v_1} E_{u_1,i+1} \ldots E_{u_s,i+1}$.*

   If $\epsilon(z) = 0$ and $\epsilon(a_j^{i+1,i}) = 1$, then $\epsilon'(A) = J_{i-1} E \epsilon(A_j) E^{-1} J^T_{i-1}$ as matrices.

(4) *Suppose I is of type (4) and the resolved birth is between strands $i+1$ and i. Then, as matrices, $\epsilon'(A)$ is obtained from $\epsilon(A_j)$ by inserting two rows (columns) of zeros after row (column) $i-1$ in $\epsilon(A_j)$ and then changing the $(i+1, i)$ entry to 1.*

Except for case (1), these matrix equations correspond to the chain maps that connect consecutive ordered chain complexes in an MCS with simple births; see Definition 3.3. Thus we see the first hint of an explicit connection between MCSs and augmentations.

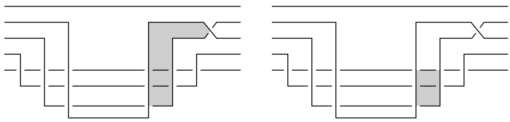


Figure 19. Disks appearing in $\partial_{i+1,i}$ with $a^{i+1,i}$ as a negative convex corner.

5c2. *Extending* $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$ *by* $\mathcal{H}_{i+1,i}$. In this section, we consider extending $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$ to an augmentation $\epsilon' \in \mathrm{Aug}(L^{d'}_\Sigma)$ such that $\epsilon'(B) = \mathcal{H}_{i+1,i}$. Recall $\mathcal{H}_{i+1,i}$ denotes a square matrix with 1 in the $(i+1, i)$ position and zeros everywhere else. During the type II move that pushes strand $i+1$ over strand i, we will choose to extend ϵ so that $\epsilon'(b^{i+1,i}) = 1$. By carefully using Lemma 5.7, we are able to determine the extended augmentation ϵ'.

Definition 5.11. Suppose $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$. We say that ϵ is *extended by* $\mathcal{H}_{i+1,i}$ if $\mu(i+1) = \mu(i)$ and after each type II move in the creation of a new dip D, we extend ϵ so that the extended augmentation $\epsilon' \in \mathrm{Aug}(L^{d'}_\Sigma)$ satisfies $\epsilon'(B) = \mathcal{H}_{i+1,i}$.

Understanding ϵ' when $\epsilon'(b^{i+1,i}) = 1$ requires that we understand all of the crossings y such that $a^{i+1,i}$ appears in $\partial_{i+1,i}\, y$. If we add the following restrictions on I, D, and the pair $(i+1, i)$, then we can identify all disks containing $a^{i+1,i}$ as a negative convex corner. Suppose

- $|b^{i+1,i}| = 0$,
- I is of type (1),
- D occurs to the immediate left of a crossing q in L_Σ between strands $i+1$ and i.

Given these conditions, $a^{i+1,i}$ only appears in $\partial_{i+1,i}q$ and $\partial_{i+1,i}a^{i+1,l}$ for $l < i$; see Figure 19. Applying Lemma 5.7, we conclude that after the type II move that pushes strand $i+1$ over strand i, the augmentation ϵ' satisfies

- $\epsilon'(b^{i+1,i}) = 1$;
- $\epsilon'(a^{i+1,i}) = \epsilon(v)$, where $\partial b = a + v$;
- $\epsilon'(q) \neq \epsilon(q)$;
- for all $l < i$, $\epsilon'(a^{i+1,l}) = \epsilon(a_j^{i+1,l})$ if and only if $\epsilon'(a^{i,l}) = 0$; and
- $\epsilon' = \epsilon$ on all other crossings.

In Section 5c1 we showed that $v = L(\partial b^{i+1,i})$ if I is an insert of type (1)–(4), where ∂ is the boundary in $L^{d'}_\Sigma$. Since I is an insert of type (1), we conclude that $\epsilon'(a^{i+1,i}) = \epsilon(a_j^{i+1,i})$.

Suppose we continue creating the dip D, and with each new type II move we extend the augmentation so that it sends the new crossing in the b-lattice to 0. By

Lemma 5.8, we need only compute the value of the extended augmentation on the new crossing in the a-lattice. Consider the type II move pushing strand k over l where $(i+1,i) \prec (k,l)$. By Lemma 5.8, we have

$$\epsilon'(a^{k,l}) = \epsilon(v) = \epsilon(L(\partial b^{k,l})) = \epsilon(a_j^{k,l}) + \sum_{l<p<k} \epsilon(a_j^{k,p})\epsilon'(b^{p,l}).$$

Since $\epsilon'(b^{p,l}) = 1$ if and only if $(p,l) = (i+1,i)$, we know that $\epsilon'(a^{k,l}) = \epsilon(a_j^{k,l})$ if and only if $l \neq i$ or $\epsilon(a_j^{k,i+1}) = 0$. Thus, if we extend $\epsilon \in \mathrm{Aug}(L_\Sigma^d)$ to $\epsilon' \in \mathrm{Aug}(L_\Sigma^{d'})$ so that $\epsilon'(B) = \mathcal{H}_{i+1,i}$, then ϵ' satisfies

$$\epsilon'(a^{k,l}) = \begin{cases} \epsilon(a_j^{i+1,l}) + \epsilon'(a^{i,l}) & \text{if } k = i+1 \text{ and } l < i, \\ \epsilon(a_j^{k,i}) + \epsilon(a_j^{k,i+1}) & \text{if } k > i+1 \text{ and } l = i, \\ \epsilon(a_j^{k,l}) & \text{otherwise.} \end{cases}$$

This equation is equivalent to $\epsilon'(A) = E_{i+1,i}\epsilon(A_j)E_{i+1,i}^{-1}$. Pulling this all together, we have the following lemma.

Lemma 5.12. *Suppose L_Σ^d is a dipped diagram of the Ng resolution L_Σ. Let q be a crossing in L_Σ^d corresponding to a resolved crossing of Σ and with $|q| = 0$. Suppose we add a dip D to the right of the existing dip D_j and just to the left of q, thus creating a new dipped diagram $L_\Sigma^{d'}$. Suppose the insert I between D and D_j is of type (1). Let $\epsilon \in \mathrm{Aug}(L_\Sigma^d)$. If ϵ is extended by $\mathcal{H}_{i+1,i}$, then the stable tame isomorphism from Section 5c maps ϵ to $\epsilon' \in \mathrm{Aug}(L_\Sigma^{d'})$, where ϵ' satisfies*

- $\epsilon'(B) = \mathcal{H}_{i+1,i}$,
- $\epsilon'(A) = E_{i+1,i}\epsilon(A_j)E_{i+1,i}^{-1}$,
- $\epsilon'(q) \neq \epsilon(q)$, *and*
- $\epsilon' = \epsilon$ *on all other crossings.*

We now have sufficient tools to begin connecting MCSs and augmentations.

6. Relating MCSs and augmentations

In this section, we construct a surjection $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ and define a simple construction that associates an MCS $\mathscr{C}$ to an augmentation $\epsilon \in \mathrm{Aug}(L_\Sigma)$ such that $\widehat{\Psi}([\mathscr{C}]) = [\epsilon]$. We also detail two algorithms that use MCS moves to place an arbitrary MCS in one of two standard forms.

6a. ***Augmentations on sufficiently dipped diagrams.*** Suppose L_Σ^d is a sufficiently dipped diagram of L_Σ with dips $D_1, \dots, D_m$ and inserts $I_1, \dots, I_{m+1}$, and let $q_i, \dots, q_M$ and $z_j, \dots, z_N$ denote the crossings found in the inserts of type (2) and (3), respectively. The subscripts on q and z correspond to the subscripts of the inserts in which the crossings appear. The formulas in Lemma 5.5 allow us to write the augmentation condition $\epsilon \circ \partial$ as a system of local equations involving the dips and inserts of L^d_Σ.

Lemma 6.1. *An algebra homomorphism $\epsilon : \mathcal{A}(L^d_\Sigma) \to \mathbb{Z}_2$ on a sufficiently dipped diagram L^d_Σ of a Ng resolution L_Σ with $\epsilon(1) = 1$ satisfies $\epsilon \circ \partial = 0$ if and only if*

(1) $\epsilon(a^{i+1,i}_{s-1}) = 0$, *where q_s is a crossing between strands $i+1$ and i,*

(2) $\epsilon(a^{k+1,k}_{r-1}) = 1$, *where z_r is a crossing between strands $i+1$ and i,*

(3) $\epsilon(A_j)^2 = 0$, *and*

(4) $\epsilon(A_j) = (I + \epsilon(B_j))\epsilon(\tilde{A}_{j-1})(I + \epsilon(B_j))^{-1}$.

We will primarily concern ourselves with the following types of augmentations.

Definition 6.2. Given a sufficiently dipped diagram L^d_Σ, we say an augmentation $\epsilon \in \mathrm{Aug}(L^d_\Sigma)$ is *occ-simple* if

- ϵ sends all of the crossings of type q_s and z_r to 0,
- either $\epsilon(B_j) = 0$ or $\epsilon(B_j) = \mathcal{H}_{k,l}$ for some $k > l$ for I_j of type (1), and
- $\epsilon(B_j) = 0$ for I_j of type (2), (3), or (4).

$\mathrm{Aug}_{occ}(L^d_\Sigma)$ denotes the set of all such augmentations in $\mathrm{Aug}(L^d_\Sigma)$. We say $\epsilon \in \mathrm{Aug}_{occ}(L^d_\Sigma)$ is *minimal occ-simple* if $\epsilon(B_j) = \mathcal{H}_{k,l}$ for some $k > l$ for all I_j of type (1). We let $\mathrm{Aug}^m_{occ}(L_\Sigma)$ denote the set of all minimal occ-simple augmentations over all possible sufficiently dipped diagrams of L_Σ.

The matrices $\epsilon(A_1), \dots, \epsilon(A_m)$ of an occ-simple augmentation determine a sequence of ordered chain complexes. This will be made explicit in Lemma 6.4.

The following result is a consequence of Lemma 6.1, Definition 6.2, and the definition of $\tilde{A}_{j-1}$ given in Section 5b. Recall that the matrices $E_{k,l}$, $P_{i+1,i}$, J_i, and $\mathcal{H}_{k,l}$ were defined in Section 3a1. The proof of Lemma 6.3 is essentially identical to the proof of Corollary 5.10.

Lemma 6.3. *If L^d_Σ is a sufficiently dipped diagram and $\epsilon \in \mathrm{Aug}_{occ}(L^d_\Sigma)$, then the following hold for each insert I_j:*

(1) *If I_j is of type (1), then either $\epsilon(A_j) = \epsilon(A_{j-1})$ or $\epsilon(A_j) = E_{k,l}\epsilon(A_{j-1})E^{-1}_{k,l}$ for some $k > l$.*

(2) *If I_j is of type (2), with crossing q between strands $i+1$ and i, then $\epsilon(A_j) = P_{i+1,i}\epsilon(A_{j-1})P^{-1}_{i+1,i}$.*

(3) *Suppose I_j is of type (3) with crossing z between strands $i+1$ and i.*
   - *Let $i+1 < u_1 < u_2 < \dots < u_s$ denote the strands at dip D_{j-1} that satisfy $\epsilon(a^{u_1,i}_{j-1}) = \dots = \epsilon(a^{u_s,i}_{j-1}) = 1$;*

- *let* $v_r < \cdots < v_1$ *denote the strands at dip* D_{j-1} *that satisfy* $\epsilon(a_{j-1}^{i+1,v_1}) = \epsilon(a_{j-1}^{i+1,v_2}) = \cdots = \epsilon(a_{j-1}^{i+1,v_r}) = 1$; *and*
- *let* $E = E_{i,v_r} \cdots E_{i,v_1} E_{u_1,i+1} \cdots E_{u_s,i+1}$.

Then $\epsilon(A_j) = J_{i-1} E \epsilon(A_{j-1}) E^{-1} J_{i-1}^T$.

(4) *Suppose* I_j *is of type* (4) *and the resolved birth is between strands* $i+1$ *and* i. *Then, as matrices,* $\epsilon(A_j)$ *is obtained from* $\epsilon(A_{j-1})$ *by inserting two rows (columns) of zeros after row (column)* $i-1$ *in* $\epsilon(A_{j-1})$ *and then changing the* $(i+1, i)$ *entry to* 1.

The equations in Lemma 6.3 are identical to those found in Definition 3.3, with the exception of the first case of a type (1) insert. Indeed, given a front diagram Σ with resolution L_Σ, MCSs on Σ correspond to minimal occ-simple augmentations of L_Σ. We assign a minimal occ-simple augmentation $\epsilon_{\mathscr{C}}$ to an MCS $\mathscr{C}$ using an argument of Fuchs and Rutherford [2008].

Lemma 6.4. *The set* $\mathrm{MCS}_b(\Sigma)$ *of MCSs of* Σ *with simple births is in bijection with the set* $\mathrm{Aug}_{occ}^m(L_\Sigma)$ *of minimal occ-simple augmentations.*

Proof. We assign an MCS to a minimal occ-simple augmentation $\epsilon \in \mathrm{Aug}(L_\Sigma^d)$, where L_Σ^d is a sufficiently dipped diagram of L_Σ. Let $\epsilon \in \mathrm{Aug}_{occ}^m(L_\Sigma)$. For each dip D_j, we form an ordered chain complex (C_j, ∂_j) as follows. Let t_j denote the x-coordinate of the vertical lines of symmetry of D_j in L_Σ^d. We label the m_j points of intersection in $L_\Sigma^d \cap (\{t_j\} \times \mathbb{R})$ by $y_1^j, \ldots, y_{m_j}^j$ and we let C_j be a $\mathbb{Z}_2$ vector space generated by $y_1^j, \ldots, y_{m_j}^j$. We label the generators $y_1^j, \ldots, y_{m_j}^j$ based on their y-coordinate so that $y_1^j > \cdots > y_{m_j}^j$. Each generator is graded by the Maslov potential of its corresponding strand in Σ. The grading condition on ϵ and the fact that $\epsilon(A_j)^2 = 0$ implies that $\epsilon(A_j)$ is a matrix representative of a differential on C_j. Thus (C_j, ∂_j) with $\mathfrak{D}_j = \epsilon(A_j)$ is an ordered chain complex. Recall that the notation $\mathfrak{D}_j$ was established in Remark 3.2. The relationship between consecutive differentials ∂_{j-1} and ∂_j is defined in Lemma 6.3. Since ϵ is minimal occ-simple, each insert I_j of type (1) satisfies $\mathfrak{D}_j = E_{k,l} \mathfrak{D}_{j-1} E_{k,l}^{-1}$, where $\epsilon(B_j) = \mathscr{H}_{k,l}$ and $k > l$. Thus, $(C_{j-1}, \partial_{j-1})$ and (C_j, ∂_j) are related by a handleslide between k and l. Since the matrix equations in Lemma 6.3 that ϵ satifies are equivalent to the matrix equations in Definition 3.3, the sequence $(C_1, \partial_1), \ldots, (C_m, \partial_m)$ forms an MCS $\mathscr{C}$ on Σ.

This process is invertible. Let $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ be a Morse complex sequence of the front projection Σ with chain complexes $(C_1, \partial_1) \cdots (C_m, \partial_m)$. We will use the marked front projection associated to $\mathscr{C}$ to define the placement of dips creating L_Σ^d. Afterwards, we define an algebra homomorphism $\epsilon_{\mathscr{C}} : \mathscr{A}(L_\Sigma^d) \to \mathbb{Z}_2$ and show that it is an augmentation. Figure 20 gives an example of this process.

Add a dip to L_Σ just to the right of the corresponding location of each resolved cusp, resolved crossing or handleslide mark. The resulting dipped diagram L_Σ^d is

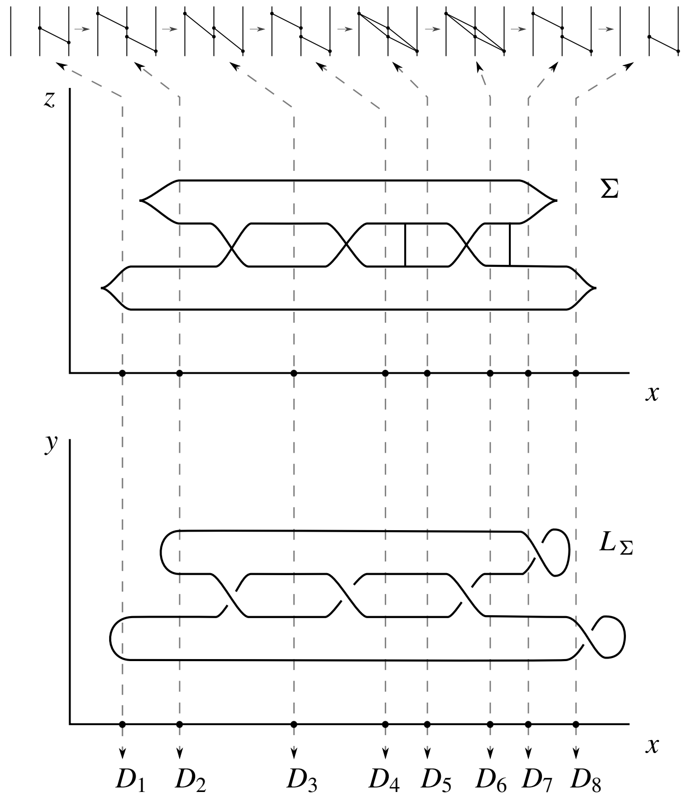


Figure 20. Assigning an augmentation to an MCS.

sufficiently dipped with m dips. We define a $\mathbb{Z}_2$-valued map $\epsilon_{\mathscr{C}}$ on the crossings of L^d_Σ as follows.

- $\epsilon_{\mathscr{C}}(q_s) = 0$ for all crossings q_s coming from a resolved crossing of Σ.
- $\epsilon_{\mathscr{C}}(z_r) = 0$ for all crossings z_r coming from a resolved right cusp.
- $\epsilon_{\mathscr{C}}(A_j) = \mathscr{D}_j$ for all j.
- If I_j is of type (1), then there exists $k > l$ such that $\mathscr{C}$ has a handleslide mark in I_j between strands k and l. Let $\epsilon_{\mathscr{C}}(B_j) = \mathscr{H}_{k,l}$.
- If I_j is of type (2), (3), or (4), then let $\epsilon_{\mathscr{C}}(B_j) = 0$.

We define $\epsilon_{\mathscr{C}}(1) = 1$ and extend $\epsilon_{\mathscr{C}}$ by linearity to an algebra homomorphism on $\mathscr{A}(L^d_\Sigma)$. If $\epsilon_{\mathscr{C}}$ is an augmentation, then it is minimal occ-simple by construction. We must show $\epsilon_{\mathscr{C}} \circ \partial = 0$ and that $\epsilon_{\mathscr{C}}(q) = 1$ implies $|q| = 0$ for all crossings q of L^d_Σ.

If $\epsilon_{\mathscr{C}}(a^{k,l}_j) = 1$, then $\langle \partial_j y^j_k \,|\, y^j_l \rangle = 1$, where the notation for the generators of (C_j, ∂_j) corresponds to the notation in Definition 3.4. Thus $\mu(k) = \mu(l) + 1$, where $\mu(i)$ denotes the Maslov potential of the strand in Σ corresponding to the generator y_i^j. Recall $|a_j^{k,l}| = \mu(k) - \mu(l) - 1$ and so $\mu(k) = \mu(l) + 1$ implies $|a_j^{k,l}| = 0$. If $\epsilon_{\mathscr{C}}(b_j^{k,l}) = 1$, then in the marked front projection of $\mathscr{C}$, a handleslide mark occurs in I_j between strands k and l. Thus $\mu(k) = \mu(l)$, and $|b_j^{k,l}| = 0$ since $|b_j^{k,l}| = \mu(k) - \mu(l)$.

It remains to show that $\epsilon_{\mathscr{C}} \circ \partial = 0$. Let q_r denote a crossing corresponding to a resolved crossing of Σ between strands $i+1$ and i. By Remark 3.6(2), the $(i+1, i)$ entry of the matrix ∂_{r-1} is 0, and hence $\epsilon_{\mathscr{C}}(a_{r-1}^{i+1,i}) = 0$. Therefore $\epsilon_{\mathscr{C}}(\partial q_r) = \epsilon_{\mathscr{C}}(a_{r-1}^{i+1,i}) = 0$. Let z_s denote a crossing corresponding to a resolved right cusp of Σ between strands $i+1$ and i. By Remark 3.6(2), the $(i+1, i)$ entry of the matrix ∂_{s-1} is 1, and hence $\epsilon_{\mathscr{C}}(a_{s-1}^{i+1,i}) = 1$. Therefore $\epsilon_{\mathscr{C}}(\partial z_s) = 1 + \epsilon_{\mathscr{C}}(a_{s-1}^{i+1,i}) = 0$. Since each $\mathscr{D}_j$ is the matrix representative of a differential, we see that $\epsilon_{\mathscr{C}}(\partial A_j) = \epsilon_{\mathscr{C}}(A_j)^2 = 0$. Verifying $\epsilon_{\mathscr{C}} \circ \partial = 0$ on each b-lattice B_j is equivalent to verifying Lemma 6.1(4) and this can be done with Lemma 6.3. Indeed, the matrix equations in Lemma 6.3 correspond to the matrix equations in Definition 3.3 relating consecutive chain complexes in $\mathscr{C}$. Thus, $\epsilon_{\mathscr{C}}(\partial B_j) = 0$ for all j and so $\epsilon_{\mathscr{C}}$ is a minimal occ-simple augmentation on L_Σ^d. □

6b. *Defining* $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$. Given $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$, Lemma 6.4 gives an explicit construction of an augmentation $\epsilon_{\mathscr{C}}$ on a sufficiently dipped diagram L_Σ^d. In this section, we will use this construction to build a map $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$. In order to do so, we must understand the possible stable tame isomorphisms between $\mathscr{A}(L_\Sigma^d)$ and $\mathscr{A}(L_\Sigma)$ coming from sequences of Lagrangian Reidemeister moves. In [2005], Kálmán proves that two sequences of Lagrangian Reidemeister moves may induce inequivalent maps between the respective contact homologies. However, if we restrict our sequences of moves to adding and removing dips, then the we can avoid this problem and give a well-defined map $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$.

The following results allow us to modify a sequence of Lagrangian Reidemeister moves by removing canceling pairs of moves and commuting pairs of moves that are far away from each other. These modifications do not change the resulting bijection on augmentation chain homotopy classes.

Proposition 6.5 [Kálmán 2005]. *If we perform a type II move followed by a type II^{-1} move that creates and then removes two crossings b and a, then the induced DGA morphism from $(\mathscr{A}(L), \partial)$ to $(\mathscr{A}(L), \partial)$ is equal to the identity. If we perform a type II^{-1} move followed by a type II move that removes and then recreates two crossings b and a, then the induced DGA morphism from $(\mathscr{A}(L), \partial)$ to $(\mathscr{A}(L), \partial)$ is chain homotopic to the identity.*

Proposition 6.6 [Kálmán 2005]. *Suppose L_1 and L_2 are related by two consecutive moves of type II or II^{-1}. We will call these moves A and B. Suppose the* *crossings involved in A and B form disjoint sets. Then the composition of DGA morphisms constructed by performing move A and then move B is chain homotopic to the composition of DGA morphisms constructed by performing move B and then move A.*

Proposition 6.6 follows from [Kálmán 2005, case 1 of Theorem 3.7].

6b1. *Dipping/undipping paths for L^d_Σ.* Suppose L^d_Σ has dips $D_1, \dots, D_m$. Let $t_1, \dots, t_m$ denote the x-coordinates of the vertical lines of symmetry of the dips $D_1, \dots, D_m$ in L^d_Σ.

Definition 6.7. A *dipping/undipping path for L^d_Σ* is a finite-length monomial w in the elements of the set $\{s_1^\pm, \dots, s_n^\pm, t_1^\pm, \dots, t_m^\pm\}$. We require that w satisfies the following:

(1) Each s_i in w denotes a point on the x-axis away from the dips $D_1, \dots, D_m$; and

(2) As we read w from left to right, the appearances of s_i alternate between s_i^+ and s_i^-, beginning with s_i^+ and ending with s_i^-. The appearances of t_i alternate between t_i^- and t_i^+, beginning with t_i^- and ending with t_i^- and each t_i is required to appear at least once.

Each dipping/undipping path w is a prescription for adding and removing dips from L^d_Σ. In particular, s_i^+ tells us to introduce a dip in a small neighborhood of s_i. The letter s_i^- tells us to remove the dip that sits in a small neighborhood of s_i. The order in which these type II^{-1} moves occur is the opposite of the order used in s_i^+. The elements t_i^+ and t_i^- work similarly. We perform these moves on L^d_Σ by reading w from left to right. The conditions we have placed on w ensure that we are left with L_Σ after performing all of the prescribed dips and undips.

Let $w_0 = t_m^- t_{m-1}^- \cdots t_1^-$. Then w_0 tells us to undip $D_1, \dots, D_m$ beginning with D_m and working to D_1. Each w determines a stable tame isomorphism ψ_w from $\mathcal{A}(L^d_\Sigma)$ to $\mathcal{A}(L_\Sigma)$ that determines a bijection $\Psi_w : \mathrm{Aug}^{ch}(L^d_\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$. We can now define a map $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$.

Definition 6.8. Define $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ by $\Psi(\mathscr{C}) = \Psi_{w_0}([\epsilon_{\mathscr{C}}])$.

The definition of Ψ is independent of dipping/undipping paths:

Lemma 6.9. *If w is a dipping/undipping paths for L^d_Σ, then $\Psi_w = \Psi_{w_0}$, and thus* $\Psi_w([\epsilon_{\mathscr{C}}]) = \Psi_{w_0}([\epsilon_{\mathscr{C}}]) = \Psi(\mathscr{C})$.

Proof. By Proposition 6.6, we may reorder type II and II^{-1} moves that are "far apart" without changing the chain homotopy type of the resulting map from $\mathcal{A}(L^d_\Sigma)$ and $\mathcal{A}(L_\Sigma)$. If s_i^+ appears in w, then the next appearance of s_i is the letter s_i^- to the right of s_i^+. The letters between s_i^- and s_i^+ represent type II and II^{-1} moves that are far away from s_i^- and s_i^+. Thus, we may commute s_i^- past the other letters so that s_i^- immediately follows s_i^+. By our ordering of the type II moves in s_i^+ and the type II^{-1} moves in s_i^-, we may remove the type II and II^{-1} moves in pairs. By Proposition 6.5, this does not change the chain homotopy type of the resulting map from $\mathscr{A}(L_\Sigma^d)$ and $\mathscr{A}(L_\Sigma)$. In this manner, we remove all pairs of letters of the form s_i^+ and s_i^- from w. By the same argument we remove pairs of letters of the form t_i^+, t_i^-. The resulting dipping/undipping path, denoted w', only contains the letters $t_1^-, \ldots, t_m^-$. We may reorder these letters so that $w' = w_0$. By Proposition 6.6, this rearrangement does not change the chain homotopy type of the resulting map from $\mathscr{A}(L_\Sigma^d)$ and $\mathscr{A}(L_\Sigma)$; thus $\Psi_w = \Psi_{w_0}$ by Lemma 4.4. $\square$

6c. *Associating an MCS to $\epsilon \in$ Aug(L_Σ).* We now prove that $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is surjective. We do so by constructing a map that assigns to each augmentation in $\mathrm{Aug}(L_\Sigma)$ an MCS in $\mathrm{MCS}_b(\Sigma)$. The construction follows from our work in Lemma 5.12 and Corollary 5.10.

Lemma 6.10. *The map $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is surjective.*

Proof. Let $[\epsilon] \in \mathrm{Aug}^{ch}(L_\Sigma)$ and let ϵ denote a representative of $[\epsilon]$. We will find an MCS mapping to $[\epsilon]$ by constructing a minimal occ-simple augmentation and applying Lemma 6.4. We add dips to L_Σ beginning at the leftmost resolved left cusp and working to the right. As we add dips, we extend ϵ using Corollary 5.10 and Lemma 5.12. In a slight abuse of notation, we will let ϵ also denote the extended augmentation at every step.

Begin by adding the dip D_1 just to the right of the leftmost resolved left cusp. This requires a single type II move. We extend ϵ by requiring $\epsilon(b_1^{2,1}) = 0$. In the dipped projection, $\partial(b_1^{2,1}) = 1 + a_1^{2,1}$. Hence, $\epsilon(a_1^{2,1}) = 1$.

Suppose we have added dips $D_1, \ldots, D_{j-1}$ to L_Σ and let x_j denote the next resolved cusp or crossing appearing to the right of D_{j-1}. We introduce the dip D_j and extend ϵ as follows.

- If x_j is a resolved left cusp, right cusp, or crossing with $\epsilon(x_j) = 0$, then we introduce D_j just to the right of x_j and extend ϵ by 0 as in Definition 5.9.
- If x_j is a resolved crossing between strands $i+1$ and i and $\epsilon(x_j) = 1$, then we introduce D_j just to the left of x_j and extend ϵ by $\mathscr{H}_{i+1,i}$ as in Definition 5.11. Note that $\epsilon(x_j) = 0$ after the extension.

In the first case, Corollary 5.10 details the relationship between $\epsilon(A_j)$ and $\epsilon(A_{j-1})$. In the second, Lemma 5.12 details the relationship between $\epsilon(A_j)$ and $\epsilon(A_{j-1})$. The resulting augmentation is minimal occ-simple by construction. Let $\tilde{\epsilon}$ denote the resulting augmentation on L_Σ^d. By Lemma 6.4, $\tilde{\epsilon}$ has an associated MCS $\mathscr{C}_\epsilon$. By definition, $\Psi(\mathscr{C}_\epsilon) = \Psi_{w_0}([\tilde{\epsilon}])$. The dipping/undipping path w_0 tells us to undip $t_1, \ldots, t_n$ in reverse order. In particular, this is the inverse of the process we used to

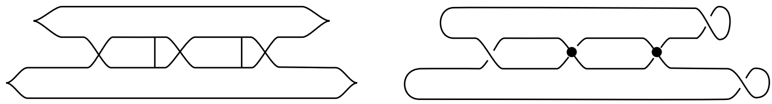


Figure 21. The dots in the right figure indicate the augmented crossings of $\epsilon \in \mathrm{Aug}(L_\Sigma)$. The left figure shows the resulting MCS $\mathscr{C}_\epsilon$ from the construction in Lemma 6.10.

create $\tilde{\epsilon}$ on L^d_Σ. Thus $\Psi(\mathscr{C}_\epsilon) = \Psi_{w_0}([\tilde{\epsilon}]) = [\epsilon]$ and so $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is surjective. □

Remark 6.11. The marked front projection for $\mathscr{C}_\epsilon$ is constructed as follows. Let $q_{j_1}, \dots, q_{j_l}$ denote the resolved crossings of L_Σ satisfying $\epsilon(q) = 1$. Then $\mathscr{C}_\epsilon$ has a handleslide mark just to the left of each of the crossings $q_{j_1}, \dots, q_{j_l}$ in Σ; see Figure 21. Each handleslide mark begins and ends on the two strands involved in the crossing.

6d. ***Defining*** $\widehat{\boldsymbol{\Psi}} : \widehat{\mathbf{MCS}}_b(\boldsymbol{\Sigma}) \to \mathbf{Aug}^{ch}(\boldsymbol{L_\Sigma})$**.** We now prove Theorem 1.3.

Theorem 6.12. *The map* $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ *defined by* $\widehat{\Psi}([\mathscr{C}]) = \Psi(\mathscr{C})$ *is a well-defined, surjective map.*

Proof. Given $\mathscr{C}_1 \sim \mathscr{C}_2$ we construct explicit chain homotopies between $\Psi(\mathscr{C}_1)$ and $\Psi(\mathscr{C}_2)$. We begin by restating the chain homotopy property $\epsilon_1 - \epsilon_2 = H \circ \partial$ as a system of local equations.

Let $\epsilon_1, \epsilon_2 \in \mathrm{Aug}(L^d_\Sigma)$, where L^d_Σ is a sufficiently dipped diagram of the Ng resolution L_Σ. Let Q denote the set of crossings of L^d_Σ. By Lemma 4.2, a map $H : Q \to \mathbb{Z}_2$ that has support on the crossings of grading -1 may be extended by linearity and the derivation product property to a map $H : (\mathscr{A}(L^d_\Sigma), \partial) \to \mathbb{Z}_2$ with support on $\mathscr{A}_{-1}(L^d_\Sigma)$. Then H is a chain homotopy between ϵ_1 and ϵ_2 if and only if $\epsilon_1 - \epsilon_2 = H \circ \partial$ on Q.

Lemma 6.13. *Let* $\epsilon_1, \epsilon_2 \in \mathrm{Aug}(L^d_\Sigma)$, *where* L^d_Σ *is a sufficiently dipped diagram of the Ng resolution* L_Σ. *Suppose the linear map* $H : (\mathscr{A}(L^d_\Sigma), \partial) \to \mathbb{Z}_2$ *satisfies the derivation product property and has support on crossings of grading* -1. *Then* H *is a chain homotopy between* ϵ_1 *and* ϵ_2 *if and only if*

(1) $\epsilon_1 - \epsilon_2 = H \circ \partial$ *for all* q_r *and* z_s,

(2) $\epsilon_1(A_j) = (I + H(A_j))\epsilon_2(A_j)(I + H(A_j))^{-1}$ *for all* A_j,

(3) $$\epsilon_1(B_j) = (I + \epsilon_1(B_j))H(A_j) + H(\tilde{A}_{j-1})(I + \epsilon_2(B_j)) + \epsilon_2(B_j) + H(B_j)\epsilon_2(A_j) + \epsilon_1(\tilde{A}_{j-1})H(B_j) \quad \textit{for all } B_j.$$

Proof. Since $\partial A_j = A_j^2$ for each A_j, the derivation property of H implies that $\epsilon_1 - \epsilon_2(A_j) = H \circ \partial(A_j)$ if and only if $\epsilon_1 - \epsilon_2(A_j) = H(A_j)\epsilon_2(A_j) + \epsilon_1(A_j)H(A_j)$. By rearranging terms, we obtain part (2).

Since $\partial B_j = (I + B_j)A_j + \tilde{A}_{j-1}(I + B_j)$ for each B_j, the derivation property of H implies $\epsilon_1 - \epsilon_2(B_j) = H \circ \partial(B_j)$ is equivalent to

$$\epsilon_1 - \epsilon_2(B_j) = (I + \epsilon_1(B_j))H(A_j) + H(\tilde{A}_{j-1})(I + \epsilon_2(B_j)) + \\ + H(B_j)\epsilon_2(A_j) + \epsilon_1(\tilde{A}_{j-1})H(B_j).$$

By rearranging terms, we obtain part (3). □

We will use the matrix equations of Lemma 6.13 to show that Ψ maps equivalent MCSs to the same augmentation class. We will restrict our attention to occ-simple augmentations and in nearly every situation build chain homotopies that satisfy $H(B_j) = 0$ for all j. These added assumptions simplify the task of constructing chain homotopies. The next corollary follows immediately from Lemma 6.13 and Definition 6.2.

Corollary 6.14. *Suppose L^d_Σ, ϵ_1, ϵ_2, and H are as in Lemma 6.13. If ϵ_1 and ϵ_2 are occ-simple and $H(B_j) = 0$ for all j, then H is a chain homotopy between ϵ_1 and ϵ_2 if and only if*

(1) $\epsilon_1 - \epsilon_2 = H \circ \partial$ *for all q_r and z_s,*
(2) $\epsilon_1(A_j) = (I + H(A_j))\epsilon_2(A_j)(I + H(A_j))^{-1}$ *for all A_j,*
(3) $\epsilon_1(B_j) + (I + \epsilon_1(B_j))H(A_j) = H(A_{j-1})(I + \epsilon_2(B_j)) + \epsilon_2(B_j)$ *for B_j with I_j of type (1),*
(4) $H(A_j) = H(\tilde{A}_{j-1})$ *for B_j with I_j of type (2), (3), or (4).*

We can now prove that $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ defined by $\widehat{\Psi}([\mathscr{C}]) = \Psi(\mathscr{C})$ is well-defined.

Lemma 6.15. *If $\mathscr{C}_1$ and $\mathscr{C}_2$ are equivalent, then $\Psi(\mathscr{C}_1) = \Psi(\mathscr{C}_2)$.*

Proof. It is sufficient to suppose $\mathscr{C}_1$ and $\mathscr{C}_2$ differ by a single MCS move. Let $\epsilon_{\mathscr{C}_1} \in \mathrm{Aug}(L^{d_1}_\Sigma)$ and $\epsilon_{\mathscr{C}_2} \in \mathrm{Aug}(L^{d_2}_\Sigma)$ be as defined in Lemma 6.4. Since each MCS move is local, we may assume $L^{d_1}_\Sigma$ and $L^{d_2}_\Sigma$ have identical dips outside of the region of the MCS move.

We compare the chain homotopy classes of $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ by extending $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ to augmentations on a third dipped diagram L^d_Σ. In the case of MCS moves 1–16, we form L^d_Σ by adding 0, 1, or 2 additional dips to $L^{d_1}_\Sigma$ and $L^{d_2}_\Sigma$. MCS move 17 may require the addition of many more dips. The dotted lines in Figures 22, 23 and 24 indicate the locations of dips in $L^{d_1}_\Sigma$ and $L^{d_2}_\Sigma$ and the additional dips needed to form L^d_Σ. The index j denotes the dip D_j. Table 1 indicates which dotted lines in Figures 22 and 23 represent new dips added to form L^d_Σ.

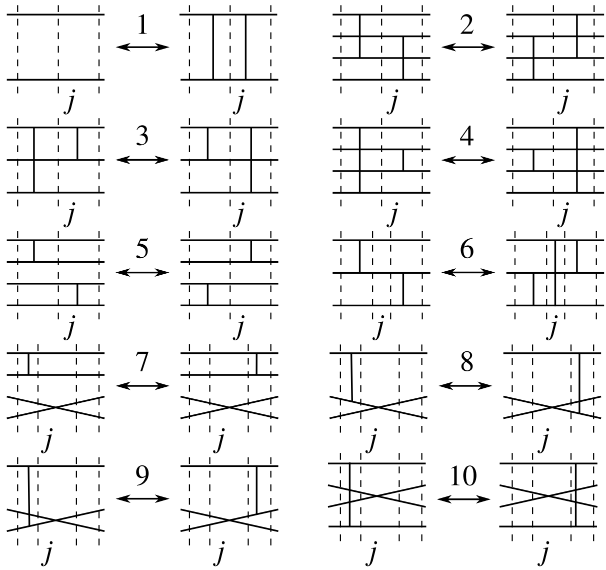


Figure 22. MCS equivalence moves 1–10 with dip locations indicated.

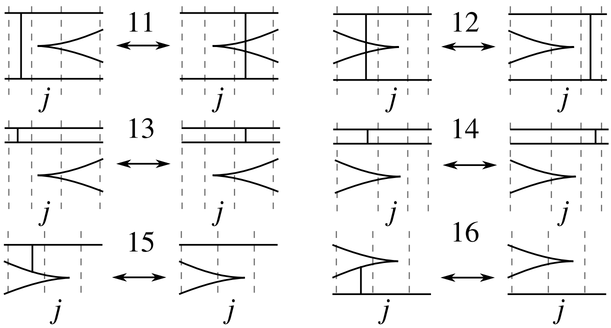


Figure 23. MCS equivalence moves 11–16 with dip locations indicated.

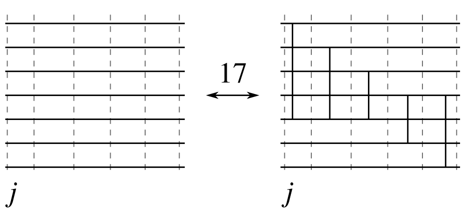


Figure 24. MCS move 17 with dip locations indicated.

As we add dips to $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$, we extend $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ by 0. By Lemma 5.8, we can keep track of the extensions of $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ after each dip. In fact, the extensions will be occ-simple. We let $\tilde{\epsilon}_{\mathscr{C}_1}, \tilde{\epsilon}_{\mathscr{C}_2} \in \mathrm{Aug}(L_\Sigma^d)$ denote the extensions

Adding dips to $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$		
Move	$L_\Sigma^{d_1}$	$L_\Sigma^{d_2}$
1	$j, j+1$	–
6	$j+1$	–
7–14	$j+2$	j
15	–	j
16	–	j

Table 1. Entries indicate the dotted lines in Figures 22 and 23 that correspond to dips added to form L_Σ^d.

of $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$. We use the matrix equations in Corollary 6.14 and Lemma 6.13 to construct chain homotopies between $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$. In particular, for all MCS moves except move 6 and 17, the chain homotopy H is given in Table 2. The chain homotopy H for MCS move 6 sends all of the crossings to 0 except in the case of A_{j-1} and A_j, where

$$H(A_{j-1}) = \epsilon_1(B_{j-1}) + \epsilon_2(B_{j-1}) + \epsilon_1(B_{j-1})\widetilde{\epsilon_2}(B_{j-1}),$$
$$H(A_j) = H(A_{j-1}) + \epsilon_2(B_{j-1}) + H(A_{j-1})\epsilon_2(B_{j-1}).$$

The case of MCS move 17 is slightly more complicated. In particular, just to the left of the region of the MCS move, the chain complex (C_j, ∂_j) in $\mathscr{C}_1$ and $\mathscr{C}_2$ has a pair of generators $y_l < y_k$ such that $|y_l| = |y_k| + 1$. Let $y_{u_1} < y_{u_2} < \cdots < y_{u_s}$ denote the generators of C_j satisfying $\langle \partial y \,|\, y_k \rangle = 1$. Let $y_{v_r} < \cdots < y_{v_1} < y_i$ denote the generators of C_j appearing in $\partial_j y_l$. MCS move 17 introduces the handleslide marks $E_{k,v_r}, \ldots E_{k,v_1}, E_{u_1,l}, \ldots, E_{u_s,l}$. We assume that $\mathscr{C}_2$ includes these marks and $\mathscr{C}_1$ does not. Then we introduce $r+s$ dips to the right of D_j in $L_\Sigma^{d_1}$. As we add dips to $L_\Sigma^{d_1}$, we extend $\epsilon_{\mathscr{C}_1}$ by 0. By Lemma 5.8 we can keep track of the extensions of $\epsilon_{\mathscr{C}_1}$ after each dip. The chain homotopy H between $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ is defined as follows. For $1 \le i \le r+s-1$, we have $H(A_{j+i}) = \sum_{k=1}^{i} \epsilon_{\mathscr{C}_2}(B_{j+k})$ and $H(B_{j+r+s}) = \mathscr{H}_{k,l}$, and all of the other crossings are sent to 0.

Since each H defined above has few nonzero entries, it is easy to check that H has support on generators with grading -1. Thus we need only check that the extension of H by linearity and the derivation product property satisfies $\tilde{\epsilon}_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2} = H \circ \partial$. In the case of MCS moves 11–16, this is equivalent to checking that H solves the matrix equations in Corollary 6.14. In the case of MCS move 17, we must check that H solves the matrix equations in Lemma 6.13. We leave this task to the reader.

The maps H given above and in Table 2 were constructed using the following process. The augmentations $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are equal on crossings outside of the region

Chain Homotopy H between ϵ_1 and ϵ_2		
Move	$H(A_j)$	$H(A_{j+1})$
1	$\epsilon_2(B_j)$	0
2	$\epsilon_1(B_j)+\epsilon_2(B_j)$	0
3	$\epsilon_1(B_j)+\epsilon_2(B_j)$	0
4	$\epsilon_1(B_j)+\epsilon_2(B_j)$	0
5	$\epsilon_1(B_j)+\epsilon_2(B_j)$	0
7	$\epsilon_1(B_j)$	$\epsilon_1(B_j)$
8	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
9	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
10	$\epsilon_1(B_j)$	$\epsilon_1(B_j)$
11	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
12	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
13	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
14	$\epsilon_1(B_j)$	$\epsilon_2(B_{j+2})$
15	$\epsilon_1(B_j)$	0
16	$\epsilon_1(B_j)$	0

Table 2. In all cases, $H = 0$ on all other crossings.

of the MCS move; thus $\tilde{\epsilon}_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2} = 0$ on these crossings. Hence, $H = 0$ satisfies $\tilde{\epsilon}_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2} = H \circ \partial$ on these crossings. Within the region of the MCS moves, we can write down explicit matrix equations relating $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$. In particular, $\tilde{\epsilon}_{\mathscr{C}_1} = \tilde{\epsilon}_{\mathscr{C}_2}$ to the left of the MCS move, and within the region of the move, $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are related by the matrix equations in Lemma 6.3. From these equations, we are able to define H.

We now show that $[\tilde{\epsilon}_{\mathscr{C}_1}] = [\tilde{\epsilon}_{\mathscr{C}_2}]$ implies $\Psi(\mathscr{C}_1) = \Psi(\mathscr{C}_2)$. Let v denote any dipping/undipping path from L^d_Σ to L_Σ. By Lemma 6.9, the definition of $\Psi : \mathrm{MCS}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is independent of dipping/undipping paths. We calculate $\Psi(\mathscr{C}_1)$ (respectively $\Psi(\mathscr{C}_2)$) using the path that travels from $L^{d_1}_\Sigma$ (respectively $L^{d_2}_\Sigma$) to L_Σ by adding dips as specified above to create L^d_Σ and then traveling along v. The first segment of this path maps $\epsilon_{\mathscr{C}_1}$ to $\tilde{\epsilon}_{\mathscr{C}_1}$, and $\epsilon_{\mathscr{C}_2}$ to $\tilde{\epsilon}_{\mathscr{C}_2}$. Since $\tilde{\epsilon}_{\mathscr{C}_1} \simeq \tilde{\epsilon}_{\mathscr{C}_2}$, the stable-tame isomorphism associated to the path v maps $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ to the same chain homotopy class. Thus, $\Psi(\mathscr{C}_1) = \Psi(\mathscr{C}_1)$, as desired. □

Finally, note that Lemma 6.15 and Lemma 6.10 imply that $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$, $[\mathscr{C}] \mapsto \Psi(\mathscr{C})$ is a well-defined, surjective map. Thus we have proved Theorem 6.12. □

6e. ***Two standard forms for MCSs.*** An MCS $\mathscr{C}$ encodes both a graded normal ruling $N_{\mathscr{C}}$ on Σ and an augmentation $\epsilon_{\mathscr{C}}$ on L^d_Σ. In this section, we formulate two

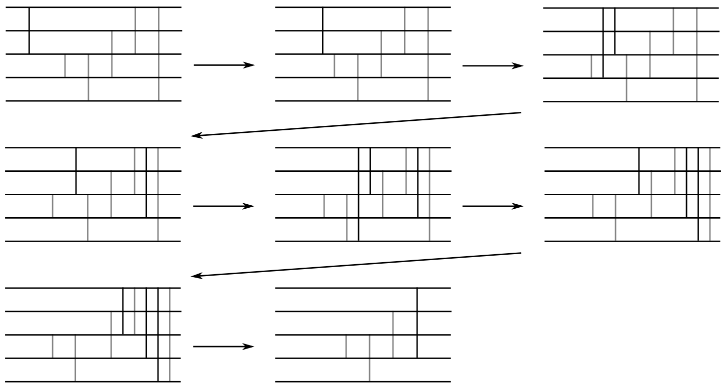


Figure 25. Sweeping a handleslide mark into V from the left. The grey marks are those contained in the original V.

algorithms using MCS moves that highlight these connections. When combined the algorithms provide a map from $\mathrm{MCS}_b(\Sigma)$ to $\mathrm{Aug}(L_\Sigma)$ that, when passed to equivalence classes, corresponds to $\widehat{\Psi}$. The upside is that this map does not require dipped diagrams or keeping track of chain homotopy classes of augmentations. We are also able to reprove the many-to-one correspondence between augmentations and graded normal rulings found in [Ng and Sabloff 2006].

6e1. *Sweeping collections of handleslide marks.* The algorithms we define sweep handleslide marks from left to right in Σ. During this process, we let V denote the collection of handleslide marks being swept. Suppose V sits away from the crossings and cusps of Σ. Near V, label the strands of Σ from bottom to top with the integers $1, \dots, n$. We define $v_{k,l} \in \mathbb{Z}_2$ to be 1 if and only if the collection V includes a handleslide mark between strands k and l, $k > l$. We abuse notation slightly by allowing $v_{k,l}$ to denote the handleslide mark as well. We order the marks in V as follows. If $k' < k$ and $v_{k,l} = v_{k',l'} = 1$, then $v_{k',l'}$ appears to the left of $v_{k,l}$ in V. If $l' < l$ and $v_{k,l} = v_{k,l'} = 1$, then $v_{k,l'}$ appears to the left of $v_{k,l}$ in V. The following moves are used in the $S\overline{R}$ and A-algorithms and describe interactions between V and handleslide marks, cusps, and crossings.

Move I (sweeping a handleslide from the left of V into/out of V). Suppose a handleslide mark h sits to the left of V between strands k and l, with $k > l$. We use MCS moves 2–6 to commute h past the handleslides in V. In order to commute h past a handleslide of the form $v_{l,i}$, we must use MCS move 6, and thus create a new handleslide mark h' between strands k and i. The ordering on V allows us to commute h' to the right so that it becomes properly ordered with the other marks in V. If $v_{k,i} = 1$, then h' cancels the handleslide $v_{k,i}$ by MCS move 1. We continue this process of commuting h past each of the strands that begin on strand l. Once

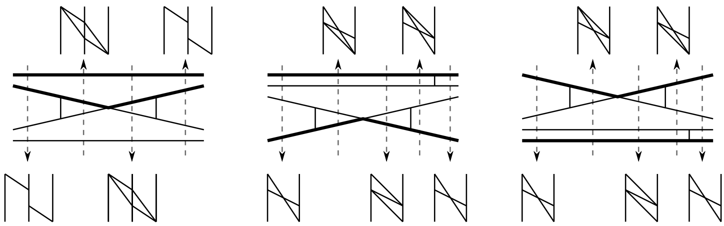


Figure 26. Simple switches.

we have done so, we can commute h past the other marks in V, without introducing new marks, until it becomes properly ordered in V; see Figure 25. Since each of MCS moves 1–6 is reversible, we are able to sweep an existing handleslide mark $v_{k,l}$ out of V to the left using the same process.

Move II (sweeping a handleslide from the right of V into/out of V). The process to sweep a handleslide from the right of V into/out of V is analogous to the process in move I.

Move III (sweeping V past a crossing q between strands $i+1$ and i assuming $v_{i+1,i}=0$). Sweep all of the handleslides of V past q using MCS moves 7–10, and if necessary, reorder the handleslide marks so that V is properly ordered.

Move IV (sweeping V past a right cusp between strands $i+1$ and i assuming there are no handleslide marks in V beginning or ending on $i+1$ or i). The handleslides in V sweep past the right cusp using MCS moves 12 and 14. V stays properly ordered during this process.

6e2. *The $S\bar{R}$-form of an MCS.* Given an MCS $\mathscr{C}$, the $S\bar{R}$-algorithm results in an MCS with handleslide marks near switched crossings and some graded returns of $N_{\mathscr{C}}$, and away from these crossings, the chain complexes are in simple form. The resulting MCS is called the *$S\bar{R}$-form of $\mathscr{C}$*. The $S\bar{R}$-form was inspired by discussions at the September 2008 AIM workshop, the work of Fuchs and Rutherford [2008] and the work of Ng and Sabloff [2006].

Definition 6.16. Given an MCS $\mathscr{C}$, we say a switch or return in $N_{\mathscr{C}}$ is *simple* if the ordered chain complexes of $\mathscr{C}$ are in simple form before and after the switch or return, and in a neighborhood of the crossing the MCS is arranged as in Figures 26 or 27. A simple return is *marked* if it corresponds to one of the three arrangements in the top row of Figure 27.

An MCS is in *$S\bar{R}$-form* if all of its switched crossings and graded returns are simple, and besides the handleslides near simple switches and returns, no other handleslides appear in $\mathscr{C}$

Theorem 6.17. *Every $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ is equivalent to an MCS in $S\bar{R}$-form.*

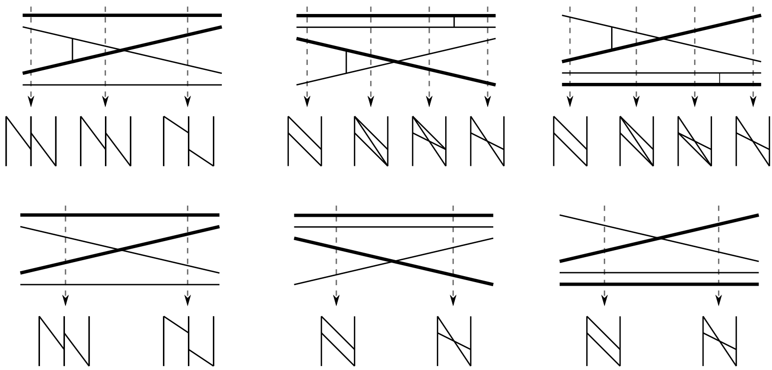


Figure 27. Simple returns.

Proof. We define an algorithm to sweep handleslide marks of $\mathscr{C}$ from left to right in the front projection, beginning at the leftmost cusp and working to the right. As we sweep handleslides marks to the right, we ensure that the MCS we leave behind is in $S\bar{R}$-form. By definition, the first chain complex of $\mathscr{C}$ is in simple form and so the collection V begins empty.

We consider each case of V encountering a handleslide mark, cusp, or crossing. In the case of a handleslide mark, we incorporate the mark into V using move II. In the case of a left cusp, we sweep V past the cusp using MCS moves 11 and 13. In either of these cases, if the MCS is in $S\bar{R}$-form to the left of V before the handleslide mark or left cusp, then it is also in $S\bar{R}$-form after we sweep past.

Moves III and IV ensure that in certain situations we may sweep V past a crossing or right cusp so that the resulting MCS is in $S\bar{R}$-form to the left of the singularity. Suppose we arrive at a crossing q between strands $i+1$ and i and $v_{i+1,i} = 1$. We use the following algorithm to ensure that q is a simple switch or simple return after we push V past q. We sweep the handleslide $v_{i+1,i}$ to the left of V using move I. We let h denote this handleslide. Now $v_{i+1,i} = 0$ and so we sweep V past q using move III. If we suppose the ordered chain complex of $\mathscr{C}$ is in simple form just before h, then around h, $\mathscr{C}$ looks like one of the 6 cases in Figure 28. The top three cases indicate that q is a switch and the bottom three cases indicate that q is a return.

Switches: If q is a switch, we use MCS move 1 to introduce two handleslides just to the right of the crossing between strands $i+1$ and i. If q is a switch of type (2) or (3), we also introduce two handleslides between the companion strands of strands $i+1$ and i. Finally, we move the righthand mark of each new pair of marks into V using move I. The resulting MCS now has a simple switch at q; see Figure 26.

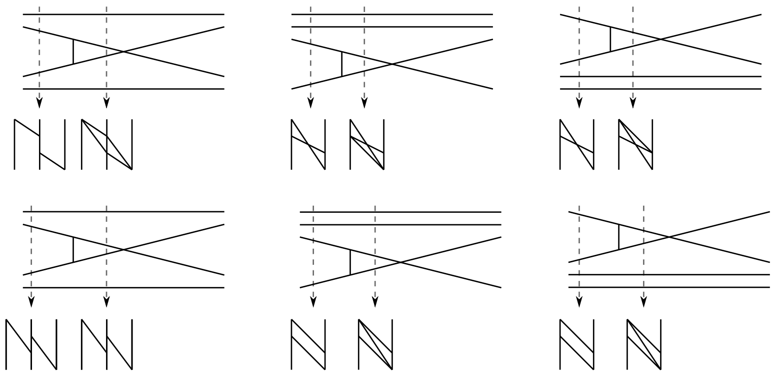


Figure 28. Sweeping V past a crossing with a handleslide immediately to the left. The top row will be switches and the bottom will be marked returns. In each row, we label the cases (1), (2) and (3) from left to right.

Returns: If q is a return of type (1), then q is a simple return as in Figure 27. If q is a return of type (2) or (3), we use MCS move 1 to introduce two handleslides just to the right of the crossing between the companion strands of strands $i+1$ and i and use move I to sweep the right-hand mark into V. The resulting MCS now has a simple return at q; see Figure 27.

Suppose V arrives at a right cusp between strands $i+1$ and i and there are handleslide marks in V beginning or ending on $i+1$ or i. Suppose also that the MCS is in $S\overline{R}$-form to the left of V. By applying move I possibly many times, we sweep the marks that end on strand i, begin on strand $i+1$, begin on strand i, or end on strand $i+1$ out of V. The order in which these moves occur corresponds to the order given in the previous sentence. The Maslov potentials on strands $i+1$ and i differ by one; thus $v_{i+1,i}=0$. As a consequence, these moves do not introduce new handleslides ending or beginning on $i+1$ or i. Next we use move IV to sweep V past the right cusp.

Now we remove the marks that have accumulated at the right cusp. We remove the marks ending on $i+1$ and beginning on i using MCS moves 15 and 16. Let $h_1, \dots, h_n$ denote the handleslides ending on strand i, ordered from left to right, and let $g_1, \dots, g_m$ denote the handleslides beginning on strand $i+1$. By assumption, the MCS is simple just to the left of h_1. Thus just before and after h_1, the pairing and MCS must look like one of the three cases in Figure 29(a). If h_1 is of type 1 or 2, we can eliminate it using MCS move 17. Suppose h_1 is of type 3, h_1 begins on strand l, and generator l is paired with generator k in the ordered chain complex just before h. Introduce a new handleslide, denoted h', between strands k and $i+1$ using MCS move 15. Move h'_1 past each of $g_1, \dots, g_m$ using

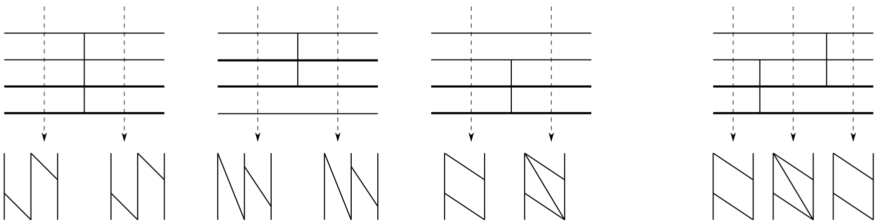


Figure 29. Left: the 3 possible local neighborhoods of the handleslide h_1. Right: MCS move 17 at a right cusp after introducing a new handleslide mark. In both (a) and (b), the two dark lines correspond to the strands entering the right cusp.

MCS move 6. Each time we perform such a move, we create a new handleslide mark which is then swept past the cusp using MCS move 12 and incorporated into V using move I. Commute h_1' past $h_2, \ldots, h_m$ using MCS moves 2–4. Now h_1 and h_1' look like Figure 29(b) and we can remove both using MCS move 17.

Using this procedure, we eliminate all of $h_1, \ldots, h_n$. The argument to eliminate $g_1, \ldots, g_m$ is essentially identical. Either we can eliminate g_1 using MCS move 17 or we can push on a handleslide using MCS move 15 and then eliminate the pair with MCS move 17. After eliminating $h_1, \ldots, h_n$ and $g_1, \ldots, g_m$, the MCS is in simple form just before and just after the right cusp. Hence, the resulting MCS is in $S\overline{R}$-form to the left of V.

As we carry out this algorithm from left to right, each time we encounter a handleslide, crossing or cusp we are able to sweep V past so that the MCS we leave behind is in $S\overline{R}$-form. □

Corollary 6.18. *Let N be a graded normal ruling on Σ with switched crossings $q_1, \ldots, q_n$ and graded returns $p_1, \ldots, p_m$. Then N is the graded normal ruling of 2^m MCSs in $S\overline{R}$-form. Hence, N is the graded normal ruling of at most 2^m MCS classes in $\widehat{\mathrm{MCS}}_b(\Sigma)$.*

Proof. By Lemmas 3.14 and 3.15, each MCS equivalence class has an associated graded normal ruling, and by Theorem 6.17, each MCS equivalence class has at least one representative in $S\overline{R}$-form. Thus, it is sufficient to show there are exactly 2^m MCSs in $S\overline{R}$-form with graded normal ruling N. The set of marked returns of any MCS in $S\overline{R}$-form with graded normal ruling N gives a subset of $\{p_1, \ldots, p_m\}$. Conversely, by adding handleslide marks to Σ as dictated by N, Figure 26, and Figure 27, any $R \subset \{p_1, \ldots, p_m\}$ will determine an MCS in $S\overline{R}$-form with graded normal ruling N and marked returns corresponding to R. □

6e3. *The A-form of an MCS.* The A-algorithm is similar to the sweeping process in the $S\overline{R}$-algorithm. However, we no longer introduce new handleslide marks after crossings. The resulting MCS, called the A-form, has marks to the left of some graded crossings and nowhere else. The set of MCSs in A-form are in bijection with the augmentations on L_Σ. In the algorithm, we still have to address the issue of mark accumulating at right cusps. If we begin with an MCS in $S\bar{R}$-form, then this is easy to do.

Definition 6.19. An MCS $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ is in A-form if

(1) outside a small neighborhood of the crossings of Σ, $\mathscr{C}$ has no handleslide marks; and

(2) within a small neighborhood of a crossing q, either $\mathscr{C}$ has no handleslide marks or it has a single handleslide mark to the left of q between the strands crossing at q.

The left figure in Figure 21 is in A-form.

Theorem 6.20. *Every $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ is equivalent to an MCS in A-form.*

Proof. Let $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ and use the $S\bar{R}$-algorithm to put $\mathscr{C}$ in $S\bar{R}$-form. The A-algorithm sweeps handleslide marks from left to right in the marked front of $\mathscr{C}$. As in the $S\bar{R}$-algorithm, we will use V to keep track of handleslide marks and the moves I - IV to sweep V past handleslides, crossings and cusps. In the case of handleslides and left cusps, the A-algorithm works the same as in the $S\bar{R}$-algorithm.

Suppose V arrives at a crossing q between strands $i+1$ and i. If $v_{i+1,i} = 0$, then we sweep V past q as described in move III. If $v_{i+1,i} = 1$, then we sweep the handleslide $v_{i+1,i}$ to the left of V and then sweep V past q using move III.

Suppose V arrives at a right cusp q between strands $i+1$ and i. If no handleslides begin or end on $i+1$ or i, then we sweep V past the right cusp using move IV. Otherwise, we sweep the handleslides that end on strand i, begin on strand $i+1$, begin on strand i, or end on strand $i+1$ out of V to the right using move II. The order in which these moves occur corresponds to the order given in the previous sentence. Let $h_1, \dots, h_n$ denote the handleslides ending on strand i, ordered from right to left, and let $g_1, \dots, g_m$ denote the handleslides beginning on strand $i+1$, also ordered from right to left. Since the MCS is in $S\bar{R}$-form to the right of h_1, we know that the chain complex between h_1 and the right cusp is simple. Thus just before and after h_1, the pairing and MCS must look like one of the three cases in Figure 30(a). If h_1 is of type (1) or (2) in Figure 30(a), we can eliminate it using MCS move 17. Suppose h_1 is of type (3), h_1 begins on strand l, and in the ordered chain complex just after h_1, generator l is paired with generator k. Use MCS move 17 to introduce two new handleslides to the left of h_1; see Figure 30(b). One new handleslide is between strands l and i. We remove h_1 and this new handleslide using MCS move 1. The second handleslide is between strands k and $i+1$ and can

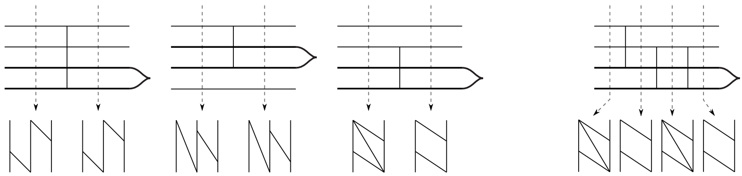


Figure 30. Left: the 3 possible local neighborhoods of the handleslide h_1. Right: MCS move 17 at a right cusp introducing two new handleslides. The two dark lines correspond to the strands entering the right cusp.

be removed using MCS move 15. We eliminate all of $h_1, \ldots, h_n$ using this process. The argument to eliminate $g_1, \ldots, g_m$ is essentially identical. After eliminating $h_1, \ldots, h_n$ and $g_1, \ldots, g_m$, we use MCS move 15 and 16 to remove all of the handleslides ending on $i+1$ and beginning on i. We sweep V past the right cusp using move IV and continue to the right.

As we progress from left to right in Σ, marks remain to the immediate left of some graded crossings. Hence, after sweeping V past the rightmost cusp, we are left with an MCS in A-form. $\square$

The construction in the proof of Lemma 6.10 assigns an MCS $\mathscr{C}_\epsilon$ to an augmentation $\epsilon \in \mathrm{Aug}(L_\Sigma)$. The MCS $\mathscr{C}_\epsilon$ is in A-form. In fact, the crossings of $\mathscr{C}_\epsilon$ with handleslide marks to their immediate left correspond to the resolved crossings in L_Σ that are augmented by ϵ. This process is invertible. Suppose $\mathscr{C}$ is in A-form with handleslide marks to the immediate left of crossings $p_1, \ldots, p_k$. From Lemma 6.4, we have an associated augmentation $\epsilon_{\mathscr{C}}$ in a sufficiently dipped diagram with dips $D_1, \ldots, D_m$. We may undip $D_1, \ldots, D_m$, beginning with D_m and working to the left, so that the resulting augmentation on L_Σ only augments the crossings corresponding to $p_1, \ldots, p_k$. As a result, we have the following corollary.

Corollary 6.21. *The set of MCSs in A-form are in bijection with the augmentations on L_Σ, and if $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ is in A-form, then $\Psi(\mathscr{C}) = [\epsilon]$, where $\epsilon(p) = 1$ if and only if p is a marked crossing in $\mathscr{C}$.*

The following corollary uses the $S\overline{R}$-form and the A-algorithm to reprove the many-to-one relationship between augmentations and graded normal ruling first noted in [Ng and Sabloff 2006].

Corollary 6.22. *Let N be a graded normal ruling on Σ with switched crossings $q_1, \ldots, q_n$ and graded returns $p_1, \ldots, p_m$. Then the 2^m MCSs in $S\overline{R}$-form with graded normal ruling N correspond to 2^m different augmentations on L_Σ.*

Proof. By Corollary 6.18, N corresponds to 2^m MCSs in $S\bar{R}$-form. In fact, two MCSs $\mathscr{C}_1$ and $\mathscr{C}_2$ in $S\bar{R}$-form corresponding to N differ only by handleslide marks around graded returns. If $\mathscr{C}_1$ and $\mathscr{C}_2$ differ at the return p, then after applying the A-algorithm, the resulting MCSs $\mathscr{C}_1'$ and $\mathscr{C}_2'$ differ at p as well. Thus the augmentations on L_Σ corresponding to $\mathscr{C}_1$ and $\mathscr{C}_2$ will differ on the resolved crossing corresponding to p. □

7. Two-bridge Legendrian knots

In this section, we prove that in the case of front projections with two left cusps, the map $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is bijective.

Definition 7.1. A front Σ of a Legendrian knot K with exactly 2 left cusps is called a 2*-bridge front projection*.

In [2001], Ng proves that every smooth knot admitting a 2-bridge knot projection is smoothly isotopic to a Legendrian knot admitting a 2-bridge front projection. Thus, the following results apply to an infinite collection of Legendrian knots.

Definition 7.2. Given a graded normal ruling N on a front Σ, we say two crossings $q_i < q_j$ of Σ, ordered by the x-axis, form a *departure-return pair* (q_i, q_j) if N has a departure at q_i and a return at q_j and the two ruling disks that depart at q_i are the same disks that return at q_j.

A departure-return pair (q_i, q_j) is *graded* if both crossings have grading 0. There are five possible arrangements of q_i and q_j in a graded departure-return pair; see, for example, Figure 31(a). We let $\nu(N)$ denote the number of graded departure-return pairs of N.

For a given fixed graded normal ruling N on any front projection Σ, each unswitched crossing is either a departure or a return and thus part of a departure-return pair. In the case of a 2-bridge front projection, we can say more.

Proposition 7.3. *Suppose Σ is a 2-bridge front projection with graded normal ruling N. For each departure-return pair (q_i, q_j) of N, no crossings or cusps of Σ may appear between q_i and q_j. In terms of the ordering of crossings by the x-axis, this says $j = i + 1$.*

Proof. Since Σ is a 2-bridge front projection, there are only two ruling disks for N. A switch or right cusp cannot appear after a departure since the two ruling disks overlap. Thus a return must immediately follow a departure. □

Definition 7.4. An MCS $\mathscr{C} \in \mathrm{MCS}_b(\Sigma)$ is in $S\bar{R}_g$*-form* if $\mathscr{C}$ is in $S\bar{R}$-form and each marked return is part of a graded departure-return pair.

Lemma 7.5. *If Σ is a 2-bridge front projection, every MCS class in $\widehat{\mathrm{MCS}}_b(\Sigma)$ has a representative in $S\bar{R}_g$-form.*

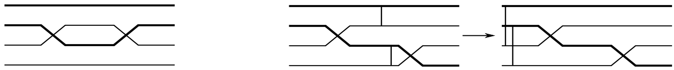


Figure 31. Left: a graded departure-return pair. Right: unmarking a graded return paired with an ungraded departure.

Proof. Let $[\mathscr{C}] \in \widehat{\mathrm{MCS}}_b(\Sigma)$, and let $\mathscr{C}$ be an $S\bar{R}$-form representative of $[\mathscr{C}]$. Suppose (q_i, q_{i+1}) is a departure-return pair of $N_{\mathscr{C}}$ such that q_{i+1} is a marked graded return with an ungraded departure q_i. We can push the handleslide mark(s) at q_{i+1} to the left, past the ungraded departure q_i; see Figure 31(b). Using MCS move 17, we may remove these handleslide mark(s). In this manner, we eliminate all of the handleslide marks at graded returns that are paired with ungraded departures. Thus, $\mathscr{C}$ is equivalent to an MCS in $S\bar{R}_g$-form. □

In fact, the $S\bar{R}_g$-form found in the previous proof is unique.

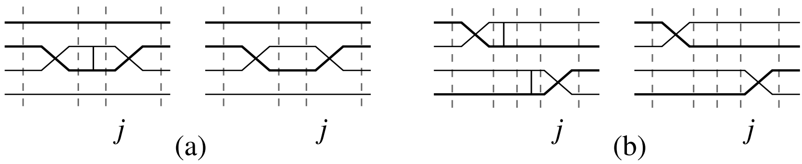
Lemma 7.6. *If Σ is a 2-bridge front projection, every MCS class in $\widehat{\mathrm{MCS}}_b(\Sigma)$ has a unique representative in $S\bar{R}_g$-form. Thus $|\widehat{\mathrm{MCS}}_b(\Sigma)| = \sum_{N \in N(\Sigma)} 2^{\nu(N)}$.*

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two representatives of $[\mathscr{C}]$ in $S\bar{R}_g$-form. Since $\mathscr{C}_1 \sim \mathscr{C}_2$, $\mathscr{C}_1$ and $\mathscr{C}_2$ induce the same graded normal ruling on Σ, which we will denote N. Thus, $\mathscr{C}_1$ and $\mathscr{C}_2$ have the same handleslide marks around switches and only differ on their marked returns. Suppose for contradiction that $\mathscr{C}_1$ and $\mathscr{C}_2$ differ at the graded departure-return pair (q_i, q_{i+1}). We assume q_{i+1} is a marked return in $\mathscr{C}_1$ and is unmarked in $\mathscr{C}_2$. We will prove that $\Psi(\mathscr{C}_1) \neq \Psi(\mathscr{C}_2)$. Thus, by Lemma 6.15 we will have the desired contradiction.

Recall that $\Psi(\mathscr{C}_1)$ and $\Psi(\mathscr{C}_2)$ are computed by mapping the augmentations $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ constructed in Lemma 6.4 to augmentations in $\mathrm{Aug}(L_\Sigma)$ using a dipping/undipping path. The augmentations $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ occur on sufficiently dipped diagrams $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$. The dipped diagrams $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$ differ by one or two dips between the resolved crossings q_i and q_{i+1}. Add these dips to $L_\Sigma^{d_2}$ and extend $\epsilon_{\mathscr{C}_2}$ by 0 using Lemma 5.8. We let $\tilde{\epsilon}_{\mathscr{C}_2}$ denote the resulting augmentation on $L_\Sigma^{d_1}$.

From Lemma 6.9, the definition of Ψ is independent of dipping/undipping paths. Thus $\Psi(\mathscr{C}_1) = \Psi(\mathscr{C}_2)$ if and only if $\epsilon_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are chain homotopic as augmentations on $L_\Sigma^{d_1}$. We will show that they are not chain homotopic. Suppose for contradiction that $H : (\mathscr{A}(L_\Sigma^{d_1}), \partial) \to \mathbb{Z}_2$ is a chain homotopy between $\epsilon_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$. We will prove a contradiction exists for two of the five possible arrangements of a graded departure-return pair. The arguments for the remaining three cases are essentially identical.

Case 1: Suppose the graded departure-return pair is arranged as in Figure 32(a). The dotted lines there indicate the location of the dips in $L_\Sigma^{d_1}$. Let $k+1$ and k



(a) (b)

Figure 32. In both (a) and (b), the MCS $\mathscr{C}_1$ is on the left and $\mathscr{C}_2$ is on the right.

denote the strands crossing at q_{i+1}. The following calculations use the formulas from Lemma 5.5 and the fact that all of the chain complexes involved are simple in the sense of Definition 3.10 or are only one handleslide away from being simple. The chain homotopy H must satisfy

(i) $H(a_j^{k+1,k}) = H(\partial q_{i+1}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(q_{i+1}) = 0$,

(ii) $H(a_j^{k+1,k}) + H(a_{j-1}^{k+1,k}) = H(\partial b_j^{k+1,k}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(b_j^{k+1,k}) = 1$, and

(iii) $H(a_{j-1}^{k+1,k}) = H(\partial b_{j-1}^{k+1,k}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(b_{j-1}^{k+1,k}) = 0$.

Combining (i) and (ii), we see that $H(a_{j-1}^{k+1,k}) = 1$, but this contradicts (iii). Thus $\epsilon_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are not chain homotopic.

Case 2: Suppose as in Case 1 that the graded departure-return pair is arranged as in Figure 32(b), with the dotted lines as before. Let $k+1$ and k denote the strands crossing at q_i, and let $l+1$ and l denote the strands crossing at q_{i+1}. The chain homotopy H must satisfy

(i) $H(a_j^{l+1,l}) = H(\partial q_{i+1}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(q_{i+1}) = 0$,

(ii) $H(a_j^{k+1,k}) = H(\partial a_j^{k+1,l}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(a_j^{k+1,l}) = 0$,

(iii) $H(a_j^{k+1,k}) + H(a_{j-1}^{k+1,k}) = H(\partial b_j^{k+1,k}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(b_j^{k+1,k}) = 0$,

(iv) $H(a_{j-1}^{k+1,k}) + H(a_{j-2}^{k+1,k}) = H(\partial b_{j-1}^{k+1,k}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(b_{j-1}^{k+1,k}) = 1$, and

(v) $H(a_{j-2}^{k+1,k}) = H(\partial b_{j-2}^{k+1,k}) = \epsilon_{\mathscr{C}_1} - \tilde{\epsilon}_{\mathscr{C}_2}(b_{j-2}^{k+1,k}) = 0$.

The second equation follows from $H(a_j^{l+1,l}) = 0$ and the formula for $\partial a_j^{k+1,l}$. By combining (ii) and (iii), we see that $H(a_{j-1}^{k+1,k}) = 0$. Combining (iv) with $H(a_{j-1}^{k+1,k}) = 0$, we see that $H(a_{j-2}^{k+1,k}) = 1$. This contradicts (v). Thus $\epsilon_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are not chain homotopic. □

Using these two lemmas, we prove Theorem 1.4.

Theorem 7.7. *If Σ is a 2-bridge front projection, then $\widehat{\Psi} : \widehat{\mathrm{MCS}}_b(\Sigma) \to \mathrm{Aug}^{ch}(L_\Sigma)$ is a bijection.*

Proof. The surjectivity of $\widehat{\Psi}$ is the content of Theorem 6.12. We need only show it is injective. Suppose $\widehat{\Psi}([\mathscr{C}_1]) = \widehat{\Psi}([\mathscr{C}_2])$ and let $\mathscr{C}_1$ and $\mathscr{C}_2$ be $S\overline{R}_g$-representatives of $[\mathscr{C}_1]$ and $[\mathscr{C}_1]$. Since each MCS class has a unique $S\bar{R}_g$-representative by Lemma 7.6, we need to show $\mathscr{C}_1 = \mathscr{C}_2$. This is equivalent to showing that $\mathscr{C}_1$ and $\mathscr{C}_2$ induce the same graded normal ruling on Σ and have the same marked returns.

Recall that $\Psi(\mathscr{C}_1)$ and $\Psi(\mathscr{C}_2)$ are computed by mapping the augmentations $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ constructed in Lemma 6.4 to augmentations in $\mathrm{Aug}(L_\Sigma)$ using a dipping/undipping path. The augmentations $\epsilon_{\mathscr{C}_1}$ and $\epsilon_{\mathscr{C}_2}$ occur on sufficiently dipped diagrams $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$. We may add dips to $L_\Sigma^{d_1}$ and $L_\Sigma^{d_2}$ and extend by 0 using Lemma 5.8 so that the resulting augmentations $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ occur on the same sufficiently dipped diagram L_Σ^d. Then $\widehat{\Psi}([\mathscr{C}_1]) = \widehat{\Psi}([\mathscr{C}_2])$ implies $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are chain homotopic as augmentations on L_Σ^d.

Let $H : (\mathscr{A}(L_\Sigma^d), \partial) \to \mathbb{Z}_2$ be a chain homotopy between $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$. Since $\tilde{\epsilon}_{\mathscr{C}_1}$ and $\tilde{\epsilon}_{\mathscr{C}_2}$ are occ-simple, Corollary 6.14 gives

$$\tilde{\epsilon}_{\mathscr{C}_1}(A_j) = (I + H(A_j))\tilde{\epsilon}_{\mathscr{C}_2}(I + H(A_j))^{-1} \quad \text{for all } j.$$

Recall that $\tilde{\epsilon}_{\mathscr{C}_1}(A_j)$ and $\tilde{\epsilon}_{\mathscr{C}_2}(A_j)$ encode the differential of a chain complex in $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively. Since $\tilde{\epsilon}_{\mathscr{C}_1}(A_j)$ and $\tilde{\epsilon}_{\mathscr{C}_2}(A_j)$ are chain isomorphic by a lower triangular matrix, the pairing of the strands of Σ determined by $\tilde{\epsilon}_{\mathscr{C}_1}(A_j)$ and $\tilde{\epsilon}_{\mathscr{C}_2}(A_j)$ agree. Thus, $\mathscr{C}_1$ and $\mathscr{C}_2$ determine the same graded normal ruling on Σ.

In the proof of Lemma 7.6 we show two MCSs in $S\bar{R}_g$-form determining the same graded normal ruling on Σ are mapped to chain homotopic augmentations only if they have the same marked returns. Since $\tilde{\epsilon}_{\mathscr{C}_1}(A_j)$ and $\tilde{\epsilon}_{\mathscr{C}_2}(A_j)$ are chain homotopic, this implies $\mathscr{C}_1$ and $\mathscr{C}_2$ have the same marked returns and thus $\mathscr{C}_1 = \mathscr{C}_2$ as desired. $\square$

Corollary 7.8. *If Σ is a 2-bridge front projection, then*

$$|\mathrm{Aug}^{ch}(L_\Sigma)| = |\widehat{\mathrm{MCS}}_b(\Sigma)| = \sum_{N \in N(\Sigma)} 2^{\nu(N)},$$

where $\nu(N)$ denotes the number of graded departure-return pairs of N.

Corollary 7.8 corresponds to Corollary 1.5 in Section 1.

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It was at the AIM workshop that I first learned of the possibility of extending graded normal rulings by including handleslide data. Petya Pushkar had outlined such a program in emails with Dmitry Fuchs in 2000. After the AIM workshop, Fuchs gave me copies of these emails together with more recent emails from Pushkar in 2008. Along with the work done at AIM, these notes directly influenced the definition of a Morse complex sequence and the equivalence relation on the set of MCSs given in this article. The results claimed by Pushkar in 2008 also directed my efforts. It is my understanding that proofs of the claims in the Pushkar emails have not yet appeared.

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MICHAEL B. HENRY
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TEXAS AT AUSTIN
1 UNIVERSITY STATION, C1200
AUSTIN, TX 78712
UNITED STATES

mbhenry@math.utexas.edu
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popa@math.ucla.edu

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jhlu@maths.hku.hk

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Department of Mathematics
University of California
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qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
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