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A FUNCTIONAL CALCULUS FOR UNBOUNDED GENERALIZED SCALAR OPERATORS ON BANACH SPACES

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We define weak and ultraweak functional calculus and construct using the Fourier transform technique an ultraweak functional calculus for an unbounded k-tuple of commuting generalized scalar operators T_j acting on a Banach space X. This functional calculus comprises the functions from the subset

$$F_\alpha = \left\{ f : \mathbb{R}^k \to \mathbb{C} \mid \hat{f} \text{ is a measure such that } \int_{\mathbb{R}^k} (1+|t|^\alpha)\, d|\hat{f}| < \infty \right\},$$

of Sobolev space $W^\alpha_\infty(\mathbb{R})$, where $\alpha > 0$. It also contains some unbounded functions. We also give examples and related results.

1. Introduction

A functional calculus for an operator A is an (in some sense) continuous homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is a topological algebra of functions and $\mathcal{B}$ is a topological algebra of operators, such that $\Phi(1) = e$ is the unit of the algebra $\mathcal{B}$ and $\Phi(x) = A$, where x denotes the identity function. Also, it is expected that the algebra $\mathcal{A}$ is generated by 1 and x, that is, that polynomials are dense in $\mathcal{A}$ (in the topology of $\mathcal{A}$).

A familiar example of a functional calculus is the analytic functional calculus on Banach algebras. It is the homomorphism $\Phi : \mathcal{H} \to \mathcal{C}$, where $\mathcal{C}$ is a Banach algebra, $A \in \mathcal{C}$ and $\mathcal{H}$ is an algebra of functions holomorphic in some neighborhood of $\sigma(A)$ (in the topology generated by uniform convergence on compact sets), defined by

$$\Phi(f) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda e - A)^{-1} d\lambda = f(A),$$

where $\Gamma \subset \operatorname{Dom}(f)$ and $\sigma(A) \Subset \operatorname{int} \Gamma$.

This mapping is continuous in the natural way.

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Another example of a functional calculus is described by the spectral theorem. Let $\mathscr{H}$ be a Hilbert space and $A \in B(\mathscr{H})$ be a normal operator. Then

$$L^\infty(\sigma(A)) \ni f \to \int_{\sigma(A)} f(\lambda) dE_\lambda = f(A)$$

(E_λ is the spectral measure) defines a homomorphism $\Phi : L^\infty \to B(\mathscr{H})$ that is continuous in the norm, because $\|f\|_\infty = \|f(A)\|$ and because the pointwise convergence of f_n to f implies $f_n(A) \to f(A)$, strongly ("continuity").

From these examples, we can see that if a functional calculus is defined for a wide class of operators, then the algebra of functions is very small. Analogously, if the algebra of functions is wide, then we must have stronger conditions on the operators.

In this note we will use the Fourier transform technique, which is not new and has been applied in many papers. McIntosh and Pryde [1987] developed the functional calculus for commuting n-tuples of bounded generalized scalar operators and functions that have Fourier transform in $L^1_s = \{f \mid \int |f(t)|(1+|t|^s)dt < +\infty\}$. A more general result was obtained by Andersson and Berndtsson [2002]. The case of a single generator of a bounded strongly continuous group was treated in [Balabane et al. 1993]. For more details and references, see [deLaubenfels 1994].

A similar idea of using Laplace transform for generators of bounded or polynomially bounded semigroups was used in [deLaubenfels and Jazar 1999] and [deLaubenfels 1995].

Another direction of development of functional calculi uses the Cauchy–Green integral formula applied to the almost holomorphic extension of a test function. The initial work of Dynkin [1972] was followed by the articles [Helffer and Sjöstrand 1989; Andersson 2003; Andersson and Sjöstrand 2004; Andersson et al. 2006]. The results obtained by this approach are mostly consequences of our results. In particular, Andersson [2003] showed that the functional calculus can be extended via the Cauchy–Green formula to the algebra of C^∞ functions with an additional condition at infinity if and only if the corresponding group is polynomially bounded, although a wider class of operators admits functional calculus for test functions. Andersson, Samuelsson and Sandberg [2006] observed that for any proper $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^k)$, the set $\{g \circ f \mid g \in \mathfrak{D}(\mathbb{R}^k)\}$ is contained in $D(\mathbb{R}^m)$; they then used this fact to construct the functional calculus (with test functions) for $f(A)$, where f is a proper C^∞ function and A is a Helffer–Sjöstrand operator, that is, an operator with real spectrum and rationally bounded resolvent on compacts.

After basic definitions in Section 2, including those of weak and ultraweak functional calculus, we introduce in Section 3 the algebra F_α and in Section 4 the algebra $F_{\alpha,\mathrm{loc}}$. In Section 4, we construct the functional calculus for the n-tuple of commuting generalized scalar operators and functions from $F_{\alpha,\mathrm{loc}}$. In

Section 5, we prove that the support of such functional calculus coincides with the spectrum of some operator closely related to the initial n-tuple and acting on the Clifford algebra derived from the initial Banach space. In Section 6, we compare the (ultra) weak functional calculus with other axiomatic approaches and find it to be more comprehensive. In other words, for very wide algebra of functions we obtain a more comprehensive theory. As shown in [Andersson 2003], this class of operators is exactly the one obtained by using the Cauchy–Green integral formula. In Section 7, we discuss relationship between some distinct definitions of commutativity of unbounded operators. In Section 8, we give some examples and comments.

2. Basic definitions

In this paper $\mathbb{R}^k \ni t = (t_1, \dots, t_k)$ and $T = (T_1, \dots, T_k)$ are k-tuples of numbers and operators, respectively. We define $t \cdot T := t_1 T_1 + \cdots + t_k T_k$, the "inner product", and $|t| := \sqrt{t_1^2 + \cdots + t_k^2}$, the Euclidean norm.

The next definition is a modification of definition given in [Vasilescu 1982].

Definition 2.1. Let $T = (T_1, \dots, T_k)$ be a k-tuple of closed densely defined operators on a Banach space X. We call the ordered couple $(\mathscr{A}, \Phi)$ a *weak functional calculus* based on $\mathbb{R}^k$ if

(1) $\mathscr{A}$ is a normed algebra of functions defined on $\mathbb{R}^k$ with a topology τ, such that
   (a) τ is weaker than the norm topology;
   (b) the addition is continuous with respect to τ;
   the multiplication is separately continuous with respect to τ;
   (c) the test functions $\mathfrak{D}$ are included and dense in $\mathscr{A}$;
   (d) embedding $\mathfrak{D} \hookrightarrow \mathscr{A}$ is continuous;

(2) $\Phi : \mathscr{A} \to B(X)$ is homomorphism, which is norm continuous and continuous with respect to τ and the weak topology on $B(X)$;

(3) for every polynomial $p(t_1, \dots, t_k)$ of degree m and for every sequence of the test functions ϑ_n tending to 1 in the natural way (increasingly and uniformly on compacts) we have $\Lambda\,(\Phi(\vartheta_n p)x) \to \Lambda(p(T_1, \dots, T_k)x)$ for all $x \in \bigcap_{\sum m_j = m} \mathfrak{D}(T_1^{m_1} \cdot \dots \cdot T_k^{m_k})$ and all $\Lambda \in X^*$.

Remark 2.2. In the condition (1), we expect continuity of multiplication only by coordinates. It would be to much to expect full continuity. Indeed, in that case from condition (2) it would follow that if $A_n \to A$ and $B_n \to B$ weakly, then $A_n B_n \to AB$ weakly, which is not true.

To extend the functional calculus to a wide set of unbounded functions, we need another condition, included in the following definition.

Definition 2.3. Let $T = (T_1, \dots, T_k)$ be a k-tuple of closed densely defined operators on a Banach space X. We call the ordered couple $(\mathcal{A}, \Phi)$ an *ultraweak functional calculus* based on $\mathbb{R}^k$ if it is a weak functional calculus and if the algebra $\mathcal{A}$ satisfies the following condition:

For an arbitrary test function $0 \leqslant \varphi \leqslant 1$ equalling 1 in some neighborhood of zero, and for any $f \in \mathcal{A}$, there holds $\varphi(x/n) f(x) \to f(x)$ as $n \to \infty$, in the topology of $\mathcal{A}$.

We will prove the existence of an ultraweak functional calculus when $\mathcal{A}$ is the subspace

$$F_\alpha = \left\{ f : \mathbb{R}^k \to \mathbb{C} \;\middle|\; \hat{f} \text{ is a measure such that } \int_{\mathbb{R}} (1 + |t|^\alpha) \, d|\hat{f}| < \infty \right\}$$

of the Sobolev space $W_\infty^\alpha(\mathbb{R}^k)$, where $\alpha = \sum_{j=1}^k \alpha_j$ is a positive real number. We assume that all operators T_j are closed and densely defined, $\sigma(T_j) \subseteq \mathbb{R}$ and

$$\|e^{itT_j}\| = O(|t|^{\alpha_j}) \quad \text{as } t \to \infty. \tag{2-1}$$

Also, we assume that T_j are mutually commuting, which means that the corresponding groups e^{itT_j} and e^{isT_k} mutually commute for any $s, t \in \mathbb{R}$ and any j, k.

Remark 2.4. We should specify what is meant by e^{itT_j}, because the set

$$\left\{ x \in X \mid x \in \bigcap_{n \geqslant 1} \mathcal{D}(T^n) \wedge \sum_{n=1}^{\infty} \frac{\|T^n x\|}{n!} < \infty \right\}$$

(the so-called analytical vectors) need not be dense in X, according to Nelson's theorem.

In this paper, e^{itT} is a one-parameter group $C(t)$, the solution of the abstract Cauchy problem

$$\frac{d}{dt} C(t) = iTC(t), \qquad C(0) = I. \tag{2-2}$$

One can prove the existence of solution of Equation (2-2) under different kinds of conditions (for example, the Hille–Yoshida conditions). Also, it is possible to give an estimate of asymptotic behavior of that solution; see [deLaubenfels 1991; 1993b; 1993a].

Remark 2.5. The condition $\sigma(T_j) \subseteq \mathbb{R}$ is actually superflous. Namely, Andersson and Berndtsson [2002, Lemma 1.2] showed that the condition (2-1) implies that the joint spectrum of the n-tuple of commuting bounded operators is a subset of $\mathbb{R}^n$. The proof presented in their Remark 1 is still valid if we omit their boundedness. However, many conditions that ensure the existence of the solution of the abstract Cauchy problem include the reality of spectrum.

Remark 2.6. One can define commutativity of unbounded operators in various ways. For instance, one can assume there is a dense subspace on which all products $T_i T_j$ are well-defined and $T_i T_j = T_j T_i$. In the literature, one also finds that the commutativity of the pair of unbounded operators is defined as the commutativity of their resolvents. In Section 7, we compare different commutativity conditions.

Remark 2.7. From the Hille–Yoshida theorem (see [Dunford and Schwartz 1958]) we can conclude that for a closed, densely defined operator A, with real spectrum such that $\|(A-\lambda I)^{-1}\| \leqslant 1/|\lambda|^m$ for nonreal λ, Equation (2-2) has a solution satisfying

$$\|C(t)\| = o(e^{\varepsilon t}) \quad \text{for all } \varepsilon > 0. \tag{2-3}$$

A natural question arises: Can we derive a polynomial estimate of $C(t)$ from (2-3)? The following example shows that this is not possible.

Example 2.8. There is an operator for which (2-1) does not hold, and its asymptotic behavior is weaker than exponential (that is, $e^{itA} = o(e^{\varepsilon|t|})$ as $|t| \to \infty$ for every $\varepsilon > 0$).

Let E be the vector space of entire functions such that there exists $C \in \mathbb{R}$ such that $|f(z)| \leqslant Ce^{\sqrt{|z|}}$. One can easily see that E is the Banach space if the norm $\|f\|$ of a function f is defined as the smallest constant C, for which the inequality above holds (that is, $\|f\| = \sup_{z\in\mathbb{C}}|f(z)|/e^{\sqrt{|z|}}$). Since

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi+z)}{\xi^2}\,d\xi,$$

it follows that operator $A = -id/dz$ is a bounded linear operator on E. Hence

$$(e^{itA}f)(z) = (e^{td/dz}f)(z) = f(z+t).$$

Since

$$\frac{|(e^{itA}f)(z)|}{e^{\sqrt{z}}} = \frac{|f(z+t)|}{e^{\sqrt{|z+t|}}}\cdot\frac{e^{\sqrt{|z+t|}}}{e^{\sqrt{|z|}}} \leqslant \|f\|\cdot e^{\sqrt{|z|+|t|}-\sqrt{|z|}} \quad \text{for every } f \in E,$$

it follows that $\|e^{itA}\| = O(e^{\sqrt{|t|}})$ as $|t| \to \infty$. Let $f_0(z) = 2\sum_{n=0}^{\infty} z^n/(2n)!$. From the above it follows that $\|e^{itA}f_0\|/\|f_0\| \leqslant e^{\sqrt{|t|}}$. Since for $x, t \geqslant 0$, we have

$$\begin{aligned}\frac{(e^{itA}f_0)(x)}{e^{\sqrt{x}}} = \frac{2\sum\limits_{n=0}^{\infty}(x+t)^n/(2n)!}{e^{\sqrt{x}}} &= \frac{2\,\mathrm{ch}(\sqrt{x+t})}{e^{\sqrt{x}}}\\ &= e^{\sqrt{x+t}-\sqrt{x}} + e^{-\sqrt{x+t}-\sqrt{x}},\end{aligned}$$

it follows that $\|e^{itA}f_0\| \sim e^{\sqrt{|t|}}$ as $t \to \infty$; therefore $\|e^{itA}f_0\| \sim e^{\sqrt{|t|}}$ as $t \to \infty$.

3. The algebra F_α

Definition 3.1. Let α be a positive real number, and let $\mathcal{M}_\alpha(\mathbb{R}^k)$ be the space of all Borel measures on $\mathbb{R}^k$ such that $\int_{\mathbb{R}^k}(1+|t|^\alpha)\,d|\mu|<\infty$ with the norm $\|\mu\|_\alpha = \int_{\mathbb{R}^k}(1+|t|^\alpha)d|\mu|$. Let $F_\alpha=\{f\in L^\infty(\mathbb{R}^k)\mid \hat{f}\in\mathcal{M}_\alpha\}$ be the normed space with the norm $\|f\|_{F_\alpha}=\|\hat{f}\|_{\mathcal{M}_\alpha}$.

Since $F_\alpha=\check{\mathcal{M}}_\alpha$, it follows that F_α is the Banach space with the norm $\|\cdot\|_{F_\alpha}$; that is

$$|\check{\mu}(x)| = \frac{1}{(2\pi)^{k/2}}\cdot\left|\int_{\mathbb{R}^k} e^{ix\cdot t}\,d\mu(t)\right| \leqslant \frac{1}{(2\pi)^{k/2}}\cdot\|\mu\|_{\mathcal{M}_\alpha}<\infty.$$

Let τ be the topology on F_α generated by the weak* topology on $\mathcal{M}_\alpha(\mathbb{R}^k)\subseteq (C_{b,\alpha}(\mathbb{R}^k))^*$, where $C_{b,\alpha}(\mathbb{R}^k)$ is the space of continuous functions such that $f(t)$ is $O(|t|^\alpha)$ as $|t|\to\infty$, with the norm $\|f\|_{C_{b,\alpha}}=\inf\{M\mid |f(t)|\leqslant M\cdot(1+|t|^\alpha)\}$. It means that the subbase of the neighborhoods of zero are

$$\mathcal{B}(g,\varepsilon)=\left\{f\in F_\alpha\;\middle|\;\left|\int_{\mathbb{R}^k} g\,d\hat{f}\right|<\varepsilon\right\}\quad \text{for } g\in C_{b,\alpha}(\mathbb{R}^k) \text{ and } \varepsilon>0;$$

see [Rudin 1973, Chapter 1].

Remark 3.2. Andersson and Berndtsson [2002] studied more general algebras $\mathcal{A}_h$, where $h:\mathbb{R}^k\to\mathbb{R}$ is a positive, continuous, subadditive function that is increasing on the rays from origin, $h(0)=0$, and $h(t)=o(|t|)$ as $t\to\infty$. The algebra F_α is a special case of $\mathcal{A}_h$, where $h(x)=\log(1+|x|^\alpha)$. The last function is not subadditive, but satisfies a somewhat weaker condition $h(x+y)\leq\log 2^\alpha+h(x)+h(y)$.

Theorem 3.3. *The space F_α is an algebra.*

Proof. F_α is a linear space, obviously. Let us prove that $f,g\in F_\alpha$ implies $f\cdot g\in F_\alpha$. The inequality

$$1+|s+t|^\alpha\leqslant 2^\alpha(1+|s|^\alpha)(1+|t|^\alpha) \tag{3-1}$$

holds. Indeed, for $|s|\leqslant|t|$ we have $1+|s+t|^\alpha\leqslant 1+(2|t|)^\alpha\leqslant 2^\alpha(1+|t|^\alpha)\leqslant 2^\alpha(1+|t|^\alpha)(1+|s|^\alpha)$ and similarly for $|s|\geqslant|t|$. Therefore, if $f,g\in F_\alpha$, then $\hat{f}$ and $\hat{g}$ are finite measures such that $\int_{\mathbb{R}^k}(1+|t|^\alpha)d|\hat{f}|<+\infty$ and $\int_{\mathbb{R}^k}(1+|t|^\alpha)d|\hat{g}|<+\infty$. Also $\widehat{f\cdot g}=\hat{f}*\hat{g}$ is well-defined and finite by the finiteness of $\hat{f}$ and $\hat{g}$, and we have

$$\begin{aligned}\int_{\mathbb{R}^k}(1+|t|^\alpha)\,d|\hat{f}*\hat{g}| &\leqslant \int_{\mathbb{R}^k}\int_{\mathbb{R}^k}(1+|s+t|^\alpha)\,d|\hat{f}|(s)d|\hat{g}|(t)\\ &\leqslant 2^\alpha\int_{\mathbb{R}^k}(1+|s|^\alpha)d|\hat{f}|(s)\int_{\mathbb{R}^k}(1+|t|^\alpha)d|\hat{g}|(t)\\ &=2^\alpha\|f\|_{F_\alpha}\|g\|_{F_\alpha}.\end{aligned}$$

□

Remark 3.4. Although such a space arises rarely in the literature, one can define the Sobolev space W^{α}_{∞} as the space of all functions $f : \mathbb{R}^k \to \mathbb{R}$ such that its fractional partial derivative $(-\Delta)^{\alpha/2} f$ belongs to L^{∞}. Here fractional derivative $(-\Delta)^{\alpha/2}$ is defined via Fourier transform. However $W^{\alpha}_{\infty} \supset F_{\alpha}$ (that is, this inclusion is proper) since Fourier transforms of functions from F_{α} have always continuous fractional partial derivative.

Remark 3.5. One can, also, define the space $F_{(\alpha_1,\dots,\alpha_n)}$ consisting of all functions f whose Fourier transform $\hat{f}$ is a measure such that $\int_{\mathbb{R}^k}(1+|t_1|^{\alpha_1}\cdot\dots\cdot|t_n|^{\alpha_n})d|\hat{f}|$ is finite. A simple application of the mean inequality shows that $F_{\alpha} \subset F_{(\alpha_1,\dots,\alpha_n)}$. From the following example we can see that this inclusion is strict.

Example 3.6. Let $n = 2$ and $(\alpha_1, \alpha_2) = (1, 1)$. Then we have

$$F_{(1,1)} = \{f \mid \int_{\mathbb{R}^2}(1+|xy|)\,d|\hat{f}| < +\infty\} \quad \text{and} \quad F_2 = \{f \mid \int_{\mathbb{R}^2}(1+x^2+y^2)\,d\hat{f} < +\infty\}.$$

Consider the function ψ defined as the inverse Fourier transform of the function

$$(x, y) \mapsto \frac{1}{1+|xy|} \cdot \frac{1}{1+x^2+y^2} \cdot \frac{1}{\log^2(1+(x^2+y^2)/2)}.$$

We claim that $\psi \in F_{(1,1)} \setminus F_2$. Indeed

$$\int_{\mathbb{R}^2}(1+|xy|)\,d\hat{\psi} = \int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} \cdot \frac{1}{\log^2(1+(x^2+y^2)/2)}\,dx\,dy < +\infty,$$

whereas

$$\begin{aligned}\int_{\mathbb{R}^2}(1+x^2+y^2)\,d\hat{\psi} &= \int_{\mathbb{R}^2} \frac{1}{1+|xy|} \cdot \frac{1}{\log^2(1+(x^2+y^2)/2)}\,dx\,dy \\ &\geqslant \int_{|x|\geqslant|y|} \frac{1}{1+|xy|} \cdot \frac{1}{\log^2(1+x^2)}\,dx\,dy \\ &= 2\int_{-\infty}^{+\infty} \frac{1}{\log^2(1+x^2)} \cdot \int_0^{|x|} \frac{d|y|}{1+|xy|}\,dx \\ &= 2\int_{-\infty}^{+\infty} \frac{dx}{|x|\log(1+x^2)} = +\infty.\end{aligned}$$

However, space $F_{(\alpha_1,\dots,\alpha_n)}$, although wider, need not be an algebra.

Example 3.7. Let $n = 2$ and $(\alpha_1, \alpha_2) = (1, 1)$. The functions $\varphi(x, y) = \sin x^2$ and $\psi(x, y) = \sin y^2$ are in $F_{(1,1)}$ since they are bounded and have bounded second order mixed derivatives $\partial^2\varphi/\partial x\partial y$ and $\partial^2\psi/\partial x\partial y$. However, their product has no bounded second order mixed derivatives. Hence $F_{(1,1)}$ is not an algebra.

Theorem 3.8. *The set $\mathfrak{D}$ is dense in the space F_{α} (with the topology τ) and the embedding $\mathfrak{D} \hookrightarrow F_{\alpha}$ is continuous.*

Proof. The sequence $([-n,n]^k)_{n\geqslant 1}$ is an increasing sequence of sets and we have $\bigcup_{n\in\mathbb{N}}[-n,n]^k=\mathbb{R}^k$. Therefore $|\mu|([-n,n]^k)\to|\mu|(\mathbb{R}^k)$ as $n\to\infty$ for $\mu\in\mathcal{M}_\alpha$. If $g\in C_{b,\alpha}(\mathbb{R}^k)$ and $M=\sup_{x\in\mathbb{R}^k}|g(x)|$, it follows that there is $n\in\mathbb{N}$ such that $|\mu|(\mathbb{R}^k\setminus[-n,n]^k)<\varepsilon/M$. Thus $\check{\mu}_n\in\check{\mu}+\mathcal{B}(g,\varepsilon)$, where $\mu_n(E)=\mu(E\cap[-n,n]^k)$ for every measurable set E. It follows that it is enough to prove the first part of our statement for inverse images $f\in F_\alpha$ of functions with compact support.

Let μ be a measure with compact support. Let $h_j(x)=j^kh(jx)$ be an approximative unit on $\mathbb{R}^k$ (that is, a sequence $(h_j)_{j\in\mathbb{N}}$ where $h\in\mathcal{D}(\mathbb{R}^k)$, $h\geqslant 0$ and $\int_{\mathbb{R}^k}h(x)dx=1$; see [Rudin 1973, 6.31]). Then we have $h_j*\mu\in\mathcal{S}$ (because $D^\alpha(h_j*\mu)=(D^\alpha h_j)*\mu$, and since h_j and μ have compact support, so does $h_j*\mu$; here $\mathcal{S}$ is the Schwartz class). If $g_1(t)=g(-t)$, it follows that

$$\begin{aligned}(h_j*\mu)(g)&=((h_j*\mu)*g_1)(0)\\&=(h_j*(\mu*g_1))(0)\to\delta*(\mu*g_1)(0)=\mu*g_1(0)=\mu(g)\end{aligned}$$

by [Rudin 1973, 6.30]. Therefore $h_j*\mu\to\mu$ as $j\to\infty$ in the weak* topology on $\mathcal{M}_\alpha$. Hence $\mathcal{S}$ is dense in the weak* topology on $\mathcal{M}_\alpha$, so it is also dense in the space F_α with the topology τ, because $\hat{S}=S$.

Therefore, it is enough to prove that $\mathcal{D}$ is dense in $\mathcal{S}$.

Since $\mathcal{D}$ is dense in $\mathcal{S}$ in the topology of the space $\mathcal{S}$,[1] for $\varphi\in\mathcal{S}$ there exists $(\varphi_n)_{n\geqslant 1}$ such that $\varphi_n\to\varphi$ in $\mathcal{S}$ as $n\to\infty$.

Let $g\in C_{b,\alpha}(\mathbb{R}^k)$. Then $\hat{g}\in\mathcal{S}'$ (see [Rudin 1973, 7.14(d)]) and therefore $\int_{\mathbb{R}^k}g(\hat{\varphi}_n-\hat{\varphi})dx=\int_{\mathbb{R}^k}(\varphi_n-\varphi)d\hat{g}\to 0$. That means that $\varphi_n\to\varphi$ as $n\to\infty$ in the topology τ.

The last part of the statement is trivial for test functions with support in some fixed compact set. Therefore by [Rudin 1973, Theorem 6.6], it is true for all $\mathcal{D}$. □

Lemma 3.9. *Pointwise addition and multiplication are continuous in* (F_α,τ).

Proof. Let $\lim_\Gamma f_\gamma=f$ and $\lim_\Gamma g_\gamma=g$ in the topology of F_α, where Γ is a net. Then $\int_{\mathbb{R}^k}h\,d\hat{f}_\gamma\to\int_{\mathbb{R}^k}h\,d\hat{f}$ and $\int_{\mathbb{R}^k}g\,d\hat{g}_\gamma\to\int_{\mathbb{R}^k}h\,d\hat{g}$ for arbitrary $h\in C_{b,\alpha}(\mathbb{R}^k)$. Then we have

$$\int_{\mathbb{R}^k}h\,d(\widehat{f_\gamma+g_\gamma})=\int_{\mathbb{R}^k}h\,d\hat{f}_\gamma+\int_{\mathbb{R}^k}h\,d\hat{g}_\gamma\to\int_{\mathbb{R}^k}h\,d\hat{f}+\int_{\mathbb{R}^k}h\,d\hat{g}=\int_{\mathbb{R}^k}h\,d(\widehat{f+g})$$

for $\gamma\in\Gamma$.

To prove the continuity of multiplication, it is enough to consider $f_\gamma\to 0$, and fixed g. Let $\mathcal{B}_{h_1,\dots,h_n,\varepsilon_1,\dots,\varepsilon_n}=\{\varphi\in F_\alpha\mid|\int_{\mathbb{R}^k}h_jd\hat{\varphi}|<\varepsilon_j,\ j=1,\dots,n\}$ be an

[1] Analogously to the procedure from the beginning of this proof, let $(h_j)_{j\in\mathbb{N}}$ be an approximative unit, such that $h(x)\leqslant 1$ and $h\equiv 1$ in some neighborhood of 0; then $\mathcal{D}\ni h_j*(\chi_{[-n,n]^k}\cdot\varphi)\to\varphi$ in the topology of $\mathcal{S}$

arbitrary basic neighborhood of zero generated by $h_j \in C_{b,\alpha}$. Using inequality (3-1), it is easy to check that the functions

$$\psi_j(x) = \int_{\mathbb{R}^k} h_j(x+y)d\hat{g}(y)$$

also belong to $C_{b,\alpha}$. If $f_\gamma \in \mathcal{B}_{\psi_1,\ldots,\psi_n,\varepsilon_1,\ldots,\varepsilon_n}$, then $f_\gamma \cdot g \in \mathcal{B}_{h_1,\ldots,h_n,\varepsilon_1,\ldots,\varepsilon_n}$. Indeed,

$$\left|\int_{\mathbb{R}^k} h_j d\hat{f}_\gamma * \hat{g}\right| = \left|\iint_{\mathbb{R}^k\times\mathbb{R}^k} h_j(x+y)d\hat{g}(y)d\hat{f}_\gamma(x)\right| = \left|\int_{\mathbb{R}^k} \psi_j d\hat{f}_\gamma\right| < \varepsilon_j.$$

In the general case, we have $f_\gamma \cdot g - fg = (f_\gamma - f)\cdot g$, and we can apply the special case. □

From Theorems 3.3 and 3.8 and Lemma 3.9 it follows that the space (F_α, τ) (from Definition 3.1) satisfies condition (1) of Definition 2.1.

Proposition 3.10. *Let φ be a test function such that $\varphi \equiv 1$ on some neighborhood of zero, and such that $0 \leqslant \varphi \leqslant 1$. Let $\chi_n(x) = \varphi(x/n)$. Then $f\chi_n \to f$ in the topology of the space F_α (as $n \to \infty$).*

Proof. We need to prove $\int_{\mathbb{R}^k} g d\widehat{f\chi_n} \to \int_{\mathbb{R}^k} g d\hat{f}$ as $n \to \infty$ for any $g \in C_{b,\alpha}(\mathbb{R}^k)$.

Let $F_n(t,s) = \int_{\mathbb{R}^k} g(t+s)d\hat{\chi}_n(s)$. Since $\hat{\chi}_n(s) \to \delta$ (the Dirac distribution) in the topology of F_α as $n \to \infty$,[2] we have $\lim_{n\to\infty} F_n(t,s) = g(t)$.

Since

$$\int_{\mathbb{R}^k} gd\widehat{f\chi_n} = \int_{\mathbb{R}^k} gd(\hat{f} * \hat{\chi}_n) = \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} g(s+t)d\hat{f}(t)d\hat{\chi}_n(s) = \int_{\mathbb{R}^k} F_n(t,s)d\hat{f}(t),$$

and $\hat{\chi}_n(s) = n^k\hat{\phi}(ns)$ (see [Rudin 1973, Theorem 7.4]), our result follows from the dominated convergence theorem, because

$$\begin{aligned}
|F_n(t,s)| &\leqslant \int_{\mathbb{R}^k} |g(t+s)|d|\hat{\chi}(s)| \\
&\leqslant \int_{\mathbb{R}^k} n^k(1+|s+t|^\alpha)d\hat{\phi}(ns) \\
&\leqslant 2^\alpha(1+|t|^\alpha)\int_{\mathbb{R}^k} n^k(1+|s|^\alpha)(1+|ns|)^{-\alpha-n-1}ds \\
&= 2^\alpha(1+|t|^\alpha)\int_{\mathbb{R}^k} \frac{(1+|u/n|^\alpha)}{(1+|u|)^{\alpha+n+1}}\,du \\
&\leqslant 2^\alpha(1+|t|^\alpha)\int_{\mathbb{R}^k} \frac{(1+|u|^\alpha)}{(1+|u|)^{\alpha+n+1}}\,du \\
&\leqslant \mathrm{const}\cdot(1+|t|^\alpha).
\end{aligned}$$

[2] $\hat{\chi}_n(s)$ belongs to the Schwartz class $\mathcal{S}$ and tends to δ in the topology of $\mathcal{S}$; see Theorem 3.8.

We used that $g \in C_{b,\alpha}(\mathbb{R}^k)$, $\hat{\phi} \in \mathcal{S}$ (so $|\hat{\phi}'(s)| \leqslant \text{const}\cdot(1+|s|)^{-m}$ for any natural m), $|u/n| \leqslant |u|$ for positive integer n, and inequality (3-1). □

The preceding lemma asserts that the algebra F_α satisfies the condition from the definition of the ultraweak functional calculus.

4. The construction of the functional calculus

Following the technique from [McIntosh and Pryde 1987] we define the functional calculus using the Fourier transform, that is, by formula

$$W_\infty^\alpha \supseteq F_\alpha \ni f \mapsto \Phi(f) = f(T) = \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} e^{-it\cdot T} d\mu_{\hat{f}}, \tag{4-1}$$

where the measure $\mu_{\hat{f}}$ is the Fourier transformation of the function f (namely, $\hat{f}(t) = (2\pi)^{-k/2} \cdot \int_{\mathbb{R}^k} e^{-its} f(s)ds$ can be divergent for some t, but this formula defines a complex Borel measure $\mu_{\hat{f}}$ and we have $\int_{-\infty}^{\infty}(1+|t|^\alpha)d|\mu_{\hat{f}}| < \infty$).

From now on, we shall abbreviate $\mu_{\hat{f}}$ to $\hat{f}$.

Theorem 4.1. *Suppose $T = (T_1, \dots, T_k)$ is a k-tuple of closed, densely defined operators such that all e^{itT_j} exist as solutions of the corresponding abstract Cauchy problem (2-2), such that*

$$\|e^{itT_j}\| = O(|t|^{\alpha_j}) \quad \text{as } t \to \infty.$$

Also, we assume that T_j are mutually commuting, which means that the corresponding groups e^{itT_j} and e^{isT_k} mutually commute for any $s, t \in \mathbb{R}$ and any j, k.

Let X be a reflexive Banach space and let $\alpha = \sum_j \alpha_j \geqslant 0$. A homomorphism $\Phi : F_\alpha \to B(X)$ is defined by formula (4-1). The integral in (4-1) exists as a weak integral and Φ is (τ, ω) continuous, where τ is the topology on F_α as the subspace of the dual space of $C_{b,\alpha}$ and ω is the weak topology on $B(X)$.

Proof. Let $x \in X$ and $\Lambda \in X^*$. Function $t \mapsto \Lambda(e^{it\cdot T}x)$ is weakly continuous. Also, we have

$$\left|\int_{\mathbb{R}^k} \Lambda(e^{it\cdot T}x) d\mu_{\hat{f}}\right| \leqslant \int_{\mathbb{R}^k} \|\Lambda\|(1+|t|^\alpha)\|x\| d|\mu_{\hat{f}}| \leqslant \|\Lambda\| \cdot \|x\| \cdot \|f\|_\alpha;$$

hence (4-1) exists as a weak integral.

The mapping Φ is obviously linear; also, we have

$$\begin{aligned}\Phi(f)\cdot\Phi(g) &= \frac{1}{(2\pi)^k}\cdot \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} e^{i(t+s)\cdot T} d\mu_{\hat{f}}(s) d\mu_{\hat{g}}(t) \\ &= \frac{1}{(2\pi)^{k/2}}\cdot\int_{\mathbb{R}^k} e^{ir\cdot T} d\mu_{\hat{f}*\hat{g}}(r) = \frac{1}{(2\pi)^{k/2}}\cdot\int_{\mathbb{R}^k} e^{ir\cdot T} d\mu_{\widehat{fg}}(r) = \Phi(fg),\end{aligned}$$

where $r = s + t$.

Finally, we need to prove the continuity.

Let $S_{\Lambda,x,\varepsilon} = \{A \mid |\Lambda(Ax)| < \varepsilon\}$ be a subbase set in $B(X)$, let $f_0 \in F_\alpha$, and let $A_f = (2\pi)^{-k/2} \cdot \int_{\mathbb{R}^k} e^{it\cdot T} d\mu_{\hat{f}}(t) \in B(X)$ for $f \in F_\alpha$.

Let

$$V = \left\{ f \;\middle|\; \frac{1}{(2\pi)^{k/2}} \cdot \left| \int_{\mathbb{R}^k} \Lambda(e^{it\cdot T}x) d\mu_{\widehat{f-f_0}} \right| < \varepsilon \right\}.$$

Since $t \mapsto \Lambda(e^{it\cdot T}x)$ is continuous, V is a neighborhood in F_α. Since

$$\Lambda[(A_f - A_{f_0})x] = \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} \Lambda(e^{it\cdot T}x) d\mu_{\widehat{f-f_0}}(t) < \varepsilon,$$

it follows that $\Phi(f_0 + V) \subseteq A_{f_0} + S_{\Lambda,x,\varepsilon}$. □

Theorem 4.2. *Let $p(t_1, t_2, \dots, t_k)$ be a polynomial of degree m and let ϑ_n be a sequence of test functions tending to 1 increasingly and uniformly on compacts. Then for all $\phi \in \bigcap_{\sum m_j = m} \operatorname{Dom} \prod_j T_j^{m_j}$ we have $\Phi(\vartheta_n p)\phi \xrightarrow{w} p(T_1, T_2, \dots, T_k)\phi$ as $n \to \infty$.*

Proof. Function $t \mapsto e^{it\cdot T}$ is strongly and weakly differentiable and satisfies (2-2). Let $D_j = -i\partial/\partial t_j$ and let δ be the Dirac distribution. Since

$$D_j^m(\hat{1}) = D_j^m((2\pi)^{k/2}\delta) = (-1)^m \widehat{t_j^m}$$

(for all $m \in \mathbb{N}_0$) and ϑ_n has compact support, from the definition of the distribution and [Rudin 1973, 6.37], we have

$$\begin{aligned}
T_1^{m_1} \cdot \dots \cdot T_k^{m_k} \phi &= \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} T_1^{m_1} \cdot \dots \cdot T_k^{m_k} e^{it\cdot T} \phi \, d((2\pi)^{k/2} \cdot \delta) \\
&= \lim_{n\to\infty} \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} D_1^{m_1} \cdot \dots \cdot D_k^{m_k} (e^{it\cdot T}) \phi \, d\mu_{(\hat{1}*\widehat{\vartheta_n})} \\
&= (-1)^{m_1+\dots+m_k} \lim_{n\to\infty} \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} e^{it\cdot T} \phi \, d\mu_{(D_1^{m_1} \cdot \dots \cdot D_k^{m_k}(\hat{1}) * \widehat{\vartheta_n})} \\
&= (-1)^{\sum_j m_j} \lim_{n\to\infty} \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} e^{it\cdot T} \phi \, d\mu_{(((-1)^{\sum_j m_j} \widehat{t_1^{m_1} \cdot \dots \cdot t_k^{m_k}}) * \widehat{\vartheta_n})} \\
&= \lim_{n\to\infty} \frac{1}{(2\pi)^{k/2}} \cdot \int_{\mathbb{R}^k} e^{it\cdot T} \phi \, d\mu_{(\widehat{t_1^{m_1} \cdot \dots \cdot t_k^{m_k} \cdot \vartheta_n})},
\end{aligned}$$

which proves our statement in the case where $p(t_1, \dots, t_k) = t_1^{m_1} \cdot \dots \cdot t_k^{m_k}$ is a monomial. In general case, the result easily follows by linearity of Φ. □

The previous results can be summarized:

Formula (4-1) defines an ultraweak functional calculus.

Next, we shall see that the possibility of constructing the operator $f(T)$ does not depend on the behavior of the function f at infinity. In fact, it depends on the local properties of the function f.

Throughout the rest of this section, φ_n (and also ψ_n and χ_n) will always denote an increasing sequence of test functions with range contained in [0, 1], and such that $\{t \in \mathbb{R}^k \mid \varphi_n(t) = 1\}$ form an increasing sequence of sets whose union covers all $\mathbb{R}^k$, unless φ_n is specified otherwise. Such sequences we shall call exhausting.

Proposition 4.3. *Let X_n denote the linear space* $\operatorname{Im}(\chi_n(T))$. *Let $E_0 = \bigcup_{n=1}^{+\infty} X_n$. Then X_n is invariant for all operators $f(T)$ and $E = \mathscr{L}(E_0)$ is dense in X.*

Proof. Let $x \in X_n$, that is, $x = \chi_n(T)y$. Then

$$f(T)x = f(T)\chi_n(T)y = \chi_n(T)f(T)y \in X_n.$$

Since $\chi_n \to 1$ in the topology of F_α, we have $x = w-\lim x_n = w-\lim \chi_n(A)x$ for any $x \in X$. Therefore the set E_0 is weakly dense, and its linear span E is also weakly dense. For linear spaces weak and strong closures coincide. □

Definition 4.4. We say that $f \in F_{\alpha,\mathrm{loc}}$ if $f\varphi \in F_\alpha$ for any $\varphi \in \mathfrak{D}$.

Proposition 4.5. *$F_\alpha \subseteq F_{\alpha,\mathrm{loc}}$ and $F_{\alpha,\mathrm{loc}}$ is an algebra.*

Proof. The first assertion follows from $\mathfrak{D} \cdot F_\alpha \subseteq F_\alpha \cdot F_\alpha \subseteq F_\alpha$. Let $f, g \in F_{\alpha,\mathrm{loc}}$ and let $\varphi \in \mathfrak{D}$ be arbitrary. Choose another $\psi \in \mathfrak{D}$ such that $\varphi\psi = \varphi$, that is, a function equals 1 in the neighborhood of the support of φ. Then $f\varphi, g\psi \in F_\alpha$ and $fg\varphi = f\varphi \cdot g\psi \in F_\alpha$. □

Proposition 4.6. *Let $f \in F_{\alpha,loc}$, let χ_n be some exhausting sequence and let $x \in E$. The sequence $(f \cdot \chi_n)(T)x$ converges strongly and its limit does not depend on the choice of an exhausting sequence.*

Proof. Let $x \in X_k$. Then $x = \psi_k(A)y$ for some $y \in X$. Pick a positive integer n_0 such that $\chi_{n_0} = 1$ on supp ψ_k. For $n \geqslant n_0$ we have $(f \cdot \chi_n)(T)x - (f \cdot \chi_{n_0})(T)x = ((f \cdot (\chi_n - \chi_{n_0}) \cdot \psi_k(T))y = 0$, implying that $(f \cdot \chi_n)(T)x$ is constant for $n \geqslant n_0$. Therefore, this sequence is constant also for $x \in E$.

To prove the independence, choose another exhausting ψ_n, and set

$$y = \lim_{n\to+\infty}(f\chi_n)(T) \text{ and } z = \lim_{n\to+\infty}(f\psi_n)(T).$$

For an arbitrary $\theta \in \mathfrak{D}$, we have

$$\theta(T)y = \lim_{n\to+\infty}\theta(T)(f\chi_n)(T)x = (\theta f)(T)x,$$

and analogously $\theta(T)z = (\theta f)(T)x$. Thus for any $\theta \in \mathfrak{D}$, we have $\theta(T)(y-z) = 0$. If θ runs through $\theta_n(x) = \varphi(x/n)$, then $\theta_n(T) \to I$ weakly, implying $y - z = 0$. □

Definition 4.7. Let $f \in F_{\alpha,\mathrm{loc}}$ and let $x \in E$. We define $f(T)x := \lim(f \cdot \chi_n)(T)x$. Thus, we defined the map $f \mapsto f(T)$

Proposition 4.8. *The map defined in Definition 4.7 is linear and multiplicative.*

Proof. The linearity is obvious. Further, let us note that the sequence $(f\chi_n)(T)x$ is not only convergent but constant for $n \geqslant n_0$. Let χ_n be an arbitrary exhausting sequence and let $x = \psi_k(T)y \in X_k$. Choose another exhausting sequence φ_n such that $\chi_n\varphi_n = \chi_n$. Then we have

$$(fg)(T)x = \lim_{n\to+\infty}(f\chi_n)(T)(g\varphi_n)(T)\psi_k(T)y. \tag{4-2}$$

Since all the operators $(f\chi_n)(T),\ (g\varphi_n)(T)$ and $\psi_k(T)$ commute, we see that both $(g\varphi_n)(T)y$ and $(f\chi_n)(T)y$ belong to X_k. Therefore $(g\varphi_n)(T)y$ and $g(T)y$ (as its limit) are in the domain of $f(T)$. In other words, the operator $f(T)g(T)$ is well-defined on X_n. By changing roles of f and g, we conclude that $g(T)f(T)$ is also well-defined on X_n. From (4-2), we get $(fg)(T) = f(T)g(T) = g(T)f(T)$. □

Proposition 4.9. *For any $f \in F_{\alpha,\mathrm{loc}}$, the operator $f(T)$ is closable.*

Proof. Let $x_n \to 0$, and let $f(T)x_n \to y$. We have

$$y = w - \lim_{n\to+\infty}\chi_n(T) = w - \lim_{n\to+\infty}\lim_{m\to+\infty}\chi_n(T)f(T)x_m = 0$$

since both $\chi_n(T)$ and $(\chi_n f)(T)$ are bounded. □

5. The support of the functional calculus

Definition 5.1. The support of the functional calculus is the smallest closed set $\operatorname{supp}\Phi \subseteq \mathbb{R}^k$ such that if $f \equiv g$ on $\operatorname{supp}\Phi$, then $\Phi(f) = \Phi(g)$.

First we need some definitions and facts concerning Clifford algebras.

Definition 5.2. Suppose F is an m-dimensional real vector space F with a basis $\{e_j \mid 1 \leqslant j \leqslant m\}$. We define the *Clifford algebra* $\mathscr{C}l(F)$ as a vector space with basis $\{e_S \mid S \subseteq \{1, 2, \dots, m\}\}$, with multiplication arising from identifications $e_\varnothing = 1$, $e_{\{j\}} = e_j$ and rules

$$e_j^2 = -1 \quad \text{and} \quad e_je_k = -e_ke_j \ \text{ if } k \neq j. \tag{5-1}$$

Remark 5.3. The rules (5-1) can be easily extended to the basis of $\mathscr{C}l(F)$ if we write $e_{j_1}e_{j_2}\cdots e_{j_s}$ for $e_S = e_{\{j_1,j_2,\dots,j_s\}}$, and further by linearity to whole $\mathscr{C}l(F)$.

Remark 5.4. For an m-dimensional real vector space F, we will write $\mathscr{C}l(m)$ for $\mathscr{C}l(F)$. It is well known that $\mathscr{C}l(1)$ is isomorphic to the field of complex numbers, and that $\mathscr{C}l(2)$ to the quaternions. For further details on Clifford algebras, see [Brackx et al. 1982; Ryan 2003].

Definition 5.5. Let X be a Banach space. The corresponding Banach module over $\mathscr{C}l(m)$ is simply $X_{(m)} = X \otimes_{\mathbb{R}} \mathscr{C}l(m) = \left\{\sum_{S\subseteq\{1,2,\dots,m\}} u_S e_S \mid u_S \in X\right\}$, with the norm

$$\Big\| \sum_{S\subseteq\{1,2,\dots,m\}} u_S e_S \Big\| = \Big(\sum_{S\subseteq\{1,2,\dots,m\}} \|u_S\|^2 \Big)^{1/2}.$$

Definition 5.6. Let $T_0, T_1, \dots, T_m$ be an $(m+1)$-tuple of operators acting on a Banach space X (bounded or unbounded). The operator $T_{\mathrm{cl}} = \sum_{j=0}^m T_j e_j$ is defined by $T_{\mathrm{cl}}(\sum_S u_S e_S) = \sum_{j=0}^m \sum_S T_j u_S e_j e_S$.

The norm of T_{cl} is $\|T_{\mathrm{cl}}\| = \sup_{\|u\|\leqslant 1} \|T_{\mathrm{cl}} u\|$.

Also we can define the resolvent set and the spectrum of T_{cl} by

$$\rho(T_{\mathrm{cl}}) = \{\lambda \in \mathbb{R}^{m+1} \mid \lambda I - T_{\mathrm{cl}} \text{ is invertible}\} \quad \text{and} \quad \sigma(T_{\mathrm{cl}}) = \mathbb{R}^{m+1} \setminus \rho(T_{\mathrm{cl}}).$$

Proposition 5.7. (a) *We have $\|u\lambda\| \leqslant 2^{m/2}\|u\|\|\lambda\|$ for all $u \in X_m$ and $\lambda \in \mathscr{C}l(m)$.*

(b) $\sigma(T_{\mathrm{cl}}) \subseteq \{\lambda \in \mathbb{R}^{m+1} \mid |\lambda| \leqslant \sqrt{m+1} \cdot \|T_{\mathrm{cl}}\|\}$.

Proof. Direct calculation. □

Proposition 5.8. *Let $T_0, T_1, \dots, T_m$ be an $(m+1)$-tuple of commuting unbounded operators. By commutativity we mean that there is a dense set $D \subseteq X$ such that $T_j T_k x = T_k T_j x$ for all $x \in D$ and j, k. Also, let $T_{\mathrm{cl}} = \sum_{j=0}^m T_j e_j$.*

(a) *The following conditions are mutually equivalent:*

 (i) *T_{cl} is invertible.*

 (ii) *$(\sum_j T_j^2) e_0$ is invertible.*

 (iii) *$\sum_j T_j^2$ is invertible in $B(X)$.*

(b) $\sigma(T_{\mathrm{cl}}) = \{\lambda \in \mathbb{R}^m \mid \sum_j (T_j - \lambda_j)^2$ *is not invertible*$\}$.

Proof. If $\sum_j T_j^2$ is invertible in $B(X)$, then $T_{\mathrm{cl}}^{-1} = (\sum_j T_j^2)^{-1} \cdot (T_0 - \sum_{j>0} T_j e_j)$, so (iii) implies (i).

Since $(T_0 - \sum_{j>0} T_j e_j)U = U T_{\mathrm{cl}}$, where $U(\sum_j u_S e_S) = \sum_j (-1)^{|S|} u_S e_S$, it follows that $(T_0 - \sum_{j>0} T_j e_j)$ is invertible; hence $(\sum_j T_j^2) e_0 = T_{\mathrm{cl}}(T_0 - \sum_{j>0} T_j e_j)$ is invertible, that is, (i) implies (ii).

That (ii) is equivalent to (iii) is obvious. Part (b) follows from the equivalence of (i) and (iii). □

Theorem 5.9. *The support of the weak functional calculus defined in Definition 2.1 is equal to $\sigma(\sum_{i=1}^m T_i e_i)$.*

Proof. Let n be an odd integer, let $n \geqslant \max\{m, 2\}$, and let $T = \sum_{i=1}^m T_i e_i$. By Proposition 5.8, and since $\operatorname{supp} \Phi \subseteq \mathbb{R}^m$, it is enough to prove that $\sigma(T) = \operatorname{supp} \Phi$. If $f = \sum_S f_S e_S$, with $\Phi_{(n)}(f) = \sum_S \Phi(f_S) e_S$, we can extend Φ to a homomorphism $\Phi_{(n)} : \mathscr{A} \otimes_{\mathbb{R}} \mathbb{R}_{(n)} \to B(X)_{(n)}$, which is continuous with respect to τ and the weak topology on $B(X)_{(n)}$ and $\operatorname{supp} \Phi_{(n)} = \operatorname{supp} \Phi$.

Let $\lambda \notin \operatorname{supp}\Phi$ and let $\vartheta_i \in \mathfrak{D}$ such that $\vartheta_i(\lambda) = 0$ and $\operatorname{supp}\vartheta_i \to \operatorname{supp}\Phi$ (increasingly and uniformly on compacts). Then the function $|x-\lambda|^{-n-1}\overline{(x-\lambda)}$ is well-defined (with value 0 for $x = \lambda$) and the function $(x-\lambda)|x-\lambda|^{n-1}$ is polynomial (since n is odd). Therefore we have

$$\begin{aligned}(T-\lambda I)|T-\lambda I|^{n-1} \cdot \lim_{i\to\infty} \Phi_{(n)}\big(\vartheta_i \cdot |x-\lambda|^{-n-1}\overline{(x-\lambda)}\big)\\ = \lim_{i\to\infty} \Phi_{(n)}\big(\vartheta_i \cdot (x-\lambda)|x-\lambda|^{n-1}\big)\Phi_{(n)}\big(\vartheta_i \cdot |x-\lambda|^{-n-1}\overline{(x-\lambda)}\big)\\ = \lim_{i\to\infty} \Phi_{(n)}(\vartheta_i^2) = I.\end{aligned}$$

Analogously, $\lim_{i\to\infty} \Phi_{(n)}(\vartheta_i \cdot |x-\lambda|^{-n-1}\overline{(x-\lambda)}) \cdot (T-\lambda I)|T-\lambda I|^{n-1} = I$, implying that $T-\lambda I$ is invertible, that is, $\lambda \notin \sigma(T)$.

For the opposite direction, suppose f is in $\mathfrak{D}$ and satisfies $\operatorname{supp} f \cap \sigma(T) = \varnothing$. Let

$$F(x) = \begin{cases} (T-xI)^{-1}|T-xI|^{-n+1}\Phi_{(n)}(f) & \text{for } x \notin \sigma(T) \\ \Phi_{(n)}\big(f(t)\cdot|t-x|^{-n-1}\overline{(t-x)}\big) & \text{for } x \notin \operatorname{supp} f.\end{cases}$$

Since $\operatorname{supp} f \cap \sigma(T) = \varnothing$, it follows that the function F is defined for all x. If $x \notin \sigma(T)$, $x \notin \operatorname{supp} f$, and ϑ_i is an element of $\mathfrak{D}$ such that $\vartheta_i \equiv 1$ on $\operatorname{supp} f$ and ϑ_i tends to 1 increasingly and uniformly on compacts, then

$$\begin{aligned}(T-xI)|T-xI|^{n-1} \cdot \Phi_{(n)}\big(f(t)\cdot|t-x|^{-n-1}\overline{(t-x)}\big)\\ = \lim_{i\to\infty} \Phi_{(n)}\big(\vartheta_i \cdot (t-x)|t-x|^{n-1}\big) \cdot \Phi_{(n)}\big(f(t)\cdot|t-x|^{-n-1}\overline{(t-x)}\big)\\ = \lim_{i\to\infty} \Phi_{(n)}(\vartheta_i f) = \Phi_{(n)}(f),\end{aligned}$$

implying F is well-defined.

Taking into account [McIntosh and Pryde 1987, Examples 5.3 and 5.4], we see the function F is an entire monogenic function. Since $f(t)\cdot|t-x|^{-n-1}\overline{(t-x)} \rightrightarrows 0$ as $|x| \to \infty$, it follows that

$$\Phi_{(n)}\big(f(t)\cdot|t-x|^{-n-1}\overline{(t-x)}\big) \xrightarrow{w} 0 \quad \text{in } B(X)_{(n)},$$

so the range of $\Phi_{(n)}$ is weakly bounded. But in locally convex space every weakly bounded set is originally bounded (see [Rudin 1973, Theorem 3.18] — actually, this theorem is consequence of the Banach–Alaoglu theorem, the Hahn–Banach theorem and the Banach–Steinhaus theorem). So the $\Phi_{(n)}(f(t)\cdot|t-x|^{-n-1}\overline{(t-x)})$ are norm bounded. By Liouville's theorem for monogenic functions (see [McIntosh and Pryde 1987, Theorem 5.1]), F is a constant function. However, a constant sequence of operators tending weakly to 0 must be identically equal to zero, that is, $F \equiv 0$. Hence $\Phi_{(n)}(f) = 0$, that is, $\operatorname{supp}\Phi_{(n)} \subseteq \sigma(T)$. □

Remark 5.10. McIntosh, Pryde and Ricker [McIntosh et al. 1988] proved that for the k-tuple of commuting bounded generalized scalar operators, the spectrum

$\sigma(T)$ coincides with many known joint spectra of the $(T_1, \dots, T_k)$, for instance Harte and Taylor spectrum. An interesting question for further research is to find whether this statement holds for a k-tuple of unbounded operators.

6. Comparison of different definitions of functional calculi

We defined the notion of weak and ultraweak functional calculus. There are at least two other definitions of functional calculi.

Definition 6.1 [deLaubenfels 1995]. Let T be an unbounded operator on a Banach space X and let $\mathcal{F}$ be an algebra of functions. Also, let $f_1(x) = x$ be the identical function, let $f_0(x) \equiv 1$, and let $g_\lambda(z)$ denote the mapping $z \mapsto (z-\lambda)^{-1}$. The mapping $\mathcal{F} \ni f \mapsto f(T) \in B(X)$ is a *$\mathcal{F}$-functional calculus* if it is a homomorphism and if the following conditions are satisfied:

(i) $f(T)T \subseteq Tf(T) = (f \cdot f_1)(T)$ whenever f and $f \cdot f_1$ belong to $\mathcal{F}$.

(ii) $f_0(T) = I_X$ whenever $f_0 \in \mathcal{F}$.

(iii) $T - \lambda I$ is injective and $(g_\lambda^k)(T) = (T - \lambda I)^{-k}$ whenever $g_\lambda^k \in \mathcal{F}$.

Definition 6.2 [Andersson et al. 2006]. We call a continuous multiplicative mapping $A\colon \mathfrak{D}(\mathbb{R}^n) \to B(X)$ is a *hyperoperator* on $\mathbb{R}^n$, writing $A \in H_{\mathfrak{D}(\mathbb{R}^n)}(X)$, if

(i) $D_A = \bigcup \operatorname{Im} A(\phi)$ is dense in X, and

(ii) $N = \bigcap \operatorname{Ker} A(\phi) = \{0\}$.

Definition 6.3 [Andersson et al. 2006]. Let $c = (c, D)$ be a linear operator mapping the dense subspace D of X into itself. Moreover, assume that c is closable and that there is a linear and multiplicative mapping $\mathscr{E}(\mathbb{R}^n) \to \text{Ł}(D)$ that extends the trivial one on polynomials, and such that $h_k(c)x \to h(c)x$ for $x \in D$ if $h_k \to h$ in $\mathscr{E}(\mathbb{R}^n)$. Then we say that c, or rather (c, D), is a *weak hyperoperator*.

In the preceding definition, $\mathscr{E}(\mathbb{R}^n)$ is the algebra of all differentiable functions on $\mathbb{R}^n$, which possess an asymptotic expansion $f(x) \sim \sum_{k=0}^\infty a_k x^{-k}$ (when $x \to \infty$), with $a_k \in \mathbb{C}$, in the sense that for every $N \in \mathbb{N}$

$$f(x) = \sum_{k=0}^{N} a_k x^{-k} + x^{-N-1} r_{N+1}(x) \quad \text{for } |x| > 1,$$

where $r_{N+1}(x)$ is bounded with all its derivatives (for more details see [Andersson and Sjöstrand 2004]) and Ł(D) denotes the set (not the space!) of all closable operators on D.

Proposition 6.4. *Any ultraweak functional calculus is an $\mathscr{A}$-functional calculus in the sense of Definition 6.1.*

Proof. (i) Let $x \in \mathcal{D}(T)$ and let $\chi_n = \varphi(x/n)$, where ϕ is a test function equal to 1 in some neighborhood of zero and satisfies $0 \leqslant \varphi \leqslant 1$. By the definition of the ultraweak functional calculus, we have $\mathcal{D} \ni ff_1\chi_n \to ff_1$ in the topology of $\mathcal{A}$. Then $(ff_1\chi_n)(T) \to ff_1(T)$ weakly. So, for x in the domain of T we have

$$f(T)Tx = w - \lim f(T)(f_1\chi_n)(T)x = w - \lim(ff_1\chi_n)(T)x = (ff_1)(T)x,$$

implying $f(T)T \subseteq (ff_1)(T)$. On the other hand, for any x such that $f(T)x$ belongs to the domain of T, we have

$$(ff_1)(T)x = w - \lim(f_1\chi_n)(T)f(T)x = Tf(T)x.$$

The set of such vectors x contains the set E from Proposition 4.3, and therefore it is dense in X. By continuity, we get $Tf(T) = (ff_1)(T)$.

(ii) Let $f_0 \in \mathcal{A}$. Choose a sequence $\chi_n \in \mathcal{D}$ tending to f_0 in the topology of $\mathcal{A}$. (Note that the definition of the weak functional calculus contains the condition that $\mathcal{D}$ is dense in $\mathcal{A}$ in the topology of $\mathcal{A}$.) Then $f_0(T)x = w - \lim \chi_n(T)x = x$ by the definition of the ultraweak functional calculus.

(iii) Let $g_\lambda^k \in \mathcal{A}$. We have $(z-\lambda)^k g_\lambda(z) = 1$. Let $\chi_n \in \mathcal{D}$ be an exhausting sequence. By the definition of the ultraweak functional calculus and by the condition (3) of Definition 2.1 (applied to the polynomial $t \mapsto t$), we have

$$x = w - \lim(\chi_n(f_1-\lambda)^k g_\lambda^k)(T)x = (T-\lambda I)^k g_\lambda(T)^k x = g_\lambda^k(T)(T-\lambda I)^k x. \quad \square$$

Proposition 6.5. *The restriction of any weak functional calculus to* $\mathcal{D}$ *is a hyperoperator.*

Proof. We only have to prove the conditions (i) and (ii) from Definition 6.2. But this easily follows from condition (3) of Definition 2.1 applied to the polynomial $f_0(x) \equiv 1$. $\square$

Remark 6.6. We think that an interesting question for further investigation is to find the relationship between an ultraweak functional calculus and the notion of weak hyperoperator.

7. On commutativity conditions

For an n-tuple of unbounded operators, it is difficult to find a commutativity condition that ensures a reasonable theory. Any of the following four properties can be used to define of commutativity of two unbounded operators on a Banach space:

(P1) There exists a dense space $\mathcal{D} \subseteq X$ such that $TSx = STx$ for any $x \in \mathcal{D}$.

(P2) Suppose that there exists a $\mathcal{D}(\mathbb{R})$ functional calculus for both T and S. For any $f, g \in \mathcal{D}(\mathbb{R})$, we have $f(T)g(S) = g(S)f(T)$.

(P3) Suppose that T and S are generators of strongly continuous groups e^{itT} and e^{itS}, respectively. For any $t, s \in \mathbb{R}$, we have $e^{itT}e^{isS} = e^{isS}e^{itT}$.

(P4) For any $\lambda \in \rho(T)$ and $\mu \in \rho(S)$, we have

$$(T - \lambda I)^{-1}(S - \mu I)^{-1} = (S - \mu I)^{-1}(T - \lambda I)^{-1}.$$

In the general case all of these properties except (P1) might be meaningless, since an unbounded operator need not have a $\mathfrak{D}(\mathbb{R})$ functional calculus, a solution of the corresponding abstract Cauchy problem, or a nonempty resolvent set.

However, if S and T are generalized scalar operators, (P1)–(P4) make sense and are mutually equivalent.

Proposition 7.1. (P1) *implies* (P4).

Proof. Obvious. □

Proposition 7.2. (P3) *implies* (P1) *if S and T are generalized scalar operators.*

Proof. For such S and T, we constructed the ultraweak functional calculus and extended it to $F_{\alpha,\mathrm{loc}}$. The function $f(x, y) = xy$ belongs to $F_{\alpha,loc}$, so the operator $f(T, S)$ is defined on the linear span of E (defined in Proposition 4.3). Examining the proof of Theorem 4.2, we see that on the set E both products TS and ST are equal to $f(T, S)$. □

Proposition 7.3. (P3) *implies* (P2) *if S and T are generalized scalar operators.*

Proof. For the pair (S, T), we constructed a functional calculus for a wider algebra than $\mathfrak{D}(\mathbb{R}^2)$, and choosing a function of the form $f(x)g(y)$, we obtain the result. □

Proposition 7.4. (P4) *implies* (P3) *if S and T are generalized scalar operators.*

Proof. S and T are generalized scalar operators that are generators of strongly continuous groups, so this statement can be derived following the standard proof of the Hille–Yoshida theorem; see [Dunford and Schwartz 1958, Theorem VIII.1.13]. The spectrum of iT is on the imaginary line, so $\lambda \in \rho(iT)$ for all $\lambda > 0$. It follows that $B_\lambda = -i\lambda[I - \lambda(\lambda I - iT)^{-1}]$ (for $\lambda > 0$) are bounded. Also, $iTx = \lim_{\lambda\to\infty} B_\lambda x$ for all $x \in \mathfrak{D}(iT) = \mathfrak{D}(T)$ and $\lim_{\lambda\to\infty} e^{tB_\lambda}x$ exists for all $x \in \mathfrak{D}(iT) = \mathfrak{D}(T)$ (of course, the value of this limit is $e^{itT}x$). Analogously, if $C_\lambda = -i\lambda[I - \lambda(\lambda I - iC)^{-1}]$ (for $\lambda > 0$), we have $iSx = \lim_{\lambda\to\infty} C_\lambda x$ for all $x \in \mathfrak{D}(iS) = \mathfrak{D}(S)$ and $\lim_{\lambda\to\infty} e^{sC_\lambda}x$ exists for all $x \in \mathfrak{D}(iS) = \mathfrak{D}(S)$.

Finally, from (P4) (the commutativity of resolvents) we have the commutativity of B_λ and C_λ (for all $\lambda > 0$), and, passing to $\lambda \to \infty$, we get (P3). □

Proposition 7.5. (P2) *implies* (P4) *if S and T are generalized scalar operators.*

Proof. The function $g_\lambda(x) = 1/(x - \lambda)$ belongs to F_α for all α and for $\operatorname{Im}\lambda \neq 0$. Indeed, it is easy to check that g_λ is the inverse Fourier transform of the function

$F(x) = i\sqrt{2\pi}e^{-i\lambda x}H(-x)$ for $\operatorname{Im}\lambda > 0$. Similarly, it is the inverse Fourier transform of the function $F(x) = -i\sqrt{2\pi}e^{-i\lambda x}H(x)$ for $\operatorname{Im}\lambda < 0$. Here $H(x)$ is the Heaviside function, equal to zero if $x < 0$ and to 1 otherwise. Hence $g_\lambda \in F_\alpha$, and $\|g_\lambda\|_{F_\alpha} = \sqrt{2\pi}(1/|\operatorname{Im}\lambda| + \Gamma(\alpha+1)/|\operatorname{Im}\lambda|^\alpha)$.

If $\varphi_n(x) = \varphi(x/n)$, where φ is a test function valued in [0, 1] and equal to 1 in a neighborhood of zero, then the sequence of test functions $\psi_{n,\lambda} = \varphi_n \cdot g_\lambda \to g_\lambda$ as $n \to +\infty$ in the topology τ defined in Definition 3.1, and therefore $\psi_{n,\lambda}(S) \to g_\lambda(S) = (S-\lambda I)^{-1}$ weakly, by Proposition 6.4. Similarly $\psi_{m,\mu}(T) \to (T-\mu I)^{-1}$ weakly. From property (P2) we have $\psi_{n,\lambda}(S)\psi_{m,\mu}(T) = \psi_{m,\mu}(T)\psi_{n,\lambda}(S)$. Taking a limit as $n \to +\infty$ and $m \to +\infty$, we derive (P4). □

8. Examples and remarks

Example 8.1. Example 2.8 can be generalized using examples from [Gorin and Karahanjan 1977] and techniques from [Evgrafov 1979].

There is an operator for which (2-1) does not hold, whose asymptotic behavior satisfies $e^{itA} \sim \rho e^{|t|^\rho}$ as $|t| \to \infty$ for every $\rho > 1/4$.

Let E be a vector space of entire functions, such that there exists $C \in \mathbb{R}$ such that $|f(z)| \leqslant Ce^{\sigma|z|^\rho}$. Like in Example 2.8, E is a Banach space if the norm $\|f\|$ of a function f is defined as the smallest constant C for which the inequality above holds. Since

$$f'(z) = \frac{1}{2\pi i}\int_{|\xi|=1} \frac{f(\xi+z)}{\xi^2}\,d\xi,$$

it follows that the operator $A = -i\,d/dz$ is a bounded linear operator on E; see [Gorin and Karahanjan 1977]. Hence

$$(e^{itA}f)(z) = (e^{td/dz}f)(z) = f(z+t).$$

It is easy to see that $\|e^{itA}\| = O(e^{|t|^\rho})$ as $|t| \to \infty$.

To see exact asymptotic behavior, we can use a function

$$f_0(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n/\rho+1)}$$

(Γ is the gamma function). Using that $f_0(x) \sim \rho e^{x^\rho}$ when $x \to \infty$ for $\rho > 1/4$ (see [Evgrafov 1979, Example 1.5]), we derive our statement.

Example 8.2. Consider the operator $A = -i\,d/dx$ acting on the dense subspace of $L^p(\mathbb{R})$ consisting of all absolutely continuous functions whose derivative also belongs to $L^p(\mathbb{R})$. It is easy to see that $e^{itA}f = f(x+t)$, and we can conclude $\|e^{itA}\| = 1$. Hence, for the operator A, we have the F_0 functional calculus.

Example 8.3. Let H be a Hilbert space and let A be an unbounded selfadjoint operator. Also, let Q be a nilpotent operator of order n, that is, $Q^n = 0$, and $Q^{n-1} \neq 0$. Let Q commute with A. Such a Q is given by the matrix

$$Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

with respect to the root subspace of some eigenvalue of A. Then we have

$$\|e^{it(A+Q)}\| = \|e^{itA}e^{itQ}\| = \|e^{itQ}\| = O(|t|^{n-1}).$$

Therefore, for the operator $A+Q$, we have the F_{n-1} functional calculus.

Example 8.4. The advection operator

$$f(x,t) \mapsto a \cdot \nabla f(x,t)$$

(here $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $x \in \mathbb{R}^k$ and $t \in \mathbb{R}$) is well known in the mechanics of fluids. It is a generalized scalar operator in which $f(x,t)$ solves the abstract Cauchy problem $\partial f(x,t)/\partial t = a \cdot \nabla f(x,t)$. This unbounded operator commutes with the Laplacian. In fact, it commute with all $\partial/\partial x_j$. So we can construct an ultraweak functional calculus for these operators and finally we can get any linear (even some nonlinear) combinations of the advection operator and the Laplacian.

Finally, we shall see that the functional calculus we established contains, in a sense, the widest algebra of functions. We deal with the one dimensional case.

Theorem 8.5. *The operator A has the F_α functional calculus if and only if*

$$\|e^{itA}\| \leqslant M(1+|t|^\alpha) \quad \text{for some } M \in \mathbb{R}.$$

Proof. If there is $M \in \mathbb{R}$ such that $\|e^{itA}\| \leqslant M(1+|t|^\alpha)$, then by Section 1, A has the F_α functional calculus.

In the opposite direction, since $d^\alpha e^{itx}/dx^\alpha = (it)^\alpha e^{itx}$, it follows that

$$\|e^{itx}\|_\alpha = \sup_{x\in\mathbb{R}} |e^{itx}| + \sup_{x\in\mathbb{R}} \left|\frac{d^\alpha}{dx^\alpha} e^{itx}\right| \leqslant 1 + |t|^\alpha,$$

and, from the existence of the functional calculus, it follows that

$$\|\Phi(e^{itx})\| \leqslant M\|e^{itx}\| \leqslant M(1+|t|^\alpha).$$

□

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