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By using the Avery–Peterson fixed point theorem, we obtain the existence of three positive solutions for a third order multipoint boundary value problem. An example illustrates the main results.

1. Introduction

We study the existence of positive solutions for the third order m-point boundary value problem

$$
\begin{aligned}
&x'''(t)+f(t,x(t),x'(t),x''(t))=0 \quad \text{for } t\in[0,1],\\
&x(1)=\sum_{i=1}^{m-2}\beta_i x(\xi_i), \quad x'(0)=\sum_{i=1}^{m-2}\alpha_i x'(\xi_i),\\
&x''(0)=0,
\end{aligned}
\tag{1-1}
$$

where $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$,

$$
\begin{aligned}
&0\le\alpha_i<1 \quad\text{and}\quad 0\le\beta_i<1 \quad \text{for } i=1,2,\dots,m-2,\\
&\sum_{i=1}^{m-2}\alpha_i<1 \quad\text{and}\quad \sum_{i=1}^{m-2}\beta_i<1
\end{aligned}
$$

and $f\in C([0,1]\times[0,+\infty)\times R^2,[0,+\infty))$.

Third order differential equations arise in various areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three layer beams, electromagnetic waves, gravity driven flows, and so on [Greguš 1987]. In recent years, much attention has focused on positive solutions of boundary value problems (BVPs for short) for third order ordinary

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differential equations. For example, Anderson [1998] established the existence of at least three positive solutions to the problem

$$\begin{aligned} -x'''(t) + f(x(t)) = 0 \quad & \text{for } t \in (0, 1), \\ x(0) = x'(t_2) = x''(1) = 0 \quad & \text{for } t_2 \in (0, 1), \end{aligned}$$

where $f : R \to [0, +\infty)$ is continuous and $1/2 \le t_2 < 1$. Palamides and Smyrlis [2008] proved that there exists at least one positive solution for the third order three-point BVP

$$\begin{aligned} & x'''(t) = a(t) f(t, x(t)) && \text{for } t \in (0, 1), \\ & x(0) = x(1) = 0, \\ & x''(\eta) = 0 && \text{for } \eta \in (0, 1). \end{aligned}$$

The results are based on the well-known Guo–Krasnoselskiĭ fixed point theorem [Guo and Lakshmikantham 1988]. Guo, Sun and Zhao [Guo et al. 2008] studied the positive solutions of the third order three-point problem

$$\begin{aligned} & x'''(t) = a(t) f(x(t)) && \text{for } t \in (0, 1), \\ & x(0) = x'(0) = 0, \\ & x'(1) = x'(\eta) && \text{for } \eta \in (0, 1), \end{aligned}$$

and obtained the existence of such solutions by using the Guo–Krasnoselskiĭ fixed point theorem. For more existence results for third order boundary value problems, see [Hopkins and Kosmatov 2007; Chu and Zhou 2006; Graef and Kong 2009; Lin et al. 2008; Pei and Chang 2007; Li 2006; Yao 2004] and references therein.

In these works they concentrate on the two- or three-point BVPs. Few papers deal with the existence of positive solutions to m-point BVPs for third order differential equations, and in those that do, first and second order derivatives are not involved in the nonlinear term.

We do allow first and second order derivatives to appear explicitly in the nonlinear term of multipoint third order boundary value problems (1-1), and consider the positive solutions of these problems. By using the Avery–Peterson fixed point theorem [2001] and analysis techniques, we prove that there exist at least three concave positive solutions of problem (1-1). The results so established are more general those of previous papers. We illustrate our results with an example.

2. Background

In this section, we present the necessary definitions from cone theory in Banach spaces and a fixed point theorem due to Avery and Peterson.

Definition 2.1. Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P \subset E$ is said to be a cone if

(1) $au \in P$ for all $u \in P$ and $a \geq 0$, and

(2) $u, -u \in P$ implies $u = 0$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a nonnegative continuous convex functional on a cone P of a real Banach space E if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y) \quad \text{for all } x, y \in P,\ t \in [0, 1].$$

Definition 2.4. A map β is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y) \quad \text{for all } x, y \in P,\ t \in [0, 1].$$

Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P and let ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c and d, we define the convex sets

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P \mid b \leq \alpha(x),\ \gamma(x) \leq d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \leq \psi(x), \gamma(x) \leq d\}.$$

Lemma 2.5. *Let P be a cone in a Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying*

$$\psi(\lambda x) \leq \lambda \psi(x) \quad \textit{for } 0 \leq \lambda \leq 1, \tag{2-1}$$

such that for some positive numbers l and d,

$$\alpha(x) \leq \psi(x) \quad \textit{and} \quad \|x\| \leq l\gamma(x) \quad \textit{for all } x \in \overline{P(\gamma, d)}. \tag{2-2}$$

Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$ such that

(S_1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \varnothing$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$,

(S_2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$,

(S_3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\begin{aligned} &\gamma(x_i) \le d \quad \text{for } i = 1, 2, 3, \\ &b < \alpha(x_1), \quad a < \psi(x_2), \quad \alpha(x_2) < b, \quad \psi(x_3) < a. \end{aligned} \tag{2-3}$$

3. Main results

Consider the problem

$$x'''(t) + y(t) = 0 \quad \text{for } t \in [0, 1], \tag{3-1a}$$

$$x''(0) = 0, \quad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \tag{3-1b}$$

Lemma 3.1. *Let*

$$\begin{aligned} &\xi_0 = 0, \quad \xi_{m-1} = 1, \quad \alpha_0 = \alpha_{m-1} = \beta_0 = \beta_{m-1} = 0, \\ &\rho = (1 - \textstyle\sum_{i=0}^{m-1} \alpha_i)(1 - \sum_{i=0}^{m-1} \beta_i) > 0. \end{aligned}$$

For $y(t) \in C[0, 1]$, the problem (3-1) has the unique solution

$$x(t) = \int_0^1 G(t, s) \int_0^s y(\tau) d\tau ds,$$

where

$$G(t, s) = \frac{1}{\rho} \begin{cases} \sum_{k=i}^{m-1} \alpha_k \big((s - t) + \sum_{k=i}^{m-1} \beta_k (t - s) + \sum_{k=0}^{i-1} \beta_k (t - \xi_k)\big) \\ \qquad + (1 - \sum_{k=0}^{i-1} \alpha_k)\big((1 - s) + \sum_{k=i}^{m-1} \beta_k (s - \xi_k)\big) & \text{if } t \le s, \\ (1 - \sum_{k=0}^{i-1} \alpha_k)\big((1 - t) + \sum_{k=0}^{i-1} \beta_k (t - s) + \sum_{k=i}^{m-1} \beta_k (t - \xi_k)\big) \\ \qquad + \sum_{k=i}^{m-1} \alpha_k \sum_{k=0}^{i-1} \beta_k (s - \xi_k) & \text{if } t \ge s, \end{cases}$$

for $\xi_{i-1} < s < \xi_i$ and $i = 1, 2, \ldots, m - 1$.

Proof. Integrating both sides of (3-1a) and considering the boundary condition $x''(0) = 0$, we have

$$-x''(t) = \int_0^t y(s) ds. \tag{3-2}$$

Let $G(t, s)$ be the Green function for the problem

$$-x''(t) = 0, \tag{3-3a}$$

$$x'(0) = \sum_{i=0}^{m-1} \alpha_i x'(\xi_i), \quad x(1) = \sum_{i=0}^{m-1} \beta_i x(\xi_i), \tag{3-3b}$$

Then for $\xi_{i-1} < s < \xi_i$ with $i = 1, 2 \dots, m-1$, we can assume

$$G(t,s) = \begin{cases} A + Bt & \text{if } t \le s, \\ C + Dt & \text{if } t \ge s. \end{cases}$$

From the definition and properties of the Green function together with (3-3b), we have

$$A + Bs = C + Ds, \qquad B = \sum_{k=0}^{i-1} \alpha_k B + \sum_{k=i}^{m-1} \alpha_k D,$$

$$B - D = 1, \qquad C + D = \sum_{k=0}^{i-1} \beta_k (A + B\xi_k) + \sum_{k=i}^{m-1} \beta_k (C + D\xi_k),$$

Hence,

$$A = \frac{1}{\rho}\Big(1 - \sum_{k=i}^{m-1} \alpha_k \Big(\sum_{k=0}^{m} \beta_k \xi_k - 1\Big) + \Big(1 - \sum_{k=0}^{m-1} \alpha_k\Big)\Big(1 - s + \sum_{k=i}^{m-1} \beta_k (s - \xi_k)\Big)\Big),$$

$$B = -\sum_{k=i}^{m-1} \alpha_k \Big/ \Big(1 - \sum_{k=0}^{m-1} \alpha_k\Big),$$

$$C = \frac{1}{\rho}\Big(-\Big(1 - \sum_{k=0}^{i-1} \alpha_k\Big)\Big(\sum_{k=0}^{i-1} \beta_k s + \sum_{k=i}^{m-1} \beta_k \xi_k - 1\Big) + \sum_{k=i}^{m-1} \alpha_k \sum_{k=0}^{i-1} \beta_k (s - \xi_k)\Big),$$

$$D = \Big(\sum_{k=0}^{i-1} \alpha_k - 1\Big) \Big/ \Big(1 - \sum_{k=0}^{m-1} \alpha_k\Big).$$

This gives the Green function explicitly. Considering (3-2) together, we obtain that (3-1) has the unique solution stated. □

Lemma 3.2. *The Green function above satisfies $G(t,s) \ge 0$ for $t, s \in [0,1]$.*

Proof. For $\xi_{i-1} \le s \le \xi_i$, with $i = 1, 2, \dots, m-1$, and $t \le s$,

$$(s - t) + \sum_{k=0}^{i-1} \beta_k (t - \xi_k) + \sum_{k=i}^{m-1} \beta_k (t - s)$$
$$\ge \sum_{k=0}^{m-1} \beta_k (s - t) + \sum_{k=0}^{i-1} \beta_k (t - \xi_k) + \sum_{k=i}^{m-1} \beta_k (t - s) = \sum_{k=0}^{i-1} \beta_k (s - \xi_k) \ge 0,$$

and

$$(1 - s) + \sum_{k=i}^{m-1} \beta_k (s - \xi_k) \ge \sum_{k=i}^{m-1} \beta_k (1 - \xi_k) \ge 0.$$

For $\xi_{i-1} \le s \le \xi_i$, with $i = 1, 2, \ldots, m-1$, and $t \ge s$,

$$(1-t) + \sum_{k=0}^{i-1} \beta_k (t-s) + \sum_{k=i}^{m-1} \beta_k (t-\xi_k) \ge \sum_{k=0}^{i-1} \beta_k (1-s) + \sum_{k=i}^{m-1} \beta_k (1-\xi_k) \ge 0.$$

These give that $G(t, s) \ge 0$ for $t, s \in [0, 1]$. □

Lemma 3.3. *If $y(t) \ge 0$ for $t \in [0, 1]$ and $u(t)$ is the solution of (3-1), then*

(1) $\min_{0\le t\le 1}|x(t)| \ge \delta \max_{0\le t\le 1}|x(t)|$ *and*

(2) $\max_{0\le t\le 1}|x(t)| \le \gamma_1 \max_{0\le t\le 1}|x'(t)|$, *where*

$$\delta = \Big(\sum_{i=1}^{m-2} \beta_i (1-\xi_i)\Big) \Big/ \Big(1 - \sum_{i=1}^{m-2} \beta_i \xi_i\Big) \quad \text{and} \quad \gamma_1 = \Big(1 - \sum_{i=1}^{m-2} \beta_i \xi_i\Big) \Big/ \Big(1 - \sum_{i=1}^{m-2} \beta_i\Big)$$

are constants.

Proof. (1) For $x'''(t) = -y(t) \le 0$ with $t \in [0, 1]$, we see that $x''(t)$ is decreasing on $[0, 1]$. Considering $x''(0) = 0$, we have $x''(t) \le 0$ for $t \in (0, 1)$. Next we claim that $x'(0) \le 0$. Otherwise, if $x'(0) > 0$, we have the contradiction

$$0 = x'(0) - x'(0) = x'(0) - \sum_{i=1}^{m-2} \alpha_i x'(\xi_i) > \sum_{i=1}^{m-2} \alpha_i (x'(0) - x'(\xi_i)) \ge 0.$$

Thus, $\max_{0\le t\le 1} x(t) = x(0)$ and $\min_{0\le t\le 1} x(t) = x(1)$. From the concavity of $x(t)$, we have

$$\xi_i (x(1) - x(0)) \le x(\xi_i) - x(0).$$

Multiplying both sides by β_i and considering $x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)$, we have

$$\Big(1 - \sum_{i=1}^{m-2} \beta_i \xi_i\Big) x(1) \ge \sum_{i=1}^{m-2} \beta_i (1-\xi_i) x(0). \tag{3-4}$$

(2) By the mean value theorem, we obtain

$$x(1) - x(\xi_i) = (1-\xi_i) x'(\eta_i) \quad \text{for } \eta \in (\xi_i, 1).$$

From the concavity of x, similar to what we did above, we see that

$$\Big(1 - \sum_{i=1}^{m-2} \beta_i\Big) x(1) < \sum_{i=1}^{m-2} \beta_i (1-\xi_i) |x'(1)|. \tag{3-5}$$

Considering (3-4) together with (3-5) we have $x(0) \le \gamma_1 |x'(1)|$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *If $y \in C[0, 1]$ for $y \ge 0$ and $\gamma_2 = 1 + (\sum_{i=0}^{m-1} \alpha_i \xi_i)/(1 - \sum_{i=0}^{m-1} \alpha_i)$ is a positive constant, then* $\max_{0\le t\le 1}|x'(t)| \le \gamma_2 \max_{0\le t\le 1}|x''(t)|$.

Proof. Since $x'(t) = x'(0) + \int_0^t x''(s)ds$, we have

$$\Big(1 - \sum_{i=0}^{m-1} \alpha_i\Big)x'(0) = \sum_{i=0}^{m-1} \alpha_i \int_0^{\xi_i} x''(s)ds \geq \sum_{i=0}^{m-1} \alpha_i \xi_i x''(1).$$

Thus

$$\Big(1 - \sum_{i=0}^{m-1} \alpha_i\Big)|x'(0)| \leq \sum_{i=0}^{m-1} \alpha_i \xi_i |x''(1)|.$$

Considering the concavity of $x(t)$ and $x'(t)$, we have

$$\max_{0\leq t\leq 1} |x'(t)| = |x'(1)| \quad \text{and} \quad \max_{0\leq t\leq 1} |x''(t)| = |x''(1)|,$$

and $x'(1) - x'(0) = x''(\eta) \geq x''(1)$ which give that

$$|x'(1)| \leq \left(1 + \frac{\sum_{i=0}^{m-1} \alpha_i \xi_i}{1 - \sum_{i=0}^{m-1} \alpha_i}\right)|x''(1)|. \qquad \square$$

Remark. Lemmas 3.3 and 3.4 ensure that

$$\max\{\max_{0\leq t\leq 1} |x(t)|, \quad \max_{0\leq t\leq 1} |x'(t)|, \quad \max_{0\leq t\leq 1} |x''(t)|\} \leq \gamma_3 \max_{0\leq t\leq 1} |x''(t)|,$$

where $\gamma_3 = \gamma_1\gamma_2 > 1$.

Let the Banach space $E = C^2[0, 1]$ be endowed with the norm

$$\|x\| = \max\Big\{\max_{0\leq t\leq 1} |x(t)|, \max_{0\leq t\leq 1} |x'(t)|, \max_{0\leq t\leq 1} |x''(t)|\Big\} \quad \text{for } x \in E.$$

We define the cone $P \subset E$ by

$$P = \Big\{x \in E \;\Big|\; x(t) \geq 0,\ x''(0) = 0,\ x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i),\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i),$$
$$x(t) \text{ is concave on } [0, 1]\Big\}.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ and θ, and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(x) = \max_{0\leq t\leq 1} |x''(t)|, \quad \theta(x) = \psi(x) = \max_{0\leq t\leq 1} |x(t)|, \quad \alpha(x) = \min_{0\leq t\leq 1} |x(t)|.$$

By Lemmas 3.3 and 3.4, the functionals defined above satisfy

$$\delta\theta(x) \leq \alpha(x) \leq \theta(x) = \psi(x) \quad \text{for } \|x\| \leq \gamma_3\gamma(x).$$

Therefore condition (2-2) of Lemma 2.5 is satisfied. Let

$$m = \int_0^1 sG(1,s)ds, \quad N = \int_0^1 sG(0,s)ds, \quad \lambda = \min\{m, \delta\gamma_3\}.$$

Assume that there exist constants $0 < a, b, d$ with $a < b < \lambda d$ such that

(A_1) $f(t,u,v,w) \le d$ for $(t,u,v) \in [0,1] \times [0,\gamma_3 d] \times [-\gamma_2 d, 0] \times [-d, 0]$;

(A_2) $f(t,u,v,w) > b/m$ for $(t,u,v) \in [0,1] \times [b, b/\delta] \times [-\gamma_2 d, 0] \times [-d, 0]$;

(A_3) $f(t,u,v,w) < a/N$ for $(t,u,v) \in [0,1] \times [0,a] \times [-\gamma_2 d, 0] \times [-d, 0]$.

Theorem 3.5. *Under assumptions (A_1)–(A_3), problem (1-1) has at least three positive solutions x_1, x_2, x_3 satisfying $\max_{0\le t\le 1}|x_i''(t)| \le d$ for $i = 1, 2, 3$, where*

$$\min_{0\le t\le 1}|x_2(t)| < b < \min_{0\le t\le 1}|x_1(t)| \quad \textit{and} \quad \max_{0\le t\le 1}|x_3(t)| \le a < \max_{0\le t\le 1}|x_2(t)|.$$

Proof. Problem (1-1) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = \int_0^1 G(t,s) \int_0^s f(\tau, x(\tau), x'(\tau), x''(\tau))d\tau ds = (Tx)(t).$$

By a simple computation, we have

$$(Tx)''(t) = -\int_0^t f(s, x, x', x'')ds.$$

For $x \in \overline{P(\gamma, d)}$, we have $\gamma(x) = \max_{0\le t\le 1}|x''(t)| \le d$. Considering Lemmas 3.3 and 3.4 and assumption (A_1), we obtain

$$f(t, x(t), x'(t), x''(\tau)) \le d,$$
$$\gamma(Tx) = |(Tx)''(1)| = \left| -\int_0^1 f(s, x, x', x'')ds \right| \le d.$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ and clearly T is a completely continuous operator.

The fact that the constant function $x(t) = b/\delta$ is in $P(\gamma, \theta, \alpha, b, c, d)$ and that $\alpha(b/\delta) > b$ implies that $\{x \in P(\gamma, \ \theta, \alpha, b, c, d \mid \alpha(x) > b)\} \ne \varnothing$. This gives that condition (S_1) of Lemma 2.5 holds.

For $x \in P(\gamma, \theta, \alpha, b, c, d)$, we have $b \le x(t) \le b/\delta$ and $|x''(t)| < d$ for $0 \le t \le 1$. From assumption (A_2), we have $f(t, x, x', x'') > b/m$. By definition of α and the cone P, we have

$$\alpha(Tx) = (Tx)(1) = \int_0^1 G(1,s) \int_0^s f(\tau, x, x', x'')d\tau ds$$
$$\ge \frac{b}{m} \int_0^1 sG(1,s)ds > \frac{b}{m} m = b,$$

which means $\alpha(Tx) > b$ for all $x \in P(\gamma, \theta, \alpha, b, b/\delta, d)$. Second, with (3-4) and $b < \lambda d$, we have

$$\alpha(Tx) \geq \delta\theta(Tx) > \delta(b/\delta) = b$$

for all $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > b/\delta$.

Thus, condition (S_2) of Lemma 2.5 holds. Finally we show that (S_3) also holds. We see that $\psi(0) = 0 < a$ and $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then by assumption (A_3),

$$\begin{aligned}\psi(Tx) = \max_{0\leq t\leq 1} |(Tx)(t)| &= \int_0^1 G(0, s) \int_0^s f(\tau, x, x', x'')\, d\tau\, ds \\ &< \frac{a}{N} \int_0^1 sG(0, s)\, ds = a.\end{aligned}$$

Thus, all conditions of Lemma 2.5 are satisfied. Hence (1-1) has at least three positive concave solutions x_1, x_2, x_3 satisfying the conditions of the theorem. □

4. Example

Consider the third order four-point boundary value problem

$$\begin{aligned}&x'''(t) + f(t, x(t), x'(t), x''(t)) = 0 \quad \text{for } t \in [0, 1], \\ &x''(0) = 0, \quad x'(0) = \tfrac{1}{4}x'(\tfrac{1}{3}) + \tfrac{1}{2}x'(\tfrac{2}{3}), \quad x(1) = \tfrac{1}{3}x(\tfrac{1}{3}) + \tfrac{1}{2}x(\tfrac{2}{3}),\end{aligned} \tag{4-1}$$

where

$$f(t, u, v, w) = \frac{1}{60}e^t + \frac{1}{60}\left(\frac{w}{1800}\right)^4 + \begin{cases} \dfrac{u^5}{10\pi} & \text{if } 0 \leq u \leq 10, \\ \dfrac{10000}{\pi} & \text{if } u \geq 10, \end{cases}$$

By a simple computation, the function $G(t, s)$ is given by

$$G(t, s) = \begin{cases} 24(5/9 - t/8 - s/24) & \text{if } 0 \leq s \leq 1/3, \quad t \leq s, \\ 24(5/9 - t/6) & \text{if } 0 \leq s \leq 1/3, \quad t \geq s, \\ 24(4/9 - s/8 - t/12) & \text{if } 1/3 \leq s \leq 2/3, \quad t \leq s, \\ 24(4/9 - t/8 - s/12) & \text{if } 1/3 \leq s \leq 2/3, \quad t \geq s, \\ 6(1 - s) & \text{if } 2/3 \leq s \leq 1, \quad t \leq s, \\ 6(1 - t/6 - 5s/6) & \text{if } 2/3 \leq s \leq 1, \quad t \geq s, \end{cases}$$

Choosing $a = 1$, $b = 4$ and $d = 1800$, we note that

$$\gamma = \frac{80}{9}, \quad \delta = \frac{7}{10}, \quad m = \int_0^1 sG(1, s)ds = \frac{35}{108}, \quad N = \int_0^1 sG(0, s)ds < 10.$$

We can check that $F = f(t, u, v, w)$ satisfies

$$
\begin{array}{ll}
F \leq 1800 & \text{in the region } [0, 1] \times [0, 16000] \times [-4800, 0] \times [-1800, 0], \\
F \geq 432/35 & \text{in the region } [0, 1] \times [4, 40/7] \times [-4800, 0] \times [-1800, 0], \\
F \leq 1/10 & \text{in the region } [0, 1] \times [0, 1] \times [-4800, 0] \times [-1800, 0].
\end{array}
$$

Then all assumptions of Theorem 3.5 are satisfied. Thus, problem (4-1) has at least three positive solutions x_1, x_2, x_3 such that

$$
\begin{gathered}
\max_{0\leq t\leq 1} |x_i'(t)| \leq 1800 \quad \text{for } i = 1, 2, 3, \\
\min_{0\leq t\leq 1} x_1(t) > 4, \qquad \max_{0\leq t\leq 1} x_2(t) > 1, \\
\min_{0\leq t\leq 1} x_2(t) < 4, \qquad \max_{0\leq t\leq 1} x_3(t) < 1.
\end{gathered}
$$

Remark. Problem (4-1) is a third order four-point BVP and the nonlinear term is involved in the first and second order derivative explicitly. Earlier results for positive solutions, to the authors' knowledge, do not apply to problem (4-1).

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