Pacific Journal of Mathematics

HOMOLOGY SEQUENCE AND EXCISION THEOREM FOR EULER CLASS GROUP

YONG YANG

Volume 249 No. 1 January 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 249, No. 1, 2011

HOMOLOGY SEQUENCE AND EXCISION THEOREM FOR EULER CLASS GROUP

YONG YANG

Let R be a Noetherian commutative ring with $\dim R = d$ and let l be an ideal of R. For an integer n such that $2n \geq d+3$, we define a relative Euler class group $E^n(R, l; R)$. Using this group, in analogy to homology sequence of the K_0-group, we construct an exact sequence

$$E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l),$$

called the homology sequence of the Euler class group. The excision theorem in K-theory has a corresponding theorem for the Euler class group. An application is that for polynomial and Laurent polynomial rings, we get short split exact sequences

$$0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and

$$0 \to E^n(R[t, t^{-1}], (t-1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(\rho)} E^n(R; R) \to 0.$$

1. Introduction

Let R be a Noetherian commutative ring of dimension d. The notion of the Euler class group of R was introduced by Nori around 1990, with the aim of developing an obstruction theory for algebraic vector bundles over smooth affine varieties [Mandal 1992]. Later, the Nori's definition was extended by S. M. Bhatwadekar and Raja Sridharan [2000]. Given a Noetherian commutative ring R with dimension $d \geq 2$, they defined an obstruction group $E^d(R; R)$ also called the Euler class group (ECG). For $\mathbb{Q} \subseteq R$ and any projective R-module P of rank d with orientation $\chi : R \cong \bigwedge^d P$, they defined an obstruction class $e(P; \chi) \in E^d(R; R)$ and proved that $P \cong Q \oplus R$ if and only if $e(P; \chi) = 0$. After that, much work

MSC2000: primary 13C10; secondary 13B25.

Keywords: Euler class group, K-theory, excision theorem.

Partially supported by NCET (NCET06-09-23), and NUDT(JC08-02-03) to Professor Lianggui Feng.

on ECGs and weak ECGs was done. K. D. Das [2003; 2006] defined the ECG $E^d(R[t]; R[t])$ for a Noetherian commutative ring R with $\dim R = d$, and proved that for a general such ring, the ECG $E^d(R; R)$ of R is a direct summand of the ECG $E^d(R[t]; R[t])$, whereas if R is a smooth affine domain over some perfect field k, then $E^d(R[t]; R[t]) \cong E^d(R; R)$. The ECG $E^d(R[t, t^{-1}]; R[t, t^{-1}])$ of a Laurent polynomial ring $R[t, t^{-1}]$ was defined in [Keshari 2007].

On the other hand, in K-theory we have a homology theory for the category of projective R-modules. The K_0-group $K_0(R)$ is closely related to the ECG of R. For example, Murthy's Chern class of the projective R-module P, which is one of the sources of Euler class theory, is defined by an element in the K_0-group. For any Noetherian commutative ring R of dimension d, there is a subgroup $F^d K_0(R)$ of $K_0(R)$ [Mandal 1998]. If R is regular and contains the field of rational numbers $\mathbb{Q}$, we have a Riemann–Roch theorem saying that $E_0^d(R) \otimes \mathbb{Q} \cong F^d K_0(R) \otimes \mathbb{Q} \cong CH^d(R) \otimes \mathbb{Q}$, in which $E_0^d(R)$ is the weak ECG of R and $CH^d(R)$ is the Chow group of codimension d of $\operatorname{Spec}(R)$ [Das and Mandal 2006].

Now let $l \subseteq R$ be an ideal of R. K-theory gives for K_0-groups the homology sequence $K_0(R, l) \to K_0(R) \to K_0(R/l)$. If the ring homomorphism $\rho : R \to R/l$ is split, then this homology sequence reduces to the short split exact sequence

$$0 \to K_0(R, l) \longrightarrow K_0(R) \longrightarrow K_0(R/l) \to 0 \ ,$$

which is said to be the *excision sequence* for K_0-groups.

Inspired by the correspondence between K-theory and ECGs, in this paper we establish ECG counterparts to the homology sequence and excision theorem of K_0-groups. These counterparts are Theorem 4.2 and Theorem 4.3, respectively.

Let R be a Noetherian commutative ring with dimension d, and let l be an ideal of R with $\dim R/l = d - m$. For any integer n such that $2n \geq d + 3$, we define in Section 3 a group homomorphism $E(\rho) : E^n(R; R) \to E^n(R/l; R/l)$, called the restriction map of the ECG. In analogy to the relative K_0-group $K_0(R, l)$ (denoted by $K_0(l)$ in [Rosenberg 1994]), we define in Section 4 the relative ECG $E^n(R, l; R)$ and the relative weak ECG $E_0^n(R, l)$. In particular, when $l = R$ the relative ECG $E^n(R, l; R)$ and the relative weak ECG $E_0^n(R, l)$ are the same as the generalized ECG $E^n(R; R)$ and the weak ECG $E_0^n(R)$, respectively. Using these groups, we construct an exact sequence

$$E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l),$$

the homology sequence of the ECG. If the ring homomorphism $\rho : R \to R/l$ has a splitting β satisfying a dimensional condition (see Theorem 4.3 and Remark 4.4), then the homology sequence above reduces to the short split exact sequence

$$0 \to E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \to 0,$$

called the excision sequence of the ECG. Under these conditions, we have an isomorphism $E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R/l; R/l)$. In Section 5, we use the results in Section 4 to get the excision sequences of the ECG for the polynomial extension and the Laurent polynomial extension:

$$0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and

$$0 \to E^n(R[t, t^{-1}], (t-1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(\rho)} E^n(R; R) \to 0.$$

Both of these are split exact; see Corollaries 5.1 and 5.3.

2. Some preliminary results

Here, recall the definition of the generalized ECG and collect some related results.

Definition 2.1. Let R be a Noetherian commutative ring, n be an integer such that $2n \geq d+3$. In [Bhatwadekar and Sridharan 2002], the *generalized Euler class group* $E^n(R; R)$ is defined as follows:

Let $J \subset R$ be an ideal of height n, such that J/J^2 is generated by n elements. Two surjections α and β from $(R/J)^n$ to J/J^2 are said to be related if and only if there exists an elementary matrix $\delta \in \mathscr{E}l_n(R/J)$ such that $\alpha\delta = \beta$. This defines an equivalence relation on the set of surjections from $(R/J)^n$ to J/J^2.

- Let G^n be the free Abelian group on the set of pairs $(J; \omega_J)$, where $J \subseteq R$ is an ideal of height n, having the property that $\mathrm{Spec}(R/J)$ is connected and J/J^2 is generated by n elements, and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ is an equivalence class of surjections.
- Now assume that $J \subseteq R$ be an ideal of height n and J/J^2 is generated by n elements. By [Bhatwadekar and Sridharan 2002, Lemma 4.1], J has a unique decomposition $J = \bigcap_{i=1}^r J_i$ where ideals J_i are pairwise comaximal and $\mathrm{Spec}(R/J_i)$ is connected. Let $\omega_J : (R/J)^n \to J/J^2$ be a surjection. Then ω_J gives rise in a natural way to surjections $\omega_{J_i} : (R/J_i)^n \to J_i/J_i^2$. By $(J; \omega_J)$ we mean the element $\sum_{i=1}^r (J_i; \omega_{J_i})$ in G^n, and $(J; \omega_J)$ is called a local orientation.
- Let H^n be the subgroup of G^n generated by set of pairs $(J; \omega_J)$, where J is an ideal of height n generated by n elements and $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ has the property that ω_J can be lifted to a surjection $\Theta : R^n \twoheadrightarrow J$. The *generalized Euler class group* $E^n(R; R)$ is defined by $E^n(R; R) = G^n/H^n$.

Let $\{e_i\}$ be the standard basis of R^n and $\alpha : R^n \twoheadrightarrow J/J^2$ be a surjection from R^n to J/J^2 that sends $\bar{e}_i$ to $\bar{a}_i$ for $1 \le i \le n$, where $a_i \in J$ and $\{\bar{a}_i\}$ generate J/J^2. In rest of this paper, we always use $(a_1, \dots, a_n)$ to denote α.

The generalized weak ECG was defined in [Mandal and Yang 2010]:

- Let L_0^n denote the set of all ideals J of height n such that $\operatorname{Spec}(J/J^2)$ is connected and there is a surjection $\alpha : (R/J)^n \twoheadrightarrow J/J^2$. Let G_0^n be the free group generated on the set L_0^n.
- For any ideal $J \subseteq R$ of height n such that J/J^2 is generated by n elements, there is a unique decomposition $J = \bigcap_{i=1}^r J_i$, where the ideals J_i are pairwise comaximal and $\operatorname{Spec}(R/J_i)$ is connected. By (J) we mean the element $\sum_{i=1}^r (J_i)$ in G_0^n.
- Let H_0^n be the subgroup of G_0^n generated by (J), where J could be generated by n elements. Then the *generalized weak Euler class group* is defined by $E_0^n = G_0^n / H_0^n$.

Theorem 2.2 [Bhatwadekar and Sridharan 2002, Theorem 4.2]. *Suppose R is a d-dimensional Noetherian commutative ring, and let n be an integer with $2n \ge d+3$. Let $J \subseteq R$ be an ideal of height n such that J/J^2 is generated by n elements, and let $\omega_J : (R/J)^n \twoheadrightarrow J/J^2$ be an equivalence class of surjections. Suppose that $(J; \omega_J)$ is zero in the ECG $E^n(R; R)$. Then, J is generated by n elements and ω_J can be lifted to a surjection $\Theta : R^n \twoheadrightarrow J$.*

The following lemma is easy to prove, so we omit the proof.

Lemma 2.3. *Let $(I; \omega_I)$ and $(J; \omega_J)$ be two elements of $E^n(R; R)$. Let surjections*

$$\omega_I : R^n \xrightarrow{(a_1,\dots,a_n)} I/I^2 \quad \text{and} \quad \omega_J : R^n \xrightarrow{(b_1,\dots,b_n)} J/J^2$$

be representatives of the equivalence classes of ω_I and ω_J, respectively. Suppose I and J are comaximal ideals of R. Then by the Chinese remainder theorem, we can find a unique surjection

$$\omega_{I\cap J} : R^n \xrightarrow{(c_1,\dots,c_n)} I \cap J/(I \cap J)^2,$$

where the c_i are elements of $I \cap J$ such that $c_i = a_i \pmod{I^2}$ and $c_i = b_i \pmod{J^2}$. Then $(I \cap J; \omega_{I\cap J}) \in E^n(R; R)$ is independent of choice representative in the equivalence classes ω_I and ω_J, and $(I; \omega_I) + (J; \omega_J) = (I \cap J; \omega_{I\cap J}) \in E^n(R; R)$.

The next lemma is an adapted version of [Mandal and Yang 2010, Lemma 4.3]. We give a proof for this new form.

Lemma 2.4 (transversal lemma). *Suppose R is a Noetherian commutative ring with* $\dim R = d$. *Assume I is an ideal of R with* height $I = n$ *and $\omega : R^n \twoheadrightarrow I/J$ is* *a surjection in which J is an ideal of R contained in I^2. Let $l_1, \ldots, l_r$ be finitely many ideals of R. Then we can find a surjective lift $v : R^n \twoheadrightarrow I \cap \mathrm{K}$ such that*

$$K + J = R, \quad \text{height } K \geq n, \quad \text{height}((K + l_i)/l_i) \geq n \quad \textit{for any } l_i \textit{ with } 1 \leq i \leq r.$$

Proof. We use standard generalized dimension theory. First, there is a lift $v_0 : R^n \to I$ of ω. Then $I = (v_0(R^n), a)$ for some $a \in J$

Let $\mathcal{P}_{n-1} \subseteq \mathrm{Spec}(R)$ be the set of all prime ideals p with height $p \leq n-1$ and $a \notin p$. For any l_i such that $1 \leq i \leq r$, let $\mathcal{Q}_{i,n-1} \subseteq \mathrm{Spec}(R)$ be the set of all prime ideals p such that $p \supseteq l_i$ and $a \notin p$, and $\text{height}(p/l_i) \leq n-1$. Write $\mathcal{P} = \bigcup_{i=1}^{r} \mathcal{Q}_{i,n-1} \cup \mathcal{P}_{n-1}$.

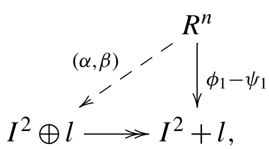
Let $d_0 : \mathcal{P}_{n-1} \to \mathbb{N}$ be the restriction of the usual dimension function and let $d_i : \mathcal{Q}_{i,n-1} \to \mathbb{N}$ be the dimension function induced by that on $\mathrm{Spec}(R_a/l_{ia})$ for $1 \leq i \leq r$. Then d_0 and d_i for $1 \leq i \leq r$ induce a generalized dimension function $d : \mathcal{P} \to \mathbb{N}$; see [Mandal 1997] or [Plumstead 1983].

Now $(v_0, a) \in R^{n*} \oplus R$ is a basic element on $\mathcal{P}$. Since $\mathrm{rank}(R^n) = n > d(p)$ for all $p \in \mathcal{P}$, there is a $\phi \in R^{n*}$ such that $v = v_0 + a\phi$ is basic on $\mathcal{P}$. Clearly, v is a lift of ω and $I = (v(R^n), a)$.

Since v is a lift of ω, we can write $v(R^n) = I \cap \mathrm{K}$, such that $K + J = R$. It is routine to check that $\text{height}(K) \geq n$ and $\text{height}((K + l_i)/l_i) \geq n$ for $1 \leq i \leq r$. □

Lemma 2.5 (avoid lemma). *Let R be a Noetherian commutative ring such that $\dim R = d$, and let $l \subseteq R$ be an ideal of R. Assume that I is an ideal of R and $\phi : R^n \twoheadrightarrow I/I^2$ is a surjective map. If there is a surjective map $\psi : R^n \twoheadrightarrow (I+l)/l$ such that $\tilde{\phi} = \psi \otimes (R/(I+l)) = \psi \otimes (\bar{R}/\bar{I})$, in which the bar denotes the reduction modulo l, and $\tilde{\phi}$ is the surjective map $\tilde{\phi} : R^n \twoheadrightarrow (I+l)/(I^2+l) \cong \bar{I}/\bar{I}^2$ induced by ϕ, then we can find a surjective lift $\hat{\phi} : R^n \twoheadrightarrow I/(I^2 l)$ of ϕ.*

Proof. Let $\phi_1 : R^n \to I$ and $\psi_1 : R^n \to I$ be lifts of ϕ and ψ, respectively. Since $\tilde{\phi} = \psi \otimes R/(I+l)$, we have $\phi_1 - \psi_1 \in \mathrm{Hom}(R^n, I^2 + l)$. Then we can find $\alpha \in \mathrm{Hom}(R^n, I^2)$ and $\beta \in \mathrm{Hom}(R^n, l)$ such that $\phi_1 - \psi_1 = \alpha + \beta$. This can be seen from the commutative diagram



in which $(\alpha, \beta) \in \mathrm{Hom}(R^n, I^2 \oplus l)$ is a lift of $\phi_1 - \psi_1$.

Now we construct a map $\phi_2 = \phi_1 - \alpha \in \mathrm{Hom}(R^n, I)$. Of course, ϕ_2 is still a lift of ϕ. Let the bar denote the reduction modulo l. Then since $\phi_2 = \phi_1 - \alpha = \psi_1 + \beta$, and $\beta \in \mathrm{Hom}(R^n, l)$, we have $\bar{\phi}_2 = \bar{\psi}_1 = \psi$. Recall that ψ is surjective, so it is clear that $\phi_2(R^n) + I \cap l = I$. Now consider the ideal $\phi_2(R^2) + I^2 l$. Since $\phi_2(R^n) + I \cap l = \phi_2(R^n) + I^2 = I$, it follows that any prime ideal p of R contains

I if and only if it contains $\phi_2(R^2) + I^2 l$. Note that since $\phi_2(R^n) + I^2 = I$, we have $(\phi_2(R^2) + I^2 l)_p = I_p$ for any prime ideal p of R containing I. So we get $\phi_2(R^n) + I^2 l = I$.

Now let $\hat{\phi} : R^n \twoheadrightarrow I/(I^2 l)$ be the map induced by ϕ_2. It's obvious that $\hat{\phi}$ is a surjective lift of ϕ. □

3. Restriction and extension map

In this section, we construct two group homomorphisms, the restriction map and extension map for the ECG.

Let $\phi : R \to A$ be a ring homomorphism and $I \subseteq R$ be an ideal of R. In the rest of the paper, $\phi(I)$ without special decorations will always denote the ideal $\phi(I)A$, which is the ideal of A generated by $\phi(I)$.

Definition 3.1 (restriction map). Let R be a Noetherian commutative ring with $\dim R = d$, and $l \subseteq R$ be an ideal of R with $\dim R/l = d - m$. Let the bar denote the reduction modulo l, and let $\rho : R \to R/l$ denote the natural ring homomorphism. For an integer n such that $2n \geq d + 3$, let $E^n(R; R)$ and $E^n(R/l; R/l)$ denote the generalized ECG of R and R/l, respectively, as defined in [Bhatwadekar and Sridharan 2002]. Then we can define a group homomorphism $E(\rho) : E^n(R; R) \to E^n(\bar{R}; \bar{R})$, called the *restriction map* of ECG, as follows:

For any element $x \in E^n(R; R)$, from the properties of the group $E^n(R; R)$, we know that x can be written as a pair of $(I; \omega_I) \in E^n(R; R)$, where I is an ideal of R with height $I \geq n$, and ω_I is an equivalence class of surjections $\omega_I : R^n \twoheadrightarrow I/I^2$. Moreover by Lemma 2.4, we can find $(I; \omega_I) = (I'; \omega_{I'}) \in E^n(R; R)$ such that height $\overline{I' + l} \geq n$ in $\bar{R}$. Then we define $E(\rho)(I; \omega_I) = (\overline{I' + l}; \omega_{\overline{I'+l}}) \in E^n(\bar{R}; \bar{R})$, in which $\omega_{\overline{I'+l}}$ is the equivalence class of induced surjection defined as

$$\omega_{\overline{I'+l}} : R^n \xrightarrow{\omega_{I'}} \frac{I'}{I'^2} \xrightarrow{\bar{\gamma}} \frac{I'+l}{I'^2+l} \cong \frac{(\overline{I'+l})}{(\overline{I'+l})^2},$$

where $\bar{\gamma}$ is the natural map from I'/I'^2 to $(I' + l)/(I'^2 + l)$, and $\omega_{I'}$ is any representative of the equivalence class $\omega_{I'}$.

(1) Since the map $\mathscr{E}l_n(R) \to \mathscr{E}l_n(R/I')$ is surjective, we know that the element $E(\rho)(I; \omega_I)$ is independent of choice of the representative of $\omega_{I'}$.

(2) If $(I; \omega_I) = 0 \in E^n(R; R)$, then $E(\rho)(I; \omega_I) = 0 \in E^n(\bar{R}; \bar{R})$.

Proof of (2). Since $(I; \omega_I) = 0 \in E^n(R; R)$, there exists by Theorem 2.2 a surjective lift of $\omega_{I'}$, denoted by $\upsilon_{I'}$. Then it is easy to check that $\bar{\upsilon}_{I'} : R^n \twoheadrightarrow I' \twoheadrightarrow (I' + l)/l$ is a surjective lift of $\omega_{\overline{I'+l}}$. So we have $E(\rho)(I; \omega_I) = 0 \in E^n(\bar{R}; \bar{R})$. □

(3) $E(\rho)(I; \omega_I)$ is independent of choice of the element $(I'; \omega_{I'})$.

Proof of (3). If there is another element $(I''; \omega_{I''}) \in E^n(R; R)$ such that $(I''; \omega_{I''}) = (I; \omega_I)$, and height $\overline{I''+l} \geq n$ in $\overline{R}$, then by Lemma 2.4 we can find $(K; \omega_K)$ in $E^n(R; R)$ such that

$$K + I = K + I' = K + I'' = R,$$
$$\text{height } \overline{K+l} \geq n,$$
$$(K; \omega_K) + (I; \omega_I) = 0 \in E^n(R; R).$$

By Lemma 2.3, $(K; \omega_K) + (I', \omega_{I'}) = (K \cap I'; \omega_{K\cap I'}) = (K; \omega_K) + (I''; \omega_{I''}) = (K \cap I''; \omega_{K\cap I''}) = 0 \in E^n(R; R)$. Then, from the properties of the ECG and the result (2) above, it can be easily checked that

$$\begin{aligned} E(\rho)(K \cap I'; \omega_{K\cap I'}) &= (\overline{K \cap I' + l}; \omega_{\overline{K\cap I'+l}}) = (\overline{K+l}; \omega_{\overline{K+l}}) + (\overline{I'+l}; \omega_{\overline{I'+l}}) \\ = E(\rho)(K; \omega_K) + E(\rho)(I'; \omega_{I'}) &= E(\rho)(K \cap I''; \omega_{K\cap I''}) = (\overline{K \cap I'' + l}; \omega_{\overline{K\cap I''+l}}) \\ &= E(\rho)(K; \omega_K) + E(\rho)(I''; \omega_{I''}) = (\overline{K+l}; \omega_{\overline{K+l}}) + (\overline{I''+l}; \omega_{\overline{I''+l}}), \end{aligned}$$

which is equal to zero. Therefore, $E(\rho)(I'; \omega_{I'}) = E(\rho)(I''; \omega_{I''})$. This shows that $E(\rho)(I, \omega_I)$ is independent of the choice of the element $(I'; \omega_{I'})$. □

(4) If $(I; \omega_I) = (J; \omega_J) \in E^n(R; R)$, then

$$E(\rho)(I; \omega_I) = E(\rho)(J; \omega_J) \in E^n(R/l; R/l).$$

(5) For any elements $x, y \in E^n(R; R)$, we have

$$E(\rho)(x) + E(\rho)(y) = E(\rho)(x + y).$$

Proof of (5). Let $x = (I; \omega_I)$, $y = (J; \omega_J)$ be two elements of $E^n(R; R)$. By the method we used above, we may further assume that $I + J = R$. Now define maps

$$\omega_I : R^n \xrightarrow{(i_1,\ldots,i_n)} I/I^2, \quad \text{where } i_r \in I \text{ for } 1 \leq r \leq n,$$

$$\omega_J : R^n \xrightarrow{(j_1,\ldots,j_n)} J/J^2, \quad \text{where } j_r \in J \text{ for } 1 \leq r \leq n,$$

Since I and J are comaximal, by Lemma 2.3, we can find a surjection

$$\omega_{I\cap J} : R^n \xrightarrow{(k_1,\ldots,k_n)} \frac{I \cap J}{(I \cap J)^2}$$

such that $k_r \in I \cap J$ and $k_r = i_r \pmod{I^2}$ and $k_r = j_r \pmod{J^2}$ for $1 \leq r \leq n$, that is, $x + y = (I \cap J; \omega_{I\cap J}) \in E^n(R; R)$. Hence, if the bar denotes reduction modulo l, we get

$$\begin{aligned}E(\rho)(x)+E(\rho)(y) &= (\bar{I};\, (\bar{i}_1,\ldots,\bar{i}_n)) + (\bar{J};\, (\bar{j}_1,\ldots,\bar{j}_n)) \\ &= (\bar{I};\, (\bar{k}_1,\ldots,\bar{k}_n)) + (\bar{J};\, (\bar{k}_1,\ldots,\bar{k}_n)) = (\overline{I\cap J};\, (\bar{k}_1,\ldots,\bar{k}_n)) \\ &= E(\rho)(I\cap J;\, \omega_{I\cap J}) = E(\rho)(x+y). \qquad \square\end{aligned}$$

(6) If $n > d-m$, the map $E(\rho)$ vanishes.

Proof of (6). This comes from the fact that in this case $E^n(R/l;\, R/l) = 0$. $\square$

By all of the above, the group homomorphism $E(\rho) : E^n(R;\, R) \to E^n(\overline{R};\, \overline{R})$ is well-defined.

Definition 3.2 (extension map). Let R and A be Noetherian commutative rings with dimension d and s, respectively. Let n be an integer with $2n \geq d+3$ and $2n \geq s+3$. If there is a ring homomorphism $\phi : R \to A$ such that

$$(*) \qquad \text{height}\, \phi(I) \geq n \text{ for any local } n\text{-orientation } \omega_I : R^n \twoheadrightarrow I/I^2.$$

then similarly to the above definition, we can construct a group homomorphism $E(\phi) : E^n(R;\, R) \to E^n(A;\, A)$, called the *extension map* of the ECG, as follows

> Let $x = (I;\, \omega_I) \in E^n(R;\, R)$ be any element, and suppose that ω_I is the surjective map
>
> $$R^n \xrightarrow{(i_1,\ldots,i_n)} I/I^2 \quad \text{in which } i_t \in I \text{ for } 1 \leq t \leq n.$$
>
> Then we define $E(\phi)(x)$ by $(\phi(I);\, \omega_{\phi(I)}) \in E^n(A;\, A)$, where $\omega_{\phi(I)}$ is the surjection
>
> $$A^n \xrightarrow{(\phi(i_1),\ldots,\phi(i_n))} \phi(I)/\phi(I)^2\ .$$

By a method similar to the one used in Definition 3.1, it can be checked that $E(\phi)$ is indeed a group homomorphism.

By forgetting the orientation in Definitions 3.1 and 3.2, we have the following for the weak ECG.

Definition 3.3. Let R and l be as in Definition 3.1. Let $\phi : R \to R/l$ be the natural ring homomorphism. For an integer n with $2n \geq d+3$, there is a group homomorphism $E_0(\phi) : E_0^n(R) \to E_0^n(R/l)$, which is called the restriction map of the weak ECG.

Similarly, let R, A and n be as in Definition 3.2, and let $\phi : R \to A$ be a ring homomorphism satisfying the condition $(*)$. Then there is a group homomorphism $E_0(\phi) : E_0^n(R) \to E_0^n(A)$, which is called the extension map of the weak ECG.

4. The relative ECG and homology sequence

In this section, in analogy to related notions in K-theory, we define the relative and relative weak ECGs. Using these groups, we construct homology sequences for the ECG, which are the counterparts of homology sequence for K_0-groups. Also we will give excision theorems for the ECG, which are the counterparts of excision theorem for K_0-groups.

Definition 4.1. Let R be a Noetherian commutative ring with $\dim R = d$, and let l be an ideal of R. Then we have the double $D(R, l)$ of R along l as the subring of the Cartesian product $R \times R$, given by

$$D(R, l) = \{(x, y) \in R \times R : x - y \in l\}.$$

Note that if p_1 denotes the projection onto the first coordinate, then there is a split exact sequence

$$0 \to l \longrightarrow D(R, l) \xrightarrow{p_1} R \to 0$$

in the sense that p_1 is split surjective (with splitting map given by the diagonal embedding of R in $D(R, l)$ and with $\ker p_1$ identified with l.)

Since $D(R, l)$ is finite over the subring R (given by the diagonal embedding), we get $D(R, l)$ that is Noetherian and is integral over the subring R. Moreover, we have $\dim D(R, l) = \dim R = d$, and height($\ker p_1$) $= 0$ (with $\ker p_1$ being regarded as an ideal of $D(R, l)$).

Then for any integer n with $2n \geq d + 3$, the relative ECG of R and l is defined by

$$E^n(R, l; R) = \ker(E(p_1) : E^n(D(R, l); D(R, l)) \to E^n(R; R)).$$

and the relative weak ECG of R and l is defined by

$$E_0^n(R, l) = \ker(E_0(p_1) : E_0^n(D(R, l)) \to E_0^n(R)).$$

in which $E(p_1)$ and $E_0(p_1)$ are the restriction map of the ECG of Definition 3.1 and the restriction map of the weak ECG of Definition 3.3, respectively.

It can be seen easily that when $l = R$, the relative ECG $E^n(R, R; R)$ and the relative weak ECG $E_0^n(R, R)$ are the same as the generalized ECG $E^n(R; R)$ and the generalized weak ECG $E_0^n(R)$, respectively.

Theorem 4.2 (homology sequence). *Let R be a Noetherian commutative ring with $\dim R = d$, and let $l \subseteq R$ be an ideal of R with $\dim R/l = d - m$. Let p_2 denote the projection from $D(R, l)$ to the second coordinate. Then, for any integer n such that $2n \geq d + 3$, we have the exact sequence*

$$E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l),$$

called the homology sequence of the ECG.

Proof. Step I: First, we check that $E(\rho)\circ E(p_2)=0$. Let $\ker p_1$ and $\ker p_2$ denote the kernels of projections p_1 and p_2. Then $\mathrm{height}(\ker p_1) = \mathrm{height}(\ker p_2) = 0$. On the other hand, we have ring homomorphisms $\rho\circ p_1, \rho\circ p_2 : D(R,l) \twoheadrightarrow R/l$. By the definition of $D(R,l)$, it can be seen easily that $\rho\circ p_1 = \rho\circ p_2$. Hence for any element $x \in E^n(R,l;R)$, by the method of used construction of the restriction map and by Lemma 2.4, we can assume that $x = (Z;\omega_Z)$, in which Z is an ideal of $D(R,l)$ with properties

$$\mathrm{height}\, p_1(Z) \geq n, \quad \mathrm{height}\, p_2(Z) \geq n \quad \text{in } R,$$
$$\mathrm{height}\, \rho\circ p_1(Z) = \mathrm{height}\, \rho\circ p_2(Z) \geq n \quad \text{in } R/l.$$

Write ω_Z as $(z_1,\dots,z_n)$, where $z_i = (x_i, y_i) \in D(R,l)$ for $1 \leq i \leq n$, and let the bar denote the reduction modulo l. Then it can be seen that

$$E(\rho)\circ E(p_1)(Z;\omega_Z) = (\overline{p_1(Z)}; (\bar{x}_1,\dots,\bar{x}_n)),$$
$$E(\rho)\circ E(p_2)(Z;\omega_Z) = (\overline{p_2(Z)}; (\bar{y}_1,\dots,\bar{y}_n)).$$

Since Z is an ideal of $D(R,l)$ and $(x_i, y_i) \in D(R,l)$ for $1 \leq i \leq n$, we have $\overline{p_1(Z)} = \overline{p_2(Z)}$ and $\bar{x}_i = \bar{y}_i$ for $1 \leq i \leq n$. On the other hand, from the definition of $E^n(R,l;R)$, we know that

$$E(\rho)\circ E(p_1)(Z;\omega_Z) = E(\rho)(p_1(Z);(x_1,\dots,x_n)) = (\overline{p_1(Z)}; (\bar{x}_1,\dots,\bar{x}_n)) = 0.$$

Thus we get

$$E(\rho)\circ E(p_2)(Z;\omega_Z) = (\overline{p_2(Z)}; (\bar{y}_1,\dots,\bar{y}_n)) = (\overline{p_1(Z)}; (\bar{x}_1,\dots,\bar{x}_n)) = 0.$$

This establishes that $E(\rho)\circ E(p_2) = 0$, that is, $\ker E(\rho) \supseteq \mathrm{Im}\, E(p_2)$.

Step II: Next we check that the kernel of $E(\rho)$ is contained in the image of $E(p_2)$, that is, $\ker E(\rho) \subseteq \mathrm{Im}\, E(p_2)$.

Let $x \in E^n(R;R)$ such that $E(\rho)(x) = 0 \in E^n(R/l;R/l)$. By the method we used in the construction of restriction map, we can assume that $x = (I;\omega_I)$, in which I is an ideal of R such that properties $\mathrm{height}\, I \geq n$, and $\mathrm{height}(I+l)/l \geq n$ in $\overline{R}$.

By the assumption that $E(\rho)(x) = 0 \in E^n(R/l;R/l)$, we have $(\bar{I};\omega_{\bar{I}}) = 0 \in E^n(R/l;R/l)$, in which $\omega_{\bar{I}} : R^n \twoheadrightarrow \bar{I}/\bar{I}^2$ is the map induced by ω_I. By [Bhatwadekar and Sridharan 2000, Theorem 4.2], there exists a surjective map $v_{\bar{I}} : R^n \twoheadrightarrow \bar{I}$ such that $v_{\bar{I}} \otimes \overline{R}/\bar{I} = \omega_{\bar{I}}$. So by Lemma 2.4, $\omega_I : R^n \twoheadrightarrow I/I^2$ can be lifted to a surjective map $\hat{\omega_I} : R^n \twoheadrightarrow I/(I^2 l)$. Then by Lemma 2.3, we can find a surjective lifting $v : R^n \twoheadrightarrow I\cap K$ of $\hat{\omega_I}$ such that $K + I^2 l = R$ and $\mathrm{height}\, K \geq n$. Since $K + I^2 l = R$, v induces a surjective map $\omega_K : R^n \twoheadrightarrow K/K^2$, which defines an element $(K;\omega_K) \in E^n(R;R)$. It can be seen easily that $(K;\omega_K) + (I;\omega_I) = 0 \in E^n(R;R)$.

Now write ω_K as $(x_1, \dots, x_n)$, where $x_i \in K$ for $1 \le i \le n$. Then we can define an element $(Z; \omega_Z) \in E^n(R, l; R)$ as follows:

- Define $Z \subseteq D(R, l)$ to be the ideal of $D(R, l)$ that is generated by pairs $(r_1, r_2) \in R \times R$ such that $r_2 \in$ K and $r_1 - r_2 \in l$.
- Define the map $\omega_Z : D(R, l)^n \xrightarrow{(z_1, \dots, z_n)} Z/Z^2$, in which $z_i = (x_i, x_i) \in Z$ for $1 \le i \le n$.

By the facts that $\ker p_1 \cap \ker p_2 = (0, l) \cap (l, 0) = 0 \subseteq D(R, l)$ and $p_1(Z) = R$ and height $p_2(Z) =$height $K \ge n$, we have height $I \ge n$.

We should check that ω_Z is surjective. Let $z = (k + l_1, k) \in Z$, where $k \in K$ and $l_1 \in l$. Since ω_K is surjective, there exist $r_i \in R$ for $1 \le i \le n$ and $k_1 \in K^2$, such that $k = \sum_{i=1}^n r_i x_i + k_1$. Since $K^2 + l^2 = R$ contains l, there exist $k_2 \in K^2$ and $l_2 \in l^2$ such that $k_2 + l_2 = l_1$. By the fact $k_2 = l_1 - l_2 \in K^2 \cap l = K^2 l$, we get $k_{3t} \in K^2$, and $l_{3t} \in l$ for $1 \le t \le m$, such that $k_2 = \sum_{t=1}^m l_{3t} k_{3t}$. Finally,

$$\begin{aligned} z = (k + l_1, k) &= \sum_{i=1}^n (r_i, r_i)(x_i, x_i) + (k_1, k_1) + (l_1, 0) \\ &= \sum_{i=1}^n (r_i, r_i)(x_i, x_i) + (k_1, k_1) + (l_2, 0) + \sum_{t=1}^m (l_{3t}, 0)(k_{3t}, k_{3t}). \end{aligned}$$

This shows that ω_Z is surjective.

Since $K + l = R$, we have $p_1(Z) = R$ by the construction of Z. This implies that $E(p_1)(Z; \omega_Z) = 0 \in E^n(R; R)$. Putting all of these together, we see $(Z; \omega_Z)$ is indeed an element of $E^n(R, l; R)$.

It can be seen easily that $E(p_2)(Z; \omega_Z) = (K; \omega_K) \in E^n(R; R)$. Now let $y \in E^n(R, l; R)$ be such that $y + (Z; \omega_Z) = 0$. Since

$$E(p_2)(y) + E(p_2)(Z; \omega_Z) = (I; \omega_I) + (K; \omega_K) = 0,$$

we see that $E(p_2)(y) = (I; \omega_I)$. This shows that $\ker E(\rho) \subseteq \mathrm{Im} E(p_2)$.

By steps I and II, the sequence is indeed an exact sequence. □

Theorem 4.3 (excision theorem). *Let R be a Noetherian commutative ring with* $\dim R = d$, *and let $l \subseteq R$ be an ideal of R with* $\dim R/l = d - m$. *Let p_2 denote the projection from $D(R, l)$ to the second coordinate. If there exists a splitting β of the ring homomorphism $\rho : R \to R/l$ such that β satisfies condition $(*)$, then for any integer n with $2n \ge d + 3$, we have the split exact sequence*

$$0 \to E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \to 0,$$

called the excision sequence of the ECG. In particular, we have an isomorphism $E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R/l; R/l)$.

Proof. Step I: First,we check that $E(\rho)$ is a split surjection. Since the ring homomorphism $\beta : R/l \to R$ satisfies the condition $(*)$, by Definition 3.2 there is a group homomorphism $E(\beta) : E^n(R/l;\ R/l) \to E^n(R;\ R)$. By the fact that β is a splitting of ρ, it is easy to check that $E(\beta)$ has the property $E(\rho) \circ E(\beta) = Id_{E^n(R/l;R/l)}$. This shows that $E(\rho)$ is split surjective.

Step II: We check that $E(p_2)$ is injective. Now we have a surjective ring homomorphism $\rho \circ p_2 : D(R,l) \to R/l$ and an exact sequence

$$0 \to l \times l \to D(R,l) \xrightarrow{\rho \circ p_2} R/l \to 0$$

in which $l \times l \subset D(R,l)$ is the ideal of $D(R,l)$ generated by elements $(l_1, l_2) \in R \times R$, where $l_1, l_2 \in l$. Then we have the restriction map $E(\rho \circ p_2) : E^n(R, l;\ R) \to E^n(R/l;\ R/l)$ of the ECG. It can be easily checked that $E(\rho) \circ E(p_2) = E(\rho \circ p_2)$.

Let $x = (Z;\ \omega_Z) \in E^n(R, l;\ R)$ be such that $E(p_2)(x) = 0 \in E^n(R;\ R)$. By the method we used in the construction of restriction map, we can assume that height $p_1(Z) \geq n$, height $p_2(Z) \geq n$ and height $\rho \circ p_2(Z) \geq n$. On the other hand, since $E(\rho \circ p_2) = E(\rho) \circ E(p_2) = 0$, by the same method we used in the proof of Theorem 4.2, we can find $(K;\ \omega_K) \in E^n(R, l;\ R)$ such that

- $(K;\ \omega_K) + (Z;\ \omega_Z) = 0$,
- $K + Z^2(l \times l) = D(R,l)$,
- height $K \geq n$, height $p_1(K) \geq n$ and height $p_2(K) \geq n$.

By the assumption that $E(p_2)(Z;\ \omega_Z) = 0$, we get $E(p_2)(K;\ \omega_K) = 0 \in E^n(R;\ R)$.

Now, write $\omega_K : D(R,l)^n \twoheadrightarrow K/K^2$ as $(k_1, \ldots, k_n)$ where $k_i = (x_i, y_i) \in K$ for $1 \leq i \leq n$. We have the following.

- $E(p_1)(K;\ \omega_K) = (p_1(K);\ \omega_{p_1(K)}) = (I_1;\ \omega_{I_1})$, where I_1 denotes the ideal $p_1(K)$, and ω_{I_1} denotes the surjection induced by ω_K, that is,
$$\omega_{p_1(K)} : R^n \xrightarrow{(x_1,\ldots,x_n)} I_1/I_1^2\ .$$
- $E(p_2)(K;\ \omega_K) = (p_2(K);\ \omega_{p_2(K)}) = (I_2;\ \omega_{I_2})$, where I_2 denotes the ideal $p_2(K)$, and ω_{I_2} denotes the surjection induced by ω_K, that is,
$$\omega_{p_2(K)} : R^n \xrightarrow{(y_1,\ldots,y_n)} I_2/I_2^2$$

Since $K + (l \times l) = D(R,l)$, we see that $I_1 + l = I_2 + l = R$.

By the fact that $E(p_1)(K;\ \omega_K) = E(p_2)(K;\ \omega_K) = 0 \in E^n(R;\ R)$, there exist surjective lifts

$$\upsilon_{I_1} : R^n \xrightarrow{(x'_1,\ldots,x'_n)} I_1 \quad \text{and} \quad \upsilon_{I_2} : R^n \xrightarrow{(y'_1,\ldots,y'_n)} I_2$$

of ω_{I_1} and ω_{I_2}, respectively, in which $x'_i \in I_1$, and $y'_i \in I_2$ for $1 \leq i \leq n$.

Since v_{I_1} is a lift of ω_{I_1}, we have $x'_i - x_i \in I_1^2$ for $1 \le i \le n$. Let $x'_i - x_i = a_i$, with $a_i \in I_1^2$ for $1 \le i \le n$. Since $I_1 = p_1(K)$, there exist $b_i \in I_2^2$ such that $(a_i, b_i) \in K^2$ for $1 \le i \le n$. Now let $y''_i = y_i + b_i$ for $1 \le i \le n$. It follows from the facts $(x_i, y_i) \in K$ and $(a_i, b_i) \in K^2$ that $(x'_i, y''_i) = (x_i + a_i, y_i + b_i) \in K$ and $(x'_i, y''_i) = (x_i + a_i, y_i + b_i) = (x_i, y_i) \pmod{K^2}$ for $1 \le i \le n$. So we obtain a surjection

$$\omega_K^1 : R^n \xrightarrow{(k'_1,\ldots,k'_n)} K/K^2$$

in which $k'_i = (x'_i, y''_i) \in K$ for $1 \le i \le n$. Clearly, $(K; \omega_K^1) = (K; \omega_K)$.

By the same method, we can get a surjection

$$\omega_K^2 : R^n \xrightarrow{(k''_1,\ldots,k''_n)} K/K^2$$

in which $k''_i = (x''_i, y'_i) \in K$ for $1 \le i \le n$, such that $(K; \omega_K^2) = (K; \omega_K)$.

Now we construct two elements $(\mathcal{I}_1; \omega_{\mathcal{I}_1})$ and $(\mathcal{I}_2; \omega_{\mathcal{I}_2})$ of $E^n(R, l; R)$. Let the bar denote reduction modulo l.

(I) Define $\mathcal{I}_1$ to be the ideal of $D(R, l)$ that is generated by pairs $(\beta(\bar{a}), b)$, where $(a, b) \in$ K.

(II) Define the map $\omega_{\mathcal{I}_1} : D(R, l)^n \xrightarrow{(\hat{k}''_1,\ldots,\hat{k}''_n)} \mathcal{I}_1/\mathcal{I}_1^2$, where $\hat{k}''_i = (\beta(\bar{x}''_i), y'_i) \in \mathcal{I}_1$ for $1 \le i \le n$.

(I*) Define $\mathcal{I}_2$ to be the ideal of $D(R, l)$ that is generated by pairs $(a, \beta(\bar{b}))$, where $(a, b) \in$ K.

(II*) Define the map $\omega_{\mathcal{I}_2} : D(R, l)^n \xrightarrow{(\hat{k}'_1,\ldots,\hat{k}'_n)} \mathcal{I}_2/\mathcal{I}_2^2$, where $\hat{k}'_i = (x'_i, \beta(\bar{y}''_i)) \in \mathcal{I}_2$ for $1 \le i \le n$.

We check that height $\mathcal{I}_1 \ge n$ and height $\mathcal{I}_2 \ge n$. By the fact that $\ker p_1 \cap \ker p_2 = (0, l) \cap (l, 0) = 0 \subset D(R, l)$ and

$$p_1(\mathcal{I}_1) = \beta(\bar{I}_1) = R \quad \text{and} \quad \text{height } p_2(\mathcal{I}_1) = \text{height } I_2 \ge n,$$

we get height $\mathcal{I}_1 \ge n$. Similarly, we have height $\mathcal{I}_2 \ge n$.

We check that $(\mathcal{I}_1; \omega_{\mathcal{I}_1})$ and $(\mathcal{I}_2; \omega_{\mathcal{I}_2})$ equate to zero in $E^n(R, l; R)$.

In fact we have a map

$$v_{\mathcal{I}_1} : D(R, l)^n \xrightarrow{(\hat{k}''_1,\ldots,\hat{k}''_n)} \mathcal{I}_1,$$

which is defined by $v_{\mathcal{I}_1}(e_i) = \hat{k}''_i \in \mathcal{I}_1$, where $\{e_i\}$ for $1 \le i \le n$ is the standard basis of R^n. It is obviously a lift of $\omega_{\mathcal{I}_1}$. Now let $(\beta(\bar{a}), b) \in \mathcal{I}_1$ for $(a, b) \in K$. Since I_2 is generated by $\{y'_i\}$, there exist $r_i \in R$ for $1 \le i \le n$ such that $\sum_{i=1}^n r_i y'_i = b$.

On the other hand, it follows from the facts $(a, b) \in K$ and $(x_i'', y_i') \in K$ that $\bar{a} = \bar{b}$ and $\bar{x}_i'' = \bar{y}_i'$ for $1 \le i \le n$. So we get

$$\beta(\bar{a}) = \beta(\bar{b}) = \sum_{i=1}^{n} \beta(\bar{r}_i)\beta(\bar{y}_i') = \sum_{i=1}^{n} \beta(\bar{r}_i)\beta(\bar{x}_i'').$$

Thus $(\beta(\bar{a}), b) = \sum_{i=1}^{n}(\beta(\bar{r}_i), r_i)(\beta(\bar{x}_i''), y_i')$. This shows that $v_{\mathcal{I}_1}$ is a surjective lift of $\omega_{\mathcal{I}_1}$. So $(\mathcal{I}_1; \omega_{\mathcal{I}_1}) = 0 \in E^n(R, l; R)$. By the same method, we can prove that $(\mathcal{I}_2; \omega_{\mathcal{I}_2}) = 0 \in E^n(R, l; R)$.

Next we check that $(\mathcal{I}_1; \omega_{\mathcal{I}_1}) + (\mathcal{I}_2; \omega_{\mathcal{I}_2}) = (K; \omega_K)$.

We first check that $\mathcal{I}_1 + \mathcal{I}_2 = R$. Since $I_1 + l = I_2 + l = R$, there exists $(i_1, i_2) \in \mathrm{K}$ such that $\beta(\bar{i}_1) = 1$. So $(1, i_2) \in \mathcal{I}_1$, and $1 - i_2 \in l$. By the same method, we can find $i_1' \in I_1$ such that $(i_1', 1) \in \mathcal{I}_2$. So $(0, 1 - i_2)(i_1', 1) = (0, 1 - i_2) \in \mathcal{I}_2$. It follows that $(1, i_2) + (0, 1 - i_2) = (1, 1) \in \mathcal{I}_1 + \mathcal{I}_2$. Hence $\mathcal{I}_1 + \mathcal{I}_2 = R$.

Second, we check that $\mathcal{I}_1 \cap \mathcal{I}_2 = K$. Let $(i_1, i_2) \in K$. Then $(\beta(\bar{i}_1), i_2) \in \mathcal{I}_1$. Now let $i_1 - \beta(\bar{i}_1) = l_1 \in l$. Since $I_1 + l = R$, there exists $(i_i', i_2') \in K$ such that $\beta(\bar{i}_1') = 1$. So we have $(l_1, 0)(\beta(\bar{i}_1'), i_2') = (l_1, 0) \in \mathcal{I}_1$. Since $(\beta(\bar{i}_1), i_2) + (l_1, 0) = (i_1, i_2) \in \mathcal{I}_1$, we get $K \subseteq \mathcal{I}_1$. Similarly, we can prove $K \subseteq \mathcal{I}_2$. Thus $K \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$.

Let $(i_1, i_2) \in \mathcal{I}_1$. Then there exist $r_{1t}, r_{2t} \in R$, with $r_{1t} - r_{2t} \in l$, and $(x_{1t}, x_{2t}) \in K$ for $1 \le t \le m$, where $m \in \mathbb{Z}$ is an integer such that

$$(i_1, i_2) = \sum_{t=1}^{m}(r_{1t}, r_{2t})(\beta(\bar{x}_{1t}), x_{2t}) = \Big(\sum_{t=1}^{m} r_{1t}\beta(\bar{x}_{1t}), \sum_{t=1}^{m} r_{2t}x_{2t}\Big).$$

So we get $i_2 \in I_2$. Thus if $(i_1, i_2) \in \mathcal{I}_1 \cap \mathcal{I}_2$, we will also have $i_1 \in I_1$. Now we have $(\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) \in K$ and $(i_1, i_2) - (\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) = (l_1, 0) \in D(R, l)$. It follows that $i_1 - \sum_{t=1}^{m} r_{1t}x_{1t} = l_1 \in l \cap I_1 = lI_1$. Hence there exist $i_{1\lambda}' \in I_1$, $i_{2\lambda}' \in I_2$ and $l_{2\lambda} \in l$ such that $l_1 = \sum_{\lambda=1}^{s} l_{2\lambda}i_{1\lambda}'$ and $(i_{1\lambda}', i_{2\lambda}') \in K$ for $1 \le \lambda \le s$, with $s \in \mathbb{Z}$. Thus $(l_1, 0) = \sum_{\lambda=1}^{s}(l_{2\lambda}, 0)(i_{1\lambda}', i_{2\lambda}') \in K$ and $(i_1, i_2) = (\sum_{t=1}^{m} r_{1t}x_{1t}, \sum_{t=1}^{m} r_{2t}x_{2t}) + (l_1, 0) \in K$. This shows that $K \supseteq \mathcal{I}_1 \cap \mathcal{I}_2$. So we get $\mathcal{I}_1 \cap \mathcal{I}_2 = K$.

Third, we check that

$$\omega_K \otimes D(R, l)/\mathcal{I}_1 = \omega_{\mathcal{I}_1} \quad \text{and} \quad \omega_K \otimes D(R, l)/\mathcal{I}_2 = \omega_{\mathcal{I}_2}.$$

Since $(K; \omega_K^1) = (K; \omega_K)$, to prove $\omega_K \otimes D(R, l)/\mathcal{I}_2 = \omega_{\mathcal{I}_2}$, we only need to show that $(x_i', y_i'') - (x_i', \beta(\bar{y}_i'')) \in \mathcal{I}_2^2$.

In fact let $(x_i', y_i'') - (x_i', \beta(\bar{y}_i'')) = (0, l_1) \in D(R, l)$, where $l_1 \in l$. Since $I_2^2 + l = R$, there exist $a \in I_1^2$ and $b \in I_2^2$, such that $(a, b) \in K^2$ and $(a, \beta(\bar{b})) = (a, 1) \in \mathcal{I}_2^2$. So we have $(0, l_1)(a, 1) = (0, l_1) \in \mathcal{I}_2^2$. This shows that $\omega_K \otimes D(R, l)/\mathcal{I}_2 = \omega_{\mathcal{I}_2}$. By the same way, we can prove $\omega_K \otimes D(R, l)/\mathcal{I}_1 = \omega_{\mathcal{I}_1}$.

By all of the results above, $(\mathcal{J}_1; \omega_{\mathcal{J}_1}) + (\mathcal{J}_2; \omega_{\mathcal{J}_2}) = (K; \omega_K) = 0$ holds in the group $E^n(R, l; R)$. This shows that $E(p_2)$ is injective.

Putting these results together with those from Step I and Step II, we have a split exact sequence

$$0 \to E^n(R, l; R) \xrightarrow{E(p_2)} E^n(R; R) \xrightarrow{E(\rho)} E^n(R/l; R/l) \to 0.$$

In particular, $E^n(R; R) \cong E^n(R, l; R) \oplus E^n(R/l; R/l)$. The proof is complete. □

Remark 4.4. The proof shows that if the ring homomorphism $\rho : R \to R/l$ has a splitting β, which may not satisfy condition $(*)$, the map $E(p_2)$ is still injective.

5. Applications for polynomial and Laurent polynomial extensions

In this section, we will use Theorem 4.3 to get excision sequences for Euler class groups of polynomial rings and Laurent polynomial rings.

Excision sequence of polynomial rings. Let R be a Noetherian commutative ring with $\dim R = d$. Let $R[t]$ be the polynomial ring over R. Then for any integer n with $2n \geq d + 4$, we get the following corollary by setting $l = (t)$ in Theorem 4.3.

Corollary 5.1. *Let R be a Noetherian commutative ring with* $\dim R = d$, *and let $R[t]$ be the polynomial ring over R. Then for any integer n with $2n \geq d + 4$, we have the short split exact sequence*

$$0 \to E^n(R[t], (t); R[t]) \xrightarrow{E(p_2)} E^n(R[t]; R[t]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and an isomorphism

$$E^n(R[t]; R[t]) \cong E^n(R[t], (t); R[t]) \oplus E^n(R; R).$$

Das [2006] constructed a map $\Psi : E^d(R[t]; R[t]) \to E^d(R; R)$ that is split surjective. In fact, this map is the map $E(\rho)$.

Proposition 5.2 [Das 2003]. *Let R be a smooth affine domain containing the field of rational numbers*, *with* $\dim R = d$. *Then there is an isomorphism*

$$\Phi : E^d(R[t]; R[t]) \cong E^d(R; R).$$

Using [Bhatwadekar and Keshari 2003, theorem 4.13], and Corollary 5.1, this result can generalized:

Corollary 5.3. *Let k be an infinite perfect field*, *and let R be a d-dimensional regular domain that is essentially of finite type over k. Let n be an integer such that $2n \geq d + 4$. Then $E^n(\rho) : E^n(R[t]; R[t]) \to E^n(R; R)$ is an isomorphism.*

Proof. By Corollary 5.1, we only need to prove that $E(p_2)(x) = 0$ for any element $x \in E^n(R[t], (t); R[t])$. Suppose $(Z; \omega_Z) \in E^n(R[t], (t); R[t])$, where Z is an ideal of $D(R[t], (t))$ and $\omega_Z : D(R[t], (t))^n \twoheadrightarrow Z/Z^2$ is a surjection. Let p_1, p_2 denote projections from $D(R[t], (t))$ to respective coordinates. We may assume further that height $p_1(Z) =$height $p_2(Z) = n$ in $R[t]$.

Let

$$E(p_1)(Z; \omega_Z) = (p_1(Z); \omega_{p_1(Z)}) \quad \text{and} \quad E(p_2)(Z; \omega_Z) = (p_2(Z); \omega_{p_2(Z)}),$$

where $\omega_{p_1(Z)}$ and $\omega_{p_2(Z)}$ are respectively surjections $R[t]^n \twoheadrightarrow p_1(Z)/p_1(Z)^2$ and $R[t]^n \twoheadrightarrow p_2(Z)/p_2(Z)^2$ induced by ω_Z. Since $(Z; \omega_Z) \in E^n(R[t], (t); R[t])$, we have $(p_1(Z); \omega_{p_1(Z)}) = 0 \in E^n(R[t]; R[t])$. For any ideal $I \subseteq R[t]$, let $I(0)$ denote the reduction modulo (t), which is the same as setting $t = 0$ in I. Now we get $(p_1(Z)(0); \omega_{p_1(Z)(0)}) = 0 \in E^n(R; R)$, where $\omega_{p_1(Z)(0)}$ is the surjection $R^n \twoheadrightarrow p_1(Z)(0)/(p_1(Z)(0))^2$ that the surjection $\omega_{p_1(Z)}$ induces by setting $t = 0$. By [Bhatwadekar and Sridharan 2002, Theorem 4.2], this means that $\omega_{p_1(Z)(0)}$ can be lifted to a surjection $\upsilon_{p_1(Z)(0)} : R^n \twoheadrightarrow p_1(Z)(0)$. On the other hand, since $(Z; \omega_Z)$ is in $E^n(R[t], (t); R[t])$, we have $p_1(Z)(0) = p_2(Z)(0)$ and $\omega_{p_1(Z)(0)} = \omega_{p_2(Z)(0)}$. So we get

$$\omega_{p_2(Z)} : R[t]^n \twoheadrightarrow p_2(Z)/p_2(Z)^2 \quad \text{and} \quad \upsilon_{p_1(Z)(0)} : R^n \twoheadrightarrow p_2(Z)(0),$$

such that $\omega_{p_2(Z)} \otimes R[t]/(t) = \upsilon_{p_1(Z)(0)} \otimes R/p_2(Z)(0)$. We can find a surjective lift of $\omega_{p_2(Z)}$ by [Bhatwadekar and Keshari 2003, Theorem 4.13]. This shows that $(p_2(Z); \omega_{p_2(Z)}) = E(p_2)(Z; \omega_Z) = 0 \in E^n(R[t]; R[t])$. □

Excision sequence of Laurent polynomial rings.

Corollary 5.4. *Let R be a Noetherian commutative ring with* $\dim R = d$, *and let n be an integer such that $2n \geq d+4$. Let $R[t, t^{-1}]$ be the Laurent polynomial ring over R. Then by setting $l = (t-1)$ in Theorem 4.3, we get the short split exact sequence*

$$0 \to E^n(R[t, t^{-1}], (t-1); R[t, t^{-1}]) \xrightarrow{E(p_2)} E^n(R[t, t^{-1}]; R[t, t^{-1}]) \xrightarrow{E(\rho)} E^n(R; R) \to 0$$

and an isomorphism

$$E^n(R[t, t^{-1}]; R[t, t^{-1}]) \cong E^n(R[t, t^{-1}], (t-1); R[t, t^{-1}]) \oplus E^n(R; R).$$

Postscript. In K-theory, we have the following results: When R is a regular Noetherian commutative ring, the homology sequences for the K_0-groups of the polynomial ring $R[t]$ and the Laurent polynomial ring $R[t, t^{-1}]$ reduce to isomorphisms $K_0(R[t]) \cong K_0(R)$ and $K_0(R[t, t^{-1}]) \cong K_0(R)$, respectively [Rosenberg 1994]. In light of this result and the correspondence between excision sequence for K-theory and the excision sequence for the ECGs, we ask if there exist isomorphisms

$$E^n(R[t];\, R[t]) \cong E^n(R;\, R) \quad \text{and} \quad E^n(R[t,t^{-1}];\, R[t,t^{-1}]) \cong E^n(R;\, R).$$

For the case of polynomial extension, Corollary 5.3 gives an affirmative answer. For the case of Laurent polynomial extension, we wonder if there is also an isomorphism $E^n(R[t,t^{-1}];\, R[t,t^{-1}]) \cong E^n(R;\, R)$ if the ring R satisfies the conditions of Corollary 5.3.

In fact, for the weak ECG, we can prove the following result using Suslin's cancellation theorem [1977] and [Murthy 1994, Theorem 2.2].

Let R be a smooth affine algebra over some algebraically closed field k, with $\dim R = d$. Let $R[t,t^{-1}]$ be the Laurent polynomial ring over R. Then we have $E_0^d(R[t,t^{-1}]) \otimes \mathbb{Q} \cong E_0^d(R) \otimes \mathbb{Q}$, which corresponds to what the Riemann–Roch theorem tells us in geometry.

Acknowledgements. I sincerely thank Professors Lianggui Feng and Satyagopal Mandal for many helpful discussions and encouragement. I also thank the referee for carefully going through the manuscript and suggesting improvements.

References

[Bhatwadekar and Keshari 2003] S. M. Bhatwadekar and M. K. Keshari, "A question of Nori: Projective generation of ideals", *K-Theory* **28**:4 (2003), 329–351. MR 2004j:13013 Zbl 1055.13008

[Bhatwadekar and Sridharan 2000] S. M. Bhatwadekar and R. Sridharan, "The Euler class group of a Noetherian ring", *Compositio Math.* **122**:2 (2000), 183–222. MR 2001g:13018 Zbl 0999.13007

[Bhatwadekar and Sridharan 2002] S. M. Bhatwadekar and R. Sridharan, "On Euler classes and stably free projective modules", pp. 139–158 in *Algebra, arithmetic and geometry* (Mumbai, 2000), edited by R. Parimala, Tata Inst. Fund. Res. Stud. Math. **16**, Tata Inst. Fund. Res., Bombay, 2002. MR 2003i:13012 Zbl 1055.13009

[Das 2003] M. K. Das, "The Euler class group of a polynomial algebra", *J. Algebra* **264**:2 (2003), 582–612. MR 2004c:13031 Zbl 1106.13300

[Das 2006] M. K. Das, "The Euler class group of a polynomial algebra, II", *J. Algebra* **299**:1 (2006), 94–114. MR 2007a:13023 Zbl 1107.13014

[Das and Mandal 2006] M. K. Das and S. Mandal, "A Riemann–Roch theorem", *J. Algebra* **301**:1 (2006), 148–164. MR 2007b:14017 Zbl 1104.14009

[Keshari 2007] M. K. Keshari, "Euler class group of a Laurent polynomial ring: Local case", *J. Algebra* **308**:2 (2007), 666–685. MR 2008i:13012 Zbl 1117.13012

[Mandal 1992] S. Mandal, "Homotopy of sections of projective modules", *J. Algebraic Geom.* **1**:4 (1992), 639–646. MR 93h:13011 Zbl 0796.14011

[Mandal 1997] S. Mandal, *Projective modules and complete intersections*, Lecture Notes in Mathematics **1672**, Springer, Berlin, 1997. MR 98k:13010 Zbl 0888.13002

[Mandal 1998] S. Mandal, "Complete intersection *K*-theory and Chern classes", *Math. Z.* **227**:3 (1998), 423–454. MR 99c:13018 Zbl 0946.19002

[Mandal and Yang 2010] S. Mandal and Y. Yang, "Intersection theory of algebraic obstructions", *J. Pure Appl. Algebra* **214**:12 (2010), 2279–2293. MR 2660913 Zbl 1195.13010

[Murthy 1994] M. P. Murthy, "Zero cycles and projective modules", *Ann. of Math.* (2) **140**:2 (1994), 405–434. MR 96g:13010 Zbl 0839.13007

[Plumstead 1983] B. Plumstead, "The conjectures of Eisenbud and Evans", *Amer. J. Math.* **105**:6 (1983), 1417–1433. MR 85d:13010 Zbl 0532.13008

[Rosenberg 1994] J. Rosenberg, *Algebraic K-theory and its applications*, Graduate Texts in Mathematics **147**, Springer, New York, 1994. MR 95e:19001 Zbl 0801.19001

[Suslin 1977] A. A. Suslin, "A cancellation theorem for projective modules over algebras", *Dokl. Akad. Nauk SSSR* **236**:4 (1977), 808–811. In Russian; translated in *Soviet Math. Dokl.* **18**:5 (1977), 1281–1284. MR 57 #5986 Zbl 0395.13003

Received April 17, 2010. Revised May 27, 2010.

YONG YANG
DEPARTMENT OF MATHEMATICS AND SYSTEMS SCIENCE
NATIONAL UNIVERSITY OF DEFENSE TECHNOLOGY
CHANGSHA, HUNAN 410073
CHINA

yangmoses@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 249 No. 1 January 2011
