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We show that any foliation transverse to a C^1 nonsingular flow ϕ on a closed 3-manifold can be detected algorithmically. We use this to describe a procedure that, for any $\delta > 0$, will determine whether or not there is a foliation whose tangent space is bounded away from the tangent space to ϕ by a distance of δ.

Introduction

An open problem in foliation theory is to determine whether a nonsingular C^1 flow ϕ on an arbitrary closed 3-manifold M has a transverse foliation. Classical results by Fried [1982] and Schwartzman [1957] state conditions for any such flow to have a transverse section, and hence a transverse foliation. Milnor [1958] and Wood [1971] found necessary and sufficiently conditions for the existence of a 2-dimensional foliation transverse to the foliation by circles of a circle bundle. Later, Naimi [1994] did the same for the foliation by circles of a Seifert fibered 3-manifold. Goodman [1986] showed that a simple topological property is, for a C^0-dense class of flows, both necessary and sufficient for the existence of a transverse foliation. However, there are flows that satisfy this property, yet do not admit a transverse foliation; for example, flows on S^3 with no periodic orbits as described in [Schweitzer 1974; Harrison 1988; Kuperberg 1994].

The subtlety of the transverse foliation problem is underscored by the Milnor–Wood result. Specifically, they showed that for circle bundles over a closed surface of positive genus, there is a foliation transverse to the fibers precisely when the Euler number of the bundle is no larger than the negative of the Euler characteristic of the surface. Since one can have a circle bundle of sufficiently small Euler number finitely covering one with a large Euler number, any property of a flow that is preserved under finite covers cannot, in general, be both necessary and sufficient for the existence of a transverse foliation.

In [Goodman and Shields 2007], we showed that when a flow ϕ has no self-return disk (that is, a disk transverse to ϕ that flows continuously into its own

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interior), a simple algorithm for modifying any branched surface transverse to ϕ will eventually produce a branched surface carrying a foliation F precisely when F is transverse to ϕ. We show in Theorem 2.2 that this algorithm also works when ϕ has self-return disks. We then find a procedure that can be used to determine whether or not any branched surface produced by our algorithm carries a foliation. In particular, we describe a process that allows us to modify any branched surface in order to produce an essential branched surface that carries a foliation if and only if the original does (Theorem 2.3). Algorithms in [Agol and Li 2003] can then be applied to determine whether or not this new branched surface carries a foliation. Hence we obtain in Theorem 2.4 an algorithmic means for detecting flows with transverse foliations.

We further show in Theorem 2.5 that any for any $\delta > 0$, one can find a positive integer K such that if the branched surface produced at the K-th stage of our algorithm does not carry a foliation transverse to ϕ, then there are no foliations that remain a bounded distance of at least δ from ϕ. If, on the other hand, this branched surface does carry a foliation, then the algorithm described in Theorem 2.4 will detect that it does.

1. Preliminaries

Throughout, M will be a closed orientable 3-manifold and $\phi : M \times \mathbb{R} \to M$ will be a C^1 nonsingular flow on M. An *orbit segment* of ϕ shall be a curve $\phi(x,t)_{t\in[a,b]}$, where $x \in M$ and $[a,b]$ is a closed interval in $\mathbb{R}$. The *forward orbit* under ϕ of a point $x = \phi(x,0)$ in M will be the set of points $\phi(x,t)_{t>0}$; the *backward orbit* consists of the points $\phi(x,t)_{t<0}$.

The foliations we consider will be C^1 and codimension one.

Branched surface construction. The branched surfaces we associate with the flow ϕ are in the class of regular branched surfaces introduced by [Williams 1974]. In particular, each is transverse to ϕ, connected, and has a set of charts defining local orientation-preserving diffeomorphisms onto one of the models in Figure 1, such that the transition maps are smooth and preserve the transverse orientation indicated by the arrows. (Each local model projects horizontally into a vertical model of $\mathbb{R}^2$ and has a smooth structure induced by $T\mathbb{R}^2$ when we pull back the local projection.) So a branched surface W is a 2-manifold except on a dimension-one subset μ (indicted by the dashed segments) called the *branch set*. The set μ is a 1-manifold except at finitely many isolated points where it intersects itself transversely. The components of $W - \mu$ are the *sectors* of W.

Given a nonsingular flow ϕ, we construct a transverse branched surface by first choosing a finite *generating set* $\Delta = \{D_i\}_{i=1,\dots n}$ *for* ϕ, consisting of pairwise disjoint disks embedded in M that satisfy the following general position requirements:

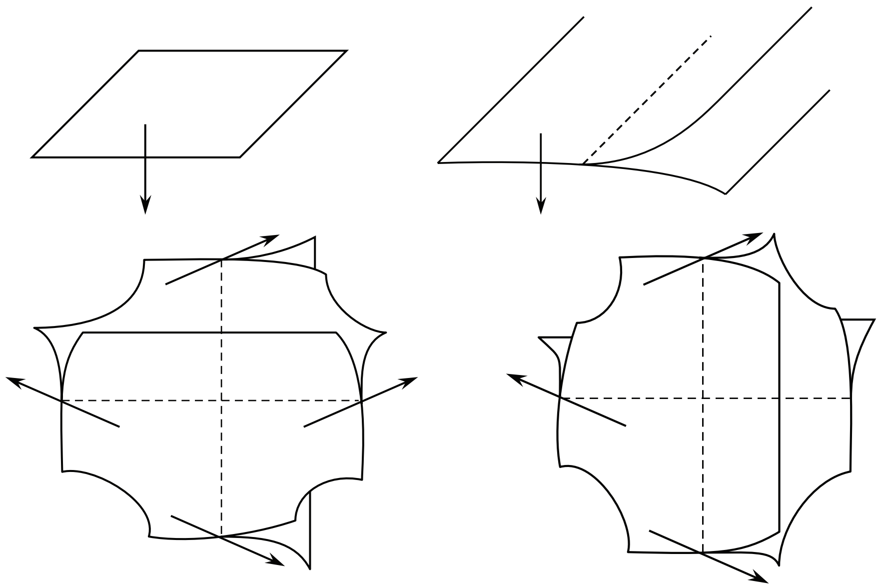


Figure 1

(i) Each D_i is transverse to ϕ.

(ii) Under ϕ, the forward and backward orbit of every point meets the interior of the generating set. In other words, the orbits all meet $\operatorname{int}\Delta = \bigcup_{i=1}^{n} \operatorname{int} D_i$.

(iii) There are only finitely many points in $\partial\Delta = \bigcup_{i=1}^{n} \partial D_i$ whose orbit, forward or backward, meets $\partial\Delta$ before meeting $\operatorname{int}\Delta$.

(iv) The forward orbit of any point in $\partial\Delta$ meets $\partial\Delta$ at most once before meeting $\operatorname{int}\Delta$.

Note that we can find such a set for any given ϕ. In particular, cover M with finitely many flow boxes for ϕ, and select a horizontal slice from each box. A slight modification of each slice can then be used to ensure that the resulting collection of disks satisfies the general position requirements above.

After choosing Δ, cut M open along the interior of each element of Δ to obtain a closed connected submanifold M^* that is transverse to ϕ (except along $\partial\Delta$) and whose boundary contains $\partial\Delta$. This can be thought of as blowing air into M to create an air pocket at each generating disk. By requirement (ii) above, the restriction of ϕ to M^* is a flow ϕ^* with the property that each orbit is homeomorphic to the unit interval $[0, 1]$. Form a quotient space by identifying points that lie on the same orbit of ϕ^*. That is, take the quotient $M^*/\sim$, where $x \sim y$ if x and y lie on the same interval orbit of ϕ^*. This quotient space can be embedded in M so that it is transverse to ϕ and locally modeled on Figure 1. Specifically, we can view the

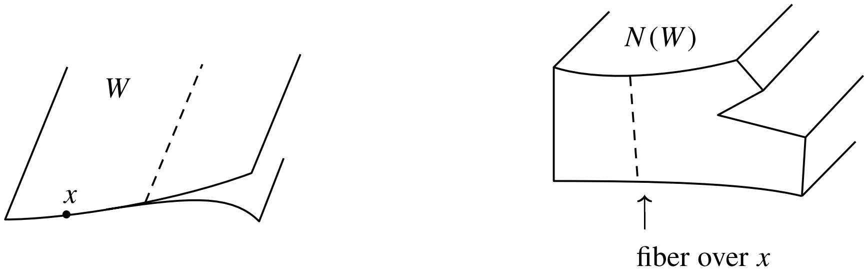


Figure 2

quotient map as enlarging the components of $M - M^*$ until each interval orbit of ϕ^* is contracted to a point in M. We refer to this embedded copy of the quotient space as *the branched surface W constructed from ϕ*.

Although there are many embeddings of the quotient $M^*/\sim$ that are transverse to ϕ, the complement of each is a union of open 3-balls. So any two embeddings of $M^*/\sim$ are diffeomorphic in M; that is, there is a diffeomorphism of M that maps one onto the other. Consequently, we only distinguish between branched surfaces transverse to ϕ up to diffeomorphism of M.

The branched surface W could have many generating sets. For example, if we flow a generating disk forward or backward slightly without allowing any of its points to pass through another point of Δ, then the quotient space described above does not change.

Also, note that we can thicken W in the transverse direction to recover M^* which, for this reason, we shall henceforth call $N(W)$, *the neighborhood of W*. In particular, $N(W)$ is obtained when we replace each point x in W with the interval orbit of ϕ^* whose quotient is x. We shall refer to these interval orbits as the *fibers of $N(W)$*. See Figure 2.

Foliations carried by a branched surface. If a foliation F is transverse to ϕ, and if there exists a generating set Δ for a branched surface W where each element of Δ is contained in a leaf of F, then *F is carried by W*. In particular, when we cut M open along Δ, the foliation F becomes a foliation of $N(W)$ whose leaves (some of which are branched) are transverse to the fibers. The branched leaves are precisely those that contain a boundary component of $N(W)$, since these are the (cut-open) leaves of F containing the elements of Δ. (They can be thought of as leaves of F with air blown into them.) Figure 3 shows a local picture of such a foliation of $N(W)$.

Conversely, each foliation of $N(W)$ that is transverse to the fibers and whose branched leaves contain the boundary components of $N(W)$ corresponds to a foliation of M that is carried by W. In particular, when we collapse the components of $M - N(W)$ (that is, the air pockets) to recover (M, ϕ), each of these foliations

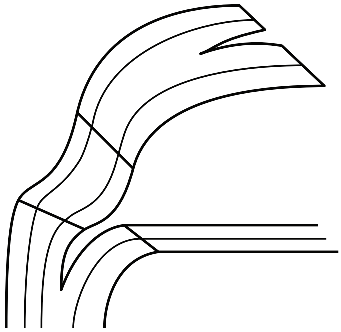
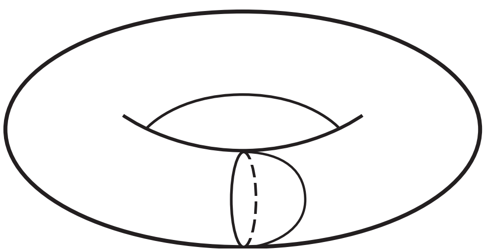


Figure 3

of $N(W)$ yields a foliation of M that is transverse to ϕ and whose leaves contain the elements of Δ. For the most part, we do not distinguish between a foliation of M carried by W and a corresponding foliation of $N(W)$.

As noted above, flowing the disks in any generating set $\Delta = \{D_i\}_{i=1,\dots n}$ for ϕ forward or backward still results in the same branched surface W, provided we do not change the relative position of any two points in $\bigcup_{i=1}^{n} D_i$ along some orbit of ϕ. It follows that W carries a foliation transverse to ϕ if and only if we can move the elements of Δ into leaves of that foliation, while preserving their relative position in the flow direction. We will use this important fact to prove Theorem 2.2.

Reeb skeletons. Given a solid torus Σ embedded in M so that $\partial\Sigma \subset W$, if $\Sigma \cap W$ carries a Reeb foliation of Σ, then we say that Σ is a *Reeb skeleton*. Such an object exists, for example, if some foliation carried by W contains a Reeb component. If a Reeb skeleton Σ contains no other Reeb skeletons, we say that Σ is *minimal*. Here is an example of a minimal Reeb skeleton:



Staircase curves. Given a nonsingular flow ϕ, let $\gamma = \tau_1 * \sigma_2 * \cdots * \tau_{k-1} * \sigma_k * \tau_k$ be a compact curve in M, where τ_1 has nonempty interior and τ_i is a positively oriented orbit segment of ϕ for any $1 \leq i \leq k$. If we can choose this decomposition of γ so that each *step* σ_i has nonempty interior and is contained in an element of some generating set Δ for ϕ, we say γ is a *staircase curve in* (Δ, ϕ). See Figure 4. The *horizontal length* $\|\gamma\|_{\mathrm{hor}}$ of γ is the sum of the lengths of its steps (that is,

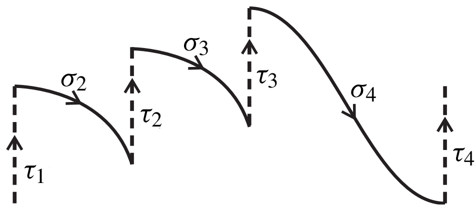


Figure 4

the lengths of the σ_i). We shall only consider staircase curves whose horizontal lengths are nonzero.

2. Main results

In [Goodman and Shields 2007], we described a procedure for successively modifying any branched surface transverse to a flow ϕ, which produces a sequence of branched surfaces $\{W_k\}$ all transverse to ϕ. We then showed the following:

Theorem 2.1. *Given a C^1 nonsingular flow ϕ on a closed orientable 3-manifold M that has no self-return disks, let W be a branched surface constructed from ϕ and let $\{W_k\}$ be a sequence of branched surfaces produced by applying the procedure to W. The flow ϕ is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$.*

Our procedure for successively modifying W specifies a particular way to break the elements of any generating set Δ for W into smaller and smaller disks. If ϕ is transverse to a foliation F, this procedure eventually produces a generating set for a branched surface that carries F. The idea is that once these disks become sufficiently small, each slides injectively along orbit segments of ϕ into a leaf of the foliation F. Moreover, the manner in which we construct these smaller generating disks ensures that this sliding can be done without changing their relative position in the ϕ-direction. So this collection of smaller disks generates a branched surface carrying F.

The proof of Theorem 2.1 requires that we carefully control the size and spacing of the new generating disks created each time we modify Δ. However, the following algorithm for modifying Δ produces the same sequence of branched surfaces (up to diffeomorphism of M).

Given $\Delta = \{D_i\}_{i=1,\dots,n}$, let T be one-third the minimal amount of the time it takes for a point in $\bigcup_{i=1}^n D_i$ to flow back into $\bigcup_{i=1}^n D_i$. For each positive integer k, find $\varepsilon_k > 0$ with the property that flowing any disk D embedded in $\bigcup_{i=1}^n D_i$ with diameter less than ε_k forward or backward for time at most T gives a disk of diameter less than $1/k$. Cover each element of Δ by disks of diameter less than

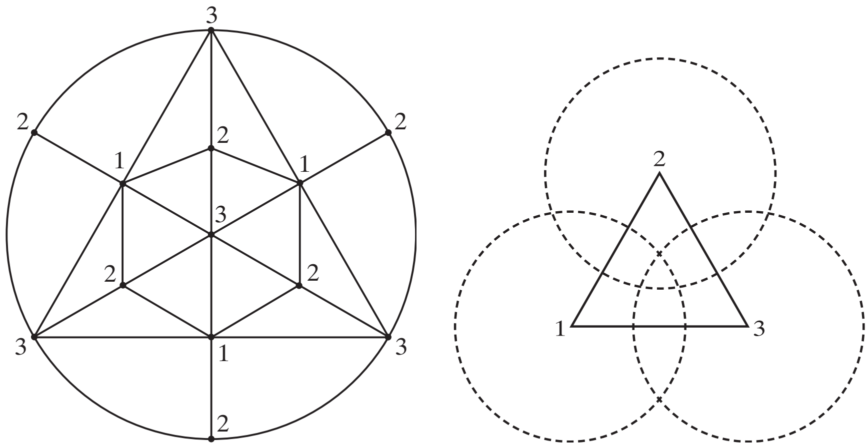


Figure 5

ε_k in the following manner: For each $D_i \in \Delta$, triangulate D_i with a graph of even valence (except along ∂D_i) so that every point in D_i is a distance of at most $\varepsilon_k/3$ from the nearest vertex. (Here we are measuring distance within D_i using the induced metric.) Cover each vertex of the graph with a disk of diameter less than ε_k so that any point $x \in D_i$ is contained in at least one and at most three disks. (Choose these disks so that their boundaries only intersect transversely.) Next, number the disks covering each $D_i \in \Delta$ 1, 2 and 3 so that no two disks of the same number meet (see Figure 5). Then lift all disks numbered 1 forward along the flow for time T and push all disks numbered 3 backward along the flow for time T. (Leave those labeled 2 fixed.) The new collection Δ_k of disks satisfies the conditions for a generating set transverse to ϕ; so Δ_k generates a branched surface W_k. If we use the same cover of Δ, but reduce the amount of time we flow its elements forward or backward, the generating set we obtain still produces the same W_k.

To prove Theorem 2.1, we showed that a flow ϕ with no self-return disks is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$. We now show this to be the case, regardless of whether or not ϕ has a self-return disk.

Theorem 2.2. *Let ϕ be a C^1 nonsingular flow on a closed orientable 3-manifold M and let W be a branched surface constructed from ϕ. The flow ϕ is transverse to a foliation F if and only if iterating the modification process above finitely many times on W yields a branched surface carrying F. Specifically, ϕ is transverse to a foliation F if and only if there exists a $K > 0$ such that W_k carries F for all $k \geq K$.*

Proof. Suppose ϕ is transverse to some foliation F. Let $\Delta = \{D_i\}_{1\leq i\leq n}$ be a generating set for a branched surface W constructed from ϕ. If W carries F, then we're done. So suppose this is not the case. As in the proof of Theorem 2.1, construct a branched surface V using another generating set X for ϕ such that each element of X is contained in a leaf of F and $X \cap \Delta = \varnothing$. So V carries F, and when we cut M open along X to obtain $N(V)$, each element of Δ becomes embedded in the interior of $N(V)$, transverse to the fibers.

Let $\{\Delta_k\}$ be a sequence of generating sets for ϕ obtained by successively applying our modification procedure to Δ. We can change the value T used in the construction of $\{\Delta_k\}$ so that it is less than one-third the minimal amount of time it takes a point in $X \cup \Delta$ to flow back into it, without affecting the corresponding sequence $\{W_k\}$ of branched surfaces. This ensures that when we cut M open along X, each Δ_k also becomes embedded in $N(V)$, transverse to the fibers.

In the proof of Theorem 2.1, we show that if none of the branched surfaces produced by our modification process carry F, then for all k sufficiently large we can find a staircase loop γ_k in $(X \cup \Delta_k, \phi)$ that is contained in $N(V)$. In addition, we can choose these loops so that $\|\gamma_k\|_{\text{hor}} \to 0$ as $k \to \infty$. (This does not require the absence of self-return disks for ϕ.) Moreover, the sequence $\{\gamma_k\}$ corresponds to a sequence $\{\gamma_k^*\}$ of staircase loops in $(X \cup \Delta, \phi)$ contained in $N(V)$ whose horizontal lengths are also decreasing to 0. This follows from the observation that for any k, each step in γ_k has a preimage in Δ (before we flow the broken pieces of Δ forward or backward). The steps of γ_k^* consist of unions of these preimages.

Now, the projection of $\partial X \cup \partial\Delta$ along fibers of $N(V)$ onto V produces a finite graph. Furthermore, each staircase loop in $(X \cup \Delta, \phi)$ that is contained in $N(V)$ corresponds to a cycle of disks from the set $X \cup \Delta$ which, when projected, gives an (possibly self-intersecting) annulus in V. Among the generators for that annulus that are contained in its boundary and hence contained in the finite graph produced above, there exists one of minimal length. It follows that there exists a lower bound on the horizontal length of staircase loops in $(X \cup \Delta, \phi)$ contained in $N(V)$. So for all k sufficiently large, W_k carrries F. □

According to Theorem 2.2, if a nonsingular flow is transverse to a foliation F, then our algorithm for successively modifying any branched surface transverse to that flow will eventually produce a branched surface that carries F. However, we still need a way to actually detect when this occurs. Our method for doing so will require the following:

Theorem 2.3. *Let ϕ be a C^1 nonsingular flow on a closed 3-manifold M and W be a branched surface constructed from ϕ. We can construct a branched surface W'' (embedded in a different manifold M'') such that W'' carries a Reebless foliation if and only if W carries a foliation.*

Proof. Let Δ be a generating set for a branched surface W transverse to ϕ. Perturb ϕ slightly, if necessary, so that inside each minimal Reeb skeleton there exists a periodic orbit that does not meet the branch set μ of W. Afterwards, if none of the periodic orbits inside the Reeb skeleton are attractors or repellors, choose one and "blow it up" so that it has a small tubular neighborhood consisting entirely of periodic orbits (which also misses μ); then perturb the flow within the tube so that it contains an attracting periodic orbit. (The new ϕ can also be used to construct W from Δ.) After all such modifications, each Reeb skeleton contains a disk, in some sector S of W, that is met by an attracting or repelling periodic orbit γ of ϕ and flows, either forward or backward, into its own interior without meeting μ. Also, there exists a corresponding self-return disk D for ϕ (or ϕ^{-1}) contained in some component of $\partial N(W)$. In other words, D projects onto our original self-return disk and is contained in some (split-open) element of Δ whose projection onto W contains S. After collapsing the complement of $N(W)$ in M, flow D slightly forward if γ is an attractor and slightly backward if γ is a repellor. Subsequently, add D to the collection Δ of generating disks for W. If γ is an attractor (repellor), then some of the original generating disks are met by forward (backward respectively) orbit segments from D back into itself. Create holes in these generating disks that are just large enough to ensure that this situation no longer occurs. See Figure 6. (As a result, our generating set no longer consists of embedded disks. However, the branched surface construction described in Section 1 can also be applied to the more general setting where Δ consists of finitely many closed planar surfaces with boundary.) These changes in Δ correspond to the insertion of a Reeb skeleton Σ through S so that the intersection of $\partial\Sigma$ with the branch set of the new W consists of finitely many meridian curves. Furthermore, all sectors branching into $\partial\Sigma$ from

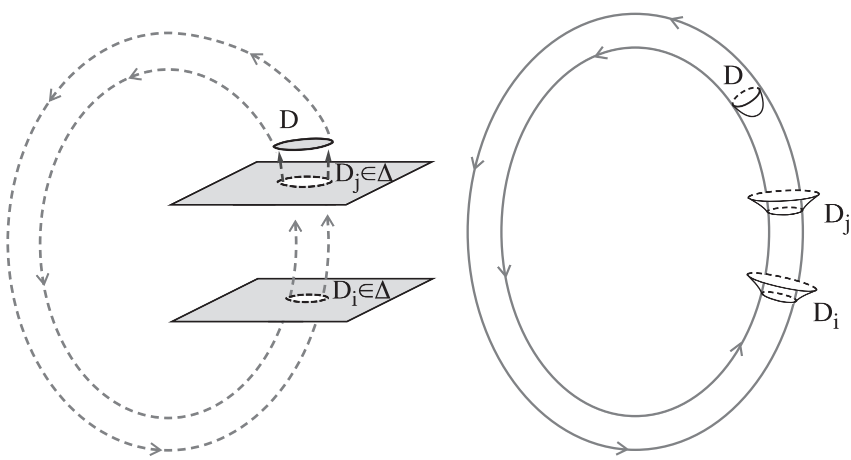


Figure 6

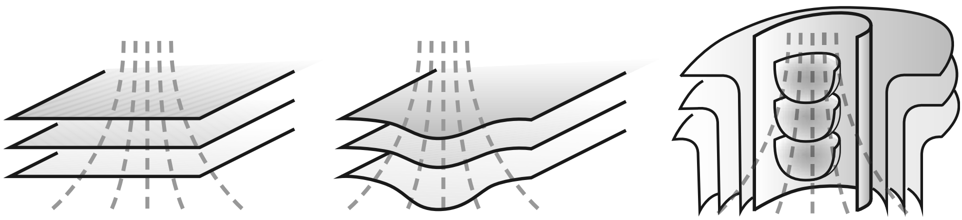


Figure 7

the exterior of Σ do so in the opposite direction than does the only sector branching into $\partial\Sigma$ from the interior of Σ. See Figure 6. So if some foliation F carried by the new W has a Reeb component carried by $\Sigma \cap W$, then F can be modified so that it has only trivial holonomy around the meridian curves of $\partial\Sigma$. This Reeb component then becomes removable in the usual sense. That is, we can modify F to eliminate this Reeb component while staying transverse to ϕ, and when we do so we get a foliation carried by the original W. Consequently, say that such a Reeb skeleton is *removable*.

Conversely, we can modify any foliation carried by the original W by inserting a Reeb component that is carried by $\Sigma \cap W$. So the modified W carries a foliation if and only if the original W carries a foliation. See Figure 7.

Continue to modify W, as above, by inserting a removable Reeb skeleton into the interior of each minimal Reeb skeleton for the original W. (These new Reeb skeletons are pairwise disjoint.) Next, excise the interior of each of the new Reeb skeletons to obtain a manifold M' with boundary. Let ϕ' and W' represent the restriction of ϕ and W, respectively, to M'. Using the identity map, glue M' to a copy of itself (on which the orientation of ϕ' has been reversed) along each of its toral boundary components $T_1, \ldots, T_N$. This produces a new manifold M'' and a new flow ϕ''. Since the flow ϕ' is transverse to $\partial M'$, the new flow is nonsingular. (It is possible that ϕ'' is not C^1 along the *seam* $\bigcup_{1\le i\le N} T_i$. Specifically, when we create M'', it is possible that some of the orbits of ϕ' in M' do not piece together smoothly with the corresponding orbits of ϕ'^{-1} in the copy of M'.)

To ensure that W' and its copy W'_c glue to give another branched surface W'', we modify W'_c slightly near each piece of its branch set contained in the seam. More precisely, the identity map used to glue each toral boundary component T_i to a copy of itself will initially yield local neighborhoods as shown in Figure 8. So we shift the location of each branching of W'_c into T_i slightly, while staying transverse to ϕ'', to obtain local neighborhoods as shown in Figure 8. We then smooth out the orbits of ϕ'' in a small neighborhood of the seam, while staying transverse to the new branched surface W'', so that ϕ'' becomes a nonsingular C^1 flow on M''.

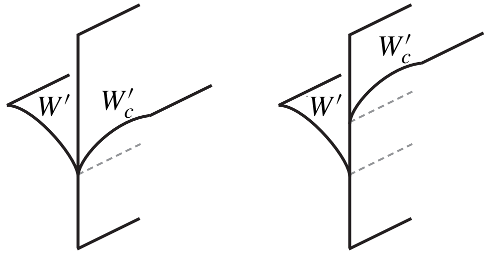


Figure 8

All sectors of W'' that branch into the same component of the seam do so in the same direction. So any smoothly embedded compact surface in W'' that intersects the seam is contained in the seam, and hence is a component of $\partial M'$. It follows that any compact surface that is smoothly embedded in W'' is also smoothly embedded in W.

As noted earlier, if the original W carries a foliation, then our modified W also carries a foliation F where T_i is a leaf contained in int $N(W)$ for each $1 \le i \le N$. In this case, W' carries a foliation of M' where each T_i is a toral leaf in the boundary of some fiber neighborhood $N(W')$. Consequently, W'' also carries a foliation where each T_i is a leaf.

Conversely, W carries a foliation if W'' does. To see this, note that we can thicken W'' to obtain $N(W'')$ so that each T_i becomes embedded in the interior of $N(W'')$. Since for every $i \le N$, all sectors of W'' branching into T_i do so from the same direction, we can isotope any foliation of $N(W'')$ so that each T_i is a leaf [Shields 1996]. So if W'' carries some foliation, then $N(W')$ has a foliation where each T_i is a leaf contained in $\partial N(W')$. See Figure 9. We can then glue Reeb skeletons back into W' along each T_i to get a branched surface transverse to ϕ and carrying a foliation F of M such that each T_i is a leaf bounding a Reeb component of F. In fact, the branched surface we obtain is the same modified W we obtained earlier by inserting removable Reeb skeletons into the original W. It follows that the original W will also carry a foliation.

All that remains is to show that W'' is Reebless. If not, there exists a solid torus Σ'' embedded in M'' so that $\partial\Sigma'' \subset W''$ and $\Sigma'' \cap W''$ carries a Reeb foliation of Σ''. Choose Σ'' so that it does not properly contain another solid torus with these properties. Since $\partial\Sigma''$ is compact and smoothly embedded in W'', either

$$\partial\Sigma'' \cap \textstyle\bigcup_{1\le i\le N} T_i = \varnothing \quad \text{or} \quad \partial\Sigma'' = T_i \quad \text{for some } 1 \le i \le n.$$

In particular, $\partial\Sigma''$ is smoothly embedded in both W' and W. Now, recall that to create M'' we removed a tube through the interior of each Reeb skeleton for W in M to get M', and then glued M' to a copy M'_c of itself. Hence, Σ'' is not contained

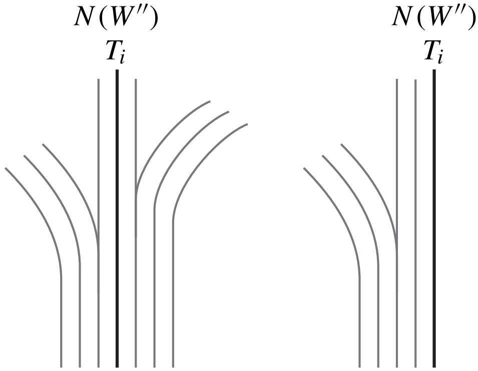


Figure 9

in M'; nor is it contained in M'_c. In other words, $T_i \subseteq \operatorname{int} \Sigma''$ for some $1 \leq i \leq N$. As noted above, all sectors of W'' branching into T_i do so in the same direction. So T_i is a leaf in the Reeb foliation carried by $\Sigma'' \cap W''$. However, this means that T_i bounds a Reeb component of this foliation that is properly contained in Σ'', contradicting the way we chose Σ''. It follows that any foliation carried by W'' is Reebless. □

Theorem 2.4. *Given a closed* 3*-manifold* M*, there is a procedure that detects when a* C^1 *nonsingular flow on* M *has a transverse foliation.*

Proof. Given a nonsingular flow ϕ, let Δ be a generating set for a branched surface W constructed from ϕ and let $\{W_k\}$ be a sequence of branched surfaces obtained by applying our algorithm to W. By Theorem 2.2, some W_k will carry a foliation if and only if ϕ is transverse to a foliation. So we describe a procedure for determining whether or not a given W_k carries a foliation.

For each branched surface W_k in our sequence, we can construct the corresponding Reebless branched surface W''_k and transverse flow ϕ'' by excising a finite nonempty collection τ_k of solid tori and gluing the resulting manifold with boundary to a copy of itself. Choose the set τ_k, as in the proof of Theorem 2.3, so that W''_k carries a Reebless foliation (where the boundary of each element of τ_k is a leaf) if and only if W_k carries a foliation.

Using the procedure described in [Agol and Li 2003, proof of Theorem 5.2, step 1], we can then determine whether the manifold M''_k created during the construction of W''_k is irreducible, prime or homeomorphic to $S^2 \times S^1$. If M''_k is prime, then it has no Reebless foliation, so there can be no such foliation carried by W''_k. If M''_k is homeomorphic to $S^2 \times S^1$, then the only Reebless foliation of M''_k is the trivial foliation by spheres. In this case, W''_k cannot carry a Reebless foliation in which the tori bounding the elements of τ_k are leaves.

So we can assume that M_k'' is irreducible. It then follows that there are no smoothly embedded spheres in W_k'' since such a sphere would be transverse to ϕ_k'' and bound a 3-ball; by Pugh's generalized Poincare index theorem [Pugh 1968], this 3-ball would necessarily contain a singularity for the flow, contradicting that ϕ_k'' is nonsingular.

Hence, Agol and Li's procedure of [2003, proofs of Theorems 2.8 and 3.9] can be used to determine whether or not W_k'' fully carries an essential lamination. In the case that it does, the method of [Gabai 1983, proof of Theorem 5.1] can be used to extend this lamination to a Reebless foliation carried by W_k''. □

We next show that if our initial generating set for ϕ is chosen carefully, then our algorithm can be used to detect whether or not there is a foliation that stays some bounded distance δ away from ϕ. To state the result more precisely, we first need some definitions.

Suppose $U = \{U_i\}_{i=1,\dots,N}$ is a covering of M by flow boxes for ϕ. For each $i \le N$, there is a homeomorphism $h_i : U_i \to I^3$, where $I = [0, 1]$ and all images of orbit segments of ϕ contained in U_i are in the vertical direction (that is, each orbit of $h_i(\phi \cap U_i)$ is of the form $(\{x_0\}X\{t\})_{0\le t\le 1}$ for some $x_0 \in I^2$). For each i, we refer to the preimage of $\partial(I^2) \times I$ under h_i as the *vertical boundary* $\partial_v U_i$ of U_i and the preimages of $I^2 \times \{0\}$ and $I^2 \times \{1\}$ as the *base* and *top*, respectively.

We say that U is a *standard covering* of M if

(1) every point of M is contained in at most three flow boxes in U,

(2) for every i and j, either $U_i \cap U_j = \varnothing$, or $\partial_v U_i$ and intersect $\partial_v U_j$ transversely along a finite number of orbit segments, and

(3) $U_i \cap U_j \cap U_k$ is connected for every i, j and k.

Theorem 2.5. *Given a C^1 nonsingular flow ϕ on a closed 3-manifold M, define $U = \{U_i\}_{i=1,\dots,N}$ to be a standard covering of M by finitely many flow boxes for ϕ such that for all $i \ne j$, the top of U_i does not intersect the top or bottom of U_j. Choose a generating set Δ for ϕ consisting of a horizontal slice from each box that does not meet the top of any box, and let $\{\Delta_k\}$ be a sequence of generating sets obtained by applying our algorithm to Δ. For any $\delta > 0$, we can find an integer $K > 0$ such that for any $k \ge K$, the branched surface generated by Δ_k carries all foliations of M that remain a bounded distance of δ from ϕ. Furthermore, K depends only on δ, ϕ, Δ and U.*

Proof. Assume $\delta > 0$ is given and choose U as in the hypotheses. Let Δ be a generating set for ϕ consisting of one horizontal slice, from each flow box, that does not meet the top of any of the flow boxes. In other words, for each $D \in \Delta$, there exists $1 \le i \le N$ and $0 \le t_0 < 1$ such that

$$D = {h_i}^{-1}(I^2 \times \{t_0\}) \quad \text{and} \quad D \cap ({h_j}^{-1}(I^2 \times \{1\})) = \varnothing$$

for all $1 \leq j \leq N$. Then for each $1 \leq i \leq N$, there exists a $t_i \in (0, 1)$ such that

$$(h_i{}^{-1}(I^2 \times [t_i, 1])) \cap \Delta = \varnothing \quad \text{and}$$
$$(h_i{}^{-1}(I^2 \times [t_i, 1])) \cap (h_j^{-1}(I^2 \times \{0\})) = \varnothing \quad \text{for all } 0 \leq j \leq N,$$

and

$$(h_i^{-1}(I^2 \times [t_i, 1])) \cap (h_j^{-1}(I^2 \times [t_j, 1])) = \varnothing \quad \text{for all } j \neq i.$$

Note that since U is finite, we can find some $d > 0$ such that the distance between any two components of $h_i(U_i \cap (\bigcup_{1\leq j\leq N} h_j^{-1}(I^2 \times [t_j, 1])))$, as well as the distance between any such component and a component of $h_i(U_i \cap (\bigcup_{1\leq j\leq N} h_j^{-1}(I^2 \times \{0\})))$, exceeds d for all $1 \leq i \leq N$.

Now suppose there exists a foliation F of M whose distance from ϕ is bounded below by δ (in that the smallest positive angle between the tangent vector to ϕ and the tangent plane to the foliation at any point exceeds δ). We can construct a branched surface V carrying F using another generating set X for ϕ, where each $C \in X$ is contained in $\bigcup_{1\leq i\leq N} h_i^{-1}(I^2 X[t_i, 1])$ and in a leaf of F. We can also ensure that each orbit segment of $\phi|_{U_i}$ meets $X \cap (h_i^{-1}(I^2 \times [t_i, 1]))$ for all $1 \leq i \leq N$. So, henceforth, we shall refer to $h_i^{-1}(I^2 \times [t_i, 1])$ as the *X-region of U_i*. Since Δ cannot intersect any of the X-regions of U, the elements of $X \cup \Delta$ are pairwise disjoint. Thus when we cut M open along X to obtain $N(V)$ (foliated by F), each element of Δ becomes embedded in the interior of $N(V)$, transverse to the fibers.

Let $\{\Delta_k\}$ be a sequence of generating sets for ϕ obtained by applying our algorithm to Δ. Recall that if we reduce the value T used in the construction of $\{\Delta_k\}$ so that it is less than one-third the minimal amount of time it takes a point in $X \cup \Delta$ to flow back into $X \cup \Delta$, we do not affect the corresponding sequence $\{W_k\}$ of branched surfaces. So we can assume that when we cut M open along X, each Δ_k also becomes embedded in $N(V)$, transverse to the fibers. (The integer K we find will not depend on X or $N(V)$. However, these objects play an important role in the proof of Theorem 2.1, which we adapt here to show that W_K carries F.)

For any $k > 0$, the branched surface W_k carries F if and only if we can flow the elements of Δ_k injectively onto disks in leaves of F without changing their relative position along orbits of ϕ (see Section 2). If we try to do so, *while staying in $N(V)$*, there are only 2 obstructions we could encounter [Goodman and Shields 2007, Lemma 2.2]. The first is the existence of a staircase loop in (Δ_k, ϕ) contained in $N(V)$. This can cause problems, for example, if all the leaves of F are compact. The other possible obstruction involves the existence of a *connecting strip*; that is, a strip embedded in the interior of $N(V)$, transverse to the fibers, with $\partial N(W)$ branching from both its ends. When such a strip is crossed with *negative index* by a staircase curve γ_k in (Δ_k, ϕ) (as in Figure 10, top), yet is crossed with *nonnegative*

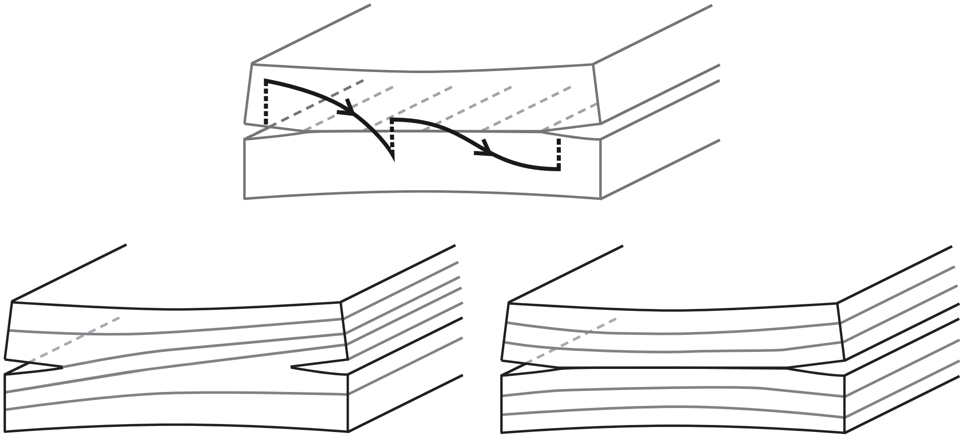


Figure 10

index by F (as in either of the bottom figures in Figure 10), we cannot move the steps of γ_k into leaves of F without either changing their relative position in the flow direction or leaving $N(V)$. (This is the only situation in which a connecting strip presents a problem.)

Both of these obstructions involve staircase curves in (Δ_k, ϕ) that miss X. As noted earlier, each such curve (or loop) γ_k corresponds to a staircase curve (or loop, respectively) γ_k^* in (Δ, ϕ) that also misses X. Also, γ_k^* crosses a connecting strip S with negative index if and only if γ_k crosses S with negative index. (For details, see [Goodman and Shields 2007, proof of Theorem 2.3, page 12].)

So we shall first consider staircase curves in (Δ, ϕ) that miss X. We show that if the horizontal length of such a curve is sufficiently small, then it cannot be a loop, nor can it cross any connecting strip with negative index that is crossed with nonnegative index by F. In other words, we find a constant η such that any staircase curve in (Δ_k, ϕ) that is involved in one of the obstructions described above corresponds to a staircase curve in (Δ, ϕ) whose horizontal length exceeds η. We then show how to find an integer K such that for every staircase curve in (Δ_K, ϕ), the horizontal length of the corresponding staircase curve in (Δ, ϕ) is less than η.

To begin, note that for any $1 \le i \le N$, we can project $(\partial\Delta \cap U_i) \cup (\partial_v U_i)$ onto the base of U_i to obtain a finite graph. We can use this to argue, as in the proof of Theorem 2.2, that there exists a lower bound λ_i on the horizontal length of staircase curves in (Δ, ϕ) contained in U_i that begin and end in the same component of $\Delta \cap U_i$.

For any $1 \le i \le N$, we can also project $U_i \cap (\bigcup_{1\le j\le n} \partial_v U_j)$ onto the base of U_i get another finite graph G_i. There exists a lower bound λ_i' on the lengths

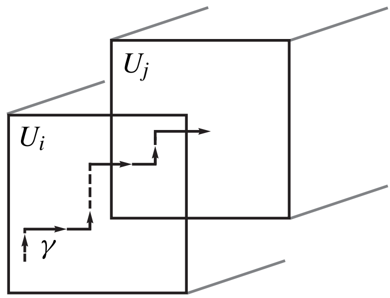


Figure 11

of paths in U_i whose projections join nonadjacent edges of G_i. Choose some $\lambda < \min(\{\lambda_i \mid 0 \le i \le N\} \cup \{\lambda_i' \mid 0 \le i \le N\})$ and let γ be a staircase curve in (Δ, ϕ) contained in $N(V)$ such that $0 < \|\gamma\|_{\text{hor}} < \lambda$.

We first show that γ cannot be a loop. If some flow box in U contains γ, then γ is not a staircase loop, since $\|\gamma\|_{\text{hor}} < \lambda$. So suppose there exist some i, $j \le N$ such that γ begins in U_i, enters U_j and then later exits U_i. In particular, choose i so that once γ exits U_i, it is no longer contained in any flow box. Choose j so that γ exits U_i while still in U_j, and so that no flow box met by γ before it enters U_j has this property. There exists a point at which γ enters U_j and remains in U_j until after its exit from U_i. Let γ' be the subcurve of γ from this point of entry into U_j to its point of exit from U_i.

For every $i \le N$, that $(h_i^{-1}(I^2 \times [t_i, 1])) \cap \Delta$ is empty means that there can be no steps of γ in the X-region of U_i. So γ cannot begin in $[h_i^{-1}(I^2 \times [t_i, 1])]$, since this would mean that the bottom of U_j intersects this X-region, contradicting the way we chose t_i. Furthermore, the way we chose X ensures that any orbit segment of ϕ that enters a X-region of U must meet X before exiting that region. So since no orbit segment in γ can flow through the X-region of U_i, the terminal point of γ' lies in $\partial_v U_i$. If the initial point of γ' lies in $\partial_v U_j$ (as in Figure 11), then projecting γ' onto the base of U_j yields a curve whose initial point and terminal point lie in adjacent edges of G_j, and whose interior does not meet G_j (since $\|\gamma\|_{\text{hor}} < \lambda_j'$). It follows that γ' is contained in U_k for some $k \ne i, j$. (Figure 12 shows the projection of γ' onto a portion of G_j.) By the way we chose i, the curve γ must then enter U_k before entering U_j, contradicting the way we chose j.

So γ enters U_j through its base, and if it subsequently exits U_j, it would have to do so through $\partial_v U_j$. However, this would mean that there exists a subcurve γ'' contained in $\gamma \cap U_j$ that begins in $\partial_v U_i$ and ends in $\partial_v U_j$. Specifically, γ'' is the portion of γ that begins at its exit from U_i and ends at its exit from U_j. Since $\|\gamma\|_{\text{hor}} < \lambda_j'$, the initial point and terminal point of the projected γ'' lie in adjacent edges of G_j, and its interior does not meet G_j. Let x be the vertex adjacent to both

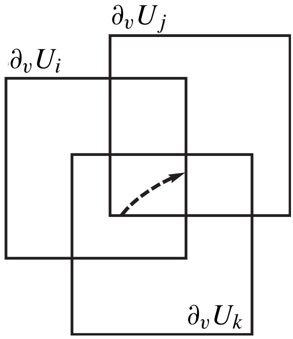


Figure 12

these edges. There exists a flow box U_l, with $l \neq i, j$, containing x. Specifically, the point at which γ enters U_j is contained in U_l. Moreover, γ cannot exit U_l before exiting U_j (by our assumption that the interior of γ'' meets no edges of G_j). So since γ exits U_i while in U_j and by the way we chose i, it enters U_l and (before entering U_j) remains there until its exit from U_i, contradicting the way we chose j. It follows that once γ enters U_j it cannot leave it. In particular, γ cannot be a loop.

Now suppose that γ crosses some connecting strip S with negative index and that F crosses S with nonnegative index. Let C' and C'' be the elements of X containing the ends of S. Specifically, the initial point $\gamma(0)$ of γ lies in C' and the terminal point $\gamma(1)$ lies in C''. Furthermore, the first step of γ intersects some fiber of $N(V)$ above C' and the last (higher) step intersects a fiber below C''. See Figure 13. Since the steps of γ are contained in Δ (which does not intersect any of the X-regions), γ must exit the X-region containing its initial point from the top, before entering the X-region containing its terminal point from the bottom.

So if γ is contained in U_i, the distance between $h_i(\gamma(0))$ and $h_i(\gamma(1))$ exceeds d. In particular, the distance in the vertical direction between the horizontal slices of I^3 containing $h_i(\gamma(0))$ and $h_i(\gamma(1))$, respectively, exceeds $d - \|h_i(\gamma)\|_{\text{hor}}$ (which

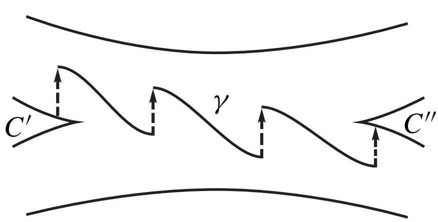


Figure 13

is possibly negative). Now, there exists a constant c such that

$$\frac{1}{c}(d(h_i(x), h_i(y)) \leq d(x, y) \leq c(d(h_i(x), h_i(y)),$$

for all $i \leq N$ and all $x, y \in M$. So the horizontal length of $h_i(\gamma)$ is less than $c\|\gamma\|_{\text{hor}}$. In particular, whenever $\|\gamma\|_{\text{hor}} < d/(2c)$, the absolute value of the smallest angle between the foliation $h_i(F|_{U_i})$ and the flow in the vertical direction must, at some point p, be less than $\arctan((2c\|\gamma\|_{\text{hor}})/(d))$ (since F crosses Σ with a nonnegative index). There also exists a constant ζ such that

$$(1/\zeta)(\angle d(h_i(v), h_i(w)) \leq \angle(v, w) \leq \zeta(\angle(h_i(v), h_i(w))$$

for all $i \leq N$ and any nonzero vectors $v, w, \in T_p(U_i)$. So, in this case, the angle between F and ϕ at $h_i^{-1}(p)$ is less than $\zeta \arctan((2c\|\gamma\|_{\text{hor}})/(d))$.

If, on the other hand, γ begins in U_i and ends in U_j (that is, one end of Σ is contained in U_i and the other is contained in U_j), then since γ enters U_j from the bottom, the lengths of both $h_i(\gamma \cap U_i)$ and $h_j(\gamma \cap U_j)$ are at least d. In particular, the distance in the vertical direction between bottom of $h_j(U_j)$ and the horizontal slice of I^3 containing $h_j(\gamma(1)$ exceeds $d - \|h_j(\gamma \cap U_j)\|_{\text{hor}}$. Likewise, the distance in the vertical direction between $h_i((h_j^{-1}(I^2 \times \{0\})) \cap U_i)$ and the horizontal slice of I^3 containing $h_i(\gamma(0))$ exceeds $d - \|h_i(\gamma \cap U_i)\|_{\text{hor}}$. Hence we can argue, as in the previous case, that whenever $\|\gamma\|_{\text{hor}} < d/(2c)$, somewhere in $U_i \cup U_j$ the angle between F and ϕ is less than $\zeta \arctan((2c\|\gamma\|_{\text{hor}})/(d))$.

Given any $\delta > 0$, we can choose an η with $0 < \eta < \min\{\lambda, d/(2c)\}$ so that $\delta > \zeta \arctan(2c\eta)/d)$. As shown above, the horizontal length of any staircase loop in (Δ, ϕ) is at least λ, and therefore exceeds η. Furthermore, since the foliation F is bounded away from ϕ by δ, the horizontal length of any staircase curve in (Δ, ϕ) that crosses a connecting strip with a different index than does F also exceeds η. So all that remains to show is that we can find an integer K such that for every staircase curve γ_K in (Δ_K, ϕ), the horizontal length of the corresponding staircase curve γ_K^* in (Δ, ϕ) is less than η.

For this, recall that to construct Δ_k, we cover each element of Δ by disks of diameter less than some number ε_k (where $\varepsilon_k \to 0$ as $k \to \infty$). We then flow some of these disks forward and some backward to obtain Δ_k. Since no two adjacent disks in same element of Δ move in the same direction, at most three consecutive steps in γ_k have preimages in the same element of Δ. Hence, if γ_k is a staircase curve in (Δ_k, ϕ), each step in the corresponding staircase curve γ_k^* in (Δ, ϕ) has length less than $3\varepsilon_k$.

Now choose K sufficiently large to ensure that $6\varepsilon_K P < \eta$, where P is the maximal number of components in $\Delta \cap U_i$ over all $i \leq N$. If γ_K^* is contained in U_i, then each of its steps is contained in a distinct element of $\Delta \cap U_i$. For suppose, to the contrary, that there exists a subcurve of γ_K^* that begins and ends in the same component of $\Delta \cap U_i$. We can choose this subcurve so that its interior does not meet any component of $\Delta \cap U_i$ more than once. This ensures that its horizontal length will then be less than $3\varepsilon_K P < \eta < \lambda < \lambda_i$, contradicting the way we chose λ_i. It follows that when γ_K^* is contained in U_i, each of its steps is contained in a distinct element of $\Delta \cap U_i$; hence $\|\gamma_K^*\|_{\mathrm{hor}} < 3\varepsilon_K P < \eta$.

If on the other hand, the initial point $\gamma_K^*(0)$ of γ_K^* lies in U_i and γ_K^* exits U_i after entering some other flow box U_j, then either γ_K^* remains in U_j or it exits U_j at some point $\gamma_K^*(s_1)$, $s_1 > 0$. In the former case, the horizontal length of γ_K^* is less than $6\varepsilon_K P < \eta$. In the latter case, this is true for the subcurve $\gamma_K^*(s)_{0 \le s \le s_1}$. But $\eta < \lambda$ and we have already shown that any staircase curve in (Δ, ϕ) with horizontal length less than λ cannot exit U_j. So this latter case cannot occur. □

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