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A metric is formal if all products of harmonic forms are again harmonic. The existence of a formal metric implies Sullivan formality of the manifold, and hence formal metrics can exist only in the presence of a very restricted topology. We show that a warped product metric is formal if and only if the warping function is constant and derive further topological obstructions to the existence of formal metrics. In particular, we determine the necessary and sufficient conditions for a Vaisman metric to be formal.

1. Introduction

A fundamental problem in algebraic topology is the reading of the homotopy type of a space in terms of cohomological data. A precise definition of this property was given by Sullivan [1977] and called *formality*. As concerns manifolds, it is known, for example, that all compact Riemannian symmetric spaces and all compact Kähler manifolds are formal. For a recent survey of topological formality, see [Papadima and Suciu 2009].

Sullivan also observed that if a compact manifold admits a metric such that the wedge product of any two harmonic forms is again harmonic, then, by Hodge theory, the manifold is formal. This motivated the following definition:

Definition 1.1 [Kotschick 2001]. A closed manifold is called *geometrically formal* if it admits a formal Riemannian metric.

In particular, the length of any harmonic form with respect to a formal metric is (pointwise) constant. This larger class of metrics having all harmonic (one-)forms of constant length naturally appears in other geometric contexts, for instance in the study of certain systolic inequalities, and has been investigated in [Nagy 2006; Nagy and Vernicos 2004].

Classical examples of geometrically formal manifolds are compact symmetric spaces. In [Kotschick and Terzić 2003; 2011] more general examples are provided,

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both of geometrically formal and of formal but nongeometrically formal homogeneous manifolds.

Geometric formality imposes strong restrictions on the (real) cohomology of the manifold. For example, it is proven in [Kotschick 2001] that a manifold admits a nonformal metric if and only if it is not a rational homology sphere.

In this note, we shall obtain further obstructions to formality. We shall see (Section 2) that if a compact manifold with $b_1 = p \geq 1$ admits a formal metric, and if there exist two vanishing Betti numbers such that the distance between them is not larger than $p+2$, then all the intermediary Betti numbers must be zero too. Also, a conformal class of metrics on an even-dimensional compact manifold with nonzero middle Betti number can contain no more than one formal metric.

Our main concern will be the formality of warped products (Section 2). We will show that a warped product metric on a compact manifold is formal if and only if the warping function is constant. On the way, we shall also provide a proof for the fact (stated in [Kotschick 2001], for instance) that a product of formal metrics is formal.

Unlike Kähler manifolds, which are known to be formal, for the time being, nothing is known about the Sullivan formality of locally conformally Kähler (in particular Vaisman) manifolds. In Section 3 of this note, we shall discuss compact Vaisman manifolds, whose universal cover is a special type of warped product, a Riemannian cone to be precise, and we shall find obstructions to the metric formality of a Vaisman metric. Several computational facts and their proofs are gathered in the Appendix.

2. Geometric formality of warped product metrics

For completeness, and as a first step in the study of geometrically formal warped products, we provide a proof for the formality of Riemannian product formal metrics.

Proposition 2.1. *If (M_1, g_1) and (M_2, g_2) are two compact Riemannian manifolds with formal metrics, then the metric $g = g_1 + g_2$ on the product manifold $M = M_1 \times M_2$ is also formal.*

Proof. Let $\gamma \in \Omega^p M$ and $\gamma' \in \Omega^q M$ be two harmonic forms on M. By Lemma A.2, γ and γ' are given by linear combinations with real coefficients of the basis elements in (A-3). Thus, it is enough to check that the exterior product of any two such basis elements is a harmonic form on M. But

$$\big(\pi_1^*(\alpha) \wedge \pi_2^*(\beta)\big) \wedge \big(\pi_1^*(\alpha') \wedge \pi_2^*(\beta')\big) = (-1)^{|\alpha'||\beta|} \pi_1^*(\alpha \wedge \alpha') \wedge \pi_2^*(\beta \wedge \beta'),$$

which is g-harmonic on M by Lemma A.2 and by the formality of g_1 and g_2 (as $\alpha \wedge \alpha'$ is again a g_1-harmonic form and $\beta \wedge \beta'$ a g_2-harmonic form). ☐

We now pass to the setting we are mainly interested in, warped products.

Theorem 2.2. *Let (B^n, g_B) and (F^m, g_F) be two compact Riemannian manifolds with formal metrics. Then the warped product metric $g = \pi^*(g_B) + (\varphi\circ\pi)^2\sigma^*(g_F)$ on $B \times_\varphi F$ is formal if and only if the warping function φ is constant.*

Proof. Let $\beta \in \Omega^p(F)$ be a g_F-harmonic form on F (as $b_m(F) = 1$, there exists at least a harmonic m-form on F). From the equalities (A-4) in the Appendix, it follows that $\sigma^*\beta$ is a g-harmonic form on the warped product $B \times_\varphi F$. If we assume the warped metric g to be formal, it follows in particular that the length of $\sigma^*\beta$ is constant. As g_F is also assumed to be formal, the length of β is constant as well. On the other hand,

$$g(\sigma^*\beta, \sigma^*\beta) = (\varphi\circ\pi)^{2p} g_F(\beta, \beta)\circ\sigma, \tag{2-1}$$

showing that the function φ must be constant.

Conversely, if φ is constant, then the warped product reduces to the Riemannian product between the Riemannian manifolds (B, g_B) and $(F, \varphi^2 g_F)$, which is geometrically formal by Proposition 2.1. □

Remark 2.3. From the above proof we see that Theorem 2.2 holds more generally for metrics having all harmonic forms of constant length.

An interesting question regarding the formal metrics is their existence in a given conformal class. Under a weak topological assumption, we prove that there may exist at most one such formal metric. More precisely, we have

Proposition 2.4. *Let M^{2n} be an even-dimensional compact manifold whose middle Betti number $b_n(M)$ is nonzero. Then, in any conformal class of metrics there is at most one formal metric* (*up to homothety*).

Proof. Let $[g]$ be a class of conformal metrics on M and suppose there are two formal metrics g_1 and $g_2 = e^{2f} g_1$ in $[g]$. The main observation is that in the middle dimension the kernel of the codifferential is invariant at conformal changes of the metric, so that there are the same harmonic forms for all metrics in a conformal class: $\mathcal{H}^n(M, g_1) = \mathcal{H}^n(M, g_2)$. As $b_n(M) \geq 1$ there exists a nontrivial g_1-harmonic (and thus also g_2-harmonic) n-form α on M. The length of α must then be constant with respect to both metrics, which are assumed to be formal and thus we get

$$g_2(\alpha, \alpha) = e^{2nf} g_1(\alpha, \alpha),$$

which shows that f must be constant. □

Using the product construction to ensure that the middle Betti number is nonzero, one can build such examples of formal metrics which are unique in their conformal class.

Other examples are provided by manifolds with "big" first Betti number, as follows from the following property of "propagation" of Betti numbers on geometrically formal manifolds proven in [Kotschick 2001, Theorem 7]: *if* $b_1(M) = p \geq 1$, *then* $b_q(M) \geq \binom{p}{q}$, *for all* $1 \leq q \leq p$. In particular, if $b_1(M^{2n}) \geq n$, then $b_n(M^{2n}) \geq 1$.

Another property of the Betti numbers of geometrically formal manifolds is this:

Proposition 2.5. *Let* M^n *be a compact geometrically formal manifold such that* $b_1(M) = p \geq 1$. *If there exist two vanishing Betti numbers* $b_k(M) = b_{k+l}(M) = 0$, *for some* k *and* l *with* $0 < k + l < n$ *and* $0 < l \leq p + 1$, *then all intermediary Betti numbers must vanish*: $b_i(M) = 0$, *for* $k \leq i \leq k + l$. *In particular*, *if there exists* $k \geq (n - p - 1)/2$ *such that* $b_k(M) = 0$, *then* $b_i(M) = 0$ *for all* $k \leq i \leq n - k$.

Proof. Let $\{\theta_1, \dots, \theta_p\}$ be an orthogonal basis of g-harmonic 1-forms, where g is a formal metric on M. We first notice that here is no ambiguity in considering the orthogonality with respect to the global scalar product or to the pointwise inner product, because, when restricting ourselves to the space of harmonic forms of a formal metric, these notions coincide. This is mainly due to [Kotschick 2001, Lemma 4], which states that the inner product of any two harmonic forms is a constant function. Thus, if two harmonic forms α and β are orthogonal with respect to the global product, we get

$$0 = (\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \, d\mathrm{vol}_g = \langle \alpha, \beta \rangle \, \mathrm{vol}(M),$$

showing that their pointwise inner product is the zero-function.

It is enough to show that $b_{k+1}(M) = 0$ and then use induction on i. Let α be a harmonic $(k + 1)$-form. By formality, $\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_{l-1} \wedge \alpha$ is a harmonic $(k + l)$-form and thus must vanish, since $b_{k+l}(M) = 0$. On the other hand,

$$\theta_j^\sharp \lrcorner \alpha = (-1)^{k(n-k-1)} * (\theta_j \wedge *\alpha)$$

is a harmonic k-form, again by formality. Since $b_k(M) = 0$, it follows that $\theta_j^\sharp \lrcorner \alpha$ vanishes for $1 \leq j \leq p$. Then, since $\{\theta_1, \dots, \theta_p\}$ are also orthogonal, we obtain

$$0 = \theta_1^\sharp \lrcorner \cdots \lrcorner \theta_{l-1}^\sharp \lrcorner (\theta_1 \wedge \cdots \wedge \theta_{l-1} \wedge \alpha) = \pm |\theta_1|^2 \cdots |\theta_{l-1}|^2 \alpha,$$

which implies that $\alpha = 0$, because each θ_j has nonzero constant length. This shows that $b_{k+1}(M) = 0$. □

3. Geometric formality of Vaisman metrics

A Vaisman manifold is a particular type of locally conformal Kähler (LCK) manifold. It is defined as a Hermitian manifold (M, J, g), of real dimension $n = 2m \geq 4$, whose fundamental 2-form ω satisfies the conditions

$$d\omega = \theta \wedge \omega, \quad \nabla \theta = 0.$$

Here θ is a (closed) 1-form, called the Lee form, and ∇ is the Levi-Civita connection of the LCK metric g (we always consider $\theta \neq 0$, to not include the Kähler manifolds among the Vaisman ones).

Locally, $\theta = df$ and the *local* metric $e^{-f}g$ is Kähler, hence the name LCK. When lifted to the universal cover, these local metrics glue to a global one, which is Kähler and acted on by homotheties by the deck group of the covering.

In the Vaisman case, the universal cover is a Riemannian cone. In fact, compact Vaisman manifolds are closely related to Sasakian ones, as the following structure theorem shows:

Theorem 3.1 [Ornea and Verbitsky 2003]. *Compact Vaisman manifolds are mapping tori over S^1. More precisely, the universal cover $\tilde{M}$ is a metric cone $N \times \mathbb{R}^{>0}$, with N compact Sasakian manifold and the deck group is isomorphic with $\mathbb{Z}$, generated by*

$$(x, t) \mapsto (\lambda(x), t + q)$$

for some $\lambda \in \mathrm{Aut}(N)$, $q \in \mathbb{R}^{>0}$.

This puts compact Vaisman manifolds into the framework of warped products and motivates their consideration here.

Vaisman manifolds are abundant. Any Hopf manifold (quotient of $\mathbb{C}^{\mathbb{N}} \setminus \{0\}$ by the cyclic group generated by a semisimple operator with subunitary eigenvalues) is such, as are its compact complex submanifolds [Verbitsky 2004, Proposition 6.5]. A complete list of compact Vaisman surfaces is given in [Belgun 2000].

On the other hand, examples of LCK manifolds (satisfying only the condition $d\omega = \theta \wedge \omega$ for a closed θ) which cannot admit any Vaisman metric are also known: for example, one type of Inoue surface and the nondiagonal Hopf surface; see [Belgun 2000]. The nondiagonal Hopf surface is particularly relevant for our discussion because it is topologically formal, as are all manifolds having the same cohomology ring as a product of odd spheres.

Being parallel and Killing [Dragomir and Ornea 1998], the Lee field $\theta^\sharp$ is real holomorphic and, together with $J\theta^\sharp$, generates a complex one-dimensional totally geodesic Riemannian foliation $\mathcal{F}$. Note that $\mathcal{F}$ is transversally Kähler, meaning that the transversal part of the Kähler form is closed (for a proof of this result, see [Vaisman 1982, Theorem 3.1]).

In the sequel, the terms *basic* (*foliate*) and *horizontal* refer to $\mathcal{F}$. We recall that a form is called *horizontal* with respect to a foliation $\mathcal{F}$ if its interior product with any vector field tangent to the foliation vanishes and is called *basic* if in addition its Lie derivative along a vector field tangent to the foliation also vanishes. Moreover, we shall use the basic versions of the standard operators acting on $\Omega^*_B(M)$, the space of basic forms: Δ_B is the basic Laplace operator, L_B is the exterior multiplication with the transversal Kähler form and Λ_B its adjoint with respect to the transversal metric. For details on these operators and their properties we refer the reader to [Tondeur 1988, Chapter 12].

Here is the main result of this section. It puts severe restrictions on formal Vaisman metrics.

Theorem 3.2. *Let* (M^{2m}, g, J) *be a compact Vaisman manifold. The metric* g *is geometrically formal if and only if* $b_p(M) = 0$ *for*

$$2 \leq p \leq 2m-2, \quad b_1(M) = b_{2m-1}(M) = 1,$$

that is, M *is a cohomological Hopf manifold.*

Proof. Let $\gamma \in \Omega^p(M)$ be a harmonic form on M for some p, $1 \leq p \leq m-1$. By [Vaisman 1982, Theorem 4.1], γ has the form

$$\gamma = \alpha + \theta \wedge \beta, \tag{3-1}$$

with α and β basic, transversally harmonic and transversally primitive.

Since α is basic, $J\alpha$ is also a basic p-form that is transversally harmonic and transversally primitive:

$$\Delta_B(J\alpha) = 0, \quad \Lambda_B(J\alpha) = 0,$$

because Δ_B and Λ_B both commute with the transversal complex structure J (as the foliation is transversally Kähler). Again from the theorem just cited, by taking $\beta = 0$, it follows that $J\alpha$ is a harmonic form on M: $\Delta(J\alpha) = 0$.

The assumption that g is geometrically formal implies that $\alpha \wedge J\alpha$ is harmonic on M, so that in particular it is coclosed: $\delta(\alpha \wedge J\alpha) = 0$. By [Vaisman 1982] (where the term *transversally effective* is used instead of transversally primitive), this implies that $\alpha \wedge J\alpha$ is transversally primitive: $\Lambda_B(\alpha \wedge J\alpha) = 0$.

Otherwise, by [Grosjean and Nagy 2009, Proposition 2.2], for primitive forms $\eta, \mu \in \Lambda^p V$, where (V, g, J) is any Hermitian vector space, the algebraic relation

$$(\Lambda)^p(\eta \wedge \mu) = (-1)^{(p(p-1))/2} p \langle \eta, J\mu \rangle, \tag{3-2}$$

holds, where J is the extension of the complex structure to $\Lambda^* V$ defined by

$$(J\eta)(v_1, \dots, v_p) := \eta(Jv_1, \dots, Jv_p), \quad \text{for all } \eta \in \Lambda^p V, v_1, \dots, v_p \in V.$$

We apply the formula above to the transversal Kähler geometry and conclude that α vanishes everywhere:

$$0 = (\Lambda_B)^p(\alpha \wedge J\alpha) = (-1)^{(p(p+1))/2} p \langle \alpha, \alpha \rangle.$$

The same argument as above applied to $\beta \in \Omega_B^{p-1}(M)$ shows that β is identically zero if $p \geq 2$. Thus, $\gamma = 0$ for $2 \leq p \leq m-1$, which proves that

$$b_2(M) = \cdots = b_{m-1}(M) = 0.$$

If $p=1$, then β is a basic function, which is transversally harmonic, so that β is a constant. Thus γ is a multiple of θ, showing that the space of harmonic 1-forms on M is 1-dimensional: $b_1(M)=1$.

It remains to show that the Betti number in the middle dimension, $b_m(M)$, also vanishes. This follows from Proposition 2.5 applied to $p=1$, $k=m-1$ and $l=2$.

The converse is clear, since the space of harmonic forms with respect to the Vaisman metric g is spanned by $\{1, \theta, *\theta, d\mathrm{vol}_g\}$ and thus the only product of harmonic forms which is not trivial is $\theta \wedge *\theta = g(\theta, \theta)\, d\mathrm{vol}_g$, which is harmonic because θ has constant length, being a parallel 1-form. □

Remark 3.3. (i) There exist Vaisman manifolds that do not admit any formal Vaisman metric. Indeed, let $f : N \hookrightarrow \mathbb{C}P^n$ be an embedded curve of genus $g > 1$ and let M be the total space of the induced Hopf bundle $f^*(S^1 \times S^{2n+1})$. Then M is Vaisman and $b_1(M) > 1$ [Vaisman 1982], hence, according to 3.2, it does not admit any formal Vaisman metric. Other examples can be found in [Belgun 2000].

(ii) On the other hand, we do not have an example of a topologically formal complex compact manifold, which admits Vaisman metrics, but does not admit geometrically formal Vaisman metrics. This seems to be a difficult open problem.

(iii) In complex dimension 2 the Vaisman condition in Theorem 3.2 is not necessary. Due to the results of Kotschick [2001], the existence of any geometrically formal metric on a non-Kähler surface implies that $b_1=1$ and $b_2=0$.

(iv) Theorem 3.2 may be considered as an analogue of the following result on the geometric formality of Sasakian manifolds.

Theorem 3.4 [Grosjean and Nagy 2009, Theorem 2.1]. *Let (M^{2n+1}, g) be a compact Sasakian manifold. If the metric g is geometrically formal, then $b_p(M)=0$ for $1 \le p \le 2n$, that is, M is a real cohomology sphere.*

Appendix: Auxiliary results

Lemma A.1 (characterization of geometric formality). *Let α and β be two harmonic forms on a compact Riemannian manifold (M^n, g). Then $\alpha \wedge \beta$ is harmonic if and only if*

$$\text{(A-1)} \qquad \sum_{i=1}^{n} (e_i \lrcorner \alpha) \wedge \nabla_{e_i} \beta = -(-1)^{|\alpha||\beta|} \sum_{i=1}^{n} (e_i \lrcorner \beta) \wedge \nabla_{e_i} \alpha,$$

where $\{e_i\}_{i=\overline{1,n}}$ is a local orthonormal basis of vector fields. Thus, the metric g is formal if and only if (A-1) holds for any two g-harmonic forms.

Proof. Since M is compact, $\alpha \wedge \beta$ is harmonic if and only if it is closed and coclosed. As $\alpha \wedge \beta$ is closed, we have to show that (A-1) is equivalent to $\alpha \wedge \beta$ being coclosed. This is implied by the following:

$$
\begin{aligned}
\delta(\alpha\wedge\beta)&\\
&=-\sum_{i=1}^{n} e_i\lrcorner\nabla_{e_i}(\alpha\wedge\beta)=-\sum_{i=1}^{n} e_i\lrcorner(\nabla_{e_i}\alpha\wedge\beta+\alpha\wedge\nabla_{e_i}\beta)\\
&=\delta\alpha\wedge\beta-(-1)^{|\alpha|}\sum_{i=1}^{n}\nabla_{e_i}\alpha\wedge(e_i\lrcorner\beta)-\sum_{i=1}^{n}(e_i\lrcorner\alpha)\wedge\nabla_{e_i}\beta+(-1)^{|\alpha|}\alpha\wedge\delta\beta\\
&=-(-1)^{|\alpha||\beta|}\sum_{i=1}^{n}(e_i\lrcorner\beta)\wedge\nabla_{e_i}\alpha-\sum_{i=1}^{n}(e_i\lrcorner\alpha)\wedge\nabla_{e_i}\beta. \qquad\square
\end{aligned}
$$

Riemannian products. Let $(M^{n+m}, g) = (M_1^n, g_1) \times (M_2^m, g_2)$. We denote by $\pi_i : M \to M_i$ the natural projections, which are totally geodesic Riemannian submersions.

One may describe the bundle of p-forms on M as follows:

$$
\Lambda^p M = \bigoplus_{k=0}^{p} \pi_1^*(\Lambda^k M_1) \otimes \pi_2^*(\Lambda^{p-k} M_2). \tag{A-2}
$$

This identification also works for the space of harmonic forms, namely the harmonic forms on (M, g) can be described in terms of the harmonic forms on the factors (M_1, g_1) and (M_2, g_2). To this end let $\mathcal{H}^k(M_i, g_i)$ be the space of harmonic k-forms on M_i and let $b_k(M_i)$ be the Betti numbers of M_i, $i = 1, 2$.

Lemma A.2. *Let $\{\alpha_1^k, \dots, \alpha_{b_k(M_1)}^k\}$ be a basis of $\mathcal{H}^k(M_1, g_1)$ and $\{\beta_1^k, \dots, \beta_{b_k(M_2)}^k\}$ a basis of $\mathcal{H}^k(M_2, g_2))$. Then the forms*

$$
\left\{\pi_1^*(\alpha_s^k) \wedge \pi_2^*(\beta_l^{p-k}) \mid 1 \le s \le b_k(M_1),\ 1 \le l \le b_{p-k}(M_2),\ 0 \le k \le p\right\} \tag{A-3}
$$

form a basis of the space of $\mathcal{H}^p(M, g)$, for each $0 \le p \le m+n$.

For a proof, see [Griffiths and Harris 1978, page 105].

Warped products. Let (B^n, g_B) and (F^m, g_F) be two Riemannian manifolds and $\varphi > 0$ be a smooth function on B. Then $M = B \times_\varphi F$ denotes the warped product with the metric $g = \pi^*(g_B) + (\varphi \circ \pi)^2 \sigma^*(g_F)$, where $\pi : M \to B$ and $\sigma : M \to F$ are the natural projections.

Let $\{e_i\}_{i=\overline{1,n}}$ be a local orthonormal basis on B and let $\{f_j\}_{j=\overline{1,m}}$ be a local orthonormal basis on F, which we lift to M and thus obtain a local orthonormal basis of M:

$$
\left\{\tilde{e}_i, \frac{1}{\varphi\circ\pi}\tilde{f}_j\right\}_{i=\overline{1,n};\, j=\overline{1,m}}.
$$

Consider the decomposition $\delta = \delta_1 + \delta_2$ of the codifferential on M, where

$$
\delta_1 := -\sum_{i=1}^{n} \tilde{e}_i \lrcorner \nabla_{\tilde{e}_i}, \qquad \delta_2 := -\frac{1}{(\varphi\circ\pi)^2}\sum_{j=1}^{m} \tilde{f}_j \lrcorner \nabla_{\tilde{f}_j}.
$$

We first determine the commutation relations between the pullback of forms on B and F with δ_1 and δ_2.

Lemma A.3. *For $\alpha \in \Omega^*(B)$ and $\beta \in \Omega^*(F)$, we have*

$$\text{(A-4)} \qquad \delta_1(\sigma^*(\beta)) = 0, \qquad \delta_2(\sigma^*(\beta)) = \frac{1}{(\varphi\circ\pi)^2}\sigma^*(\delta^{g_F}(\beta)),$$

$$\text{(A-5)} \qquad \delta_1(\pi^*(\alpha)) = \pi^*(\delta^{g_B}(\alpha)), \qquad \delta_2(\pi^*(\alpha)) = -\frac{m}{\varphi\circ\pi}\operatorname{grad}(\varphi\circ\pi)\lrcorner\pi^*(\alpha).$$

Proof. Let $\beta \in \Omega^{p+1}(F)$. For any tangent vector fields $X_1, \dots, X_p$ to M we obtain

$$\begin{aligned}
&\delta_1(\sigma^*(\beta))(X_1,\dots,X_p)\\
&\quad= -\sum_{i=1}^{n}(\tilde e_i \lrcorner \nabla_{\tilde e_i}(\sigma^*\beta))(X_1,\dots,X_p)\\
&\quad= -\sum_{i=1}^{n}\tilde e_i\big(\beta(\sigma_*\tilde e_i,\sigma_*X_1,\dots,\sigma_*X_p)\circ\sigma\big) + \sum_{i=1}^{n}\beta(\sigma_*(\nabla_{\tilde e_i}\tilde e_i),\sigma_*X_1,\dots,\sigma_*X_p)\\
&\qquad+\sum_{i=1}^{n}\Big(\beta\big(\sigma_*\tilde e_i,\sigma_*(\nabla_{\tilde e_i}X_1),\dots,\sigma_*X_p\big)+\dots+\beta\big(\sigma_*\tilde e_i,\sigma_*X_1,\dots,\sigma_*(\nabla_{\tilde e_i}X_p)\big)\Big)\\
&\quad= 0,
\end{aligned}$$

since $\sigma_*\tilde e_i = 0$, because $\tilde e_i$ is the lift of a vector field on B and also

$$\sigma_*(\nabla_{\tilde e_i}\tilde e_i) = \sigma_*(\widetilde{\nabla^{g_B}_{e_i}e_i}) = 0.$$

This proves that $\delta_1(\sigma^*(\beta)) = 0$.

The commutation rule in (A-4) is shown as follows:

$$\begin{aligned}
&(\varphi\circ\pi)^2\delta_2(\sigma^*(\beta))(X_1,\dots,X_p)\\
&\quad= -\sum_{j=1}^{m}(\tilde f_j \lrcorner \nabla_{\tilde f_j}(\sigma^*\beta))(X_1,\dots,X_p)\\
&\quad= -\sum_{j=1}^{m}\tilde f_j(\beta(\sigma_*\tilde f_j,\sigma_*X_1,\dots,\sigma_*X_p)\circ\sigma) + \sum_{j=1}^{m}\beta(\sigma_*(\nabla_{\tilde f_j}\tilde f_j),\sigma_*X_1,\dots,\sigma_*X_p)\circ\sigma\\
&\qquad+\sum_{j=1}^{m}\big(\beta(\sigma_*\tilde f_j,\sigma_*(\nabla_{\tilde f_j}X_1),\dots,\sigma_*X_p)\\
&\qquad\qquad+\dots+\beta(\sigma_*\tilde f_j,\sigma_*X_1,\dots,\sigma_*(\nabla_{\tilde f_j}X_p))\big)\circ\sigma\\
&\quad= -\sum_{j=1}^{m} f_j(\beta(f_j,\sigma_*X_1,\dots,\sigma_*X_p))\circ\sigma\\
&\qquad+\sum_{j=1}^{m}\beta(\sigma_*(\widetilde{\nabla^{g_F}_{f_j}f_j} - \frac{g(\tilde f_j,\tilde f_j)}{\varphi\circ\pi}\operatorname{grad}(\varphi\circ\pi)),\sigma_*X_1,\dots,\sigma_*X_p)\circ\sigma\\
&\qquad+\sum_{j=1}^{m}\big(\beta(f_j,\sigma_*(\nabla_{\tilde f_j}X_1),\dots,\sigma_*X_p)+\dots+\beta(f_j,\sigma_*X_1,\dots,\sigma_*(\nabla_{\tilde f_j}X_p))\big)\circ\sigma,
\end{aligned}$$

where we may again assume, without loss of generality, that X_i are lifts of vector fields Z_i on F: $X_i = \tilde{Z}_i$ for $i = 1, \dots, p$. For a tangent vector field Y to B, each of the above terms vanishes, since $\sigma_*(Y) = 0$. We then get

$$
\begin{aligned}
&(\varphi\circ\pi)^2\delta_2(\sigma^*(\beta))(X_1,\dots,X_p)\\
&\qquad= -\sum_{j=1}^m f_j(\beta(f_j, Z_1,\dots,Z_p))\circ\sigma + \sum_{j=1}^m \beta(\nabla^{g_F}_{f_j} f_j, Z_1,\dots,Z_p)\circ\sigma\\
&\qquad\quad+\sum_{j=1}^m [\beta(f_j, \nabla^{g_F}_{f_j} Z_1,\dots,\sigma_* X_p)+\dots+\beta(f_j, Z_1,\dots,\nabla^{g_F}_{f_j} Z_p)]\circ\sigma\\
&\qquad= \sigma^*(\delta^{g_F}(\beta))(X_1,\dots,X_p).
\end{aligned}
$$

The relations (A-5) can be obtained by similar computations, which we omit here. □

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