Pacific Journal of Mathematics

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Volume 251 No. 1 May 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 251, No. 1, 2011

THE DIRICHLET PROBLEM FOR CONSTANT MEAN CURVATURE GRAPHS IN $\mathbb{H} \times \mathbb{R}$ OVER UNBOUNDED DOMAINS

ABIGAIL FOLHA AND SOFIA MELO

We study graphs of constant mean curvature H in $\mathbb{H} \times \mathbb{R}$, where $\mathbb{H}$ is the hyperbolic plane. When $0 < H < \frac{1}{2}$, we find necessary and sufficient conditions for the existence of these graphs over unbounded domains in $\mathbb{H}$, having prescribed, possibly infinite, boundary data.

1. Introduction

This work deals with graphs in $\mathbb{H} \times \mathbb{R}$, where $\mathbb{H}$ is the hyperbolic plane, having constant mean curvature H defined over unbounded domains in $\mathbb{H}$. In the Euclidean space $\mathbb{R}^3$, Finn [1963; 1965] and Jenkins and Serrin [1966] studied the existence of a function whose graph over a bounded domain $\mathfrak{D} \subset \mathbb{R}^2$ is minimal and has prescribed boundary data. Finn studied the behavior of graphs in $\mathbb{R}^3$ over bounded convex domains in $\mathbb{R}^2$ having constant mean curvature $H = 0$ and established criteria to determine when a graph tends to infinity over a boundary arc of the domain. Jenkins and Serrin showed that necessary conditions for the existence of graphs over a domain $D \subset \mathbb{R}^2$ having unbounded boundary values given by the flux (see Section 5 for precise definition) on D are also sufficient.

The work of Jenkins and Serrin inspired many extensions to other ambient spaces and some of their ideas are present in these extensions. In $\mathbb{H} \times \mathbb{R}$ the existence theorem was proved by Nelli and Rosenberg [2002]. Collin and Rosenberg [2010] treated the case in which the domain $\mathfrak{D}$ in $\mathbb{H}$ is unbounded and Mazet, Rodríguez and Rosenberg [2008] dealt with a more general setting. Spruck [1972] extended the theorem of Jenkins and Serrin to constant mean curvature graphs in $\mathbb{R}^3$ over bounded domains of $\mathbb{R}^2$. Spruck's work introduced an important idea for the case $H \neq 0$: the reflection of the curves in order to get values $-\infty$ over boundary arcs. The case of graphs of constant mean curvature over bounded domains in $\mathbb{H}$

Folha was supported by FAPERJ. Melo was supported by CAPES.
MSC2000: 53A10, 53C42.
Keywords: constant mean curvature, unbounded domain, graph, Dirichlet problem.

was considered by Hauswirth, Rosenberg and Spruck [2009]. There are other articles about this theory; see, for example, [Rosenberg 2002; Pinheiro 2009; Gálvez and Rosenberg 2010].

It is a well known fact that there is no entire graph for H greater than $1/2$ in $\mathbb{H}\times\mathbb{R}$; moreover, Hauswirth, Rosenberg and Spruck [2008] prove that a complete graph with $H=1/2$ in $\mathbb{H}\times\mathbb{R}$ is an entire graph. Hence, we consider in this work values of $H>0$ less than $1/2$. We take a convex domain $\mathfrak{D}$ whose boundary $\partial\mathfrak{D}$ is composed of ideal arcs $\{A_i\}$, $\{B_j\}$ and $\{C_k\}$ such that the curvatures of the arcs with respect to the domain are $\kappa(A_i)=2H$, $\kappa(B_j)=-2H$ and $\kappa(C_k)\geq 2H$. We give necessary and sufficient conditions on the geometry of the domain $\mathfrak{D}$ which assure the existence of a function u defined in $\mathfrak{D}$, whose graph has constant mean curvature and u assumes the value $+\infty$ on each A_i, $-\infty$ on each B_j and prescribed continuous data on each C_k. The conditions, as in Jenkins and Serrin's work [1966], will be considered in terms of the lengths and the areas of inscribed polygons. Since these quantities are infinite in general, the formulation of the conditions is somewhat delicate. For an example, the reader may look at Section 8. In order to control lengths we do the same as Collin and Rosenberg [2010]; however, the new and key idea appears when we consider the area and we split it in two parts, one finite and the other infinite (see Section 3).

This paper is organized as follows. In Section 2, we introduce notation. In Section 3, we state the main theorems, which will be proved in Section 7. Sections 4 and 5 contain general maximum principles and the flux formulas, which are useful tools to prove preliminary results and the necessary conditions of the main theorems. In Section 6, we state results about divergence lines, which are essential to prove the sufficient conditions of the main theorems. Finally, in Section 8, we construct an example.

2. Notation

Let $\mathbb{H}$ be the hyperbolic plane, and $\mathbb{H}\times\mathbb{R}$ be given the product metric. Let $u:D\subset\mathbb{H}\to\mathbb{R}$ be a function in $C^2(D)$, where D is a simply connected domain. Denote the graph of u by $S=\mathrm{Graph}(u)=\{\,(p,u(p))\mid p\in D\,\}$. Since S is a graph, there are two choices for the unit normal vector $N(P)$ to S at a point $P=(p,u(p))$, $p\in D$. We choose

$$N(P)=\frac{-\nabla u+\partial_t}{\sqrt{1+\|\nabla u\|^2}},$$

that is, the normal vector pointing up.

Let $\overrightarrow{H}(P)$ be the mean curvature vector of S at P. The mean curvature function of S at a point P is defined by $H(P)=\langle N,\overrightarrow{H}\rangle(P)$. Consider graphs with $0<H(P)<\frac{1}{2}$ for all $P\in S$; in particular, $\overrightarrow{H}$ points up.

The graph S has constant mean curvature H if $H(P)=H$ for all $P\in S$. This means u satisfies the equation

$$Mu := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 2H, \tag{1}$$

where the divergence and gradient are taken with respect to the metric on $\mathbb{H}$. A function that satisfies this equation in D is called a *solution in* D. We will use the notation $X_u=\nabla u/W_u$, where $W_u=\sqrt{1+|\nabla u|^2}$.

Let $E\subset\mathbb{H}$ be a smooth curve. Denote by $\kappa(p)$ the (nonnegative) curvature of E at a point $p\in E$ and when $\kappa(p)=K$ for all $p\in E$, we will say $\kappa(E)=K$. When E is a boundary arc of a domain D, we will often let $\kappa(p)$, $p\in E$, denote the algebraic curvature of E at p with respect to D, that is, $\kappa(p)\geq 0$ if E is convex with respect to D, and $\kappa(p)<0$ otherwise.

We will consider ideal domains in $\mathbb{H}$ whose asymptotic boundary is composed only of a finite number of isolated points. Domains mean a connected, simply connected open set. The boundary of an ideal domain will be called *ideal polygon*.

3. Main theorems

In this section, we state the theorems that give necessary and sufficient conditions for the existence of constant mean curvature graphs which take the boundary values $+\infty$ on certain arcs A_i, $-\infty$ on arcs B_i and continuous data on arcs C_i.

Definition 3.1 (admissible domain). We say that an unbounded domain $\mathfrak{D}$ in $\mathbb{H}$ is admissible if it is simply connected and $\partial\mathfrak{D}$ is an ideal polygon with sides $\{A_i\}$, $\{B_i\}$ and $\{C_i\}$ satisfying $\kappa(A_i)=2H$, $\kappa(B_i)=-2H$ and $\kappa(C_i)\geq 2H$, respectively (with respect to the interior of $\mathfrak{D}$). Suppose that no two of the arcs A_i and no two of the arcs B_i have a common endpoint. Moreover, all the sides of $\partial\mathfrak{D}$ are contained in $\mathbb{H}$ and all the vertices of $\partial\mathfrak{D}$ are in the asymptotic boundary of $\mathbb{H}$.

Definition 3.2 (Dirichlet problem). Let $\mathfrak{D}$ be an admissible domain and fix $0<H<\frac{1}{2}$. The generalized Dirichlet problem is to find a solution of (1) in $\mathfrak{D}$ of mean curvature H, which assumes the value $+\infty$ on each A_i, $-\infty$ on each B_i and prescribed continuous data on each C_i.

Definition 3.3 (admissible inscribed polygon). Let $\mathfrak{D}$ be an admissible domain. We say that $\mathcal{P}$ is an admissible inscribed polygon if $\mathcal{P}\subset\mathfrak{D}\cup\partial\mathfrak{D}$, its sides have curvature $\pm 2H$ and all the vertices of $\mathcal{P}$ are vertices of $\mathfrak{D}$.

In [Hauswirth et al. 2009], the Dirichlet problem was solved for bounded admissible domains. The necessary and sufficient conditions in this case are in terms of the lengths and areas of inscribed polygons. When the domain is unbounded, these quantities can be infinite. Using the ideas in [Collin and Rosenberg 2010], we control the lengths as follows.

Let $\mathcal{P}$ be an inscribed polygon in $\mathcal{D}$ and let $\{d_i\}$ be the vertices of $\mathcal{P}$. Consider the set

$$\Theta = \{\mathcal{H}_i \mid \mathcal{H}_i \text{ is a horocycle at } d_i,\, \mathcal{H}_i \cap \mathcal{H}_j = \varnothing,\ i \neq j, \\ \text{and these horocycles satisfy condition (5)}\}.$$

Remark 3.1. We define condition (5) in Section 7. This is a technical condition which is always satisfied for sufficiently "small" horocycles at the vertices d_i. Throughout we only consider horocycles $\mathcal{H}_i$ contained in this set Θ.

Let F_i be the convex horodisk with boundary $\mathcal{H}_i$. Each A_i meets exactly two horodisks. Denote by $\widetilde{A}_i$ the compact arc of A_i which is the part of A_i outside the two horodisks; we define $|A_i|$ as the length of $\widetilde{A}_i$. For each arc $\eta_j \in \mathcal{P}$ we define $\widetilde{\eta}_j$ and $|\eta_j|$ in the same way.

We define

$$\alpha(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} |A_i|, \quad \beta(\mathcal{P}) = \sum_{B_i \in \mathcal{P}} |B_i| \quad \text{and} \quad l(\mathcal{P}) = \sum_j |\eta_j|$$

where $\mathcal{P} = \bigcup_j \eta_j$.

Now, let $\gamma_i = \mathcal{H}_i \cap (\mathcal{D} \cup \partial\mathcal{D})$. Consider γ_i^* the geodesic reflection of γ_i about the geodesic joining the endpoints of γ_i.

Denote by Ω the domain bounded by $\mathcal{P}$ and $\widetilde{\Omega} = \bigcup_j (\Omega \cap F_j)$, where the area $\mathcal{A}(\Omega \cap F_j)$ is finite.

Let $\mathcal{H} = \{\mathcal{H}_i\}_{i=1,\dots,n}$ be a family of horocycles.

For each family $\mathcal{H}$, we define

$$\widetilde{\mathcal{A}}(\Omega) := \mathcal{A}(\Omega_{\mathcal{H}}) + \mathcal{A}(\widetilde{\Omega}),$$

where

$$\mathcal{A}(\Omega_{\mathcal{H}}) = \mathcal{A}\big(\Omega - \big(\textstyle\bigcup_i (\Omega \cap F_i)\big)\big)$$

for all i. This definition plays an important role in this work – actually, this is the key idea which we need to extend previous results of [Collin and Rosenberg 2010; Hauswirth et al. 2009] to our setting. In Section 7, we will point out where this definition is used.

Notice that the definitions of $\alpha(\mathcal{P})$, $\beta(\mathcal{P})$ and $l(\mathcal{P})$ can be extended to the boundary of $\mathcal{D}$ and $\widetilde{\mathcal{A}}(\Omega)$ to $\mathcal{D}$.

Remark 3.2. When $\partial\mathcal{D}$ only has sides of type A_i and B_i, we have that $\widetilde{\mathcal{A}}(\mathcal{D}) = \mathcal{A}(\mathcal{D})$, because $\mathcal{A}(\mathcal{D} \cap F_i)$ is finite for all i (this may be infinite when there are arcs C_i present). Also, in this case, for all admissible polygons $\mathcal{P}$ in $\mathcal{D}$ we have $\widetilde{\mathcal{A}}(\Omega) = \mathcal{A}(\Omega)$.

With these definitions we can state the main theorems.

Theorem 3.1. *Consider the Dirichlet problem in an admissible domain $\mathcal{D}$ and suppose the family $\{C_i\}$ is empty. Then, there exists a solution to the Dirichlet problem if and only if for some choice of the horocycles* (*in* Θ) *at the vertices*,

$$\alpha(\partial\mathcal{D}) = \beta(\partial\mathcal{D}) + 2H\widetilde{\mathcal{A}}(\mathcal{D}) \tag{2}$$

and for all admissible polygons $\mathcal{P}$,

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\widetilde{\mathcal{A}}(\Omega) \quad \textit{and} \quad 2\beta(\mathcal{P}) < l(\mathcal{P}) - 2H\widetilde{\mathcal{A}}(\Omega). \tag{3}$$

Now we remove the hypothesis that $\{C_i\}$ is empty from Theorem 3.1.

Theorem 3.2. *Consider the Dirichlet problem in an admissible domain $\mathcal{D}$ and suppose the family $\{C_i\}$ is nonempty. Then there exists a solution to the Dirichlet problem if and only if for some choice of the horocycles* (*in* Θ) *at the vertices*,

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\widetilde{\mathcal{A}}(\Omega) \quad \textit{and} \quad 2\beta(\mathcal{P}) < l(\mathcal{P}) - 2H\widetilde{\mathcal{A}}(\Omega) \tag{4}$$

for all admissible polygons $\mathcal{P}$.

4. Maximum principles

The next results are general maximum principles for sub- and supersolutions of the constant mean curvature operator for boundary data having a finite number of discontinuities. The first one is in a bounded domain and the second one is in an unbounded domain. First we state a local lemma whose proof is in [Hauswirth et al. 2009].

Lemma 4.1. *Let u^1 and u^2 be functions in $C^2(D)$, $D \subset \mathbb{H}$. Then*

$$\left\langle \nabla u^1 - \nabla u^2, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle \geq 0,$$

with equality at a point if and only if $\nabla u^1 = \nabla u^2$. Here $W_i = W(\nabla u^i)$, $W(p) = \sqrt{1+|p|^2}$, $i = 1, 2$.

Theorem 4.1 (general maximum principle 1). *Let u^1 and u^2 satisfy $Mu^1 \geq 2H \geq Mu^2$ in a bounded domain $D \subset \mathbb{H}$. Suppose that* $\liminf(u^2 - u^1) \geq 0$ *for any approach to ∂D with the possible exception of a finite number of points of ∂D. Then $u^2 \geq u^1$ with strict inequality unless $u^2 \equiv u^1$.*

Theorem 4.2 (general maximum principle 2). *Let D be a domain with ∂D an ideal polygon. Let $W \subset D$ be a domain and let $u^1, u^2 \in C^0(\overline{W})$ be two solutions of (1) in W with $u^1 \leq u^2$ on ∂W. Suppose that for each vertex p of ∂D,* $\liminf \operatorname{dist}_{\mathbb{H}}(\Gamma_1, \Gamma_2) \to 0$ *as one converges to p, where Γ_1, Γ_2 are the curves on ∂D with p as vertex. Then $u^1 \leq u^2$ in W.*

The proof of Theorem 4.1 is given in [Hauswirth et al. 2009]. The proof of Theorem 4.2 is analogous to the one of Theorem 2 in [Collin and Rosenberg 2010] using Lemma 4.1.

We will see examples of barriers which will enable us to control convergence of solutions on ∂D, when we know they converge in D. Then the limit of the sequence on the boundary is the limit of the boundary values and the limit solution extends continuously to the boundary. The following examples can be found in [Hauswirth et al. 2009].

Example 4.1. Let $B \subset \mathbb{H}$ be a ball of radius δ centered at p. Let p_1 and p_2 be "antipodal" points on ∂B. We choose points d_1, d_2 on ∂B symmetric with respect to the geodesic through $p_1 p p_2$. Now let B_1 be an arc of curvature $-2H$ (as seen from p) joining d_1, d_2 and set $A_1 = B_1^*$, where B_1^* is the geodesic reflection of B_1. Let B_2 be the reflection of B_1 with respect to the geodesic orthogonal to $p_1 p p_2$ through p, and set $A_2 = B_2^*$. For δ small compared with H, there is a solution u^+ in B^+, the connected domain bounded by A_1, A_2 and arcs of ∂B such that u^+ is $+\infty$ on A_1 and A_2 and a constant $M > 0$ on the rest of ∂B^+. Similarly, there is a solution u^- in B^-, the domain bounded by B_1, B_2 and parts of ∂B such that u^- is $-\infty$ on B_1 and B_2 and a constant $-M$, $M > 0$ on the rest of ∂B^-.

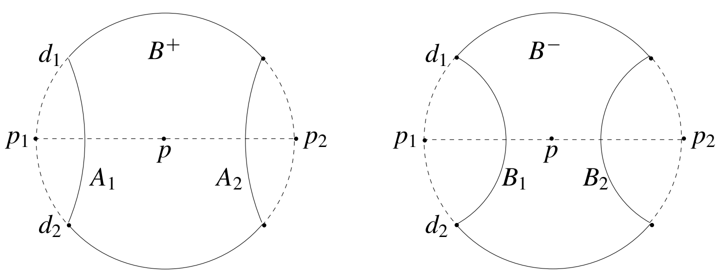

Figure 1. Domains of the solutions u^+ and u^- in Example 4.1.

5. Flux formulas

In this section, we state some results about the flux of a solution. As in [Jenkins and Serrin 1966], the flux will give us the necessary conditions, which also will be sufficient, to the existence of solutions having infinite boundary values. Finn [1963] proved that if a minimal solution in Euclidean space tends to $+\infty$ or $-\infty$ over a boundary arc Γ, then Γ is a line. The flux formula gives the requirement on the curvature of the boundary arcs of an admissible domain.

Let $u \in C^2(D)\cap C^1(\bar{D})$ be a solution in the bounded domain D. Then integrating (1) over D, we have

$$2H\mathscr{A}(D) = \int_{\partial D}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds,$$

where $\mathscr{A}(D)$ is the area of D and ν is the outer normal to ∂D. This integral is called *the flux of u across ∂D*. Let η be a subarc of ∂D (homeomorphic to [0, 1]). Even if u is not differentiable on η we can define the flux of u across η as follows; see [Hauswirth et al. 2009].

Definition 5.1. Choose Υ to be an embedded smooth curve in D so that $\eta\cup\Upsilon$ bounds a simply connected domain Δ_Υ. We then define the flux of u across η as

$$F_u(\eta) = 2H\mathscr{A}(\Delta_\Upsilon) - \int_{\Upsilon}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds.$$

The last integral is well defined, and $F_u(\eta)$ does not depend in the choice of Υ.

With this definition we can remove the condition $u \in C^2(D)\cap C^1(\bar{D})$ and state important flux formulas, whose proofs are in [Hauswirth et al. 2009].

Theorem 5.1. *Let u be a solution in D.*

(i) *If ∂D is a compact cycle, we have $F_u(\partial D) = 2H\mathscr{A}(D)$.*

(ii) *If D is bounded in part by a C^1 arc η, then:*

(a) *If u tends to $+\infty$ on η, we have $\kappa(\eta) = 2H$ and*

$$\int_{\eta}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds = |\eta|.$$

(b) *If u tends to $-\infty$ on η, we have $\kappa(\eta) = -2H$ and*

$$\int_{\eta}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds = -|\eta|.$$

(c) *If η is C^2, $\kappa(\eta)\geq 2H$ and u is continuous on η, we have*

$$\left|\int_{\eta}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds\right| < |\eta|.$$

Lemma 5.1. *Let D be a domain bounded in part by an arc η with $\kappa(\eta) = 2H$. We take a sequence of solutions $\{u_n\}$ in D with each u_n continuous on η. Then if the sequence diverges to $-\infty$ uniformly on compact subsets of D while remaining uniformly bounded on compact subsets of η, we have*

$$\lim_{n\to\infty}\int_{\Upsilon}\left\langle \frac{\nabla u}{W}, \nu\right\rangle ds = |\eta|.$$

The next lemma is almost a converse of the above Theorem 5.1. We follow the ideas in [Mazet et al. 2008].

Lemma 5.2. *Let u be a solution in D. Let $\tilde{\eta} \subset \partial D$ be an arc with $\kappa(\tilde{\eta}) = 2H$ ($\kappa(\tilde{\eta}) = -2H$) such that $F_u(\eta) = |\eta|$ ($F_u(\eta) = -|\eta|$), for every compact arc $\eta \subset \tilde{\eta}$. Then u takes boundary value $+\infty$ ($-\infty$) on $\tilde{\eta}$.*

Proof. Suppose that $\kappa(\tilde{\eta}) = 2H$. Let η be a compact arc as in the lemma, small enough so that the domain Δ bounded by η and η^* (the geodesic reflection of η) is contained in D. Consider the solution v which takes values $+\infty$ on η and $v = u$ on η^*; this solution exists by [Hauswirth et al. 2009, Theorem 7.11]. We need to show that $u = v$. If this is not the case, the set $O = \{u - v < \epsilon\}$ is nonempty, where $\epsilon > 0$ is a regular value of $u - v$. Let D' be the connected component of the complement of O in Δ which has $\partial\Delta - \eta$ in its boundary and let O' be the complement of D' in Δ, so $O \subset O'$ and $\partial O' \subset \partial O$. Let q be a point in $\partial O' - \eta$. For $\mu > 0$, let $O'(\mu)$ be the set defined by $O'(\mu) = \{p \in O' \mid \mathrm{dist}_{\mathbb{H}}(p, \eta) > \mu\}$. Let q_1, q_2 be the endpoints of the connected component of $\partial O' \cap \partial O'(\mu)$ which contains q. Let p_i be the projection of q_i on η. Let $\widetilde{O}(\mu)$ be the domain bounded by the segments $[p_1, q_1]$, $[p_2, q_2]$, the arc $[p_1, p_2] \subset \eta$ and the boundary component of $O'(\mu)$ between q_1, q_2, which is denoted by $\Gamma(\mu)$. On $\Gamma(\mu)$ the vector $X_u - X_v$ points outside $\widetilde{O}(\mu)$. Calculating the flux of $u - v$ across $\partial O'$ gives

$$0 = F_{u-v} = \int_{\Gamma(\mu)} \langle X_u - X_v, \nu\rangle + \int_{[p_1,q_1]\cup[p_2,q_2]} \langle X_u - X_v, \nu\rangle + \int_{[p_1,p_2]} \langle X_u - X_v, \nu\rangle \,.$$

So applying the flux formula, we have

$$\begin{aligned} 0 < \int_{\Gamma(\mu)} \langle X_u - X_v, \nu\rangle &= -\int_{[p_1,q_1]\cup[p_2,q_2]} \langle X_u - X_v, \nu\rangle - \int_{[p_1,p_2]} \langle X_u - X_v, \nu\rangle \\ &\le 4\mu, \end{aligned}$$

since the last term in the first line vanishes by the hypothesis on u and Theorem 5.1 applied to v. Note that the integral on $\Gamma(\mu)$ increases when $\mu \to 0$. So this inequality cannot occur.

If $\kappa(\tilde{\eta}) = -2H$, we consider the domain Δ which is bounded by η and an arc η' of curvature greater than $2H$ (with respect to the domain Δ) contained in D having the same endpoints as η. Then we consider v the solution on Δ with values $-\infty$ on η and $v = u$ on η'; this solution exists by [Hauswirth et al. 2009, Theorem 7.11]. Then the same argument made in the case $\kappa(\eta) = 2H$ can be applied. □

6. Divergence lines

In this section, we will study some characteristics of the sets where a sequence of solutions in a domain D converges or diverges. Jenkins and Serrin [1966] studied the convergence of a sequence (monotone) using a maximum principle. They also presented the structure of the divergence set of this sequence. Here, we study the convergence of a sequence defined over bounded or unbounded domains (not necessarily monotone) without the aid of a maximum principle. Nevertheless, the structure of the set where such a sequence converges is the same one found by Jenkins and Serrin. Many ideas found here were inspired by [Mazet et al. 2008].

Definition 6.1. Let D be a domain with piecewise smooth boundary, and u_n a sequence of solutions in D. We define the convergence set as

$$\mathcal{U}=\left\{\, p\in D \mid \{\|\nabla u_n(p)\|\} \textit{ is bounded independent of } n \,\right\}$$

and the divergence set as

$$\mathcal{V}=D-\mathcal{U}.$$

In this section, D denotes a domain in $\mathbb{H}$ with piecewise smooth boundary.

Lemma 6.1. *Let $p\in D$ and u_n be a sequence of solutions in the domain D. If $p\in\mathcal{U}$, there is a subsequence of $\{v_n\}$ with $v_n=u_n-u_n(p)$ converging uniformly to a solution in a neighborhood of p in D. If $p\in\mathcal{V}$, there is a compact arc $L_p(\tilde{\delta})$ of curvature $2H$ containing p such that, after passing to a subsequence, $\{N_{v_n}(p)\}$ converges to a horizontal vector which is orthogonal to $L_p(\tilde{\delta})$ having the same direction as the curvature vector $\vec{\kappa}$ of $L_p(\tilde{\delta})$, where $N_{v_n}(p)$ is the upward unit normal vector to the graph of v_n at $(p,0)$.*

Remark 6.1. All the vectors $\{N_{u_n}(p)\}$ can be thought as vectors at $(p,0)$ by vertical translation, with the identification $N_{u_n}(p)=N_{v_n}(p)$.

Proof of Lemma 6.1. Denote by $G(v_n)$ the graph of v_n over D. Note that $N_{u_n}(q)=N_{v_n}(q)$, and the convergence and divergence sets are the same for $\{u_n\}$ and $\{v_n\}$.

The curvature estimates (see [Zhang 2005]) give us a $\delta>0$ independent of n (in fact δ depends only on the distance from p to ∂D) such that a neighborhood of $P=(p,v_n(p))=(p,0)$ in $G(v_n)$ is a graph, in geodesic coordinates, with height and slope uniformly bounded over the disk $\mathbb{D}^n_\delta(P)$ of radius δ centered at the origin of $T_P G(v_n)$. We call this graph $G_P(v_n,\delta)$.

If $p\in\mathcal{U}$ the sequence $\{\|\nabla u_n\|\}$ is bounded, so there is a subsequence of $\{N_{v_n}(p)\}$, still called $\{N_{v_n}(p)\}$, which converges to a nonhorizontal vector and consequently the tangent planes associated to this subsequence converge to a nonvertical plane Π. Then, since the graphs $G_P(v_n,\delta)$ have height and slope uniformly bounded, there is a subsequence of $\{v_n\}$ such that these graphs converge to a graph $G_P(\delta)$ with constant mean curvature H over a disk of radius δ centered at the origin of Π. Since this plane Π is a nonvertical plane, there is $\tilde{\delta}$, $0<\tilde{\delta}\le\delta$ such that $G_P(\delta)$ is a graph over a geodesic ball in D centered at p of radius $\tilde{\delta}$. We conclude that there is a neighborhood of $p\in D$ such that a subsequence of $\{v_n\}$ converges to a solution in this neighborhood.

Now, suppose that $p \in \mathcal{V}$. Since $\{\|\nabla u_n\|\}$ is unbounded, there is a subsequence of $\{N_{v_n}(p)\}$ that converges to a horizontal vector N_P, so (for this subsequence) the tangent planes $T_P G(v_n)$ converge to a vertical plane Π and the graphs $G_P(v_n, \delta)$ converge to a constant mean curvature H graph $G_P(\delta')$ over a disk of radius $\delta' \leq \delta$ centered at the origin of Π. By the choice of the direction of the normal vector and the choice of $H > 0$, the limit of the curvature vectors of $G_P(v_n, \delta)$ has the same direction as the normal limit.

Take the curve $L_p \subset D$ passing through p orthogonal to N_P, with curvature $2H$ and the curvature vector at p having the same direction as N_P. We want to prove that $G_P(\delta') \subset (L_p \times \mathbb{R})$.

Since $G_P(\delta')$ is tangent to $L_p \times \mathbb{R}$ at P, if $G_P(\delta')$ is on one side of $L_p \times \mathbb{R}$, by the maximum principle, we have that $G_P(\delta') \subset (L_p \times \mathbb{R})$. If this is not the case, $G_P(\delta') \cap (L_p \times \mathbb{R})$ is composed of $k \geq 2$ curves passing through p, meeting transversely at p. So in a neighborhood of p these curves separate $G_P(\delta')$ in $2k$ components and the adjacent components lie in alternate sides of $L_p \times \mathbb{R}$. Moreover, the curvature vector alternates from pointing down to pointing up when one goes from one component to the other. This implies that the normal vector to $G_P(\delta)$ points down and up. So, for n large enough, the normal vector to $G_P(v_n, \delta)$ would point down and up, which does not occur.

Let $L_p(\tilde{\delta}) \subset D$, $\delta' \geq \tilde{\delta}$, be the curve contained in $G_P(\delta') \cap (L_p \times \{0\})$ which contains p and has length $2\tilde{\delta}$. Since $G_P(\delta') \subset (L_p \times \mathbb{R})$, we have that for all $q \in L_p(\tilde{\delta})$ the normal vector to $G_P(\delta')$ at q is a horizontal vector normal to $L_p(\tilde{\delta})$ having the same direction as the curvature vector of $L_p(\tilde{\delta})$ at q. $\square$

Remark 6.2. Lemma 6.1 shows that the convergence set is a domain.

Lemma 6.2. *Let $\{u_n\}$ be a sequence of solutions in D. Given $p \in \mathcal{V}$, there is a curve $L \subset D$ of curvature $2H$ which passes through p and such that, after passing to a subsequence, the sequence of normal vectors $\{N_{u_n}|_L\}$ converges to a horizontal vector normal to L having the same direction as the curvature vector of L. This curve L contains the compact arc $L_p(\tilde{\delta})$ given in Lemma 6.1.*

Proof. Let L be the curve of constant curvature $2H$ in D which contains $L_p(\tilde{\delta})$ joining the points of ∂D ($L_p(\tilde{\delta})$ is given in Lemma 6.1). Given $p, q \in D$, denote by $\overline{pq}$ the compact arc in L between p, q. We define

$$\Lambda = \{ q \in L \mid \text{there is a subsequence of } \{u_n\} \text{ such that } \{N_{u_n}|_{\overline{pq}}\} \text{ becomes horizontal, orthogonal to } L \text{ having the same direction as the curvature vector of } L\}.$$

We want to prove that $\Lambda = L$. Since $p \in \Lambda$, Λ is nonempty. We will prove that Λ is open and closed. First, we will prove that Λ is open. Let q be a point in Λ. Denote $\{u_{\Lambda(n)}\}$ the subsequence associated to Λ. Since $\Lambda \subset \mathcal{V}$, Lemma 6.1 gives us a curve

$L_q(\delta)$ through q such that, after passing to a subsequence, $\{N_{u_{\Lambda(n)}}|_{L_q(\delta)}\}$ becomes horizontal and having the same direction as the curvature vector of $L_q(\delta)$. Note that this subsequence of $\{N_{u_{\Lambda(n)}}|_{L_q(\delta)}\}$ converges to a horizontal vector normal to $L_q(\delta)$ and to L simultaneously, so $L_q(\delta)\subset L$, then Λ is open.

Now we will prove that Λ is closed. We take a convergent sequence $\{q_n\}$ in Λ, $q_n\to q\in L$. We will show that $q\in\Lambda$. For each m, there is a subsequence of $\{u_{\Lambda(n)}\}$ such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq}_m}\}$ becomes horizontal with the same direction as the curvature vector in $\overline{pq}_m$. By the diagonal process we obtain a subsequence of $\{u_{\Lambda(n)}\}$ such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq}_m}\}$ converges to a horizontal vector having the same direction as the curvature vector of L in $\overline{pq}_m$ for all m. Then by Lemma 6.1, we can find a curve $L_{q_m}(\delta)$ having constant curvature $2H$ through q_m, (for m large, δ depends only on the distance from q to ∂D) such that $\{N_{u_{\Lambda(n)}}|_{\overline{pq}_m}\}$ converges to a horizontal vector having the same direction as the curvature vector to $L_{q_m}(\delta)$. So $L_{q_m}(\delta)\subset L$ and since $q_m\to q$, we have that, for all m large enough, $q\in L_{q_m}(\delta)$. Consequently, $q\in\Lambda$. □

An important conclusion of this lemma is that the divergence set is given by $\mathcal{V}=\bigcup_{i\in I}L_i$, where L_i is a curve, called a *divergence line*, having curvature $2H$.

Lemma 6.3. *Let $\{u_n\}$ be a sequence of solutions in D. Suppose that the divergence set $\mathcal{V}$ of $\{u_n\}$ is composed of a countable number of divergence lines. Then there is a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that*

(1) *the divergence set of $\{u_n\}$ is composed of a countable number of pairwise disjoint divergence lines*;
(2) *for any connected component $\mathcal{U}'$ of $\mathcal{U}=D-\mathcal{V}$ and for any $p\in\mathcal{U}'$, the sequence $\{u_n-u_n(p)\}$ converges uniformly on compact subsets of $\mathcal{U}'$ to a solution in $\mathcal{U}'$.*

Proof. Suppose that $\mathcal{V}\neq\varnothing$ and let L_1 be a divergence line of $\{u_n\}$. Lemma 6.1 guarantees that, after passing to a subsequence, $\{N_{u_n}(q)\}$ converges to a horizontal vector orthogonal to L_1 at q for all q in L_1. The divergence set of this subsequence is contained in the divergence set of the original sequence, so the divergence set associated to this subsequence has only a countable number of lines. This subsequence is still denoted by $\{u_n\}$ and its divergence set by $\mathcal{V}$. If there is a divergence line $L_2\neq L_1$ in $\mathcal{V}$, we can find a subsequence such that $\{N_{u_n}(q)\}$ converges to a horizontal vector orthogonal to L_2 at q for each $q\in L_2$. This implies that $L_1\cap L_2=\varnothing$. In fact, if this does not occur, we take a point $q\in L_1\cap L_2$ so the sequence $\{N_{u_n}(q)\}$ converges to a horizontal vector orthogonal to L_1 and L_2 at q having the same direction as the curvature vector of L_1 and L_2. Then the uniqueness of a curve through q having curvature $2H$ with a given tangent vector shows that $L_1=L_2$. We continue this process to get a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, whose divergence set is composed of a countable number of pairwise disjoint divergence lines.

Lemma 6.1 shows that there is a subsequence of $\{u_n\}$ and a neighborhood of each point $p \in \mathcal{U}$ such that the sequence $\{u_n - u_n(p)\}$ converges to a constant mean curvature graph H, and this convergence is uniform on compact subsets of this neighborhood. Then taking a countable dense sequence $\{p_i\}$ in $\mathcal{U}'$, by the diagonal process we obtain a subsequence of $\{u_n\}$ such that $\{u_n - u_n(p)\}$ converges uniformly on compact subsets of $\mathcal{U}'$ for all $p \in \mathcal{U}'$. $\square$

Lemma 6.4. *Let $\{u_n\}$ be a sequence of solutions in D such that its divergence set is composed of a countable number of pairwise disjoint divergence lines. Suppose that $\{u_n\}$ converges to a solution u in a connected set $\mathcal{U}' \subset D$. Let γ be a compact arc in $\partial\mathcal{U}'$ included in a divergence line of $\{u_n\}$ such that $X_{u_n} \to \nu$ along γ, where ν is the outer conormal to γ with respect to $\mathcal{U}'$. Then if $p \in \mathcal{U}'$ and $q \in \gamma$, we have*

$$\lim_{n\to\infty} (u_n(q) - u_n(p)) = +\infty.$$

Proof. We choose p, q as in the hypothesis of the lemma. Since $X_{u_n} \to \nu$ we have $F_{u_n}(\gamma) \to |\gamma|$, where $F_{u_n}(\gamma)$ is the flux of u_n across γ. So Lemma 5.2 ensures that $u|_\gamma = +\infty$.

Claim 6.1. *There is an $\epsilon > 0$ such that $\partial u_n/\partial t \geq 0$ on $\{\Upsilon(t) \mid -\epsilon < t \leq 0\}$, where $\Upsilon(t)$ $(-\theta < t \leq 0,\ \theta \geq \epsilon)$ is the geodesic in $\mathcal{U}'$ such that $\Upsilon(0) = (q, 0)$ and $\Upsilon'(0) = \nu$. The inequality is strict on $\{\Upsilon(t) \mid -\epsilon < t < 0\}$.*

Using Lemma 6.1 and the fact that $u|_\gamma = +\infty$, we obtain a $\epsilon > 0$ such that $\partial u/\partial t \geq 1$ in $\{\Upsilon(t) \mid -\epsilon < t < 0\}$. The convergence $u_n \to u$ implies that $\partial u_n/\partial t > 0$ in $\{\Upsilon(t) \mid -\epsilon < t < -\eta\}$, for every $0 < \eta < \epsilon$ and $n \geq n_0(\eta)$.

If the claim is not true, considering a subsequence if necessary, there is a sequence $\{q_n\}$ in $\{\Upsilon(t) \mid -\eta \leq t \leq 0\}$ such that $q_n \to q$ and $(\partial u_n/\partial t)(q_n) = 0$.

If the sequence $\{\|\nabla u_n(q_n)\|\}$ is bounded, we have from the curvature estimates that $\{\|\nabla u_n\|\}$ is uniformly bounded on a disk D_n of radius independent of n, centered at q_n. Since $q_n \to q$, the sequence $\{\|\nabla u_n(q)\|\}$ is bounded, because for n large enough, $q \in D_n$. This contradicts that q is contained in the divergence set.

If the sequence $\{\|\nabla u_n(q_n)\|\}$ is unbounded, consider the sequence $\{u_n - u_n(q_n)\}$ and $\mathbb{D}_n^1$ the disk of radius δ in the graph of $\{u_n - u_n(q_n)\}$ centered at $(q_n, 0)$ given by the curvature estimates, δ independent of n. Since $(\partial u_n/\partial t)(q_n) = 0$, the disks $\mathbb{D}_n^1$ converge to a δ vertical disk centered at $(q, 0)$ in $\widetilde{\Upsilon} \times \mathbb{R}$, where $\widetilde{\Upsilon}$ is a curve having constant curvature $2H$ through q orthogonal to γ. Let $\mathbb{D}_n^2$ be the disk of radius δ centered at $(q, 0)$ in the graph of $\{u_n - u_n(q)\}$. Since γ is contained in a divergence line, $\{\mathbb{D}_n^2\}$ converges to a vertical disk centered at $(q, 0)$ in $\gamma \times \mathbb{R}$. Then, for n large enough, these disks $\mathbb{D}_n^1$ and $\mathbb{D}_n^2$ intersect transversally, but this is impossible because the normal vectors to $\mathbb{D}_n^1$ and $\mathbb{D}_n^2$ only depend on the gradient of u_n, so they are the same vector (on domains where both sequences are defined) for the two sequences. This proves Claim 6.1.

Let $q_t \in \mathcal{U}'$ be the point $q_t = \Upsilon(t),\ t < 0$, for t small enough. Claim 6.1 ensures that for n large,

$$\begin{aligned} u_n(q) - u_n(p) &\geq u_n(q_t) - u_n(p) \\ &\geq u(q_t) - u(p) - 1. \end{aligned}$$

The second inequality comes from the convergence of $\{u_n\}$ to u. The third term is as large as we want, because $u|_\gamma = +\infty$. □

Lemma 6.5. *Let $E \subset \partial D$ be a smooth arc having $\kappa(E) \geq 2H$. Consider a sequence of solutions $\{u_n\}$ in $\mathfrak{D}$ such that $\lim_{n\to\infty} u_n|_E = f$ for f a continuous function. Then a divergence line cannot finish at an interior point of E.*

Proof. Let $p \in E$ be an interior point. If $\kappa(E) > 2H$ at p, Lemma 4.9 in [Hauswirth et al. 2009] (see also the lemma on page 139 of [Finn 1965]) shows that $\{u_n\}$ is uniformly bounded in a neighborhood of p in D. Then, a divergence line cannot end at p.

If $\kappa(E) = 2H$ at p, by [Hauswirth et al. 2009, Lemma 4.9], we have that the sequence $\{u_n\}$ does not diverge to $+\infty$ in a neighborhood of p. Suppose there is one divergence line L leaving p. Then there is a subset $V \subset D$ which contains a subarc (containing p) of E in its boundary, and the sequence diverges to $-\infty$ on V. Consider a point $q \in E \cap \partial V$, and denote by $\overline{pq}$ the arc contained in E joining the points p and q. Let s be a point in L and $\overline{ps}$ the arc in L joining p and s. Denote by $\overline{sq}$ the geodesic joining s and q, suppose that q is as close to s as necessary, in order to guarantee $\overline{sq} \subset V$. We choose this "triangle" T so that the sequence $\{u_n\}$ diverges to $-\infty$ in the domain $\Delta_T \subset V$ bounded by T. By the flux formulas,

$$2H\mathcal{A}(\Delta_T) = F_{u_n}(\overline{ps}) + F_{u_n}(\overline{pq}) + F_{u_n}(\overline{sq}).$$

We have

$$\lim_{n\to+\infty} F_{u_n}(\overline{pq}) = |\overline{pq}|.$$

Since $\overline{ps} \subset L$, either

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}| \quad \text{or} \quad \lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}|.$$

First, suppose that

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}|.$$

Then,

$$\begin{aligned} \lim_{n\to+\infty} 2H\mathcal{A}(\Delta_T) &= \lim_{n\to+\infty} F_{u_n}(\overline{ps}) + \lim_{n\to+\infty} F_{u_n}(\overline{pq}) + \lim_{n\to+\infty} F_{u_n}(\overline{sq}) \\ &\geq |\overline{ps}| + |\overline{pq}| - |\overline{sq}| \end{aligned}$$

which implies

$$\frac{2H\mathscr{A}(\Delta_T)}{|\overline{sq}|} \geq \frac{|\overline{ps}|+|\overline{pq}|}{|\overline{sq}|} - 1.$$

We move q to q' and s to s' so that $|\overline{pq}'| = \lambda|\overline{pq}|$ and $|\overline{ps}'| = \lambda|\overline{ps}|$. When $\lambda \to 0$, the inequality

$$\frac{2H\mathscr{A}(\Delta_T)}{|\overline{sq}|} \geq \frac{|\overline{ps}|+|\overline{pq}|}{|\overline{sq}|} - 1$$

tends to zero on the left side, but is bounded from zero in the right side; a contradiction.

Now we consider the case where

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}|.$$

By Lemma 6.4 we have that $\{u_n\}$ diverges to $-\infty$ on a subset of $D - V$ which has L and a subarc of E in its boundary. Then applying the same argument as above, we get a contradiction.

Now, suppose that there are two or more divergence lines leaving from p. We fix two divergence lines, L_1, L_2. The point $p \in E$ divides E in two curves E_1, E_2, and we orient L_1, L_2, E_1, E_2 such that W_1 is the domain bounded in part by $L_1 \cup E_1$ and not containing L_2, W_2 is the domain bounded in part by $E_2 \cup L_2$ and not containing L_1 and finally W_3 is the domain bounded in part by $L_1 \cup L_2$ and not containing $E_1 \cup E_2$. Let $q \in L_1$, $s \in L_2$, $p_1 \in E$, $p_2 \in E$ be points. Denote by $\overline{pq}$ the segment in L_1 joining p and q, by $\overline{ps}$ the segment in L_2 joining p and s, by $\overline{sq} \subset W_3$ the segment of the geodesic joining q to s, by $\overline{qp_1} \subset W_1$ the segment of the geodesic joining q and p_1, and by $\overline{sp_2} \subset W_2$ the segment of the geodesic joining s and p_2. In some of these subsets W_i, $i = 1, 2, 3$, the sequence $\{u_n\}$ diverges to $-\infty$. Suppose that in W_3 the sequence diverges to $-\infty$, and that $\overline{sq} \subset W_3$.

If either

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}| \quad \text{or} \quad \lim_{n\to+\infty} F_{u_n}(\overline{pq}) = |\overline{pq}|,$$

with respect to W_3, applying the flux formulas to the triangle formed by $\overline{ps}$, $\overline{pq}$ and $\overline{sq}$, we obtain a contradiction as before.

If, with respect to W_3, either

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}| \quad \text{or} \quad \lim_{n\to+\infty} F_{u_n}(\overline{pq}) = -|\overline{pq}|,$$

then doing as we have done before to the triangle formed by $\overline{qp_1}$, $\overline{pq}$ and $\overline{p_1 p}$, if $\lim_{n\to+\infty} F_{u_n}(\overline{pq}) = -|\overline{pq}|$, or to the triangle formed by $\overline{ps}$, $\overline{pp_2}$ and $\overline{sp_2}$ if $\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}|$, we obtain a contradiction. $\square$

7. Proof of the main theorems

Before the proof of the theorems we need to show that the conditions of the hypothesis make sense, that is, we have to show that they are preserved for smaller horocycles.

Let $\mathcal{H}_i$ be an horocycle at d_i. Suppose that the conditions of Theorems 3.1 and 3.2 are satisfied for a family of horocycles $\mathcal{H}=\{\mathcal{H}_i\}_{i=1,\dots,n}$. These conditions are

(i) $\alpha(\partial\mathcal{D})-\beta(\partial\mathcal{D})=2H\widetilde{\mathcal{A}}(\mathcal{D})$,

and for all admissible polygons $\mathcal{P}\neq\partial\mathcal{D}$,

(ii) $2\alpha(\mathcal{P})<l(\mathcal{P})+2H\widetilde{\mathcal{A}}(\Omega)$,

(iii) $2\beta(\mathcal{P})<l(\mathcal{P})-2H\widetilde{\mathcal{A}}(\Omega)$.

Fixing $s\in\{1,\dots,n\}$, we will show that these conditions are also true for a family $\mathcal{H}'=\{\mathcal{H}_i\}_{i\neq s}\cup\{\mathcal{H}'_s\}$, where $\mathcal{H}'_s$ is contained in the horodisk F_s bounded by $\mathcal{H}_s$. We are interested in "smaller" horocycles because in this way we have an exhaustion of $\mathcal{P}$. To prove this we will use subindices T and T' to clarify the dependence of $\alpha(\mathcal{P})$, $\beta(\mathcal{P})$ and $l(\mathcal{P})$ with respect to $\mathcal{H}$ and $\mathcal{H}'$ respectively.

First, consider condition (i). We observe that when we change the family of horocycles, the left side of (i) does not change. So our definition for $\widetilde{\mathcal{A}}$ should not change. This is the first reason for the definition of $\widetilde{\mathcal{A}}$.

Note that

$$\alpha(\partial\mathcal{D}_{T'})-\beta(\partial\mathcal{D}_{T'})=\alpha(\partial\mathcal{D}_{T})-\beta(\partial\mathcal{D}_{T})=\text{constant}.$$

Thus, if (i) is true for $\mathcal{H}$, then it is also true for $\mathcal{H}'$.

Condition (ii) is equivalent to

$$2\alpha(\mathcal{P})-l(\mathcal{P})<2H\widetilde{\mathcal{A}}(\Omega).$$

When we change from family $\mathcal{H}$ to family $\mathcal{H}'$ the left side of the above inequality is nonincreasing and the right side is nondecreasing, so the inequality is preserved.

Finally, we handle the inequality of condition (iii).

There are two distinct situations. The first one is when the horocycle $\mathcal{H}_s$ meets sides E_1,E_2 where $\kappa(E_1)=-2H,\ \kappa(E_2)=2H$. The second one is when $\mathcal{H}_s$ meets sides E_1,E_2 with $\kappa(E_1)=2H,\ \kappa(E_2)=2H$.

In the first case, the area $\widetilde{\mathcal{A}}(\Omega)$ does not change when we change from the family $\mathcal{H}$ to $\mathcal{H}'$, and $2\beta(\mathcal{P})-l(\mathcal{P})$ is nonincreasing, so the inequality is preserved.

The second case is the most delicate one. Here, it will be necessary to have horocycles small enough.

More precisely, we consider the half-space model of $\mathbb{H}$. We can suppose that the vertices of $\mathcal{P}$ are $d_j=(x_j,0)$ for all $j\neq l$ and $d_l\in\{\partial\mathbb{H}-\{y=0\}\}$. We choose the family $\{\mathcal{H}_i\}$ of horocycles at the vertices d_i. We define

$$(0, M_l) = \mathcal{H}_l \cap \{x = 0\}.$$

The necessary condition is

$$M_l > \frac{2H(|x_{l-1}| + |x_{l+1}|)}{2\sqrt{1-4H^2}} \quad \text{for all } l = 1, \dots, n. \tag{5}$$

Remark 7.1. This is always the case for sufficiently small horocycles.

With this hypothesis on the horocycles, we can finish that the inequality in (iii) is preserved for the family $\mathcal{H}'$.

Suppose that $\mathcal{H}_s$ meets sides E_1 and E_2, where $\kappa(E_1) = \kappa(E_2) = 2H$. We point out that this is the case where we use (5) and also the definition of $\widetilde{\mathcal{A}}$, since $\widetilde{\mathcal{A}}$ should have the right behavior as the area is infinite.

Note that

$$2\beta(\mathcal{P}_{T'}) = 2\beta(\mathcal{P}_T) < l(\mathcal{P}_T) - 2H\widetilde{\mathcal{A}}(\Omega_T).$$

We will show

$$l(\mathcal{P}_T) - 2H\widetilde{\mathcal{A}}(\Omega_T) < l(\mathcal{P}_{T'}) - 2H\widetilde{\mathcal{A}}(\Omega_{T'}),$$

that is,

$$\big(l(\mathcal{P}_{T'}) - l(\mathcal{P}_T)\big) - \big(2H\widetilde{\mathcal{A}}(\Omega_{T'}) - 2H\widetilde{\mathcal{A}}(\Omega_T)\big) > 0. \tag{6}$$

In fact, we show that $l(\mathcal{P}_T) - 2H\widetilde{\mathcal{A}}(\Omega_T)$ increases when $\mathcal{H}$ decreases.

Consider the half-space model of $\mathbb{H}$. We can assume that $d_s = (0,0) \in \partial_\infty \mathbb{H}$. Using an inversion I with respect to the geodesic centered at $(0,0)$ of radius 1, we have that $\mathcal{H}_s$ and $\mathcal{H}'_s$ are taken to the horizontal straight lines through $(0, M)$ and $(0, y_0)$, respectively, and the sides A and E are taken to tilted lines leaving the points $(-x_0, 0)$ and $(x_1, 0)$ and making an angle θ with the vertical, where $\sin\theta = 2H$, $x_0 > 0$ and $x_1 > 0$; see Figure 2.

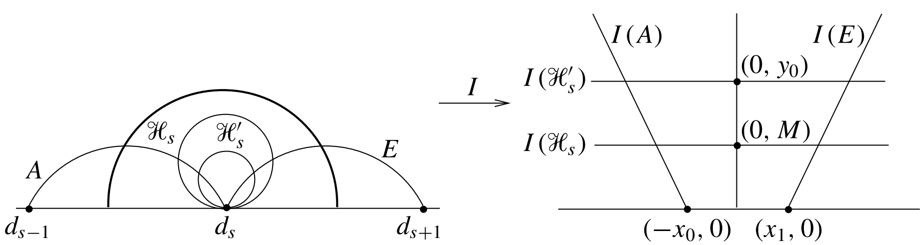

Figure 2. Using the inversion I.

Now, we calculate the length of the arcs of $I(A)$ and $I(E)$ bounded by $I(\mathcal{H}'_s)$ and $I(\mathcal{H}_s)$, denoted by $l(A_{\mathcal{H},\mathcal{H}'})$ and $l(E_{\mathcal{H},\mathcal{H}'})$, and the area limited by $I(A)$, $I(E)$, $I(\mathcal{H}_s)$ and $I(\mathcal{H}'_s)$, denoted by $\mathcal{A}(\Omega_{\mathcal{H},\mathcal{H}'})$.

Then,

$$l(A_{\mathcal{H},\mathcal{H}'}) = l(E_{\mathcal{H},\mathcal{H}'}) = \int_M^{y_0} \frac{\sec\theta}{y}\,dy = \sec\theta \ln y\Big|_M^{y_0}$$

and the area satisfies

$$\begin{aligned}\mathcal{A}(\Omega_{\mathcal{H},\mathcal{H}'}) &= \int_M^{y_0}\int_{-x_0-y\tan\theta}^{x_1+y\tan\theta} \frac{dx\,dy}{y^2}\\ &= \int_M^{y_0}\Big(\frac{2\tan\theta}{y} + \frac{(x_1+x_0)}{y^2}\Big)\,dy\\ &= 2\tan\theta\ln y\Big|_M^{y_0} - (x_1+x_0)\frac{1}{y}\Big|_M^{y_0}.\end{aligned}$$

Therefore,

$$\begin{aligned}&l(A_{\mathcal{H},\mathcal{H}'}) + l(E_{\mathcal{H},\mathcal{H}'}) - 2H\mathcal{A}(\Omega_{\mathcal{H},\mathcal{H}'})\\ &\qquad= 2(\sec\theta - 2H\tan\theta)\ln y\Big|_M^{y_0} + 2H\frac{(x_1+x_0)}{y}\Big|_M^{y_0}\\ &\qquad= 2\Big(\frac{1-\sin^2\theta}{\cos\theta}\Big)\ln y\Big|_M^{y_0} + \frac{2H(x_1+x_0)}{y}\Big|_M^{y_0}\\ &\qquad= 2\cos\theta\ln y_0 + \frac{2H(x_1+x_0)}{y_0} - 2\cos\theta\ln M - \frac{2H(x_1+x_0)}{M}.\end{aligned}$$

Then, to prove the inequality (6) it suffices to show that the function of y_0 above is increasing, because when $y_0 = M$, it is zero. We show that its derivative is greater than zero.

Differentiating we have

$$\frac{2\cos\theta}{y_0} - \frac{2H(x_1+x_0)}{y_0^2}.$$

So

$$\frac{2\cos\theta}{y_0} - \frac{2H(x_1+x_0)}{y_0^2} > 0 \quad\Longleftrightarrow\quad 2y_0\cos\theta - 2H(x_1+x_0) > 0,$$

that is,

$$y_0 > \frac{2H(x_1+x_0)}{2\cos\theta}.$$

But our family $\mathcal{H}$ satisfies

$$M > \frac{2H(x_1+x_0)}{2\cos\theta}.$$

Thus, we have the inequality (6) as desired, and consequently the inequality in (iii) is satisfied.

We fix some notation which will be useful in the proof of the theorems. Let $\{d_i = (x_i, y_i)\}$ be the set of vertices of $\partial\mathcal{D}$. For each i, let $\mathcal{H}_i(n)$ be a horocycle asymptotic to d_i such that $\mathcal{H}_i(n)$ belongs to Θ for all i, n. We choose $\mathcal{H}_i(n)$ such that $\mathcal{H}_i(n+1) \subset F_i(n)$, where $F_i(n)$ is the convex horodisk bounded by $\mathcal{H}_i(n)$. Let $\mathcal{D}(n) \subset \mathcal{D}$ be the domain bounded by

$$\partial\mathcal{D}(n) = \big(\partial\mathcal{D} - \big(\bigcup_i F_i(n)\big)\big) \cup \big(\bigcup_i \gamma_i(n)\big),$$

where $\gamma_i(n) = \mathcal{H}_i(n) \cap (\partial\mathcal{D} \cup \mathcal{D})$. Let $\mathcal{D}^*(n) \subset \mathcal{D}$ be the domain bounded by

$$\partial\mathcal{D}^*(n) = \big(\partial\mathcal{D} - \big(\bigcup_i F_i(n)\big)\big) \cup \big(\bigcup_i \gamma_i^*(n)\big),$$

where $\gamma_i^*(n)$ is the geodesic reflection of $\gamma_i(n)$. Similarly, we define $\Omega(n)$ as the domain whose boundary is

$$\mathcal{P}(n) = \big(\mathcal{P} - \big(\bigcup_i F_i(n)\big)\big) \cup \big(\bigcup_i \gamma_i(n) \cap \Omega(n)\big)$$

and $\Omega^*(n)$ as the domain bounded by

$$\partial\Omega^*(n) = \big(\mathcal{P} - \big(\bigcup_i F_i(n)\big)\big) \cup \big(\bigcup_i (\gamma_i^*(n) \cap \Omega^*(n))\big).$$

Finally, given an arc $\eta \subset \mathcal{P}$, we define $\eta(n) = \eta \cap \mathcal{P}(n)$.

Proof of Theorem 3.1. Suppose that the conditions (2) and (3) are true for all polygons in $\mathcal{D}$.

Claim 7.1. *There is a solution in $\mathcal{D}$ which boundary values*

$$u_n = \begin{cases} n & \text{on } \bigcup_k A_k, \\ -n & \text{on } \bigcup_l B_l^*. \end{cases}$$

Assume this Claim is true and take $\{u_n\}$ a sequence of solutions in $\mathcal{D}$, where u_n is defined as in the Claim. Then, this sequence has, or does not have, a divergence line.

First, we assume that there is some divergence line, and we will obtain a contradiction. By Lemma 6.5, the endpoints of these lines are among vertices of $\mathcal{D}$. Since $\partial\mathcal{D}$ has only a finite number of vertices, we can suppose that the divergence set is composed of a finite number of disjoint divergence lines. These lines separate the domain $\mathcal{D}$ in at least two connected components, and the interior of these components belongs to the convergence domain. By Lemma 6.4, in some connected components of the convergence set, the sequence $\{u_n\}$, $p \in \mathcal{D}$, diverges to $+\infty$ or $-\infty$. Suppose that in some connected component of the convergent set $\mathcal{U}'$, the sequence diverges to $+\infty$ (the case $-\infty$ is similar).

Since $\mathcal{U}' \subset \mathcal{U}$, where $\mathcal{U}$ is the convergence domain, we have that the sequence $\{u_n - u_n(p)\}$, $p \in \mathcal{U}'$, converges uniformly on compact subsets of $\mathcal{U}'$ to a solution u in $\mathcal{U}'$. On the other hand, by the choice of $\mathcal{U}'$ we have $u_n(p) \to +\infty$, $p \in \mathcal{U}'$. Moreover, we note that $\partial\mathcal{U}' = \mathcal{P}$ is an admissible polygon, we can choose $\mathcal{P}$ satisfying the next Claim.

Claim 7.2. *One can choose $\mathcal{P}$ so that*

$$F_u\Big(\mathcal{P}(n) - \Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big)\Big) = -l\Big(\mathcal{P}(n) - \Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big)\Big),$$

where $\partial\mathcal{U}' = \mathcal{P}$.

See [Mazet et al. 2008] for a proof.

We are supposing that there is a divergence line, so $\mathcal{P} \neq \partial\mathcal{D}$. By Claim 7.2 and the flux formulas

$$\begin{aligned} F_u(\mathcal{P}(n)) &= 2H\mathcal{A}(\mathcal{U}'(n)) \\ &= F_u\Big(\mathcal{P}(n) - \Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big)\Big) \\ &\qquad + F_u\Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big) \\ &\le -l\Big(\mathcal{P}(n) - \Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big)\Big) \\ &\qquad + l\Big(\big(\bigcup_i A_i(n)\big) \cup \big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big)\Big) \\ &= 2\alpha(\mathcal{P}) - l(\mathcal{P}) + l\big(\bigcup_i (\gamma_i(n) \cap \mathcal{U}')\big). \end{aligned}$$

When $n \to \infty$, the area $\mathcal{A}\big(\mathcal{D} \cap \big(\bigcup_j F_j\big)\big)$ tends to zero, so

$$2H\mathcal{A}(\mathcal{U}') \le 2\alpha(\mathcal{P}) - l(\mathcal{P}),$$

contradicting the hypothesis. So the sequence $\{u_n\}$ has no divergence lines.

Since the sequence $\{u_n\}$ does not have any divergence lines, $\mathcal{D}$ is the convergence domain, so there is a subsequence of $\{u_n - u_n(p)\}$, $p \in \mathcal{D}$ which converges to a solution u on $\mathcal{D}$. If the sequence $\{u_n\}$ is bounded at the point $p \in \mathcal{D}$, u has the boundary values as desired, that is, $u|_{A_k} = +\infty$ and $u|_{B_l} = -\infty$. We will show that even if the sequence $\{u_n\}$ is unbounded, the solution u has the boundary values as prescribed.

Suppose the sequence $\{u_n(p)\}$ tends to $-\infty$. By the flux formulas,

$$\begin{aligned} \lim_{n\to\infty} F_{u_n}(\mathcal{P}(m)) &= 2H\mathcal{A}(\mathcal{D}(m)) \\ &= 2H\widetilde{\mathcal{A}}(\mathcal{D}) - 2H\mathcal{A}\big(\mathcal{D} \cap \big(\bigcup_i F_i(m)\big)\big) \\ &= \sum \lim_{n\to\infty} F_{u_n}(A_i(m)) + \sum \lim_{n\to\infty} F_{u_n}(B_i(m)) \\ &\qquad + \sum \lim_{n\to\infty} F_{u_n}(\gamma_i(m)) \\ &\ge \alpha(\mathcal{P}) - \beta(\mathcal{P}) - \sum |\gamma_i(m)| \end{aligned}$$

which implies

$$2\beta(\mathcal{P}) \ge l(\mathcal{P}) - 2H\mathcal{A}(\Omega).$$

The hypothesis does not allow $2\beta(\mathcal{P}) > l(\mathcal{P}) - 2H\mathcal{A}(\Omega)$. Then equality holds: $2\beta(\mathcal{P}) = l(\mathcal{P}) - 2H\mathcal{A}(\Omega)$. This implies that $\lim_{n\to\infty} F_{u_n}(B_l(m)) = |B_l(m)|$. So $\{u_n - u_n(p)\}$ tends to $-\infty$ on B_l for all l.

Suppose the sequence $\{u_n(p)\}$ tends to $+\infty$. By the flux formulas,

$$\begin{aligned}\lim_{n\to\infty} F_{u_n}(\mathcal{P}(m)) = 2H\mathcal{A}(\mathfrak{D}(m)) &= 2H\widetilde{\mathcal{A}}(\mathfrak{D}) - 2H\mathcal{A}\big(\mathfrak{D}\cap\big(\textstyle\bigcup_i F_i(m)\big)\big)\\ &= \sum \lim_{n\to\infty} F_{u_n}(A_i(m)) + \sum \lim_{n\to\infty} F_{u_n}(B_i(m)) \lim_{n\to\infty} F_{u_n}(\gamma_i(m))\\ &\le \alpha(\mathcal{P}) - \beta(\mathcal{P}) + \sum |\gamma_i(m)|,\end{aligned}$$

which implies

$$2\alpha(\mathcal{P}) \ge l(\mathcal{P}) + 2H\mathcal{A}(\Omega).$$

Since we cannot have $2\alpha(\mathcal{P}) > l(\mathcal{P}) + 2H\mathcal{A}(\Omega)$, we have $2\alpha(\mathcal{P}) = l(\mathcal{P}) + 2H\mathcal{A}(\Omega)$, which implies $\lim_{n\to\infty} F_{u_n}(A_k(m)) = |A_k(m)|$. Then $\{u_n - u_n(p)\}$ tends to $+\infty$ on A_k for all k.

Proof of Claim 7.1. By the existence theorem for continuous boundary values and bounded domains [Hauswirth et al. 2009], for each m in $\mathfrak{D}^*(m)$ there is a solution with boundary values

$$u_m = \begin{cases} n & \text{on } \bigcup_k A_k(m),\\ -n & \text{on } \bigcup_l B_l^*(m),\\ 0 & \text{on } \bigcup_i \gamma_i^*(m).\end{cases}$$

Fix m_0. For all $m > m_0$, we have that $\{u_m|_{\mathfrak{D}^*(m_0)}\}$ is a sequence of solutions in $\mathfrak{D}^*(m_0)$. If there were any divergence lines, we would find a divergence set which would contradict the hypothesis, as in the proof of Theorem 3.1. Moreover, as there are no divergence lines, either this sequence is bounded or it is not bounded. If this sequence is not bounded, say $u_m(p) \to +\infty$, $p \in \mathfrak{D}^*(m_0)$, a subsequence $\{u_m|_{\mathfrak{D}^*(m_0)} - u_m(p)\}$ converges to a solution in $\mathfrak{D}^*(m_0)$ and tends to $-\infty$ on each arc $A_i(m_0)$, which cannot occur. If $\{u_m(p)\} \to -\infty$, $p \in \mathfrak{D}^*(m_0)$, some subsequence of $\{u_m|_{\mathfrak{D}^*(m_0)} - u_m(p)\}$ converges to a solution in $\mathfrak{D}^*(m_0)$ and tends to $+\infty$ on each arc $A_i(m_0)$, $B_l^*(m_0)$. Taking $m_0 \to \infty$ we again get a contradiction, since two arcs with the same vertex point have values $+\infty$. So this sequence is bounded and some subsequence is convergent, by the boundary values of the $\{u_m\}$, we have $u_m|_{A_k(m_0)} = n$ and $u_m|_{B_l(m_0)} = -n$. By the diagonal process, we have in $\mathfrak{D}$ a solution u_n given by

$$u_n = \begin{cases} n & \text{on } \bigcup_k A_k,\\ -n & \text{on } \bigcup_l B_l^*,\end{cases}$$

which completes the proof. $\square$

We return to the proof of Theorem 3.1 and prove the necessary conditions. Suppose there is a solution u in $\mathcal{D}$ of the Dirichlet problem. Applying the flux formulas to $\mathcal{P}(n) = \partial\mathcal{D}(n)$, and remembering that, in this case, $\widetilde{\mathcal{A}} = \mathcal{A}$, we have

$$\begin{aligned} F_u(\mathcal{P}(n)) = 2H\mathcal{A}(\mathcal{D}(n)) &= 2H\widetilde{\mathcal{A}}(\mathcal{D}) - 2H\mathcal{A}\big(\mathcal{D}\cap\big(\textstyle\bigcup_i F_i(n)\big)\big) \\ &= \textstyle\sum F_u(A_i(n)) + \sum F_u(B_i(n)) + \sum F_u(\gamma_i(n)). \end{aligned}$$

Since $\widetilde{\mathcal{D}} = \mathcal{D}\cap\big(\bigcup_i F_i(n)\big)$,

$$\begin{aligned} \textstyle\sum |A_i(n)| - \sum |B_i(n)| - \sum |\gamma_i(n)| &\le 2H\widetilde{\mathcal{A}}(\mathcal{D}) - 2H\mathcal{A}(\widetilde{\mathcal{D}}) \\ &\le \textstyle\sum |A_i(n)| - \sum |B_i(n)| + \sum |\gamma_i(n)|. \end{aligned}$$

It follows that

$$\alpha(\mathcal{D}) - \beta(\mathcal{D}) - \textstyle\sum |\gamma_i(n)| \le 2H\widetilde{\mathcal{A}}(\mathcal{D}) - 2H\mathcal{A}(\widetilde{\mathcal{D}}) \le \alpha(\mathcal{D}) - \beta(\mathcal{D}) + \sum |\gamma_i(n)|.$$

When $n\to\infty$, we have $|\gamma_i|\to 0$ and $\mathcal{A}(\widetilde{\mathcal{D}})\to 0$, so $\alpha(\mathcal{D})-\beta(\mathcal{D}) = 2H\widetilde{\mathcal{A}}(\mathcal{D})$. Now, we prove the inequalities (3). Applying the flux formulas to the polygon $\mathcal{P}(n)$, and denoting its interior arcs by E_m, we have

$$\begin{aligned} F_u(\mathcal{P}(n)) &= 2H\mathcal{A}(\Omega(n)) \\ &= \sum_k F_u(A_k(n)) + \sum_l F_u(B_l(n)) + \sum_m F_u(E_m(n)) + \sum_j F_u(\gamma_j(n)\cap\Omega(n)) \\ &\ge \sum_k |A_k(n)| - \sum_l |B_l(n)| + \delta - \textstyle\sum |E_m(n)| - \displaystyle\sum_j |\gamma_j(n)\cap\Omega(n)| \\ &= 2\alpha(\mathcal{P}) - l(\mathcal{P}) + \delta - \sum_j |\gamma_j(n)\cap\Omega(n)|. \end{aligned}$$

We see that $\widetilde{\mathcal{A}}(\Omega) > \mathcal{A}(\Omega(n))$ and $\sum_j |\gamma_j(n)\cap\Omega(n)| - \delta < 0$ for n large enough, so

$$2\alpha(\mathcal{P}) < l(\mathcal{P}) + 2H\widetilde{\mathcal{A}}(\Omega).$$

Similarly,

$$\begin{aligned} F_u(\mathcal{P}(n)) &= 2H\mathcal{A}(\Omega(n)) \\ &= \sum_k F_u(A_k(n)) + \sum_l F_u(B_l(n)) + \sum_m F_u(E_m(n)) + \sum_j F_u(\gamma_j(n)\cap\Omega) \\ &\le \sum_k |A_k(n)| - \sum_l |B_l(n)| - \delta + \sum_m |E_m(n)| + \sum_j |\gamma_j(n)\cap\Omega| \\ &= -2\beta(\mathcal{P}) + l(\mathcal{P}) - \delta; \end{aligned}$$

that is, for n sufficiently large,

$$\begin{aligned} 2\beta(\mathcal{P}) &\le l(\mathcal{P}) - 2H\mathcal{A}(\Omega(n)) - \delta \\ &< l(\mathcal{P}) - 2H\widetilde{\mathcal{A}}(\Omega). \end{aligned}$$

□

Proof of Theorem 3.2. This is similar to the proof of Theorem 3.1.

Claim 7.3. *There is a solution on $\mathfrak{D}$ having boundary values*

$$u_n = \begin{cases} n & \text{on } A_k, \\ -n & \text{on } B_l^*, \\ f_n & \text{on } C_m, \end{cases}$$

where $f_n = \varphi \circ f$ for $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } -n \le x \le n, \\ -n & \text{if } x < -n, \\ n & \text{if } x > n. \end{cases}$$

Assume that Claim 7.3 is true and take a sequence $\{u_n\}$ on $\mathfrak{D}$ given by this claim.

Suppose that $\{u_n\}$ has a divergence line. By Lemma 6.5, we can suppose that the divergence set is composed of a finite number of disjoint divergence lines. These lines separate the domain $\mathfrak{D}$ in at least two connected components, and the interior of these components belongs to the convergence domain. By Lemma 6.4, in connected components of the convergence set the sequence $\{u_n\}$, $p \in \mathfrak{D}$, diverges to $+\infty$ or $-\infty$. We observe that if there is some arc $C \subset \partial\mathfrak{D}$ having $\kappa(C) > 2H$, Lemma 4.9 in [Hauswirth et al. 2009] ensures that in a neighborhood of this arc the sequence $\{u_n\}$ is bounded.

As in the proof of Theorem 3.1 we will work on subdomains of $\mathfrak{D}$ where the sequence diverges to $+\infty$ or $-\infty$, so the boundary of these domains only has arcs of curvature $2H$. This means that the boundary of these domains are admissible polygons. From now on, the proof is similar to the proof of Theorem 3.1.

Proof of Claim 7.3. The only difference between Claim 7.3 and Claim 7.1 is found in the construction of solutions over bounded domains. Let $\{d_i\}$ be the vertices points of $\mathfrak{D}$, after some isometry of the hyperbolic plane, we can assume that each d_i belongs to $\{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. Let $\sigma_i[m]$ be geodesics which are semicircles centered at d_i with radius $1/m$. The hypothesis on the curvature of the arcs C_i enables us to conclude that, if m is big enough, $\sigma_i[m]$ divides $\mathfrak{D}$ in exactly two components, one of them having d_i in its asymptotic boundary. Let $\varrho_i[m]$ be the arc of the equidistant curve to $\sigma_i[m]$ having curvature $2H$ joining points of the boundary of $\mathfrak{D}$. Then $\varrho_i[m]$ divides $\mathfrak{D}$ in exactly two components, one having d_i in its asymptotic boundary. We chose the curvature vector of $\varrho_i[m]$ pointing to the component of $\mathfrak{D}$ which does not have d_i on its boundary. Now we can find a solution with prescribed boundary values using the existence theorem of [Hauswirth et al. 2009]. Let $A_i[m]$ be the compact arcs contained in A_i bounded by the endpoints of $\{\varrho_i[m]\}$, $B_i[m]$ be the compact arcs contained in B_i bounded by the endpoints of $\{\varrho_i[m]\}$ and $C_i[m]$ be the compact arcs contained in C_i bounded by the endpoints of $\{\varrho_i[m]\}$. So there exists

$$u_n = \begin{cases} n & \text{on } A_i[m], \\ -n & \text{on } B_i^*[m], \\ f_n & \text{on } C_i[m], \\ 0 & \text{on } \varrho_i[m], \end{cases}$$

where $f_n = \varphi \circ f$, for $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} x & -n \leq x \leq n, \\ -n & x < -n, \\ n & x > n. \end{cases}$$

From now on, the same procedure as in Claim 7.1 enables us to conclude the existence of a solution over $\mathscr{D}$ as desired in Claim 7.3. $\square$

Now, we go back to the proof of Theorem 3.2. Suppose that there is a solution u for the Dirichlet problem. Let Ω be the domain bounded by the admissible polygon $\mathscr{P}$ and $\Omega(n), \mathscr{P}(n)$ as found in the notation at the beginning of this section. Applying the flux formulas,

$$\begin{aligned}
&F_u(\mathscr{P}(n)) \\
&= 2H\mathscr{A}(\Omega(n)) \\
&= \sum_k F_u(A_k(n)) + \sum_l F_u(B_l(n)) + \sum_p F_u(C_p(n)) + \sum_m F_u(E_m(n)) + \sum_j F_u(\gamma_j(n)\cap\Omega) \\
&\geq \sum_k |A_k(n)| - \sum_l |B_l(n)| - \sum_p |C_p(n)| + \delta - \sum |E_m(n)| - \sum_j |\gamma_j(n)\cap\Omega| \\
&= 2\alpha(\mathscr{P}) - l(\mathscr{P}) + \delta - \sum_j |\gamma_j(n)\cap\Omega|.
\end{aligned}$$

Either $\mathscr{A}(\mathscr{D}) < \infty$, or $\mathscr{A}(\mathscr{D}) = \infty$. If $\mathscr{A}(\mathscr{D}) < \infty$, since $\widetilde{\mathscr{A}}(\Omega) > \mathscr{A}(\Omega_{\mathscr{H}})$ and $|\gamma_j(n)| \to 0$ for all j, we have

$$2\alpha(\mathscr{P}) < l(\mathscr{P}) + 2H\mathscr{A}(\Omega(n)) < l(\mathscr{P}) + 2H\widetilde{\mathscr{A}}(\Omega).$$

If $\mathscr{A}(\mathscr{D}) = \infty$, we have

$$2H\widetilde{\mathscr{A}}(\Omega) \geq 2H\mathscr{A}(\Omega_{\mathscr{H}}) > 2\alpha(\mathscr{P}) - l(\mathscr{P}) - \sum_j |\gamma_j(n)\cap\Omega|,$$

Then,

$$2H\widetilde{\mathscr{A}}(\Omega) + l(\mathscr{P}) - 2\alpha(\mathscr{P}) > -\sum_j |\gamma_j(n)\cap\Omega|.$$

Remembering that $l(\mathscr{P}) - 2\alpha(\mathscr{P})$ is nondecreasing, we have that the left side of this inequality is increasing and tends to $+\infty$, when the horocycles tend to the vertices.

Therefore, we can suppose

$$2H\widetilde{\mathcal{A}}(\Omega)+l(\mathcal{P})-2\alpha(\mathcal{P})>0.$$

Similarly,

$$\begin{aligned}
&F_u(\mathcal{P}(n))\\
&=2H\mathcal{A}(\Omega_{\mathcal{H}})\\
&=\sum_k F_u(A_k(n))+\sum_l F_u(B_l(n))+\sum_p F_u(C_p(n))+\sum_m F_u(E_m(n))+\sum_j F_u(\gamma_j(n)\cap\Omega)\\
&\quad\le \sum_k |A_k(n)|-\sum_l |B_l(n)|+\sum_p |C_p(n)|-\delta+\sum_m |E_m(n)|+\sum_j |\gamma_j(n)\cap\Omega|\\
&= -2\beta(\mathcal{P})+l(\mathcal{P})-\delta+\sum_j |\gamma_j(n)\cap\Omega|.
\end{aligned}$$

Then, if $\mathcal{A}(\mathfrak{D})<\infty$,

$$2\beta(\mathcal{P})<l(\mathcal{P})-2H\mathcal{A}(\Omega_{\mathcal{H}})-\frac{\delta}{2}\le l(\mathcal{P})-2H\widetilde{\mathcal{A}}(\Omega),$$

since we can choose $2H\mathcal{A}(\Omega\cap(\cup_i F_i))\le\frac{\delta}{2}$ and $\sum_j |\gamma_j(n)\cap\Omega|<\frac{\delta}{2}$.

If $\mathcal{A}(\mathfrak{D})=\infty$,

$$2H\widetilde{\mathcal{A}}(\Omega)+2\beta(\mathcal{P})-l(\mathcal{P})<2H\mathcal{A}(\Omega(n))+2\beta(\mathcal{P})-l(\mathcal{P})\le\sum_j |\gamma_j(n)|,$$

because we can choose $2H\mathcal{A}(\widetilde{\Omega})<\delta$. Since $2H\widetilde{\mathcal{A}}(\Omega)+2\beta(\mathcal{P})-l(\mathcal{P})$ tends to $-\infty$ when the horocycles converge to vertices, we can suppose

$$2H\widetilde{\mathcal{A}}(\Omega)+2\beta(\mathcal{P})-l(\mathcal{P})<0. \qquad \square$$

8. Example

Consider a domain $\mathfrak{D}$ whose boundary has sides A_1, B_1, A_2 and B_2 and vertices $d_1=(x_{d_1},0)$, $d_2=(x_{d_2},0)$, $d_3=(x_{d_3},0)$ and $d_4\in\{\partial_\infty\mathbb{H}-y=0\}$ with $x_{d_1}<x_{d_2}<x_{d_3}$. Suppose that the vertices of A_1 are d_4 and d_1, the vertices of B_1 are d_1 and d_2, the vertices of A_2 are d_2 and d_3 and the vertices of B_2 are d_3 and d_4. So A_1, B_2 are tilted lines and B_1 and A_2 are contained in Euclidean circles; see Figure 3.

Denote by $2\mu=x_{d_2}-x_{d_1}$, $2\omega=x_{d_3}-x_{d_2}$, and $0<\theta<\frac{\pi}{2}$ the angle such that $2H=\sin\theta$. This domain is not defined for all values of μ,ω,θ. We have to suppose that $B_1\cap B_2=\varnothing$.

Claim 8.1. *With the notation above, for*

$$2H<\sqrt{\frac{\omega}{\omega+\mu}},$$

the domain $\mathfrak{D}$ is well defined.

Proof. Since B_2 is a tilted line making angle θ with vertical, we can write

$$B_2(y) = (x_{d_3} - y\tan(\theta),\, y).$$

The curve B_1 satisfies $(x-(x_{d_1}+\mu))^2 + (y-\mu\tan\theta)^2 = \left(\frac{\mu}{\cos\theta}\right)^2$ for $y > 0$. Since $x_{d_3} = x_{d_1} + 2\mu + 2\omega$, we have $B_1 \cap B_2 \neq \varnothing$ if

$$y > 0 \quad \text{and} \quad (\mu + 2\omega - y\tan\theta)^2 + (y - \mu\tan\theta)^2 = \left(\frac{\mu}{\cos\theta}\right)^2.$$

Then $B_1 \cap B_2 = \varnothing$ if

$$2H = \sin(\theta) < \sqrt{\frac{\omega}{\omega+\mu}}\,. \qquad \square$$

We will assume that the domain $\mathfrak{D}$ is well defined. We will show that the conditions of Theorem 3.1 are true for some choice of the horocycles at the vertices of $\mathfrak{D}$, provided that $2H < \sqrt{2}/2$.

Suppose that B_1 and A_2 are contained in Euclidean circles centered at $(x_{d_1}+\mu,\, h)$ and $(x_{d_2}+\omega,\, R_A)$, respectively, where $R_A = \omega/\cos\theta$, $R_B = \mu/\cos\theta$ are the Euclidean radii of these circles and $l = \omega\tan\theta$, $h = \mu\tan\theta$; see Figure 3.

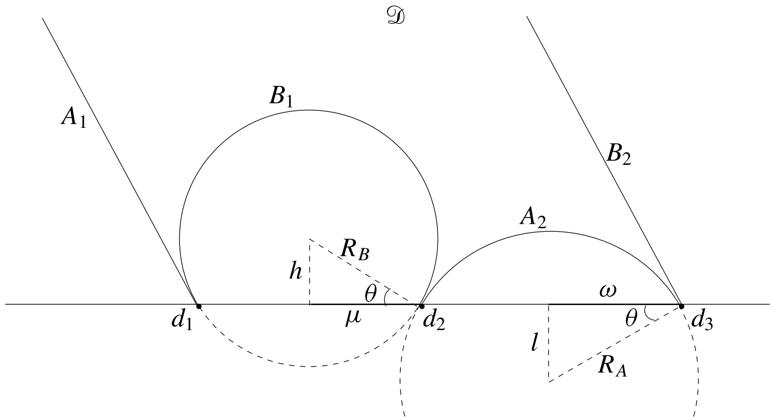

Figure 3. The domain $\mathfrak{D}$.

On each vertex d_i we put horocycles $\mathcal{H}_i$, $\mathcal{H}_i \cap \mathcal{H}_j = \varnothing$, $i \neq j$. Since this domain does not have inscribed polygons we will verify only condition (3) of Theorem 3.1. When $\mu = \omega$ and $2H < \sqrt{2}/2$ we have, for this choice of horocycles, that $\alpha(\partial\mathfrak{D}) = \beta(\partial\mathfrak{D})$, so condition (2) of Theorem 3.1 can't occur. The next proposition shows that there is a choice of ω such that this condition is satisfied for $2H < \sqrt{2}/2$.

Proposition 8.1. *With the notation above, given $\mu \geq 3$ and $2H < \sqrt{2}/2$, there is $\omega_0 \geq \mu$ such that the condition $\alpha(\partial\mathfrak{D}) - \beta(\partial\mathfrak{D}) = 2H\mathcal{A}(\mathfrak{D})$ is satisfied.*

Proof. First, we calculate the area $\mathscr{A}(\mathscr{D})$. Since the arc B_1 satisfies the equation $(x-(x_{d_1}+\mu))^2+(y-h)^2=R_B^2$ and the arc A_2 satisfies the equation $(x-(x_{d_2}+\omega))^2+(y+l)^2=R_A^2$, we have

$$\mathscr{A}(\mathscr{D})=\lim_{a\to 0}\frac{2(\mu+\omega)}{a}-2\lim_{a\to 0^+}\int_a^{R_B+h}\int_{x_{d_1}+\mu}^{\sqrt{R_B^2-(y-h)^2}+x_{d_1}+\mu}\frac{dxdy}{y^2}$$
$$-2\lim_{a\to 0^+}\int_a^{R_A-l}\int_{x_{d_2}+\omega}^{\sqrt{R_A^2-(y+l)^2}+x_{d_2}+\omega}\frac{dxdy}{y^2},$$

where the first term is the area between the arcs A_1, B_2 and straight line segment joining d_1, d_2, d_3.

Then

$$\mathscr{A}(\mathscr{D})=2\pi+2\tan\theta\ln\frac{2\omega^2(R_A-l)}{R_A^2-lR_A}+2\tan\theta\ln\frac{R_B^2+hR_B}{2\mu^2(R_B+h)}=2\Big(\pi+\ln\frac{\omega}{\mu}\Big).$$

Now, we are interested in the difference $\alpha(\partial\mathscr{D})-\beta(\partial\mathscr{D})$. We can suppose the horocycles $\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3$ are the same, that is, they differ by a horizontal translation. With this choice of the horocycles, we have $\alpha(\partial\mathscr{D})-\beta(\partial\mathscr{D})=|A_2|-|B_1|$, where $|A_2|$ and $|B_1|$ are the lengths of the compact arcs of A_2, B_1, respectively, which are outside of the horodisks bounded by $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$. Moreover, we will suppose that $\omega\geq\mu$ and that $\mathscr{H}_i\cap\Upsilon_i=(x_{d_i},\mu/2)$, where Υ_i is the vertical geodesic through x_{d_i}. It is possible to show that the intersection of B_1 and $\mathscr{H}_1$ occurs at $(x_{d_1},0)$ and at

$$(x_0,y_0)=\Big(-\sqrt{R_B^2-(y_0-h)^2}+x_{d_1}+\mu,\ \frac{8\mu^3}{17\mu^2+16h^2-8h\mu}\Big),$$

where B_1 and $\mathscr{H}_1$ satisfy the equations $(x-(x_{d_1}+\mu))^2+(y-h)^2=R_B^2$ and $(x-x_{d_1})^2+(y-\mu/4)^2=\mu^2/16$ respectively.

Similarly, the intersection of A_2 and $\mathscr{H}_2$ occurs at $(x_{d_2},0)$ and at

$$(x_1,y_1)=\Big(-\sqrt{R_A^2-(y_1+l)^2}+x_{d_2}+\omega,\ \frac{8\omega^2\mu}{16\omega^2+\mu^2+16l^2+8\mu l}\Big),\tag{7}$$

where A_2 and $\mathscr{H}_2$ satisfy the equations $(x-(x_{d_2}+\omega))^2+(y+l)^2=R_A^2$ and $(x-x_{d_2})^2+(y-\mu/4)^2=\mu^2/16$, respectively.

Then, the length of B_1 with respect to the horocycles $\mathscr{H}_1, \mathscr{H}_2$ is

$$|B_1|=2\int_{y_0}^{R_B+h}\frac{R_B}{y\sqrt{R_B^2-(y-h)^2}}\,dy$$
$$=\frac{2}{\cos\theta}\Big(-\ln R_B-\ln y_0+\ln\big(\mu\sqrt{R_B^2-(y_0-h)^2}+\mu^2+hy_0\big)\Big).$$

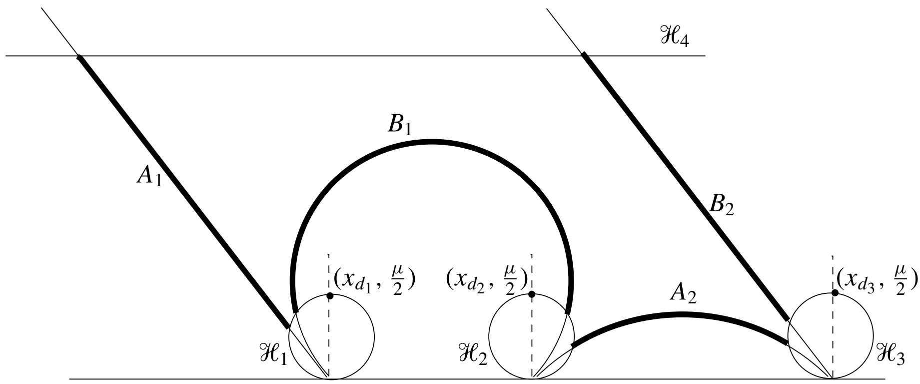

Figure 4. The domain $\mathcal{D}$ with the horocycles.

Analogously, the length of A_2 with respect to the horocycles $\mathcal{H}_2$, $\mathcal{H}_3$ is

$$\begin{aligned}|A_2| &= 2\int_{y_1}^{R_A-l} \frac{R_A}{y\sqrt{R_A^2-(y+l)^2}}\,dy \\ &= \frac{2}{\cos\theta}\Big(-\ln R_A - \ln y_1 + \ln\big(\omega\sqrt{R_A^2-(y_1+l)^2}+\omega^2-ly_1\big)\Big).\end{aligned}$$

So $\alpha(\partial\mathcal{D})-\beta(\partial\mathcal{D})-2H\mathcal{A}(\mathcal{D})$ only depends on μ and ω, because θ also depends on μ or ω. Thus consider, for each $\mu\in\mathbb{R}$, $\mu\geq 3$ fixed, the function

$$F(\omega)=\alpha(\partial\mathcal{D})-\beta(\partial\mathcal{D})-2H\mathcal{A}(\mathcal{D}).$$

We will show that at any moment this function is zero. We know for $\mu=\omega$ that $F(\omega)=-2H\mathcal{A}(\mathcal{D})<0$; thus we must show that for ω large enough, $F(\omega)>0$, so there exists a ω_0 such that $F(\omega_0)=0$ for each $\mu\geq 3$ fixed. We have

$$\begin{aligned}F(\omega) &= \frac{2}{\cos\theta}\Big(-\ln R_A-\ln y_1+\ln\big(\omega\sqrt{R_A^2-(y_1+l)^2}+\omega^2-ly_1\big). \\ &\qquad +\ln R_B+\ln y_0-\ln\big(\mu\sqrt{R_B^2-(y_0-h)^2}+\mu^2+hy_0\big)\Big) \\ &\qquad -4H\Big(\pi+\ln\frac{\omega}{\mu}\Big) \\ &= \frac{2}{\cos\theta}\left(\ln\Big(\frac{1}{R_Ay_1}\big(\omega\sqrt{R_A^2-(y_1+l)^2}+\omega^2-ly_1\big)\Big)\right. \\ &\qquad \left.+\ln\frac{R_By_0}{\mu\sqrt{R_B^2-(y_0-h)^2}+\mu^2+hy_0}\right) \\ &\qquad -4H\pi-2\sin\theta\ln\frac{\omega}{\mu}.\end{aligned}$$

The second logarithmic term in the big parentheses is constant, because we are supposing μ fixed. As for the remaining terms, we substitute the value of y_1 from (7) and find that the difference

$$\frac{2}{\cos\theta}\ln\Big(\frac{1}{R_A y_1}\big(\omega\sqrt{R_A^2-(y_1+l)^2}+\omega^2-l y_1\big)\Big)-2\sin\theta\ln\frac{\omega}{\mu}$$

is strictly positive and increasing, so the function F is increasing and unbounded. Thus there is a ω_0 such that $F(\omega_0)=0$. $\square$

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Received March 16, 2010.

Abigail Folha
Instituto de Matemática - Departamento de Geometria
Universidade Federal Fluminense
R. Mário Santos Braga, S/N
Campus do Valonguinho
24020-140 Niterói RJ
Brazil

abigailfolha@vm.uff.br

Sofia Melo
Departamento de Matemática – ICE
Universidade Federal de Juiz de Fora
Cidade Universitária
36036-330 Juiz de Fora, MG
Brazil

sofia.melo@ufjf.edu.br

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chari@math.ucr.edu

Darren Long
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
long@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Alexander Merkurjev
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
merkurev@math.ucla.edu

Jonathan Rogawski
Department of Mathematics
University of California
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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

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PACIFIC JOURNAL OF MATHEMATICS

Volume 251 No. 1 May 2011