Pacific
Journal of
Mathematics

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Volume 251 No. 1 May 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 251, No. 1, 2011

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Recently, Cappell and Miller extended the classical construction of the analytic torsion for de Rham complexes to coupling with an arbitrary flat bundle and the holomorphic torsion for $\bar{\partial}$-complexes to coupling with an arbitrary holomorphic bundle with compatible connection of type $(1, 1)$. Cappell and Miller also studied the behavior of these torsions under metric deformations. On the other hand, Mathai and Wu generalized the classical construction of the analytic torsion to the twisted de Rham complexes with an odd degree closed form as a flux and later, more generally, to the $\mathbb{Z}_2$-graded elliptic complexes. Mathai and Wu also studied the properties of analytic torsions for the $\mathbb{Z}_2$-graded elliptic complexes, including the behavior under metric and flux deformations. In this paper we define the Cappell–Miller holomorphic torsion for the twisted Dolbeault-type complexes and the Cappell–Miller analytic torsion for the twisted de Rham complexes. We obtain variation formulas for the twisted Cappell–Miller holomorphic and analytic torsions under metric and flux deformations.

1. Introduction

Ray and Singer, in the celebrated works [1971; 1973], defined the analytic torsion for de Rham complexes and the holomorphic torsion for $\bar{\partial}$-complexes of complex manifolds. They studied properties of these torsions, including the behavior under metric deformations and coupled the Riemannian Laplacian and the $\bar{\partial}$-Laplacian with unitary flat vector bundles and obtained self-adjoint operators. Hence, the analytic torsion and holomorphic torsion are real numbers in the acyclic cases considered by Ray and Singer and are expressed as elements of real determinant line in the nonacyclic case.

Recently, Cappell and Miller [2010] extended the classical construction of the analytic torsion to coupling with an arbitrary flat bundle and the holomorphic torsion to coupling with an arbitrary holomorphic bundle with compatible connection

MSC2000: 58J52.
Keywords: determinant, analytic torsion.

of type $(1,1)$; see Definition 3.1. This includes both unitary and flat (not necessarily unitary) bundles as special cases. However, in this general setting, the associated operators are not necessarily self-adjoint and the torsions are complex-valued. Cappell and Miller also studied the behavior of these torsions under metric deformations.

Mathai and Wu [2008; 2010b] generalized the classical construction of the Ray–Singer torsion for de Rham complexes to the twisted de Rham complex with an odd degree closed differential form H as a flux. Later, in [Mathai and Wu 2010a], they extended this to $\mathbb{Z}_2$-graded elliptic complexes. The definitions use pseudo-differential operators and residue traces. Mathai and Wu also studied the properties of analytic torsion for $\mathbb{Z}_2$-graded elliptic complexes, including the behavior under the variation of metric and flux.

Let E be a holomorphic bundle with a compatible type-$(1,1)$ connection D (see Definition 3.1) over a complex manifold W of complex dimension n and $H \in A^{0,\bar{1}}(W,\mathbb{C})$ be a $\bar{\partial}$-closed differential form of type $(0,\text{odd})$. In Definition 3.5, for each p, $1 \leq p \leq n$, we define the twisted Cappell–Miller holomorphic torsion $\tau_{\text{holo},p}(W,E,H)$ as a nonvanishing element of the determinant line:

$$\tau_{\text{holo},p}(W,E,H) \in \operatorname{Det} H^{p,\bullet}_{\bar{\partial}_E}(W,E,H) \otimes \big(\operatorname{Det} H^{\bullet,n-p}_{D^{1,0}}(W,E,H)\big)^{(-1)^{n+1}}.$$

We show that the variation of the twisted Cappell–Miller holomorphic torsion $\tau_{\text{holo},p}(W,E,H)$ under the deformation of the metric is given by a local formula; see Theorem 3.8. We also show that along any deformation of H that fixes the cohomology class $[H]$ and the natural identification of determinant lines, the variation of the twisted Cappell–Miller holomorphic torsion $\tau_{\text{holo},p}(W,E,H)$ under the deformation of the flux is given by a local formula; see Theorem 3.12.

Let $\mathscr{E}$ be a complex flat vector bundle over a closed manifold M endowed with a flat connection ∇ and let $\mathscr{H}$ be an odd degree flux form. Then the Cappell–Miller analytic torsion $\tau(\nabla,\mathscr{H})$ (see Definition 4.2) for the twisted de Rham complexes is an element of $\operatorname{Det} H^\bullet(M,\mathscr{E}\oplus\mathscr{E}',\mathscr{H})$, where $\mathscr{E}'$ is the dual of the vector bundle $\mathscr{E}$. We show that the variation of the twisted Cappell–Miller analytic torsion $\tau(\nabla,\mathscr{H})$ under the deformation of the metric is given by a local formula; see Theorem 4.3. We also show that along any deformation of $\mathscr{H}$ that fixes the cohomology class $[\mathscr{H}]$ and the natural identification of determinant lines, the variation of the twisted Cappell–Miller analytic torsion $\tau(\nabla,\mathscr{H})$ under the deformation of the flux is given by a local formula; see Theorem 4.4. In particular, we show that if the manifold M is an odd-dimensional closed oriented manifold, then the twisted Cappell–Miller analytic torsion is independent of the Riemannian metric and the representative $\mathscr{H}$ in the cohomology class $[\mathscr{H}]$. See also [Su 2011, Section 6]. We also compare the twisted Cappell–Miller analytic torsion with the twisted refined analytic torsion [Huang 2010]; see Theorem 4.5.

In the paper just cited we defined and studied the refined analytic torsion of Braverman and Kappeler [2007; 2008b] for the twisted de Rham complexes. Later, Su [2011] defined and studied the Burghelea–Haller [2007; 2008; 2010] analytic torsion for the twisted de Rham complexes and compared the twisted Burghelea–Haller torsion with the twisted refined analytic torsion. Su [2011] also briefly discussed the twisted Cappell–Miller analytic torsion when the dimension of the manifold is odd.

The rest of the paper is organized as follows. In Section 2, we define and calculate the Cappell–Miller torsion for the $\mathbb{Z}_2$-graded finite-dimensional bigraded complex. In Section 3, we first define the Dolbeault-type bigraded complexes twisted by a flux form and its (co)homology groups. We then define the Cappell–Miller holomorphic torsion for the twisted Dolbeault-type bigraded complexes. We prove variation theorems for the twisted Cappell–Miller holomorphic torsion under metric and flux deformations. In Section 4, we first define the de Rham bigraded complex twisted by a flux form and its (co)homology groups. Then we define the Cappell–Miller analytic torsion for the twisted de Rham bigraded complex. We prove variation theorems for the twisted Cappell–Miller analytic torsion under metric and flux deformations.

Throughout this paper, a bar over an integer means taking the value modulo 2.

2. The Cappell–Miller torsion for a $\mathbb{Z}_2$-graded finite-dimensional bigraded complex

In this section we define and calculate the Cappell–Miller torsion for the $\mathbb{Z}_2$-graded finite-dimensional bigraded complex. For the $\mathbb{Z}$-graded case, see [Cappell and Miller 2010, Section 6]. Throughout this section $\boldsymbol{k}$ is a field of characteristic zero.

Determinant lines of a $\mathbb{Z}_2$-graded finite-dimensional bigraded complex. Given a $\boldsymbol{k}$-vector space V of dimension n, the determinant line of V is the line $\mathrm{Det}(V) := \bigwedge^n V$, where $\bigwedge^n V$ denotes the n-th exterior power of V. By definition, we set $\mathrm{Det}(0) := \boldsymbol{k}$. Further, we denote by $\mathrm{Det}(V)^{-1}$ the dual line of $\mathrm{Det}(V)$. Let

$$C^{\bar{0}} = C^{\mathrm{even}} = \bigoplus_{i=0}^{[m/2]} C^{2i},$$
$$C^{\bar{1}} = C^{\mathrm{odd}} = \bigoplus_{i=0}^{[(m-1)/2]} C^{2i+1},$$

where C^i, $i = 0, \dots, m$, are finite-dimensional $\boldsymbol{k}$-vector spaces. Let

$$(C^\bullet, d): \quad \cdots \xrightarrow{d} C^{\bar{0}} \xrightarrow{d} C^{\bar{1}} \xrightarrow{d} C^{\bar{0}} \xrightarrow{d} \cdots \tag{2-1}$$

be a $\mathbb{Z}_2$-graded cochain complex of finite dimensional $\boldsymbol{k}$-vector spaces. Denote by $H^\bullet(d) = H^{\bar{0}}(d) \oplus H^{\bar{1}}(d)$ its cohomology. Set

$$\begin{aligned} \operatorname{Det}(C^\bullet) &:= \operatorname{Det}(C^{\bar{0}}) \otimes \operatorname{Det}(C^{\bar{1}})^{-1}, \\ \operatorname{Det}(H^\bullet(d)) &:= \operatorname{Det}(H^{\bar{0}}(d)) \otimes \operatorname{Det}(H^{\bar{1}}(d))^{-1}. \end{aligned} \tag{2-2}$$

Assume that $C^\bullet$ has another differential $d^* : C^{\bar{k}} \to C^{\overline{k-1}}$ giving the complex

$$(C^\bullet, d^*) : \quad \cdots \xleftarrow{d^*} C^{\bar{0}} \xleftarrow{d^*} C^{\bar{1}} \xleftarrow{d^*} C^{\bar{0}} \xleftarrow{d^*} \cdots .$$

Denote its homology by $H_\bullet(d^*) = H_{\bar{0}}(d^*) \oplus H_{\bar{1}}(d^*)$. Set

$$\operatorname{Det}(H_\bullet(d^*)) := \operatorname{Det}(H_{\bar{0}}(d^*)) \otimes \operatorname{Det}(H_{\bar{1}}(d^*))^{-1}.$$

The fusion isomorphisms. (See [Braverman and Kappeler 2007, Section 2.3].) For two finite-dimensional $\boldsymbol{k}$-vector spaces V and W, we denote by $\mu_{V,W}$ the canonical fusion isomorphism

$$\mu_{V,W} : \operatorname{Det}(V) \otimes \operatorname{Det}(W) \to \operatorname{Det}(V \oplus W). \tag{2-3}$$

For $v \in \operatorname{Det}(V)$, $w \in \operatorname{Det}(W)$, we have

$$\mu_{V,W}(v \otimes w) = (-1)^{\dim V \cdot \dim W} \mu_{W,V}(w \otimes v). \tag{2-4}$$

By a slight abuse of notation, denote by $\mu_{V,W}^{-1}$ the transpose of the inverse of $\mu_{V,W}$.

Similarly, if $V_1, \ldots, V_r$ are finite-dimensional $\boldsymbol{k}$-vector spaces, we define an isomorphism

$$\mu_{V_1,\ldots,V_r} : \operatorname{Det}(V_1) \otimes \cdots \otimes \operatorname{Det}(V_r) \to \operatorname{Det}(V_1 \oplus \cdots \oplus V_r). \tag{2-5}$$

The isomorphism between determinant lines. For $k = 0, 1$, fix a direct sum decomposition

$$C^{\bar{k}} = B^{\bar{k}} \oplus H^{\bar{k}} \oplus A^{\bar{k}}, \tag{2-6}$$

such that $B^{\bar{k}} \oplus H^{\bar{k}} = (\operatorname{Ker} d) \cap C^{\bar{k}}$ and $B^{\bar{k}} = d(C^{\overline{k-1}}) = d(A^{\overline{k-1}})$. Then $H^{\bar{k}}$ is naturally isomorphic to the cohomology $H^{\bar{k}}(d)$ and d defines an isomorphism $d : A^{\bar{k}} \to B^{\overline{k+1}}$.

Fix $c_{\bar{k}} \in \operatorname{Det}(C^{\bar{k}})$ and $x_{\bar{k}} \in \operatorname{Det}(A^{\bar{k}})$. Let $d(x_{\bar{k}}) \in \operatorname{Det}(B^{\overline{k+1}})$ be the image of $x_{\bar{k}}$ under the map $\operatorname{Det}(A^{\bar{k}}) \to \operatorname{Det}(B^{\overline{k+1}})$ induced by the isomorphism $d : A^{\bar{k}} \to B^{\overline{k+1}}$. Then there is a unique element $h_{\bar{k}} \in \operatorname{Det}(H^{\bar{k}})$ such that

$$c_{\bar{k}} = \mu_{B^{\bar{k}},H^{\bar{k}},A^{\bar{k}}}\big(d(x_{\overline{k-1}}) \otimes h_{\bar{k}} \otimes x_{\bar{k}}\big), \tag{2-7}$$

where $\mu_{B^{\bar{k}},H^{\bar{k}},A^{\bar{k}}}$ is the fusion isomorphism; see (2-5) and [Braverman and Kappeler 2007, Section 2.3].

Define the canonical isomorphism

$$\phi_{C^\bullet} = \phi_{(C^\bullet,d)} : \mathrm{Det}(C^\bullet) \longrightarrow \mathrm{Det}(H^\bullet(d)) \tag{2-8}$$

by the formula

$$\phi_{C^\bullet} : c_{\bar{0}} \otimes c_{\bar{1}}^{-1} \mapsto h_{\bar{0}} \otimes h_{\bar{1}}^{-1}. \tag{2-9}$$

Following the sign convention of [Braverman and Kappeler 2007, (2-14)], Equation (2.10) of [Huang 2010] introduced a sign-refined version of the canonical isomorphism (2-8). Here we follow the sign convention of [Cappell and Miller 2010, Section 6].

Similarly, for $k = 0, 1$, fix a direct sum decomposition

$$C^{\bar{k}} = B_{\bar{k}} \oplus H_{\bar{k}} \oplus A_{\bar{k}}, \tag{2-10}$$

such that $B_{\bar{k}} \oplus H_{\bar{k}} = (\operatorname{Ker} d^*) \cap C^{\bar{k}}$ and $B_{\bar{k}} = d^*(C^{\overline{k+1}}) = d^*(A_{\overline{k+1}})$. Then $H_{\bar{k}}$ is naturally isomorphic to the homology $H_{\bar{k}}(d^*)$ and d^* defines an isomorphism $d^* : A_{\bar{k}} \to B_{\overline{k-1}}$.

Similarly, fix $c_{\bar{k}} \in \mathrm{Det}(C^{\bar{k}})$ and $y_{\bar{k}} \in \mathrm{Det}(A_{\bar{k}})$. Let $d^*(y_{\bar{k}}) \in \mathrm{Det}(B_{\overline{k-1}})$ denote the image of $y_{\bar{k}}$ under the map $\mathrm{Det}(A_{\bar{k}}) \to \mathrm{Det}(B_{\overline{k-1}})$ induced by the isomorphism $d^* : A_{\bar{k}} \to B_{\overline{k-1}}$. Then there is a unique element $h'_{\bar{k}} \in \mathrm{Det}(H^{\bar{k}})$ such that

$$c_{\bar{k}} = \mu_{B_{\bar{k}},H_{\bar{k}},A_{\bar{k}}}\big(d^*(y_{\overline{k+1}}) \otimes h'_{\bar{k}} \otimes y_{\bar{k}}\big), \tag{2-11}$$

where $\mu_{B_{\bar{k}},H_{\bar{k}},A_{\bar{k}}}$ is the fusion isomorphism; see (2-5) and [Braverman and Kappeler 2007, Section 2.3].

Define the canonical isomorphism

$$\phi'_{C^\bullet} = \phi'_{(C^\bullet,d^*)} : \mathrm{Det}(C^\bullet) \longrightarrow \mathrm{Det}(H_\bullet(d^*)) \tag{2-12}$$

by the formula

$$\phi'_{C^\bullet} : c_{\bar{0}} \otimes c_{\bar{1}}^{-1} \mapsto h'_{\bar{0}} \otimes {h'_{\bar{1}}}^{-1}. \tag{2-13}$$

The Cappell–Miller torsion for a $\mathbb{Z}_2$-graded finite-dimensional bigraded complex. Let $C^\bullet = C^{\bar{0}} \oplus C^{\bar{1}}$ and $\widetilde{C}^\bullet = \widetilde{C}^{\bar{0}} \oplus \widetilde{C}^{\bar{1}}$ be finite-dimensional $\mathbb{Z}_2$-graded $\boldsymbol{k}$-vector spaces. The fusion isomorphism

$$\mu_{C^\bullet,\widetilde{C}^\bullet} : \mathrm{Det}(C^\bullet) \otimes \mathrm{Det}(\widetilde{C}^\bullet) \to \mathrm{Det}(C^\bullet \oplus \widetilde{C}^\bullet)$$

is defined by the formula

$$\mu_{C^\bullet,\widetilde{C}^\bullet} := (-1)^{\mathcal{M}(C^\bullet,\widetilde{C}^\bullet)} \mu_{C^{\bar{0}},\widetilde{C}^{\bar{0}}} \otimes \mu^{-1}_{C^{\bar{1}},\widetilde{C}^{\bar{1}}}, \tag{2-14}$$

where

$$\mathcal{M}(C^\bullet, \widetilde{C}^\bullet) := \dim C^{\bar{1}} \cdot \dim \widetilde{C}^{\bar{0}}. \tag{2-15}$$

Consider the element $c := c_{\bar{0}} \otimes c_{\bar{1}}^{-1}$ of $\mathrm{Det}(C^\bullet)$. Then, for the bigraded complex $(C^\bullet, d, d^*)$, the Cappell–Miller torsion is the algebraic torsion invariant

$$\tau(C^\bullet, d, d^*) := (-1)^{S(C^\bullet)} \phi_{C^\bullet}(c)(\phi'_{C^\bullet}(c))^{-1} \in \mathrm{Det}(H^\bullet(d)) \otimes \mathrm{Det}(H_\bullet(d^*))^{-1}, \tag{2-16}$$

where $(-1)^{S(C^\bullet)}$ is defined by the formula

$$S(C^\bullet) := \sum_{k=0,1} \big(\dim B_{\overline{k-1}} \cdot \dim B^{\overline{k+1}} + \dim B^{\overline{k+1}} \cdot \dim H_{\bar{k}} + \dim B_{\overline{k-1}} \cdot \dim H^{\bar{k}}\big). \tag{2-17}$$

Calculation of the $\mathbb{Z}_2$-graded Cappell–Miller torsion. We first compute the torsion in the case that the combinatorial Laplacian $\Delta := d^*d + dd^*$ is bijective.

For $k = 0, 1$, define

$$C^{\bar{k}}_+ := \mathrm{Ker}\, d^* \cap C^{\bar{k}}, \quad C^{\bar{k}}_- := \mathrm{Ker}\, d \cap C^{\bar{k}}. \tag{2-18}$$

The proof of the following proposition is similar to the proof of the $\mathbb{Z}$-graded case [Cappell and Miller 2010, Section 6.2, Claim B].

Proposition 2.1. *Suppose that the combinatorial Laplacian Δ has no zero eigenvalue. Then the cohomology group $H^\bullet(d) = 0$ and the homology group $H_\bullet(d^*) = 0$. Moreover,*

$$\tau(C^\bullet, d, d^*) = \mathrm{Det}(d^*d|_{C^{\bar{0}}_+}) \cdot \mathrm{Det}(d^*d|_{C^{\bar{1}}_+})^{-1}, \tag{2-19}$$

Proof. The proof of the first assertion that $H^\bullet(d) = 0$ and $H_\bullet(d^*) = 0$ is standard, so we skip the proof.

To compute $\tau(C^\bullet, d, d^*)$ (see (2-16)), we first compute $\phi'_{C^\bullet}(c)$. For each $k = 0, 1$, we now have the direct sum decomposition

$$C^{\bar{k}} = d^* C^{\overline{k+1}} \oplus d C^{\overline{k-1}}. \tag{2-20}$$

We also have the isomorphisms

$$d : d^* C^{\overline{k+1}} \cong d C^{\bar{k}}, \quad d^* : d C^{\bar{k}} \cong d^* C^{\overline{k+1}}. \tag{2-21}$$

By (2-18), (2-20) and (2-21), we know that

$$C^{\bar{k}}_+ = d^* C^{\overline{k+1}}, \quad C^{\bar{k}}_- = d C^{\overline{k-1}}. \tag{2-22}$$

By (2-6), (2-10), (2-21), (2-22) and the first assertion we know that

$$C^{\bar{k}}_+ = B_{\bar{k}} \cong A^{\bar{k}}, \quad C^{\bar{k}}_- = B^{\bar{k}} \cong A_{\bar{k}}. \tag{2-23}$$

Let $\{\, d^* y_{\overline{k+1},i} \mid 1 \le i \le \dim B_{\bar{k}} \,\}$ be a basis for $B_{\bar{k}} = d^* C^{\overline{k+1}} \cong A^{\bar{k}}$. Since

$$d^*d : d^* C^{\overline{k+1}} \to d^* C^{\overline{k+1}}$$

is an isomorphism, there is a unique vector

$$x_{\bar{k},i} \in B_{\bar{k}} = d^* C^{\overline{k+1}}$$

such that

$$d^* d x_{\bar{k},i} = d^* y_{\overline{k+1},i}. \tag{2-24}$$

Then $\{ x_{\bar{k},i} \mid 1 \le i \le \dim B_{\bar{k}} \}$ is also a basis for $B_{\bar{k}} \cong A^{\bar{k}}$. Since $d : d^* C^{\overline{k+1}} \to dC^{\bar{k}}$ is an isomorphism, it follows that $\{ dx_{\bar{k},i} \mid 1 \le i \le \dim B_{\bar{k}} \}$ is a basis for $B^{\overline{k+1}} = dC^{\bar{k}} \cong A_{\overline{k+1}}$. Hence, in view of the decomposition (2-20), we conclude that

$$\{ d^* y_{\overline{k+1},i} \mid 1 \le i \le \dim B_{\bar{k}} \} \cup \{ dx_{\overline{k-1},i} \mid 1 \le i \le \dim B_{\overline{k-1}} \}$$

forms a basis for $C^{\bar{k}}$. In particular, by the first assertion and (2-6), we have

$$\dim B^{\bar{k}} = \dim B_{\overline{k-1}}. \tag{2-25}$$

With this particular choice of basis, we set

$$\begin{aligned} y_{\overline{k+1}} &:= y_{\overline{k+1},1} \wedge \cdots \wedge y_{\overline{k+1},\dim B_{\bar{k}}} \in \operatorname{Det}(A_{\overline{k+1}}) \\ x_{\overline{k-1}} &:= x_{\overline{k-1},1} \wedge \cdots \wedge x_{\overline{k-1},\dim B_{\overline{k-1}}} \in \operatorname{Det}(B_{\overline{k-1}}). \end{aligned}$$

Let $d^* y_{\overline{k+1}}$ and $dx_{\overline{k-1}}$ be the induced elements in $\operatorname{Det}(B_{\bar{k}})$ and $\operatorname{Det}(A_{\bar{k}})$. Set

$$c_{\bar{k}} = \mu_{B_{\bar{k}},A_{\bar{k}}}(d^* y_{\overline{k+1}} \otimes dx_{\overline{k-1}}). \tag{2-26}$$

To compute $\phi'_{C^\bullet}(c)$ (see (2-13)), we need to compute $h'_{\bar{k}} \in \operatorname{Det}(H_{\bar{k}}(d^*)) \cong \boldsymbol{k}$.

If L is a complex line and $x, y \in L$ with $y \neq 0$, we denote by $[x : y] \in \boldsymbol{k}$ the unique number such that $x = [x : y]y$. Then

$$\begin{aligned} h'_{\bar{k}} &= [c_{\bar{k}} : \mu_{B_{\bar{k}},A_{\bar{k}}}(d^* y_{\overline{k+1}} \otimes dx_{\overline{k-1}})] && \text{by (2-24)} \\ &= [\mu_{B_{\bar{k}},A_{\bar{k}}}(d^* y_{\overline{k+1}} \otimes dx_{\overline{k-1}}) : \mu_{B_{\bar{k}},A_{\bar{k}}}(d^* y_{\overline{k+1}} \otimes dx_{\overline{k-1}})] && \text{by (2-26)} \\ &= 1. \end{aligned} \tag{2-27}$$

We next compute $\phi_{C^\bullet}(c)$. By (2-9), we need to compute $h_{\bar{k}}$. By our choice of basis, we have

$$\begin{aligned} h_{\bar{k}} &= [c_{\bar{k}} : \mu_{B^{\bar{k}},A^{\bar{k}}}(dx_{\overline{k-1}}) \otimes x_{\bar{k}}] \\ &= [\mu_{B_{\bar{k}},A_{\bar{k}}}(d^* y_{\overline{k+1}} \otimes dx_{\overline{k-1}}) : \mu_{B^{\bar{k}},A^{\bar{k}}}(dx_{\overline{k-1}}) \otimes x_{\bar{k}}] && \text{by (2-26)} \\ &= [\mu_{B_{\bar{k}},A_{\bar{k}}}(d^* dx_{\bar{k}} \otimes dx_{\overline{k-1}}) : \mu_{A_{\bar{k}},B_{\bar{k}}}(dx_{\overline{k-1}}) \otimes x_{\bar{k}}] && \text{by (2-23), (2-24)} \\ &= (-1)^{\dim B_{\bar{k}} \dim A_{\bar{k}}} [\mu_{B_{\bar{k}},A_{\bar{k}}}(d^* dx_{\bar{k}} \otimes dx_{\overline{k-1}}) : \mu_{B_{\bar{k}},A_{\bar{k}}}(x_{\bar{k}} \otimes dx_{\overline{k-1}})] \\ &= (-1)^{\dim B_{\overline{k-1}} \dim B^{\overline{k+1}}} \operatorname{Det}(d^* d|_{C^{\bar{k}}_+}), && \text{by (2-23), (2-25).} \end{aligned} \tag{2-28}$$

Combining (2-16), (2-17), (2-27), (2-28) with the first assertion gives (2-19). □

We now compute the torsion in the case that the combinatorial Laplacian $\Delta := d^*d + dd^*$ is *not* bijective. For simplicity, we restrict to the case $\boldsymbol{k} = \mathbb{C}$ for the rest of discussion in this section. The operator Δ maps $C^{\bar{k}}$ into itself. For an arbitrary interval $\mathcal{I} \subset [0, \infty)$, let $C^{\bar{k}}_{\mathcal{I}} \subset C^{\bar{k}}$ denote the linear span of the generalized eigenvectors of the restriction of Δ to $C^{\bar{k}}$, corresponding to eigenvalue λ with $|\lambda| \in \mathcal{I}$. Since both d and d^* commute with Δ, we have $d(C^{\bar{k}}_{\mathcal{I}}) \subset C^{\overline{k+1}}_{\mathcal{I}}$ and $d^*(C^{\bar{k}}_{\mathcal{I}}) \subset C^{\overline{k-1}}_{\mathcal{I}}$. Hence, we obtain a subcomplex $C^\bullet_{\mathcal{I}}$ of $C^\bullet$. We denote by $H^\bullet_{\mathcal{I}}(d)$ the cohomology of the complex $(C^\bullet_{\mathcal{I}}, d_{\mathcal{I}})$ and $H_{\bullet,\mathcal{I}}(d^*)$ the homology of the complex $(C^\bullet_{\mathcal{I}}, d^*_{\mathcal{I}})$. Denote by $d_{\mathcal{I}}$ and $d^*_{\mathcal{I}}$ the restrictions of d and d^* to $C^{\bar{k}}_{\mathcal{I}}$ and denote by $\Delta_{\mathcal{I}}$ the restriction of Δ to $C^{\bar{k}}_{\mathcal{I}}$. Then $\Delta_{\mathcal{I}} = d^*_{\mathcal{I}} d_{\mathcal{I}} + d_{\mathcal{I}} d^*_{\mathcal{I}}$. For $k = 0, 1$, we also denote by $C^{\bar{k}}_{\pm,\mathcal{I}}$ the restrictions of $C^{\bar{k}}_{\pm}$ to $C^{\bar{k}}_{\mathcal{I}}$.

For each $\lambda \geq 0$, we have $C^\bullet = C^\bullet_{[0,\lambda]} \oplus C^\bullet_{(\lambda,\infty)}$. Then $H^\bullet_{(\lambda,\infty)}(d) = 0$ whereas $H^\bullet_{[0,\lambda]}(d) \cong H^\bullet(d)$ and $H_{\bullet,(\lambda,\infty)}(d^*) = 0$ whereas $H_{\bullet,[0,\lambda]}(d^*) \cong H_\bullet(d^*)$. Hence there are canonical isomorphisms

$$\begin{aligned} \Phi_\lambda &: \mathrm{Det}(H^\bullet_{(\lambda,\infty)}(d)) \to \mathbb{C}, & \Psi_\lambda &: \mathrm{Det}(H^\bullet_{[0,\lambda]}(d)) \to \mathrm{Det}(H^\bullet(d)) \\ \Phi'_\lambda &: \mathrm{Det}(H_{\bullet,(\lambda,\infty)}(d^*)) \to \mathbb{C}, & \Psi'^*_\lambda &: \mathrm{Det}(H_{\bullet,[0,\lambda]}(d^*))^{-1} \to \mathrm{Det}(H_\bullet(d^*))^{-1}. \end{aligned}$$

In the sequel, we will write t for $\Phi_\lambda(t) \in \mathbb{C}$ and t' for $\Phi'_\lambda(t') \in \mathbb{C}$.

Proposition 2.2. *Let $(C^\bullet, d, d^*)$ be a $\mathbb{Z}_2$-graded bigraded complex of finite-dimensional $\boldsymbol{k}$-vector spaces. Then, for each $\lambda \geq 0$,*

$$\tau(C^\bullet, d, d^*) = \mathrm{Det}(d^*d|_{C^{\bar{0}}_{+,(\lambda,\infty)}}) \cdot \mathrm{Det}(d^*d|_{C^{\bar{1}}_{+,(\lambda,\infty)}})^{-1} \cdot \tau(C^\bullet_{[0,\lambda]}, d, d^*), \tag{2-29}$$

where we view $\tau(C^\bullet_{[0,\lambda]}, d, d^*)$ as an element of $\mathrm{Det}(H^\bullet(d)) \otimes \mathrm{Det}(H_\bullet(d^*))^{-1}$ *via the canonical isomorphism* $\Psi_\lambda \otimes \Psi'^*_\lambda : \mathrm{Det}(H^\bullet_{[0,\lambda]}(d)) \otimes \mathrm{Det}(H_{\bullet,[0,\lambda]}(d^*))^{-1} \to \mathrm{Det}(H^\bullet(d)) \otimes \mathrm{Det}(H_\bullet(d^*))^{-1}$.

In particular, the right side of (2-29) *is independent of* $\lambda \geq 0$.

Proof. Recall the natural isomorphisms

$$\begin{aligned} \mathrm{Det}(H^{\bar{k}}_{[0,\lambda]}(d) \otimes H^{\bar{k}}_{(\lambda,\infty)}(d)) &\cong \mathrm{Det}(H^{\bar{k}}_{[0,\lambda]}(d) \oplus H^{\bar{k}}_{(\lambda,\infty)}(d)) \\ &= \mathrm{Det}(H^{\bar{k}}(d)), \end{aligned} \tag{2-30}$$

$$\begin{aligned} \mathrm{Det}(H_{\bar{k},[0,\lambda]}(d^*) \otimes H_{\bar{k},(\lambda,\infty)}(d^*)) &\cong \mathrm{Det}(H_{\bar{k},[0,\lambda]}(d^*) \oplus H_{\bar{k},(\lambda,\infty)}(d^*)) \\ &= \mathrm{Det}(H_{\bar{k}}(d^*)). \end{aligned} \tag{2-31}$$

From (2-16), Proposition 2.1, (2-30) and (2-31) we obtain the result. □

3. Twisted Cappell–Miller holomorphic torsion

In this section we first review the $\bar{\partial}$-Laplacian for a holomorphic bundle with compatible type (1,1) connection introduced in [Cappell and Miller 2010]. Then we define the Dolbeault-type bigraded complexes twisted by a flux form and its cohomology and homology groups. We define the Cappell–Miller holomorphic torsion for the twisted Dolbeault-type bigraded complexes. We also prove variation theorems for the twisted Cappell–Miller holomorphic torsion under metric and flux deformations.

The $\bar{\partial}$-Laplacian for a holomorphic bundle with compatible type $(1, 1)$ connection. In this section we review some materials from [Cappell and Miller 2010]; see also [Liu and Yu 2010].

Let (W, J) be a complex manifold of complex dimension n with the complex structure J and let g^W be any Hermitian metric on TW. Let $E \to W$ be a holomorphic bundle over W endowed with a linear connection D and let h^E be a Hermitian metric on E.

The complex structure J induces a splitting $TW \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}W \oplus T^{(0,1)}W$, where $T^{(1,0)}W$ and $T^{(0,1)}W$ are eigenbundles of J corresponding to eigenvalues i and $-i$, respectively. Let $T^{*(1,0)}W$ and $T^{*(0,1)}W$ be the corresponding dual bundles. For $0 \le p, q \le n$, let

$$A^{p,q}(W, E) = \Gamma\big(W, \textstyle\bigwedge^p(T^{*(1,0)}W) \otimes \bigwedge^q(T^{*(0,1)}W) \otimes E\big)$$

be the space of smooth (p, q)-forms on W with values in E. Set

$$A^{\bullet,\bullet}(W, E) = \bigoplus_{p,q=0}^{n} A^{p,q}(W, E).$$

Let $\bar{\partial} : A^{p,q}(W, \mathbb{C}) \to W^{p,q+1}(W, \mathbb{C})$ and $\partial : A^{p,q}(W, \mathbb{C}) \to A^{p+1,q}(W, \mathbb{C})$ be the standard operators obtained by decomposing by type the exterior derivative

$$d = \bar{\partial} + \partial$$

acting on complex-valued smooth forms of type (p, q). From $d^2 = 0$, we have $\bar{\partial}^2 = 0$, $\partial^2 = 0$.

Since E is holomorphic, the operator $\bar{\partial}$ on $A^{\bullet,\bullet}(W, \mathbb{C})$ has a unique natural extension to $A^{\bullet,\bullet}(W, E)$ (see [Cappell and Miller 2010, page 139])

$$\bar{\partial}_E : A^{p,q}(W, E) \to W^{p,q+1}(W, E).$$

Under the splitting $\Gamma(W, (T^*W \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E) = A^{1,0}(W, E) \oplus A^{0,1}(W, E)$, the connection D decomposes as a sum $D = D^{1,0} \oplus D^{0,1}$ with

$$D^{1,0} : \Gamma(W, E) \to A^{1,0}(W, E), \quad D^{0,1} : \Gamma(W, E) \to A^{0,1}(W, E).$$

Extend the connection D on $\Gamma(W, E)$ in a unique way to $A^{\bullet,\bullet}(W, E)$ by the Leibniz formula [Berline et al. 2004, page 21]. The extended D again decomposes as a sum $D = D^{1,0} + D^{0,1}$ also satisfying the Leibniz formula [Berline et al. 2004, page 131].

Recall the following definition from [Cappell and Miller 2010, pages 139–140] or [Liu and Yu 2010, Definition 2.1].

Definition 3.1. The connection D is said to be *compatible* with the holomorphic structure on E if $D^{0,1} = \bar{\partial}_E$. The connection D is said to be of *type* (1, 1) if the curvature D^2 is of type (1, 1), that is, $(D^{1,0})^2 = 0$ and $(D^{0,1})^2 = 0$.

The complex Hodge star operator $\star$ acting on forms is a complex conjugate linear mapping

$$\star : A^{p,q}(W, \mathbb{C}) \to A^{n-p,n-q}(W, \mathbb{C})$$

induced by a conjugate linear bundle isomorphism; see [Cappell and Miller 2010, page 141] for this and other statements on this page.

The natural conjugate mapping

$$\mathrm{conj} : A^{p,q}(W, \mathbb{C}) \to A^{q,p}(W, \mathbb{C})$$

is a complex linear mapping induced by the bundle automorphism

$$T^*W \otimes_{\mathbb{R}} \mathbb{C} \to T^*W \otimes_{\mathbb{R}} \mathbb{C},\ v \otimes \lambda \mapsto v \otimes \bar{\lambda}, \qquad v \in T^*W,\ \lambda \in \mathbb{C},$$

of the complexified cotangent bundle. Define $\hat{\star} := \mathrm{conj}\,\star$. Then

$$\hat{\star} = \mathrm{conj}\,\star : A^{p,q}(W, \mathbb{C}) \to A^{n-q,n-p}(W, \mathbb{C})$$

is a complex linear mapping. Clearly, $\hat{\star} = \mathrm{conj}\,\star = \star\,\mathrm{conj}$.

As pointed out by Cappell and Miller, since $\hat{\star}$ is complex linear, it may be coupled to a complex linear bundle mapping, for example, the identity mapping. We also denote by $\hat{\star}$ the complex linear mapping

$$\hat{\star} : A^{p,q}(W, E) \to A^{n-q,n-p}(W, E).$$

Recall that the adjoint $\bar{\partial}^*$ of $\bar{\partial}$ with respect to the chosen Hermitian inner product on TW is given by

$$\bar{\partial}^* = -\star\,\bar{\partial}\,\star.$$

In particular,

$$\bar{\partial}^* = -\hat{\star}\ \mathrm{conj}\ \bar{\partial}\ \mathrm{conj}\ \hat{\star} = -\hat{\star}\,\partial\,\hat{\star}.$$

Let D be a compatible (1, 1) connection. Following Cappell and Miller, we define

$$\bar{\partial}^*_{E,D^{1,0}} = -\hat{\star}\,D^{1,0}\,\hat{\star}$$

and the $\bar{\partial}$-Laplacian for the holomorphic bundle E with compatible type-(1, 1) connection D by

$$\Box_{E,\bar{\partial}} = \bar{\partial}_E \bar{\partial}^*_{E,D^{1,0}} + \bar{\partial}^*_{E,D^{1,0}} \bar{\partial}_E.$$

Note that $(\bar{\partial}^*_{E,D^{1,0}})^2 = 0$, since $(D^{1,0})^2 = 0$ and $\hat{\star}^2 = \star^2 = \pm 1$.

Denote by δ_E the adjoint of the $\bar{\partial}$-operator $\bar{\partial}_E$ with respect to the inner product $\langle\,\cdot\,,\cdot\,\rangle_E$ on $A^{\bullet,\bullet}(W, E)$ induced by the Hermitian metrics g^W and h^E. Then the associated self-adjoint $\bar{\partial}$-Laplacian is defined as

$$\square_E = (\bar{\partial}_E + \delta_E)^2 = \bar{\partial}_E \delta_E + \delta_E \bar{\partial}_E.$$

Recall that, in general, the operator $\square_{E,\bar{\partial}}$ is not self-adjoint with respect to the inner product $\langle\,\cdot\,,\cdot\,\rangle_E$ on $A^{\bullet,\bullet}(W, E)$, but has the same leading symbol as the operator $\square_E$; see [Cappell and Miller 2010, Section 3]. When the connection on E is compatible with the Hermitian inner product $\langle\,\cdot\,,\cdot\,\rangle_E$ on $A^{\bullet,\bullet}(W, E)$, the operator $\square_{E,\bar{\partial}}$ recovers the self-adjoint operators considered in [Bismut 1993; Bismut et al. 1988a; 1988b; 1988c; 1990; Bismut and Lebeau 1989; 1991]. When the bundle E is unitary flat, the operator $\square_{E,\bar{\partial}}$ recovers the self-adjoint operators of [Ray and Singer 1973]. For more details about the operator $\square_{E,\bar{\partial}}$, see [Cappell and Miller 2010].

Twisted Dolbeault-type cohomology and homology groups. For each $0 \le p \le n$, denote by $A^{p,\bar{0}}(W, E) := A^{p,\text{even}}(W, E)$ and $A^{p,\bar{1}}(W, E) := A^{p,\text{odd}}(W, E)$. Let $H \in A^{0,\bar{1}}(W, \mathbb{C})$ and $\bar{\partial}^H_E := \bar{\partial}_E + H \wedge \cdot\,$. We assume that $\bar{\partial} H = 0$. Then, as in the de Rham case, $(\bar{\partial}^H_E)^2 = 0$. Hence, we can consider the twisted complex

$$(A^{p,\bullet}(W, E), \bar{\partial}^H_E):\ \cdots \xrightarrow{\bar{\partial}^H_E} A^{p,\bar{0}}(W, E) \xrightarrow{\bar{\partial}^H_E} A^{p,\bar{1}}(W, E) \xrightarrow{\bar{\partial}^H_E} A^{p,\bar{0}}(W, E) \xrightarrow{\bar{\partial}^H_E} \cdots .$$

Define the twisted Dolbeault-type cohomology groups of $(A^{p,\bullet}(W, E), \bar{\partial}^H_E)$ as

$$H^{p,\bar{k}}_{\bar{\partial}_E}(W, E, H) := \frac{\operatorname{Ker}(\bar{\partial}^H_E : A^{p,\bar{k}}(W, E) \to A^{p,\overline{k+1}}(W, E))}{\operatorname{Im}(\bar{\partial}^H_E : A^{p,\overline{k-1}}(W, E) \to A^{p,\bar{k}}(W, E))}, \quad k = 0, 1.$$

Define $\overline{H} := \operatorname{conj} H$. Let $D^{1,0}_{\overline{H}} := D^{1,0} + \overline{H} \wedge \cdot\,$. Then $(D^{1,0}_{\overline{H}})^2 = 0$. Hence, we can also consider the twisted complex

$$(A^{\bullet,p}(W, E), D^{1,0}_{\overline{H}}):\ \cdots \xrightarrow{D^{1,0}_{\overline{H}}} A^{\bar{0},p}(W, E) \xrightarrow{D^{1,0}_{\overline{H}}} A^{\bar{1},p}(W, E) \xrightarrow{D^{1,0}_{\overline{H}}} A^{\bar{0},p}(W, E) \xrightarrow{D^{1,0}_{\overline{H}}} \cdots .$$

Define the twisted Dolbeault-type cohomology groups of $(A^{\bullet,p}(W, E), D^{1,0}_{\overline{H}})$ as

$$H^{\bar{k},p}_{D^{1,0}}(W, E, H) := \frac{\operatorname{Ker}(D^{1,0}_{\overline{H}} : A^{\bar{k},p}(W, E) \to A^{\overline{k+1},p}(W, E))}{\operatorname{Im}(D^{1,0}_{\overline{H}} : A^{\overline{k-1},p}(W, E) \to A^{\bar{k},p}(W, E))}, \quad k = 0, 1.$$

Define $\bar{\partial}^{*,H}_{E,D^{1,0}} := -\hat{\star}\,(D^{1,0} + \operatorname{conj} H \wedge \cdot\,)\,\hat{\star} = -\hat{\star}\, D^{1,0}_{\overline{H}}\,\hat{\star}$. Then $(\bar{\partial}^{*,H}_{E,D^{1,0}})^2 = 0$. Again we can consider the twisted complex

$$\begin{aligned}(A^{p,\bullet}(W, E), \bar{\partial}^{*,H}_{E,D^{1,0}}):\ \cdots \xleftarrow{\bar{\partial}^{*,H}_{E,D^{1,0}}} A^{p,\bar{0}}(W, E) \xleftarrow{\bar{\partial}^{*,H}_{E,D^{1,0}}} A^{p,\bar{1}}(W, E)& \\ \xleftarrow{\bar{\partial}^{*,H}_{E,D^{1,0}}} A^{p,\bar{0}}(W, E) \xleftarrow{\bar{\partial}^{*,H}_{E,D^{1,0}}} \cdots .&\end{aligned}$$

Define the twisted Dolbeault-type homology groups of $(A^{p,\bullet}(W,E), \bar\partial^{*,H}_{E,D^{1,0}})$ as

$$H_{\bar k}(A^{p,\bullet}(W,E), \bar\partial^{*,H}_{E,D^{1,0}}) := \frac{\operatorname{Ker}(\bar\partial^{*,H}_{E,D^{1,0}} : A^{p,\bar k}(W,E) \to A^{p,\overline{k-1}}(W,E))}{\operatorname{Im}(\bar\partial^{*,H}_{E,D^{1,0}} : A^{p,\overline{k+1}}(W,E) \to A^{p,\bar k}(W,E))}, \; k=0,1.$$

The operator $\hat\star$ induces a $\mathbb{C}$-linear isomorphism from $(A^{p,\bullet}(W,E), \bar\partial^{*,H}_{E,D^{1,0}})$ to $(A^{\overline{n-\bullet},n-p}(W,E), \pm D^{1,0}_{\bar H})$. Hence, as in the $\mathbb{Z}$-graded case (see [Cappell and Miller 2010, page 151] or [Liu and Yu 2010, (2.19)]), we have the isomorphism

$$H^{\overline{n-k},n-p}_{D^{1,0}}(W,E,H) \cong H_{\bar k}(A^{p,\bullet}(W,E), \bar\partial^{*,H}_{E,D^{1,0}}), \quad k=0,1. \tag{3-1}$$

ζ-function and ζ-regularized determinant. In this section we briefly recall some definitions of ζ-regularized determinants of non-self-adjoint elliptic operators. See [Braverman and Kappeler 2007, Section 6] for more details. Let F be a complex (respectively, holomorphic) vector bundle over a closed smooth (respectively, complex) manifold N. Let $D : C^\infty(N,F) \to C^\infty(N,F)$ be an elliptic differential operator of order $m \geq 1$. Assume that θ is an Agmon angle; see, for example, [Braverman and Kappeler 2007, Definition 6.3]. Let $\Pi : L^2(N,F) \to L^2(N,F)$ denote the spectral projection of D corresponding to all nonzero eigenvalues of D. The ζ-function $\zeta_\theta(s,D)$ of D is defined as

$$\zeta_\theta(s,D) = \operatorname{Tr} \Pi D_\theta^{-s}, \quad \operatorname{Re} s > \frac{\dim N}{m}. \tag{3-2}$$

Seeley [1967] (see also [Shubin 2001]) showed that $\zeta_\theta(s,D)$ has a meromorphic extension to the whole complex plane and that 0 is a regular value of $\zeta_\theta(s,D)$.

Definition 3.2. The ζ-regularized determinant of D is defined by the formula

$$\operatorname{Det}'_\theta(D) := \exp\Bigl(-\frac{d}{ds}\Big|_{s=0} \zeta_\theta(s,D)\Bigr).$$

Define

$$\operatorname{LDet}'_\theta(D) = -\frac{d}{ds}\Big|_{s=0} \zeta_\theta(s,D).$$

Let Q be a 0-th order pseudo-differential projection, that is, a 0-th order pseudo-differential operator satisfying $Q^2 = Q$. Set

$$\zeta_\theta(s,Q,D) = \operatorname{Tr} Q\Pi D_\theta^{-s}, \quad \operatorname{Re} s > \frac{\dim M}{m}. \tag{3-3}$$

The function $\zeta_\theta(s,Q,D)$ also has a meromorphic extension to the whole complex plane and by [Wodzicki 1984, Section 7], it is regular at 0.

Definition 3.3. Suppose that Q is a 0-th order pseudo-differential projection commuting with D. Then $V := \operatorname{Im} Q$ is D invariant subspace of $C^\infty(M,E)$. The ζ-regularized determinant of the restriction $D|_V$ of D to V is defined by the formula

$$\operatorname{Det}'_\theta(D|_V) := e^{\operatorname{LDet}'_\theta(D|_V)},$$

where

$$\mathrm{LDet}'_\theta(D|_V) = -\frac{d}{ds}\Big|_{s=0}\zeta_\theta(s, Q, D). \tag{3-4}$$

Remark 3.4. The prime in Det'_θ and LDet'_θ indicates that we ignore the zero eigenvalues of the operator in the definition of the regularized determinant. If the operator is invertible we usually omit the prime and write Det_θ and LDet_θ instead.

Twisted Cappell–Miller holomorphic torsion. For each $0 \le p \le n$, the twisted flat $\bar\partial$-Laplacian, defined as

$$\square^H_{E,\bar\partial} := (\bar\partial^H_E + \bar\partial^{*,H}_{E,D^{1,0}})^2,$$

maps $A^{p,\bar k}(W, E)$, $k = 0, 1$, into itself. Suppose that $\mathcal{I}$ is an interval of the form $[0, \lambda]$, $(\lambda, \mu]$ or $(\lambda, \infty)(\mu > \lambda \ge 0)$. Denote by $\Pi^{E,\mathcal{I}}$ the spectral projection of $\square^H_{E,\bar\partial}$ corresponding to the set of generalized eigenvalues, whose absolute values lie in $\mathcal{I}$. Set

$$A^{p,\bar k}_{\mathcal{I}}(W, E) := \Pi^{E,\mathcal{I}}(A^{p,\bar k}(W, E)) \subset A^{p,\bar k}(W, E), \quad k = 0, 1.$$

If the interval $\mathcal{I}$ is bounded, then for each $0 \le p \le n$, the space $A^{p,\bar k}_{\mathcal{I}}(W, E)$, $k = 0, 1$, is finite-dimensional. The differentials $\bar\partial^H_E$ and $\bar\partial^{*,H}_{E,D^{1,0}}$ commute with $\square^H_{E,\bar\partial}$, so the subspace $A^{p,\bar k}_{\mathcal{I}}(W, E)$ is a subcomplex of the twisted bigraded complex $(A^{p,\bullet}(W, E), \bar\partial^H_E, \bar\partial^{*,H}_{E,D^{1,0}})$. Clearly, for each $\lambda \ge 0$, the complex $A^{p,\bar k}_{(\lambda,\infty)}(W, E)$ is doubly acyclic, that is,

$$H^{\bar k}(A^{p,\bullet}_{(\lambda,\infty)}(W, E), \bar\partial^H_E) = 0 \quad \text{and} \quad H_{\bar k}(A^{p,\bullet}_{(\lambda,\infty)}(W, E), \bar\partial^{*,H}_{E,D^{1,0}}) = 0.$$

Since

$$A^{p,\bar k}(W, E) = A^{p,\bar k}_{[0,\lambda]}(W, E) \oplus A^{p,\bar k}_{(\lambda,\infty)}(W, E),$$

we have the isomorphisms

$$H^{\bar k}(A^{p,\bullet}_{[0,\lambda]}(W, E), \bar\partial^H_E) \cong H^{p,\bar k}_{\bar\partial_E}(W, E, H)$$

and, by (3-1),

$$H_{\bar k}(A^{p,\bullet}_{[0,\lambda]}(W, E), \bar\partial^{*,H}_{E,D^{1,0}}) \cong H^{\overline{n-k}}(A^{\bullet,n-p}_{[0,\lambda]}(W, E), \pm D^{1,0}_{\bar H}) \cong H^{\overline{n-k},n-p}_{D^{1,0}}(W, E, H).$$

In particular, we have the isomorphisms

$$\mathrm{Det}\, H^\bullet(A^{p,\bullet}_{[0,\lambda]}(W, E), \bar\partial^H_E) \cong \mathrm{Det}\, H^{p,\bullet}_{\bar\partial_E}(W, E, H), \tag{3-5}$$

$$\mathrm{Det}\, H_\bullet(A^{p,\bullet}_{[0,\lambda]}(W, E), \bar\partial^{*,H}_{E,D^{1,0}}) \cong \mathrm{Det}\, H^{\overline{n-\bullet},n-p}_{D^{1,0}}(W, E, H). \tag{3-6}$$

For any $\lambda \ge 0$, $0 \le p \le n$, let $\tau_{p,[0,\lambda]}$ denote the Cappell–Miller torsion of the twisted bigraded complex $(A^{p,\bar k}_{[0,\lambda]}(W, E), \bar\partial^H_E, \bar\partial^{*,H}_{E,D^{1,0}})$; see (2-16). Then, by (3-5) and (3-6), we can view $\tau_{p,[0,\lambda]}$ as an element of the determinant line

$$\begin{aligned}\tau_{p,[0,\lambda]} &\in \operatorname{Det} H^{p,\bullet}_{\bar\partial_E}(W,E,H) \otimes \big(\operatorname{Det} H^{\overline{n-\bullet},n-p}_{D^{1,0}}(W,E,H)\big)^{-1} \\ &\cong \operatorname{Det} H^{p,\bullet}_{\bar\partial_E}(W,E,H) \otimes \big(\operatorname{Det} H^{\bullet,n-p}_{D^{1,0}}(W,E,H)\big)^{(-1)^{n+1}}.\end{aligned} \tag{3-7}$$

For each $k = 0, 1$ and each $0 \le p \le n$, set

$$\begin{aligned}A^{p,\bar k}_{+,\mathcal{J}}(W,E) &:= \operatorname{Ker}(\bar\partial^H_E \bar\partial^{*,H}_{E,D^{1,0}}) \cap A^{p,\bar k}_{\mathcal{J}}(W,E),\\ A^{p,\bar k}_{-,\mathcal{J}}(W,E) &:= \operatorname{Ker}(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E) \cap A^{p,\bar k}_{\mathcal{J}}(W,E).\end{aligned}$$

Clearly,

$$A^{p,\bar k}_{\mathcal{J}}(W,E) = A^{p,\bar k}_{+,\mathcal{J}}(W,E) \oplus A^{p,\bar k}_{-,\mathcal{J}}(W,E), \text{ if } 0 \notin \mathcal{J}.$$

Let $\theta \in (0, 2\pi)$ be an Agmon angle of the operator $\square^H_{E,\bar\partial}$; see, for example, [Braverman and Kappeler 2007, Section 6]. Since the leading symbol of the operator $\square^H_{E,\bar\partial}$ is positive definite, the ζ-regularized determinant

$$\operatorname{Det}_\theta(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E)|_{A^{p,\bar k}_{+,\mathcal{J}}(W,E)}$$

is independent of the choice of the Agmon angle θ of the operator $\square^H_{E,\bar\partial}$.

For any $0 \le \lambda \le \mu \le \infty$, one easily sees that

$$\begin{aligned}&\prod_{k=0,1} \big(\operatorname{Det}_\theta(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)^{(-1)^k} \\ &\qquad = \bigg(\prod_{k=0,1} \big(\operatorname{Det}_\theta(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\mu)}(W,E)}\big)^{(-1)^k}\bigg) \\ &\qquad\qquad \cdot \bigg(\prod_{k=0,1} \big(\operatorname{Det}_\theta(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E)|_{A^{p,\bar k}_{+,(\mu,\infty)}(W,E)}\big)^{(-1)^k}\bigg)\end{aligned} \tag{3-8}$$

By Proposition 2.2 and (3-8), we know that the element

$$\tau_{\text{holo},p}(W,E,H) := \tau_{p,[0,\lambda]} \cdot \prod_{k=0,1} \big(\operatorname{Det}_\theta(\bar\partial^{*,H}_{E,D^{1,0}} \bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)^{(-1)^k} \tag{3-9}$$

is independent of the choice of λ. It is also independent of the choice of the Agmon angle $\theta \in (0, 2\pi)$ of the operator $\square^H_{E,\bar\partial}$.

Definition 3.5. The nonvanishing element of the determinant

$$\tau_{\text{holo},p}(W,E,H) \in \operatorname{Det} H^{p,\bullet}_{\bar\partial_E}(W,E,H) \otimes \big(\operatorname{Det} H^{\bullet,n-p}_{D^{1,0}}(W,E,H)\big)^{(-1)^{n+1}}$$

defined in (3-9) is called the twisted Cappell–Miller holomorphic torsion.

Twisted Cappell–Miller holomorphic torsion under metric deformation. Let g_u^W, $u \in \mathbb{R}$, be a smooth family of Hermitian metrics on the complex manifold W. Denote by $\star_u$ the Hodge star operators associated to the metrics g_u^W and denote by

$$\bar{\partial}^{*,H}_{E,D^{1,0},u} := -\hat{\star}_u\,(D^{1,0} + \operatorname{conj} H \wedge \cdot\,)\,\hat{\star}_u.$$

Let $\square^H_{E,\bar{\partial},u} = (\bar{\partial}^H_E + \bar{\partial}^{*,H}_{E,D^{1,0},u})^2$ be the flat Laplacian operators associated to the metrics g_u^W.

Fix $u_0 \in \mathbb{R}$ and choose $\lambda \geq 0$ so that there are no eigenvalues of $\square^H_{E,\bar{\partial},u}$ whose absolute values are equal to λ. Then there exists $\delta > 0$ such that the same is true for all $u \in (u_0 - \delta, u_0 + \delta)$. In particular, if we denote by $A^{p,\bullet}_{[0,\lambda],u}(W, E)$ the span of the generalized eigenvectors of $\square^H_{E,\bar{\partial},u}$ corresponding to eigenvalues with absolute value $\leq \lambda$, then $\dim A^{p,\bullet}_{[0,\lambda],u}(W, E)$ is independent of $u \in (u_0 - \delta, u_0 + \delta)$.

For any $\lambda \geq 0$, $0 \leq p \leq n$, let $\tau_{p,[0,\lambda],u}$ denote the Cappell–Miller torsion of the twisted bigraded complex $(A^{p,\bullet}_{[0,\lambda]}(W, E), \bar{\partial}^H_E, \bar{\partial}^{*,H}_{E,D^{1,0},u})$. Set

$$\alpha_u = \star_u^{-1} \cdot \frac{d}{du}\star_u = \hat{\star}_u^{-1} \cdot \frac{d}{du}\hat{\star}_u.$$

Let $Q_{p,\bar{k}}$ be the spectral projection onto $A^{p,\bar{k}}_{[0,\lambda]}(W, E)$. The proof of the following lemma is similar to the proof of [Cappell and Miller 2010, Lemma 7.1], where the untwisted case was treated.

Lemma 3.6. *Under the assumptions above*, *we have*

$$\frac{d}{du}\tau_{p,[0,\lambda],u} = -\sum_{k=0,1}(-1)^k \operatorname{Tr}(\alpha_u Q_{p,\bar{k}}) \cdot \tau_{p,[0,\lambda],u}.$$

Lemma 3.7. *Under the assumptions above*, *we have*

$$\begin{aligned}\frac{d}{du}\Biggl(\sum_{k=0,1}(-1)^k \operatorname{LDet}_\theta(\bar{\partial}^{*,H}_{E,D^{1,0},u}\bar{\partial}^H_E)|_{A^{p,\bar{k}}_{+,(\lambda,\infty)}(W,E)}\Biggr)&\\ = \sum_{k=0,1}(-1)^k \operatorname{Tr}(\alpha_u Q_{p,\bar{k}}) &+ \sum_{k=0,1}(-1)^k \int_W b_{n,p,\bar{k},u},\end{aligned}$$

where $b_{n,p,\bar{k},u}$ is given by a local formula.

Proof. Set

$$\begin{aligned}f(s,u) &= \sum_{k=0,1}(-1)^k \int_0^\infty t^{s-1}\operatorname{Tr}\Bigl(\exp\bigl(-t\,(\bar{\partial}^{*,H}_{E,D^{1,0},u}\bar{\partial}^H_E)|_{A^{p,\bar{k}}_{+,(\lambda,\infty)}(W,E)}\bigr)\Bigr)\,dt \\ &= \Gamma(s)\sum_{k=0,1}(-1)^k \zeta\bigl(s, (\bar{\partial}^{*,H}_{E,D^{1,0},u}\bar{\partial}^H_E)|_{A^{p,\bar{k}}_{+,(\lambda,\infty)}(W,E)}\bigr).\end{aligned}\tag{3-10}$$

The equality

$$\frac{d}{du}\bar{\partial}^{*,H}_{E,D^{1,0},u}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)} = -\bigl[\alpha_u, \bar{\partial}^{*,H}_{E,D^{1,0},u}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\bigr],\tag{3-11}$$

follows easily from $\bar\partial^{*,H}_{E,D^{1,0},u} := -\hat\star_u\,(D^{1,0}+\operatorname{conj} H\wedge\cdot\,)\,\hat\star_u$ and the equality

$$\star_u^{-1}\cdot\frac{d}{du}\star_u = -\frac{d}{du}\star_u\cdot\star_u^{-1}.$$

If A is of trace class and B is a bounded operator, it is well known that $\operatorname{Tr}(AB)=\operatorname{Tr}(BA)$. By this and the semigroup property of the heat operator, we have

(3-12)

$$\begin{aligned}
&\operatorname{Tr}\Big(\bar\partial^{*,H}_{E,D^{1,0},u}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\alpha_u\bar\partial^H_E|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\exp\big(-t(\bar\partial^{*,H}_{E,D^{1,0},u}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)\Big)\\
&\quad=\operatorname{Tr}\Big(\exp\Big(-\frac{t}{2}(\bar\partial^{*,H}_{E,D^{1,0},u}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\Big)\bar\partial^{*,H}_{E,D^{1,0},u}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\\
&\qquad\qquad\cdot\alpha_u\bar\partial^H_E|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\exp\Big(-\frac{t}{2}(\bar\partial^{*,H}_{E,D^{1,0},u}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\Big)\Big)\\
&\quad=\operatorname{Tr}\Big(\alpha_u\bar\partial^H_E|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\exp\Big(-\frac{t}{2}(\bar\partial^{*,H}_{E,D^{1,0},u}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\Big)\\
&\qquad\qquad\cdot\exp\big(-\frac{t}{2}(\bar\partial^{*,H}_{E,D^{1,0},u}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)\bar\partial^{*,H}_{E,D^{1,0},u}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\Big)\\
&\quad=\operatorname{Tr}\Big(\alpha_u(\bar\partial^H_E\bar\partial^{*,H}_{E,D^{1,0},u})|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\exp\big(-t(\bar\partial^H_E\bar\partial^{*,H}_{E,D^{1,0},u})|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\big)\Big)
\end{aligned}$$

Now, by (3-10), (3-11) and (3-12), we have

(3-13)

$$\begin{aligned}
\frac{d}{du}f(s,u)&=\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\operatorname{Tr}\Big(t\big[\alpha_u,\bar\partial^{*,H}_{E,D^{1,0}}|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\big]\\
&\qquad\qquad\times\exp\big(-t(\bar\partial^{*,H}_{E,D^{1,0}}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)\Big)\,dt\\
&=\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\\
&\quad\times\operatorname{Tr}\Big(t\alpha_u\Big((\bar\partial^{*,H}_{E,D^{1,0}}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\exp\big(-t(\bar\partial^{*,H}_{E,D^{1,0}}\bar\partial^H_E)|_{A^{p,\bar k}_{+,(\lambda,\infty)}(W,E)}\big)\\
&\quad-(\bar\partial^H_E\bar\partial^{*,H}_{E,D^{1,0},u})|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\exp\big(-t(\bar\partial^H_E\bar\partial^{*,H}_{E,D^{1,0},u})|_{A^{p,\overline{k+1}}_{-,(\lambda,\infty)}(W,E)}\big)\Big)\Big)\,dt\\
&=\sum_{k=0,1}(-1)^k\int_0^\infty t^s\operatorname{Tr}\Big(\alpha_u(\Box^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{(\lambda,\infty)}(W,E)}\\
&\qquad\qquad\times\exp\big(-t(\Box^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{(\lambda,\infty)}(W,E)}\big)\Big)\,dt\\
&=-\sum_{k=0,1}(-1)^k\int_0^\infty t^s\frac{d}{dt}\operatorname{Tr}\big(\alpha_u\exp\big(-t(\Box^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{(\lambda,\infty)}(W,E)}\big)\big)\,dt\\
&=s\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\operatorname{Tr}\big(\alpha_u\exp\big(-t(\Box^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{(\lambda,\infty)}(W,E)}\big)\big)\,dt,
\end{aligned}$$

where the second equality holds by (3-12) and we used integration by parts for the last equality. Since $\square^H_{E,\bar\partial,u}$ is an elliptic operator, the dimension of $A^{p,\bullet}_{[0,\lambda]}(W,E)$ is finite. Then we can rewrite (3-13) as

$$\begin{aligned}\frac{d}{du}f(s,u) &= s\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)\,dt\\ &\quad+s\sum_{k=0,1}(-1)^k\int_1^\infty t^{s-1}\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)\,dt\\ &\quad-s\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{[0,\lambda]}(W,E)}\bigr)\bigr)\,dt\\ &\quad-s\sum_{k=0,1}(-1)^k\int_1^\infty t^{s-1}\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{[0,\lambda]}(W,E)}\bigr)\bigr)\,dt.\end{aligned}\tag{3-14}$$

Now $\dim W=2n$ is even, so for small time asymptotic expansion for

$$\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)$$

has a term $a_{n,p,\bar k,u}t^0$ in its expansion about $t=0$. That means

$$\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)-a_{n,p,\bar k,u}t^0$$

does not contain a constant term as $t\downarrow 0$. Hence, the integrals

$$\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)-a_{n,p,\bar k,u}t^0\,dt$$

do not have poles at $s=0$. But the integrals

$$\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}a_{n,p,\bar k,u}t^0\,dt$$

have poles of order 1 with residue $a_{n,p,\bar k,u}$, $k=0,1$. And, because of the exponential decay of $\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}(W,E)}\bigr)\bigr)$ and $\operatorname{Tr}\bigl(\alpha_u\exp\bigl(-t(\square^H_{E,\bar\partial,u})|_{A^{p,\bar k}_{[0,\lambda]}(W,E)}\bigr)\bigr)$ for large t, the integrals of the second and fourth terms on the right-hand side of (3-14) are entire functions in s. Hence we have

$$\begin{aligned}\frac{d}{du}\Big|_{s=0}f(s,u) &= -s\biggl(\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\bigl(\operatorname{Tr}[\alpha_u Q_{p,\bar k}]-a_{n,p,\bar k,u}\bigr)\,dt\biggr)\Big|_{s=0}\\ &= -\sum_{k=0,1}(-1)^k\operatorname{Tr}[\alpha_u Q_{p,\bar k}]+\sum_{k=0,1}(-1)^k a_{n,p,\bar k,u}.\end{aligned}\tag{3-15}$$

Hence, the result follows. $\square$

By combining Lemma 3.6 with Lemma 3.7, we obtain the main theorem of this section. For the untwisted case, see [Cappell and Miller 2010, Theorem 4.4].

Theorem 3.8. *Let W be a complex manifold of complex dimension n and let E be a holomorphic bundle with connection D that is compatible and of type* $(1,1)$ *over W. Suppose that $H \in A^{0,\bar{1}}(W,\mathbb{C})$ and $\bar{\partial} H = 0$. Let g_u^W, $u \in (u_0 - \delta, u_0 + \delta)$, be a smooth family of Riemannian metrics on the complex manifold W. Then the corresponding twisted Cappell–Miller holomorphic torsion $\tau_{\mathrm{holo},p,u}(W,E,H)$ varies smoothly and the variation of $\tau_{\mathrm{holo},p,u}(W,E,H)$ is given by a local formula*

$$\frac{d}{du}\tau_{\mathrm{holo},p,u}(W,E,H) = \left(\sum_{k=0,1}(-1)^k \int_W b_{n,p,\bar{k},u}\right)\cdot \tau_{\mathrm{holo},p,u}(W,E,H).$$

We have the following corollary. See also [Mathai and Wu 2010a, Theorem 5.3, Corollary 7.1] for the case of analytic torsion on $\mathbb{Z}_2$-graded elliptic complexes.

Corollary 3.9. *Let W be a complex manifold of complex dimension n and let E be a holomorphic bundle with connection D that is compatible and of type* $(1,1)$ *over W. Suppose that $H \in A^{0,\bar{1}}(W,\mathbb{C})$ and $\bar{\partial} H = 0$. Let F_1, F_2 be two flat complex bundles over W of the same dimension. Then*

$$\tau_{\mathrm{holo},p}(W, E\otimes F_1, H) \otimes \left(\tau_{\mathrm{holo},p}(W, E\otimes F_2, H)]^{-1}\right)$$

in the tensor product of determinant lines

$$\Big(\mathrm{Det}\, H^{p,\bullet}_{\bar{\partial}_E}(W, E\otimes F_1, H) \otimes \left(\mathrm{Det}\, H^{\bullet,n-p}_{D^{1,0}}(W, E\otimes F_1, H)\right)^{(-1)^{n+1}}\Big) \\ \otimes \Big(\mathrm{Det}\, H^{p,\bullet}_{\bar{\partial}_E}(W, E\otimes F_2, H) \otimes \left(\mathrm{Det}\, H^{\bullet,n-p}_{D^{1,0}}(W, E\otimes F_2, H)\right)^{(-1)^{n+1}}\Big)^{-1}$$

is independent of the Hermitian metric g^W chosen.

This follows from the fact that the two bundles $E\otimes F_1$ and $E\otimes F_2$ are locally identical as bundles. For the untwisted case, see [Cappell and Miller 2010, Corollary 4.5].

Twisted Cappell–Miller holomorphic torsion under flux deformation. Suppose that the flux form H is deformed smoothly along a one-parameter family with parameter $v \in \mathbb{R}$ in such a way that the cohomology class $[H] \in H^{0,\bar{1}}(W,\mathbb{C})$ is fixed. Then $(d/dv)H = -\bar{\partial}B$ for some form $B \in A^{0,\bar{0}}(W,\mathbb{C})$ that depends smoothly on v. Let $\beta = B \wedge \cdot$. Fix $v_0 \in \mathbb{R}$ and choose $\lambda > 0$ such that there are no eigenvalues of $\Box^H_{E,\bar{\partial},v_0}$ of absolute value λ. Then there exists $\delta > 0$ small enough that the same holds for the spectrum of $\Box^H_{E,\bar{\partial},v}|_{A^{p,\bar{k}}_{(\lambda,\infty)}(W,E)}$ for $v \in (v_0-\delta, v_0+\delta)$. For simplicity, we omit the parameter v in the notation in the following discussion. Recall that $Q_{p,\bar{k}}$ is the spectral projection onto $A^{p,\bar{k}}_{[0,\lambda]}(W,E)$.

The proof of the following lemma is similar to the proof of [Mathai and Wu 2008, Lemma 3.7]; see also [Huang 2010, Lemma 4.7].

Lemma 3.10. *Under the assumptions above, we have*

$$\frac{d}{dv}\tau_{p,[0,\lambda]} = -\sum_{k=0,1}(-1)^k \operatorname{Tr}[\beta Q_{p,\bar{k}}]\cdot\tau_{p,[0,\lambda]},$$

upon identification of determinant lines under the deformation.

Lemma 3.11. *Under the assumptions above, we have*

$$\frac{d}{dv}\Bigg(\sum_{k=0,1}(-1)^k \operatorname{LDet}_\theta(\bar{\partial}^{*,H}_{E,\nabla^{1,0}}\bar{\partial}^H_E)|_{A^{p,\bar{k}}_{+,(\lambda,\infty)}(W,E)}\Bigg) = \sum_{k=0,1}(-1)^k \operatorname{Tr}[\beta Q_{p,\bar{k}}] + \sum_{k=0,1}(-1)^k\int_W c_{n,p,\bar{k}},$$

where $c_{n,p,\bar{k}}$ is given by a local formula.

Proof. Under the deformation, we have

$$\frac{d}{dv}\bar{\partial}^H_E = [\beta, \bar{\partial}^H_E], \qquad \frac{d}{dv}\bar{\partial}^{*,H}_{E,D^{1,0}} = -[\beta, \bar{\partial}^{*,H}_{E,D^{1,0}}].$$

Following the proof of [Mathai and Wu 2008, Lemma 3.5], we obtain the desired variation formula. □

By combining Lemma 3.10 with Lemma 3.11, we obtain the main theorem of this section.

Theorem 3.12. *Let W be a complex manifold of complex dimension n and let E be a holomorphic bundle with connection D that is compatible and of type* (1, 1) *over W. Along any one parameter deformation of H that fixes the cohomology class $[H]$ and the natural identification of determinant lines, we have the variation formula*

$$\frac{d}{dv}\tau_{\text{holo},p}(W, E, H) = \Bigg(\sum_{k=0,1}(-1)^k\int_W c_{n,p,\bar{k}}\Bigg)\cdot\tau_{\text{holo},p}(W, E, H).$$

As with Corollary 3.9, we have the following corollary. See also [Mathai and Wu 2010a, Corollary 7.1] for the case of analytic torsion on $\mathbb{Z}_2$-graded elliptic complexes.

Corollary 3.13. *Let W be a complex manifold of complex dimension n and let E be a holomorphic bundle with connection D that is compatible and of type* (1, 1) *over W. Suppose that $H \in A^{0,\bar{1}}(W, \mathbb{C})$ and $\bar{\partial}H = 0$. Let F_1, F_2 be two flat complex bundles over W of the same dimension. Then*

$$\tau_{\text{holo},p}(W, E\otimes F_1, H)\otimes\big(\tau_{\text{holo},p}(W, E\otimes F_2, H)\big)^{-1}$$

is invariant under any deformation of H by an $\bar{\partial}$-exact form, up to natural identification of the determinant lines.

4. Twisted Cappell–Miller analytic torsion

In this section we first define the de Rham bigraded complex twisted by a flux form H and its (co)homology groups. Then we define the Cappell–Miller analytic torsion for the twisted de Rham bigraded complex. We obtain the variation theorems of the twisted Cappell–Miller analytic torsion under metric and flux deformations. Su, in a recent preprint [2011], also briefly discussed the twisted Cappell–Miller analytic torsion when dimension of the manifold M is odd.

The twisted de Rham complexes. Suppose M is a closed oriented m-dimensional smooth manifold and let $\mathscr{E}$ be a complex vector bundle over M endowed with a flat connection ∇. We denote by $\Omega^p(M, \mathscr{E})$ the space of p-forms with values in the flat bundle $\mathscr{E}$, that is, $\Omega^p(M, \mathscr{E}) = \Gamma(M, \bigwedge^p(T^*M)_{\mathbb{R}} \otimes \mathscr{E})$ and by

$$\nabla : \Omega^\bullet(M, \mathscr{E}) \to \Omega^{\bullet+1}(M, \mathscr{E})$$

the covariant differential induced by the flat connection on $\mathscr{E}$. Fix a Riemannian metric g^M on M and let $\star : \Omega^\bullet(M, \mathscr{E}) \to \Omega^{m-\bullet}(M, \mathscr{E})$ denote the Hodge $\star$ operator. We choose a Hermitian metric $h^{\mathscr{E}}$ so that together with the Riemannian metric g^M we can define a scalar product $\langle \cdot, \cdot \rangle_M$ on $\Omega^\bullet(M, \mathscr{E})$. Define the chirality operator $\Gamma = \Gamma(g^M) : \Omega^\bullet(M, \mathscr{E}) \to \Omega^\bullet(M, \mathscr{E})$ by [Braverman and Kappeler 2007, (7-1)]

$$\Gamma\omega := i^r(-1)^{q(q+1)/2} \star \omega, \quad \omega \in \Omega^q(M, \mathscr{E}), \tag{4-1}$$

where $r = (m+1)/2$ if m is odd and $r = m/2$ if m is even. The numerical factor in (4-1) has been chosen so that $\Gamma^2 = \mathrm{Id}$; see [Berline et al. 2004, Proposition 3.58].

Assume $\mathscr{H}$ is an odd degree closed differential form on M. Let $\Omega^{\bar{0}/\bar{1}}(M, \mathscr{E}) := \Omega^{\text{even}/\text{odd}}(M, \mathscr{E})$ and $\nabla^{\mathscr{H}} := \nabla + \mathscr{H} \wedge \cdot$. Assume that $\mathscr{H}$ does not contain a 1-form component, which can be absorbed in the flat connection ∇.

It is not difficult to check that $(\nabla^{\mathscr{H}})^2 = 0$. Clearly, for each $k = 0, 1$, we have $\nabla^{\mathscr{H}} : \Omega^{\bar{k}}(M, \mathscr{E}) \to \Omega^{\overline{k+1}}(M, \mathscr{E})$. Hence we can consider the twisted de Rham complex

$$(\Omega^\bullet(M, \mathscr{E}), \nabla^{\mathscr{H}}) : \quad \cdots \xrightarrow{\nabla^{\mathscr{H}}} \Omega^{\bar{0}}(M, \mathscr{E}) \xrightarrow{\nabla^{\mathscr{H}}} \Omega^{\bar{1}}(M, \mathscr{E}) \xrightarrow{\nabla^{\mathscr{H}}} \Omega^{\bar{0}}(M, \mathscr{E}) \xrightarrow{\nabla^{\mathscr{H}}} \cdots . \tag{4-2}$$

We define the *twisted de Rham cohomology group of* $(\Omega^\bullet(M, \mathscr{E}), \nabla^{\mathscr{H}})$ as

$$H^{\bar{k}}(M, \mathscr{E}, \mathscr{H}) := \frac{\mathrm{Ker}(\nabla^{\mathscr{H}} : \Omega^{\bar{k}}(M, \mathscr{E}) \to \Omega^{\overline{k+1}}(M, \mathscr{E}))}{\mathrm{Im}(\nabla^{\mathscr{H}} : \Omega^{\overline{k-1}}(M, \mathscr{E}) \to \Omega^{\bar{k}}(M, \mathscr{E}))}, \quad k = 0, 1.$$

The groups $H^{\bar{k}}(M, \mathscr{E}, \mathscr{H})$, $k = 0, 1$, are independent of the choice of the Riemannian metric on M or the Hermitian metric on $\mathscr{E}$. Replacing $\mathscr{H}$ by $\mathscr{H}' = \mathscr{H} - d\mathscr{B}$ for some $\mathscr{B} \in \Omega^{\bar{0}}(M)$ gives an isomorphism $\varepsilon_{\mathscr{B}} := e^{\mathscr{B}} \wedge \cdot : \Omega^\bullet(M, \mathscr{E}) \to \Omega^\bullet(M, \mathscr{E})$

satisfying

$$\varepsilon_{\mathcal{B}} \circ \nabla^{\mathcal{H}} = \nabla^{\mathcal{H}'} \circ \varepsilon_{\mathcal{B}}.$$

Therefore $\varepsilon_{\mathcal{B}}$ induces an isomorphism on the twisted de Rham cohomology. Also denote by $\varepsilon_{\mathcal{B}}$ the map

$$\varepsilon_{\mathcal{B}} : H^{\bullet}(M, \mathcal{E}, \mathcal{H}) \to H^{\bullet}(M, \mathcal{E}, \mathcal{H}'). \tag{4-3}$$

Denote by $\nabla^{\mathcal{H},*}$ the adjoint of $\nabla^{\mathcal{H}}$ with respect to the scalar product $\langle \cdot , \cdot \rangle_M$. Then the Laplacian

$$\Delta^{\mathcal{H}} := \nabla^{\mathcal{H},*}\nabla^{\mathcal{H}} + \nabla^{\mathcal{H}}\nabla^{\mathcal{H},*}$$

is an elliptic operator and therefore the complex (4-2) is elliptic. By Hodge theory, we have the isomorphism $\operatorname{Ker} \Delta^{\mathcal{H}} \cong H^{\bullet}(M, \mathcal{E}, \mathcal{H})$. For more details of the twisted de Rham cohomology, see, for example, [Mathai and Wu 2008].

Now denote by ∇' the connection on $\mathcal{E}$ dual to the connection ∇ with respect to the Hermitian metric $h^{\mathcal{E}}$ [Braverman and Kappeler 2007, Section 10.1]. Denote by $\mathcal{E}'$ the flat bundle $(\mathcal{E}, \nabla')$, referring to $\mathcal{E}'$ as the dual of the flat vector bundle $\mathcal{E}$. We emphasize that, similar to the untwisted case [Braverman and Kappeler 2007, (10-8); Cappell and Miller 2010, (8.4)],

$$\nabla^{\mathcal{H},*} = \Gamma \nabla'^{\mathcal{H}} \Gamma,$$

where $\nabla'^{\mathcal{H}} = \nabla' + \mathcal{H} \wedge \cdot$.

Let $\nabla^{\mathcal{H},\sharp} := \Gamma \nabla^{\mathcal{H}} \Gamma$. Then $(\nabla^{\mathcal{H},\sharp})^2 = 0$. Clearly, $\nabla^{\mathcal{H},\sharp} : \Omega^{\bar{k}}(M, \mathcal{E}) \to \Omega^{\overline{k-1}}(M, \mathcal{E})$. Hence we can consider the twisted de Rham complex

$$(\Omega^{\bullet}(M, \mathcal{E}), \nabla^{\mathcal{H},\sharp}) : \quad \cdots \xleftarrow{\nabla^{\mathcal{H},\sharp}} \Omega^{\bar{0}}(M, \mathcal{E}) \xleftarrow{\nabla^{\mathcal{H},\sharp}} \Omega^{\bar{1}}(M, \mathcal{E}) \xleftarrow{\nabla^{\mathcal{H},\sharp}} \Omega^{\bar{0}}(M, \mathcal{E}) \xleftarrow{\nabla^{\mathcal{H},\sharp}} \cdots . \tag{4-4}$$

We also define the homology group of the complex $(\Omega^{\bullet}(M, \mathcal{E}), \nabla^{\mathcal{H},\sharp})$ as

$$H_{\bar{k}}(\Omega^{\bullet}(M, \mathcal{E}), \nabla^{\mathcal{H},\sharp}) := \frac{\operatorname{Ker}(\nabla^{\mathcal{H},\sharp} : \Omega^{\bar{k}}(M, \mathcal{E}) \to \Omega^{\overline{k-1}}(M, \mathcal{E}))}{\operatorname{Im}(\nabla^{\mathcal{H},\sharp} : \Omega^{\overline{k+1}}(M, \mathcal{E}) \to \Omega^{\bar{k}}(M, \mathcal{E}))}, \quad k = 0, 1.$$

Similarly, the groups $H_{\bar{k}}(\Omega^{\bullet}(M, \mathcal{E}), \nabla^{\mathcal{H},\sharp})$, $k = 0, 1$, are independent of the choice of the Riemannian metric on M or the Hermitian metric on $\mathcal{E}$. Suppose that $\mathcal{H}$ is replaced by $\mathcal{H}'' = \mathcal{H} - \delta\mathcal{B}'$ for some $\mathcal{B}' \in \Omega^{\bar{0}}(M)$ and δ the adjoint of d with respect to the scalar product induced by the Riemannian metric g^M. Then there is an isomorphism $\varepsilon_{\mathcal{B}'} := e^{\mathcal{B}'} \wedge \cdot : \Omega^{\bullet}(M, \mathcal{E}) \to \Omega^{\bullet}(M, \mathcal{E})$ satisfying

$$\varepsilon_{\mathcal{B}'} \circ \nabla^{\mathcal{H},\sharp} = \nabla^{\mathcal{H}'',\sharp} \circ \varepsilon_{\mathcal{B}'}.$$

Therefore $\varepsilon_{\mathscr{B}'}$ induces an isomorphism on the twisted de Rham homology. Also denote by $\varepsilon_{\mathscr{B}'}$ the map

$$\varepsilon_{\mathscr{B}'} : H_\bullet(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp}) \to H_\bullet(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H}'',\sharp}). \tag{4-5}$$

Denote by $\nabla^{\mathscr{H},\sharp,*}$ the adjoint of $\nabla^{\mathscr{H},\sharp}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_M$. Then we have the equalities

$$\nabla^{\mathscr{H},\sharp,*} = \nabla'^{\mathscr{H}}, \quad \Delta'^{\mathscr{H}} := \nabla'^{\mathscr{H},*}\nabla'^{\mathscr{H}} + \nabla'^{\mathscr{H}}\nabla'^{\mathscr{H},*} = \nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H},\sharp,*} + \nabla^{\mathscr{H},\sharp,*}\nabla^{\mathscr{H},\sharp}.$$

Again the Laplacian $\Delta'^{\mathscr{H}}$ is an elliptic operator and thus the complex (4-4) is elliptic. By Hodge theory, we have the isomorphism $\operatorname{Ker}\Delta'^{\mathscr{H}} \cong H_\bullet(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp})$. In particular, for $k = 0, 1$,

$$H_{\bar{k}}(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp}) \cong H^{\bar{k}}(M,\mathscr{E}',\mathscr{H}). \tag{4-6}$$

Definition of twisted Cappell–Miller analytic torsion. The flat Laplacian

$$\Delta^{\mathscr{H},\sharp} := (\nabla^{\mathscr{H}} + \nabla^{\mathscr{H},\sharp})^2$$

maps $\Omega^{\bar{k}}(M,\mathscr{E})$ into itself. Suppose $\mathscr{I}$ is an interval of the form $[0,\lambda]$, $(\lambda,\mu]$, or (λ,∞) $(\mu > \lambda \geq 0)$. Denote by $\Pi^{\sharp,\mathscr{I}}$ the spectral projection of $\Delta^{\mathscr{H},\sharp}$ corresponding to the set of generalized eigenvalues, whose absolute values lie in $\mathscr{I}$. Set

$$\Omega^{\bar{k}}_{\mathscr{I}}(M,\mathscr{E}) := \Pi^{\sharp,\mathscr{I}}(\Omega^{\bar{k}}(M,\mathscr{E})) \subset \Omega^{\bar{k}}(M,\mathscr{E}).$$

If the interval $\mathscr{I}$ is bounded, then the space $\Omega^{\bar{k}}_{\mathscr{I}}(M,\mathscr{E})$ is finite dimensional. Since $\nabla^{\mathscr{H}}$ and $\nabla^{\mathscr{H},\sharp}$ commute with $\Delta^{\sharp}_{\mathscr{H}}$, the subspace $\Omega^\bullet_{\mathscr{I}}(M,\mathscr{E})$ is a subcomplex of the twisted de Rham bi-complex $(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H}}, \nabla^{\mathscr{H},\sharp})$. Clearly, for each $\lambda \geq 0$, the complex $\Omega^\bullet_{(\lambda,\infty)}(M,\mathscr{E})$ is doubly acyclic, that is, $H^{\bar{k}}(\Omega^\bullet_{(\lambda,\infty)}(M,\mathscr{E}), \nabla^{\mathscr{H}}) = 0$ and $H_{\bar{k}}(\Omega^\bullet_{(\lambda,\infty)}(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp}) = 0$. Since

$$\Omega^{\bar{k}}(M,\mathscr{E}) = \Omega^{\bar{k}}_{[0,\lambda]}(M,\mathscr{E}) \oplus \Omega^{\bar{k}}_{(\lambda,\infty)}(M,\mathscr{E}), \tag{4-7}$$

the homology $H_{\bar{k}}(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp})$ of the complex $(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp})$ is naturally isomorphic to the homology $H_{\bar{k}}(\Omega^\bullet(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp}) \cong H^{\bar{k}}(M,\mathscr{E}',\mathscr{H})$ (see (4-6)), and the cohomology $H^{\bar{k}}(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H}})$ of $(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H}})$ is naturally isomorphic to the cohomology $H^{\bar{k}}(M,\mathscr{E},\mathscr{H})$.

Similar to the $\mathbb{Z}$-graded case [Cappell and Miller 2010, Section 8], the chirality operator Γ establishes a complex linear isomorphism of the homology groups with cohomology groups

$$H_{\bar{k}}(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H},\sharp}) \cong H^{\overline{m-k}}(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}), \nabla^{\mathscr{H}}) \cong H^{\overline{m-k}}(M,\mathscr{E},\mathscr{H}).$$

In particular, we have the isomorphism

$$\begin{aligned}\operatorname{Det} H_\bullet(\Omega^\bullet(M,\mathscr{E}),\nabla^{\mathscr{H},\sharp}) &\cong \operatorname{Det} H_\bullet(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}),\nabla^{\mathscr{H},\sharp})\\ &\cong (\operatorname{Det} H^\bullet(M,\mathscr{E},\mathscr{H}))^{(-1)^m}.\end{aligned} \tag{4-8}$$

Using Poincaré duality, we also have the isomorphism

$$\operatorname{Det} H^{\overline{m-k}}(M,\mathscr{E},\mathscr{H})^{-1} \cong \operatorname{Det} H^{\bar{k}}(M,\mathscr{E}',\mathscr{H}), \tag{4-9}$$

where $\mathscr{E}'$ is the dual vector bundle of the vector bundle E. Therefore, we have

$$\begin{aligned}&\operatorname{Det} H^\bullet(M,\mathscr{E},\mathscr{H})\otimes \operatorname{Det} H^{\overline{m-\bullet}}(M,\mathscr{E},\mathscr{H})^{-1}\\ &\cong \operatorname{Det} H^\bullet(M,\mathscr{E},\mathscr{H})\otimes \operatorname{Det} H^\bullet(M,\mathscr{E}',\mathscr{H}) \quad \text{by (4-9)}\\ &\cong \operatorname{Det} H^\bullet(M,\mathscr{E}\oplus\mathscr{E}',\mathscr{H}).\end{aligned} \tag{4-10}$$

For $k=0,1$, set

$$\begin{aligned}\Omega^{\bar{k}}_{+,\mathscr{I}}(M,\mathscr{E}) &:= \operatorname{Ker}(\nabla^{\mathscr{H}}\nabla^{\mathscr{H},\sharp})\cap \Omega^{\bar{k}}_{\mathscr{I}}(M,\mathscr{E}),\\ \Omega^{\bar{k}}_{-,\mathscr{I}}(M,\mathscr{E}) &:= \operatorname{Ker}(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})\cap \Omega^{\bar{k}}_{\mathscr{I}}(M,\mathscr{E}).\end{aligned} \tag{4-11}$$

Clearly,

$$\Omega^{\bar{k}}_{\mathscr{I}}(M,\mathscr{E}) = \Omega^{\bar{k}}_{+,\mathscr{I}}(M,\mathscr{E})\oplus\Omega^{\bar{k}}_{-,\mathscr{I}}(M,\mathscr{E}) \quad \text{if } 0\notin\mathscr{I}.$$

Let $\theta\in(0,2\pi)$ be an Agmon angle; see [Shubin 2001]. Since the leading symbol of $\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}}$ is positive definite, the ζ-regularized determinant $\operatorname{Det}_\theta(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})|_{\Omega^{\bar{k}}_{+,\mathscr{I}}(M,\mathscr{E})}$ is independent of the choice of θ.

For any $0\le\lambda\le\mu\le\infty$, one easily sees that

$$\begin{aligned}&\prod_{k=0,1}\big(\operatorname{Det}_\theta(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})|_{\Omega^{\bar{k}}_{+,(\lambda,\infty)}(M,\mathscr{E})}\big)^{(-1)^k}\\ &\qquad= \bigg(\prod_{k=0,1}\big(\operatorname{Det}_\theta(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})|_{\Omega^{\bar{k}}_{+,(\lambda,\mu)}(M,\mathscr{E})}\big)^{(-1)^k}\bigg)\\ &\qquad\qquad\cdot\bigg(\prod_{k=0,1}\big(\operatorname{Det}_\theta(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})|_{\Omega^{\bar{k}}_{+,(\mu,\infty)}(M,\mathscr{E})}\big)^{(-1)^k}\bigg).\end{aligned}$$

For any $\lambda\ge 0$, denote by $\tau_{[0,\lambda]}$ the Cappell–Miller torsion of the twisted de Rham bigraded complex $\big(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}),\nabla^{\mathscr{H}},\nabla^{\mathscr{H},\sharp}\big)$. Via the isomorphisms

$$\begin{aligned}H_\bullet(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}),\nabla^{\mathscr{H},\sharp}) &\cong H_\bullet(\Omega^\bullet(M,\mathscr{E}),\nabla^{\mathscr{H},\sharp}),\\ H^\bullet(\Omega^\bullet_{[0,\lambda]}(M,\mathscr{E}),\nabla^{\mathscr{H}}) &\cong H^\bullet(M,\mathscr{E},\mathscr{H}),\end{aligned}$$

and (4-10), we can view $\tau_{[0,\lambda]}$ as an element of $\operatorname{Det} H^\bullet(M,\mathscr{E}\oplus\mathscr{E}',\mathscr{H})$. In particular, if m is odd, then, up to an isomorphism,

$$\tau_{[0,\lambda]}\in \operatorname{Det} H^\bullet(M,\mathscr{E},\mathscr{H})\otimes \operatorname{Det} H^\bullet(M,\mathscr{E},\mathscr{H}) \cong \operatorname{Det} H^\bullet(M,\mathscr{E}\oplus\mathscr{E}',\mathscr{H}). \tag{4-12}$$

The proof of the following lemma is similar to the proof of [Cappell and Miller 2010, Theorem 8.3].

Lemma 4.1. *The element*

$$\tau_{[0,\lambda]} \cdot \prod_{k=0,1} \big(\mathrm{Det}_\theta(\nabla^{\mathcal{H},\sharp}\nabla^{\mathcal{H}})|_{\Omega^{\bar{k}}_{+,(\lambda,\infty)}(M,\mathcal{E})}\big)^{(-k)}$$

is independent of the choice of λ.

We now define the Cappell–Miller analytic torsion for the de Rham complex twisted by a flux.

Definition 4.2. Let $(\mathcal{E}, \nabla)$ be a complex vector bundle over a connected oriented m-dimensional closed Riemannian manifold M and $\mathcal{H}$ be a closed odd degree form (not a 1-form). Further, let

$$\nabla^{\mathcal{H}} = \nabla + \mathcal{H} \wedge \cdot \quad \text{and} \quad \nabla^{\mathcal{H},\sharp} = \Gamma \nabla^{\mathcal{H}} \Gamma.$$

Let $\theta \in (0, 2\pi)$ be an Agmon angle for the operator $\Delta^{\mathcal{H},\sharp} := (\nabla^{\mathcal{H}} + \nabla^{\mathcal{H},\sharp})^2$. The Cappell–Miller torsion $\tau(\nabla, \mathcal{H})$ for the twisted de Rham bigraded complex $(\Omega^\bullet(M,\mathcal{E}),\nabla^{\mathcal{H}},\nabla^{\mathcal{H},\sharp})$ is an element of $\mathrm{Det}\, H^\bullet(M,\mathcal{E},\mathcal{H})\otimes\big(\mathrm{Det}\, H^\bullet(M,\mathcal{E},\mathcal{H})\big)^{(-1)^{m+1}}$ defined as

$$\tau(\nabla, \mathcal{H}) := \tau_{[0,\lambda]} \cdot \prod_{k=0,1} \big(\mathrm{Det}_\theta(\nabla^{\mathcal{H},\sharp}\nabla^{\mathcal{H}})|_{\Omega^{\bar{k}}_{+,(\lambda,\infty)}(M,\mathcal{E})}\big)^{(-k)}. \tag{4-13}$$

Twisted Cappell–Miller analytic torsion under metric and flux deformations. In this section we obtain the variation formulas for the twisted Cappell–Miller analytic torsion $\tau(\nabla, \mathcal{H})$ under the metric and flux deformations. In particular, we show that if the manifold M is an odd-dimensional closed oriented manifold, then the twisted Cappell–Miller analytic torsion is independent of the Riemannian metric and the representative $\mathcal{H}$ in the cohomology class $[\mathcal{H}]$. See also [Su 2011].

The proof of the following theorem is similar to the proof of Theorem 3.8.

Theorem 4.3. *Let $(\mathcal{E}, \nabla)$ be a complex vector bundle over a m-dimensional connected oriented closed Riemannian manifold M and $\mathcal{H}$ be a closed odd degree form (not a 1-form). Let g^M_v, $a \le v \le b$, be a smooth family of Riemannian metrics on M. Then the corresponding twisted Cappell–Miller analytic torsion $\tau_v(\nabla, \mathcal{H})$ varies smoothly and the variation of $\tau_v(\nabla, \mathcal{H})$ is given by a local formula*

$$\frac{d}{dv}\tau_v(\nabla, \mathcal{H}) = \bigg(\sum_{k=0,1}(-1)^k\int_M b_{m/2,\bar{k},v}\bigg)\cdot \tau_v(\nabla, \mathcal{H}).$$

In particular, if the dimension of the manifold M is odd, then twisted Cappell–Miller analytic torsion $\tau(\nabla, \mathcal{H})$ is independent of the Riemannian metric g^M.

For the untwisted case considered in [Bismut and Zhang 1992], the variation of the torsion can be integrated to an anomaly formula.

The proof of the following is similar to that of [Mathai and Wu 2010a, Theorem 6.1]. See also [Mathai and Wu 2008, Theorem 3.8].

Theorem 4.4. *Let $(\mathscr{E}, \nabla)$ be a complex vector bundle over a m-dimensional connected oriented closed Riemannian manifold M and $\mathscr{H}$ be a closed odd degree form (not a 1-form). Under the natural identification of determinant lines and along any one parameter deformation $\mathscr{H}_v$ of $\mathscr{H}$ that fixes the cohomology class $[\mathscr{H}]$, we have the variation formula*

$$\frac{d}{dv}\tau(\nabla, \mathscr{H}_v) = \left(\sum_{k=0,1}(-1)^k \int_M c_{m/2,\bar{k},v}\right)\cdot \tau(\nabla, \mathscr{H}_v).$$

In particular, if the dimension of the manifold M is odd, then, under the natural identification of determinant lines, the twisted Cappell–Miller analytic torsion $\tau(\nabla, \mathscr{H})$ is independent of any deformation of $\mathscr{H}$ that fixes the cohomology class $[\mathscr{H}]$.

Relationship with the twisted refined analytic torsion. In this section we assume that M is a closed compact oriented manifold of odd dimension. Recall that in [Huang 2010, (3.13)], for each $\lambda > 0$, we define the twisted refined torsion $\rho_{\Gamma_{[0,\lambda]}}$ of the twisted finite-dimensional complex $(\Omega^\bullet_{[0,\lambda]}(M, \mathscr{E}), \nabla^{\mathscr{H}})$ corresponding to the chirality operator $\Gamma_{[0,\lambda]}$. In our setting, as in the $\mathbb{Z}$-graded case [Braverman and Kappeler 2008a, (5.1)], the twisted Cappell–Miller torsion can be described as (see (4-12))

$$\tau_{[0,\lambda]} := \rho_{\Gamma_{[0,\lambda]}} \otimes \rho_{\Gamma_{[0,\lambda]}} \in \operatorname{Det} H^\bullet(M, \mathscr{E}, \mathscr{H}) \otimes \operatorname{Det} H^\bullet(M, \mathscr{E}, \mathscr{H}). \tag{4-14}$$

By combining (3.14), (3.20), (5.28) and Definition 4.5 of [Huang 2010], the twisted refined analytic torsion can be written as

$$\begin{aligned}\rho_{\mathrm{an}}(\nabla^{\mathscr{H}}) = \pm\rho_{\Gamma_{[0,\lambda]}} \cdot \prod_{k=0,1} &\big(\operatorname{Det}_\theta(\nabla^{\mathscr{H},\sharp}\nabla^{\mathscr{H}})|_{\Omega^{\bar{k}}_{+,(\lambda,\infty)}(M,\mathscr{E})}\big)^{-k/2} \\ &\cdot \exp\big(-i\pi(\eta(\mathscr{B}^{\mathscr{H}}_0(\nabla^{\mathscr{H}})) - \operatorname{rank}\mathscr{E}\cdot\eta_{\mathrm{trivial}})\big),\end{aligned} \tag{4-15}$$

where $\eta(\mathscr{B}^{\mathscr{H}}_0(\nabla^{\mathscr{H}})) - \operatorname{rank}\mathscr{E}\cdot\eta_{\mathrm{trivial}}$ is the ρ-invariant of the twisted odd signature operator $\mathscr{B}^{\mathscr{H}}_0(\nabla^{\mathscr{H}})$ defined in [Huang 2010, (3.2)].

By combining (4-13), (4-14) with (4-15), we have the following comparison theorem of the twisted Cappell–Miller analytic torsion and twisted refined analytic torsion.

Theorem 4.5. *Let $(\mathscr{E}, \nabla)$ be a complex vector bundle over a connected oriented odd-dimensional closed Riemannian manifold M and $\mathscr{H}$ be a closed odd degree* *form* (*not a* 1-*form*). *Further*, *let* $\nabla^{\mathscr{H}} = \nabla + \mathscr{H} \wedge \cdot$. *Then*

$$\tau(\nabla, \mathscr{H}) \cdot \exp\bigl(-2i\pi(\eta(\mathscr{B}_0^{\mathscr{H}}(\nabla^{\mathscr{H}})) - \operatorname{rank}\mathscr{E} \cdot \eta_{\text{trivial}})\bigr) = \rho_{\text{an}}(\nabla^{\mathscr{H}}) \otimes \rho_{\text{an}}(\nabla^{\mathscr{H}}).$$

Su in [2011, Theorem 5.1] compared the twisted Burghelea–Haller analytic torsion which he introduced with the twisted refined analytic torsion. By combining [Su 2011, Theorem 5.1] with Theorem 4.5, we can also obtain the comparison theorem of the twisted Burghelea–Haller torsion and the twisted Cappell–Miller analytic torsion. We skip the details.

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Received April 1, 2010.

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E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

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