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A CONSTANT MEAN CURVATURE ANNULUS TANGENT TO TWO IDENTICAL SPHERES IS DELAUNEY

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We show that a compact embedded annulus of constant mean curvature in $\mathbb{R}^3$ tangent to two spheres of the same radius along its boundary curves and having nonvanishing Gaussian curvature is part of a Delaunay surface. In particular, if the annulus is minimal, it is part of a catenoid. We also show that a compact embedded annulus of constant mean curvature with negative meeting a sphere tangentially and a plane at a constant contact angle $\geq \pi/2$ (in the case of positive Gaussian curvature) or $\leq \pi/2$ (in the negative case) is part of a Delaunay surface. Thus, if the contact angle is $\geq \pi/2$ and the annulus is minimal, it is part of a catenoid.

Delaunay surfaces are rotational surfaces (surfaces of revolution) of constant mean curvature in $\mathbb{R}^3$. Besides cylinders and spheres, they are divided into unduloids, nodoids, and (allowing the case of zero mean curvature in the definition, for convenience) the catenoid, recognized long ago [Bonnet 1860] as the only nonplanar minimal surface of rotation in $\mathbb{R}^3$.

Thus a Delaunay surface meets every plane perpendicular to the axis of rotation under a constant angle. Conversely, if a compact surface of constant mean curvature meets two parallel planes in constant contact angles, it is part of a Delaunay surface. This can be proved by using Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] with planes perpendicular to the parallel planes.

A compact immersed minimal annulus meeting two parallel planes in constant contact angles is also part of a catenoid. This result is not true when the constant mean curvature is nonzero: Wente [1995] constructed examples of immersed constant mean curvature annuli in a slab or in a ball meeting the boundary planes or the boundary sphere perpendicularly. Compared to the above first case, we may ask whether a compact minimal annulus or a compact embedded constant mean curvature annulus meeting two spheres in constant contact angles is part of a catenoid or of a plane. In [Park and Pyo $\geq$ 2011], it is shown that if a compact embedded minimal annulus meets two concentric spheres perpendicularly then the minimal annulus is part of a plane.

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In this paper, we show that a compact embedded constant mean curvature annulus $\mathscr{A}$ in $\mathbb{R}^3$ meeting two spheres S_1 and S_2 of the same radius ρ tangentially and having nonvanishing Gaussian curvature K is part of a Delaunay surface. More precisely, depending on the values of K and the mean curvature H we have three cases: (i) $K < 0$ and $H > -1/\rho$, in which case $\mathscr{A}$ is part of a unduloid if $H < 0$, part of a catenoid if $H = 0$ and part of a nodoid if $H > 0$, (ii) $K > 0$ and $-1/\rho < H < -1/2\rho$, in which case $\mathscr{A}$ is part of a unduloid, and (iii) $K > 0$ and $H < -1/\rho$, in which case $\mathscr{A}$ is part of a nodoid. In the first two cases, $\mathscr{A}$ stays outside of the balls B_1 and B_2 bounded by S_1 and S_2. If (iii) holds, then $\mathscr{A} \subset B_1 \cap B_2$.

We also show that a compact embedded constant mean curvature annulus $\mathscr{B}$ in $\mathbb{R}^3$ with negative (respectively, positive) Gaussian curvature meeting a unit sphere tangentially and a plane in constant contact angle $\geq \pi/2$ (respectively, $\leq \pi/2$) is part of a Delaunay surface. In particular, a compact embedded minimal annulus in $\mathbb{R}^3$ meeting a sphere tangentially and a plane in constant contact angle $\geq \pi/2$ is part of a catenoid.

To prove Theorems 3.1 and 3.2, we use the $-\rho$-parallel surface $\tilde{\mathscr{A}}$ of $\mathscr{A}$ (respectively, $\widetilde{\mathscr{B}}$ of $\mathscr{B}$), that is, the parallel surface of $\mathscr{A}$ (respectively, of $\mathscr{B}$) with distance ρ in the direction to the centers of the spheres. We use Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] to prove that $\tilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ are rotational. Since $\tilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ are the parallel surfaces of $\mathscr{A}$ and $\mathscr{B}$ respectively, $\mathscr{A}$ and $\mathscr{B}$ are also rotational and, hence, are part of a Delaunay surface or part of a catenoid.

1. Constant mean curvature annulus meeting spheres tangentially

In the following, we may assume that the spheres have radius 1. Let $\mathscr{A}$ be a compact embedded annulus with constant mean curvature H meeting two unit spheres S_1 and S_2 tangentially along the boundary curves γ_1 and γ_2. We fix the unit normal N of $\mathscr{A}$ in such a way that N points away from the center of S_i along each γ_i. Let $Y : A(1, R) \to \mathbb{R}^3$ be a conformal parametrization of $\mathscr{A}$ from an annulus $A(1, R) = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} \leq R\}$. We define X by $X = Y \circ \exp$ on the strip $B = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\}$. Then X is periodic with period 2π. Let $z = u + iv$ and $\lambda^2 := |X_u|^2 = |X_v|^2$ with $\lambda > 0$.

Let h_{ij}, $i, j = 1, 2$, be the coefficients of the second fundamental form of X with respect to N. Note that the Hopf differential $\phi(z)\, dz^2 = (h_{11} - h_{22} - 2ih_{12})\, dz^2$ is holomorphic for constant mean curvature surfaces [Hopf 1989]. The theorem of Joachimsthal [do Carmo 1976] says that γ_1 and γ_2 are curvature lines of $\mathscr{A}$. Hence $h_{12} \equiv 0$ on $u = 0$ and $u = \log R$. Since h_{12} is harmonic and periodic, we have $h_{12} \equiv 0$ on B. This implies that z is a conformal curvature coordinate and $h_{11} - h_{22}$ is constant [McCuan 1997]. Let $c = h_{11} - h_{22}$. If $\mathscr{A}$ is minimal, then we have $K < 0$ and $c = 2h_{11} > 0$ by the choice of N. When $H = -1$, $\mathscr{A}$ is part of the unit sphere $S_1 = S_2$ by the boundary comparison principle for the mean curvature operator [Gilbarg and Trudinger 2001]. We assume that $H \neq -1$ in the following. The principal curvatures of $\mathscr{A}$ are

$$\kappa_1 = H + \frac{c}{2\lambda^2} \quad \text{and} \quad \kappa_2 = H - \frac{c}{2\lambda^2}. \tag{1}$$

We use for γ_1 and γ_2 the parametrizations $\gamma_1(v) = X(0, v)$ and $\gamma_2(v) = X(\log R, v)$, for $v \in [0, 2\pi)$. In the following, we assume that $\mathscr{A}$ has nonzero Gaussian curvature.

Lemma 1.1. *Each $\gamma_i(v)$, $i = 1, 2$, has constant speed $\sqrt{c/2(1+H)}$ and κ_2 is -1 on γ_1 and γ_2. As spherical curves, γ_1 and γ_2 are convex. On $\mathscr{A} \setminus \partial\mathscr{A}$, we have $\lambda^2 < c/2(1+H)$ when $K < 0$ and $\lambda^2 > c/2(1+H)$ when $K > 0$.*

Proof. The curvature vector of $\gamma_1(v)$ is

$$\begin{aligned} \vec{\kappa} &= \frac{1}{|X_v|}\frac{d}{dv}\left(\frac{X_v}{|X_v|}\right) = \frac{1}{|X_v|^2}X_{vv} - \frac{X_v}{|X_v|^4}(X_v \cdot X_{vv}) \\ &= \frac{1}{\lambda^2}\left(-\frac{\lambda_u}{\lambda}X_u + h_{22}N\right). \end{aligned} \tag{2}$$

Let the center of S_1 be the origin of $\mathbb{R}^3$. Since $\mathscr{A}$ is tangential to S_1 along γ_1, we have $N(0, v) = X(0, v) = \gamma_1(v)$ on γ_1. Since γ_1 is on the unit sphere S_1, the curvature vector $\vec{\kappa}$ of γ_1 satisfies $(\vec{\kappa} \cdot \gamma_1)(v) = -1$. Hence we have $\kappa_2 = h_{22}/\lambda^2 = -1$ on γ_1. Since $\lambda^2 = |\gamma_{1_v}|^2$ on γ_1, we have $|\gamma_{1_v}| = \sqrt{c/2(1+H)}$ from (1). By choosing the center of S_2 as the origin of $\mathbb{R}^3$, we get the results for γ_2.

The Gaussian curvature K satisfies

$$\Delta \log \lambda = -K\lambda^2,$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$. We can rewrite this equation as

$$\lambda\Delta\lambda = |\nabla\lambda|^2 - K\lambda^4. \tag{3}$$

Since $\lambda_v(0, v) = 0$ and $\lambda_v(\log R, v) = 0$ and $K \neq 0$, λ does not have interior maximum when $K < 0$, and does not have interior minimum when $K > 0$. Since $\lambda^2 = c/2(1+H)$ on γ_1 and γ_2, it follows that $\lambda^2 < c/2(1+H)$ on $\mathscr{A} \setminus \partial\mathscr{A}$ when $K < 0$ and $\lambda^2 > c/2(1+H)$ when $K > 0$. Moreover we have $\lambda_u \le 0$ on $u = 0$ and $\lambda_u \ge 0$ on $u = \log R$ when $K < 0$ and $\lambda_u \ge 0$ on $u = 0$ and $\lambda_u \le 0$ on $u = \log R$ when $K > 0$. Since $X_u/|X_u| \in TS_i$ is perpendicular to γ_i, the geodesic curvature of γ_i as a spherical curve is $\vec{\kappa} \cdot (X_u/|X_u|) = -\lambda_u/\lambda^2$. Hence γ_1 and γ_2 are convex as spherical curves. □

Remark 1.2. If $\lambda^2 \equiv c/2(1+H)$ on $\mathscr{A}$, then $K \equiv 0$ and $\mathscr{A}$ is part of a cylinder.

2. The −1-parallel surface

The -1-parallel surface $\tilde{\mathscr{A}}$ of $\mathscr{A}$ is defined by

$$\tilde{X} = X - N.$$

The image of γ_1 (respectively, of γ_2) in $\tilde{\mathscr{A}}$ is a point corresponding to the center of S_1 (respectively, of S_2). We denote the centers of S_1 and S_2 by O and O_2 for simplicity. We fix the unit normal $\tilde{N}$ of $\tilde{\mathscr{A}}$ to be N. Since $z = u + iv$ is a curvature coordinate of X, we have

$$\tilde{X}_u = \left(1 + \frac{h_{11}}{\lambda^2}\right) X_u \quad \text{and} \quad \tilde{X}_v = \left(1 + \frac{h_{22}}{\lambda^2}\right) X_v. \tag{4}$$

Since $\kappa_2 = -1$ on γ_i by Lemma 1.1, $\tilde{X}$ is singular for $u = 0$ and $u = \log R$. By Lemma 1.1, we have $\lambda^2 \neq c/2(1+H)$ on $\mathscr{A} \setminus \partial\mathscr{A}$, which implies that $1 + \kappa_2 \neq 0$ on $\mathscr{A} \setminus \partial\mathscr{A}$. When $K < 0$, we have $\kappa_1 > 0$ on $\mathscr{A} \setminus \partial\mathscr{A}$. Hence $\tilde{X}$ is regular for $0 < u < \log R$ and we have $H > -1$.

Now suppose that $K > 0$. Since $\kappa_2 = -1$ on γ_i by Lemma 1.1, we have $\kappa_1 < 0$ and $H < -1/2$. We consider two cases separately: $H < -1$ and $-1 < H < -1/2$. If $H < -1$, then $c < 0$ from $\lambda^2 = c/2(1+H) > 0$ on γ_i. Hence we have $\kappa_1 < -1$, which implies that $\tilde{X}$ is regular for $0 < u < \log R$. If $-1 < H < -1/2$, then we must have $c > 0$. This implies that $1 + \kappa_1 \neq 0$. Otherwise we have $0 < 2\lambda^2(1+H) = -c$, which contradicts $c > 0$. Hence $\tilde{X}$ is regular for $0 < u < \log R$.

Remark 2.1. When $K < 0$ or $K > 0$ and $-1 < H < -1/2$, $\mathscr{A}$ stays outside of the balls B_1 and B_2 bounded by S_1 and S_2. If $K > 0$ and $H < -1$, then $\mathscr{A} \subset B_1 \cap B_2$.

Lemma 2.2. *The mean curvature $\tilde{H}$ and the Gaussian curvature $\tilde{K}$ of $\tilde{\mathscr{A}}$ satisfies $(1+H)\tilde{K} = (1+2H)\tilde{H} - H$. On $\tilde{\mathscr{A}} \setminus \{O, O_2\}$, we have the following*:

(i) *If $K < 0$ and $H > -1$, then $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$.*

(ii) *If $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1+H) < \min\{1, -H/(1+H)\}$, $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1+H)$ and $\tilde{H} < H/(1+H)$.*

(iii) *If $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1+H) < 1$, $\tilde{\kappa}_1 > (1+2H)/2(1+H)$, $\tilde{\kappa}_2 > H/(1+H)$ and $\tilde{H} > H/(1+H)$.*

Proof. Since

$$\tilde{h}_{12} = N \cdot \tilde{X}_{uv} = \left(1 + \frac{h_{11}}{\lambda^2}\right)(N \cdot X_{uv}) = 0,$$

(u, v) is a curvature coordinate (not conformal) for $\tilde{\mathscr{A}}$ except for O and O_2. We have

$$\tilde{h}_{11} = N \cdot \tilde{X}_{uu} = \left(1 + \frac{h_{11}}{\lambda^2}\right) h_{11}, \quad \tilde{h}_{22} = N \cdot \tilde{X}_{vv} = \left(1 + \frac{h_{22}}{\lambda^2}\right) h_{22}.$$

The principal curvatures of $\tilde{\mathscr{A}}$ are

$$\tilde{\kappa}_1 = \frac{\kappa_1}{1+\kappa_1} = \frac{H/(1+H) + \big(c/2\lambda^2(1+H)\big)}{1+\big(c/2\lambda^2(1+H)\big)},$$
$$\tilde{\kappa}_2 = \frac{\kappa_2}{1+\kappa_2} = \frac{H/(1+H) - \big(c/2\lambda^2(1+H)\big)}{1-\big(c/2\lambda^2(1+H)\big)}.$$

From $\kappa_1+\kappa_2 = 2H$, we have $H = \dfrac{\tilde{H}-\tilde{K}}{1-2\tilde{H}-\tilde{K}}$ or $(1+H)\tilde{K} = (1+2H)\tilde{H} - H$. It is straightforward to see that

$$\tilde{H} = \frac{H/(1+H) - \big(c/2\lambda^2(1+H)\big)^2}{1-\big(c/2\lambda^2(1+H)\big)^2}.$$

Note that $\kappa_2 < 0$ on $\mathscr{A}$. First suppose that $K < 0$. Then we have $\kappa_1 > 0$, which implies that $\tilde{\kappa}_1 = \kappa_1/(1+\kappa_1) > 0$. Since $c/2\lambda^2(1+H) > 1$ by Lemma 1.1, we have $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$.

When $K > 0$, we have $\kappa_1 = H + c/2\lambda^2 < 0$. If $-1 < H < -1/2$, then we have $c > 0$ because $\lambda^2 = c/2(1+H) > 0$ on γ_i. It follows that $c/2\lambda^2(1+H) < -H/(1+H)$. By Lemma 1.1, we also have $c/2\lambda^2(1+H) < 1$. Therefore $0 < c/2\lambda^2(1+H) < \min\{1, -H/(1+H)\}$. It is easy to see that $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1+H) < 0$ and $\tilde{H} < H/(1+H) < 0$.

When $K > 0$ and $H < -1$, we have $c < 0$ and $0 < c/2\lambda^2(1+H) < 1$. It is straightforward to see that $\tilde{\kappa}_1 > (1+2H)/(1+H)$, $\tilde{\kappa}_2 > H/(1+H)$ and $\tilde{H} > H/(1+H)$. □

This lemma says that $\tilde{\mathscr{A}}$ is a linear Weingarten surface with two singular points O and O_2 and is positively curved outside O and O_2.

Lemma 2.3. *$\tilde{\mathscr{A}}$ is embedded.*

Proof. Let $\nu(v) = (X_u/|X_u|)(0, v)$. Note that ν is a closed curve in the unit sphere S_1. We claim that ν is *convex as a spherical curve*. Otherwise, there is a great circle η intersecting the image of ν at no less than 3 points $\nu(v_1), \dots, \nu(v_n)$. (It is possible that ν maps an interval $(v_a, v_b) \subset [0, 2\pi)$ into a single point. We choose the v_i's in such a way that ν maps no two v_i's to the same point.) Each $\nu(v_i)$ determines a great circle $\mathbb{S}^1_{v_i} \subset S_1$ contained in the plane perpendicular to $\nu(v_i)$. At each $\gamma_1(v_i)$, γ_1 is tangent to $\mathbb{S}^1_{v_i}$. Since η and $\mathbb{S}^1_{v_i}$ are perpendicular, γ_1 cannot be convex when $n \geq 3$. Hence ν intersect every geodesic of S_1 at no more than two points. This shows that ν is convex as a spherical curve. Similarly, $(X_u/|X_u|)(\log R, v)$ is also convex as a spherical curve.

Since $\tilde{\mathscr{A}}$ is a parallel surface of $\mathscr{A}$, the tangent cone $\mathrm{Tan}(O, \tilde{\mathscr{A}})$ of $\tilde{\mathscr{A}}$ at O is the cone formed by rays from O through ν. Since ν is a convex spherical curve, $\mathrm{Tan}(O, \tilde{\mathscr{A}})$ is convex. This shows that a small neighborhood of O in $\tilde{\mathscr{A}}$ is embedded and nonnegatively curved as a metric space [Alexandrov 1948]. Similarly, there is a neighborhood of O_2 in $\tilde{\mathcal{A}}$ which is embedded and nonnegatively curved as a metric space.

Hadamard showed that a closed surface S in $\mathbb{R}^3$ with strictly positive Gaussian curvature is the boundary of a convex body [Hopf 1989]. In particular, S is embedded. Alexandrov [1948] generalized Hadamard's theorem to nonnegatively curved metric spaces. Since $\tilde{\mathcal{A}}$ is a nonnegatively curved closed metric space, $\tilde{\mathcal{A}}$ is embedded. □

Remark 2.4. We have $\nu_v = (\lambda_u/\lambda^2)X_v$. At points where $\lambda_u \neq 0$, the curvature vector of ν is

$$\vec{\kappa}_\nu = \frac{1}{\lambda_u}\left(-\frac{\lambda_u}{\lambda}X_u + h_{22}N\right).$$

The geodesic curvature of ν as a spherical curve $\vec{\kappa}_\nu \cdot N = h_{22}/\lambda_u$.

3. Main results

We use Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] to prove the theorems.

Theorem 3.1. *A compact embedded constant mean curvature annulus $\mathcal{A}$ with nonvanishing Gaussian curvature meeting two spheres S_1 and S_2 of the same radius tangentially is part of a Delaunay surface. In particular, if $\mathcal{A}$ is minimal, then $\mathcal{A}$ is part of a catenoid.*

Proof. We suppose that the radius of S_1 and S_2 is 1. By Lemma 2.2 and Lemma 2.3, $\tilde{\mathcal{A}}$ is a compact embedded surface with two singular points O and O_2 and satisfying $(1+H)\tilde{K} = (1+2H)\tilde{H} - H$ at regular points. A small neighborhood of a regular point of $\tilde{\mathcal{A}}$ can be represented as the graph of a function $f(x, y)$ satisfying

$$2(1+H)(f_{xx}f_{yy} - f_{xy}^2) + 2H(1+f_x^2+f_y^2)^2 = (1+2H)\big((1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}\big)(1+f_x^2+f_y^2)^{1/2}. \tag{5}$$

This equation can be rewritten as

$$\det\big(2(1+H)D^2f + A(Df)\big) = W^4, \tag{6}$$

where

$$A(Df) = -(1+2H)\begin{pmatrix}(1+f_x^2)W & f_xf_yW \\ f_xf_yW & (1+f_y^2)W\end{pmatrix} \quad \text{and} \quad W = \sqrt{1+f_x^2+f_y^2}.$$

Equation (6) is elliptic with respect to f if $2(1+H)D^2f + A(Df)$ is positive definite. Since $\det\big(2(1+H)D^2f + A(Df)\big) = W^4 > 0$, this happens if

$$\operatorname{Tr}\big(2(1+H)D^2f + A(Df)\big) = 2(1+H)\Delta f - (1+2H)(2+f_x^2+f_y^2)W \tag{7}$$

is strictly positive.

First we consider the case $K < 0$. Since $\tilde{H} > 1$ by Lemma 2.2, we have

$$\Delta f + f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} > 2W^{3/2}, \tag{8}$$

for f representing $\tilde{\mathscr{A}}$. We may assume that f is defined on $B(0,\epsilon) \subset T_p\tilde{\mathscr{A}}$ so that $\nabla f(0) = \vec{0}$ and $D^2 f$ is diagonal. For sufficiently small $\epsilon = \epsilon(p)$, (8) implies that (7) is strictly positive. Hence (6) is elliptic with respect to f representing $\tilde{\mathscr{A}}$.

When $-1 < H < -1/2$, (7) is automatically satisfied.

Now we consider the case $K > 0$ and $H < -1$. Since $\tilde{H} > H/(1+H)$ by Lemma 2.2, we have

$$\Delta f + f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} > \frac{2H}{1+H} W^{3/2}. \tag{9}$$

Assuming that f is defined on $B(0,\epsilon) \subset T_p\tilde{\mathscr{A}}$ with $\nabla f(0) = \vec{0}$ and $D^2 f$ is diagonal, (9) implies that

$$\Delta f - \frac{1+2H}{2(1+H)}(2 + f_x^2 + f_y^2)W$$

is strictly positive for sufficiently small ϵ. So $\det\big(-2(1+H)D^2 f - A(Df)\big) = W^4$ is elliptic for f representing $\widetilde{A}$. The ellipticity of (6) for f representing $\widetilde{A}$ enables us to use the maximum principle and the boundary point lemma [Gilbarg and Trudinger 2001].

Since $\tilde{\mathscr{A}}$ is convex and embedded, we can use Alexandrov's moving plane argument [Alexandrov 1962; Hopf 1989] to show that $\tilde{\mathscr{A}}$ is rotational as follows. Let Π_θ be the plane containing the line segment $\overline{OO_2} \subset \mathbb{R}^3$ and making angle θ with a fixed vector $\vec{E}$ which is perpendicular to $\overline{OO_2}$. Fix a positive constant L such that each plane Π_θ^L that is parallel to Π_θ with distance L from Π_θ does not meet $\tilde{\mathscr{A}}$ for all θ. Let Π_θ^l be the plane between Π_θ^L and Π_θ with distance l from Π_θ. When Π_θ^l intersects $\tilde{\mathscr{A}}$, we reflect the Π_θ^L side part of $\tilde{\mathscr{A}}$ about Π_θ^l. Denote this reflected surface by $\tilde{\mathscr{A}}_{l,\theta}^{\text{ref}}$. As we decrease l from L, there might be a first $l_\theta \geq 0$ for which $\tilde{\mathscr{A}}_{l_\theta,\theta}^{\text{ref}}$ is tangent to $\tilde{\mathscr{A}}$ at an interior point or at a boundary point of $\partial\tilde{\mathscr{A}}_{l_\theta,\theta}^{\text{ref}}$. We call this point the *first touch point*. If there is no nonnegative l with the first touch point, we repeat the process for $\Pi_{\theta+\pi}^L$ to find $l_{\theta+\pi}$, which must be positive. At the first touch point, we apply the comparison principles for (5) to see that the part of $\tilde{\mathscr{A}}$ in the Π_θ side and $\tilde{\mathscr{A}}_{l_\theta,\theta}^{\text{ref}}$ are identical and, hence, $l_\theta = 0$. This implies that Π_θ is a symmetry plane for $\tilde{\mathscr{A}}$. Since θ can be chosen arbitrarily, $\tilde{\mathscr{A}}$ should be rotational and, hence, $\mathscr{A}$ is also rotational. Since the Delaunay surfaces and the catenoid are the only nonplanar rotational minimal and constant mean curvature surfaces, $\mathscr{A}$ is part of a Delaunay surface or part of a catenoid. □

We used the embeddedness of $\mathscr{A}$ to prove that $\tilde{\mathscr{A}}$ is embedded. Whether there is a nonembedded minimal or constant mean curvature annulus meeting two unit spheres tangentially is an interesting question. Moreover we raise the following questions.

(1) Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere perpendicularly or in constant contact angles part of a catenoid or part of a Delaunay surface? Nitsche showed that an immersed disk type minimal or constant mean curvature surface meeting a sphere in constant contact angle is either a flat disk or a spherical cap [Nitsche 1985].

(2) Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting two spheres in constant contact angles part of a catenoid or a plane or part of a Delaunay surface?

(3) Is a compact immersed minimal or constant mean curvature annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere and a plane in constant contact angles part of a catenoid or part of a Delaunay surface? We give an affirmative answer to this problem in a special case in the following.

Theorem 3.2. *A compact embedded constant mean curvature annulus $\mathcal{B}$ with negative* (*respectively*, *positive*) *Gaussian curvature meeting a sphere tangentially and a plane in constant contact angle $\geq \pi/2$* (*respectively*, $\leq \pi/2$) *is part of a Delaunay surface. In particular*, *if $\mathcal{B}$ is minimal and the constant contact angle is $\geq \pi/2$ then $\mathcal{B}$ is part of a catenoid.*

The angle is measured between the outward conormal of $\mathcal{B}$ and the outward conormal of the bounded domain in Π bounded by the boundary curve. Since the proof of this theorem is similar to that of Theorem 3.1, we omit some previously proved details.

Proof. Denote the sphere by S_2 and the plane by Π. We may assume that the radius of S_2 is 1. Let α be the constant contact angle between $\mathcal{B}$ and Π. If $\alpha = \pi/2$, then we can reflect $\mathcal{B}$ about Π to get a constant mean curvature annulus meeting two unit spheres tangentially. Hence $\mathcal{B}$ is part of a catenoid or a Delaunay surface by Theorem 3.1.

In the following, we assume that $\alpha \neq \pi/2$. As in the case for $\mathcal{A}$ in Section 1, there is a conformal parametrization X of $\mathcal{B}$ from a strip $\{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\}$ for which $z = u + iv$ is a curvature coordinate. We fix the normal N of $\mathcal{B}$ to point away from the center of S_2. Let $c_1(v) = X(0, v)$ be on Π and $c_2(v) = X(\log R, v)$ be on S_2 with $\partial X_3/\partial u > 0$ along c_1. As in Lemma 1.1, c_2 has constant speed $\sqrt{c/2(1 + H)}$ and $\kappa_2 = -1$ along c_2. Since $K \neq 0$ on $\mathcal{B}$ and $z = u + iv$ is a curvature coordinate, we have $\kappa_2 < 0$ on c_1. The curvature of c_1 is $|\vec{\kappa}| = -\kappa_2/\sin\alpha > 0$, which shows that c_1 is locally convex. Since c_1 is a Jordan curve, it is convex.

First, we assume that $K < 0$ and $\alpha > \pi/2$. Since $(\vec{\kappa}/|\vec{\kappa}|)\cdot(X_u/X_u|) = \cos\alpha < 0$ on c_1, it follows from (2) that $\lambda_u > 0$ on c_1. Since $\lambda_v(\log R, v) = 0$ (see Lemma 1.1), it follows from (3) that $\lambda_u \geq 0$ on c_2. Otherwise, λ will have an interior maximum, which contradicts (3). Hence we have $\lambda^2 < c/2(1+H)$ on $\mathcal{B}\setminus c_2$. Note that $\kappa_1 > 0$ and $\kappa_2 < 0$ in $\mathcal{B}$. From $\lambda_u \leq 0$ on c_2, we see that c_2 is convex as a spherical curve (see Lemma 1.1). Arguing as in the proof of Lemma 2.3, we see that $(X_u/|X_u|)(\log R, v)$ is also convex as a spherical curve.

When $K > 0$ and $\alpha < \pi/2$, we have $(\vec{\kappa}/|\vec{\kappa}|)\cdot(X_u/|X_u|) = \cos\alpha > 0$ on c_1. Hence $\lambda_u < 0$ on c_1. Since $\lambda_v(\log R, v) = 0$, it follows from (3) that λ does not have interior minimum. Then we have $\lambda_u \leq 0$ on c_2 and $\lambda^2 > c/2(1+H)$ on $\mathcal{B}\setminus c_2$. Note that $\kappa_1 < 0$ and $\kappa_2 < 0$ in $\mathcal{B}$. From $\lambda_u \leq 0$ on c_2, it follows that c_2 is convex as a spherical curve. Moreover $(X_u/|X_u|)(\log R, v)$ is convex as a spherical curve (see Lemma 2.3).

Let $\widetilde{\mathcal{B}}$ be the -1-parallel surface of $\mathcal{B}$. As in Section 2, we can show that $\widetilde{\mathcal{B}}$ is regular except for O_2: the image of c_2, and $H > -1$ when $K < 0$ and $H < -1/2$ when $K > 0$. As in Lemma 2.2, we see that mean curvature $\tilde{H}$ and the Gaussian curvature $\tilde{K}$ of $\widetilde{\mathcal{B}}$ satisfies $(1+H)\tilde{K} = (1+2H)\tilde{H} - H$ and (i) if $K < 0$ and $H > -1$, then $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$, (ii) if $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1+H) < \min\{1, -H/(1+H)\}$, $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1+H)$ and $\tilde{H} < H/(1+H)$, and (iii) if $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1+H) < 1$, $\tilde{\kappa}_1 > (1+2H)/2(1+H)$, $\tilde{\kappa}_2 > H/(1+H)$ and $\tilde{H} > H/(1+H)$.

The convexity of $(X_u/|X_u|)(\log R, v)$ as a spherical curve implies that there is a neighborhood of O_2 in $\widetilde{\mathcal{B}}$ which is embedded and nonnegatively curved as a metric space. Let $\widetilde{\Pi}$ be the plane parallel to Π and containing $\tilde{c}_1$. The curvature of $\tilde{c}_1$ is $|\tilde{\kappa}_2|/\sin\alpha$, which does not vanish. Hence $\tilde{c}_1$ is locally convex. Using the orthogonal projection onto $\widetilde{\Pi}$, $\tilde{c}_1$ may be considered as a $(\sin\alpha)$-parallel curve of c_1 in $\widetilde{\Pi}$. Hence $\tilde{c}_1$ is also a convex Jordan curve.

Suppose that $K < 0$ and $\alpha > \pi/2$. Since $\kappa_1 > 0$, $\tilde{X}_u$ is a positive multiple of X_u by (4). The positivity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\widetilde{\mathcal{B}}$ meets $\widetilde{\Pi}$ in constant angle $\pi - \alpha$. Suppose that $K > 0$ and $\alpha < \pi/2$. If $-1 < H < -1/2$, then we have $c > 0$ and $\kappa_1 > -1$. Hence $\tilde{X}_u$ is a positive multiple of X_u by (4). The negativity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\widetilde{\mathcal{B}}$ meets $\widetilde{\Pi}$ in constant angle α. When $K > 0$ and $H < -1$, we have $c < 0$ and $\kappa_1 < -1$. Hence $\tilde{X}_u$ is negative multiple of X_u by (4). In this case, $\widetilde{\mathcal{B}}$ lies below $\widetilde{\Pi}$ and $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are both positive. It is straightforward to see that $\widetilde{\mathcal{B}}$ meets $\widetilde{\Pi}$ in constant angle α.

Let $\breve{\mathcal{B}}$ be the singular surface obtained from $\widetilde{\mathcal{B}}$ by attaching the disk in $\widetilde{\Pi}$ bounded by $\tilde{c}_1$ to $\widetilde{\mathcal{B}}$. Since $\widetilde{\mathcal{B}}$ meets $\widetilde{\Pi}$ in acute angle, $\breve{\mathcal{B}}$ is a nonnegatively curved metric space. By Alexandrov's generalization [1948] of Hadamard's theorem, $\breve{\mathcal{B}}$ is the boundary of a convex body. Therefore $\breve{\mathcal{B}}$ is embedded. Note again that $\tilde{H}$, $\tilde{K}$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ satisfy the statements of Lemma 2.2. Hence (5) is elliptic for functions representing $\widetilde{\mathcal{B}}$ locally. We can apply Alexandrov's moving plane argument to $\widetilde{\mathcal{B}}$ using planes perpendicular to $\widetilde{\Pi}$ as in the proof of Theorem 3.1 to see that $\widetilde{\mathcal{B}}$ is rotational. Hence $\mathcal{B}$ is rotational and, as a result, is part of a Delaunay surface or part of a catenoid. □

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