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A BEURLING–HÖRMANDER THEOREM ASSOCIATED WITH THE RIEMANN–LIOUVILLE OPERATOR

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We establish an analogue of the Beurling theorem associated with the Riemann–Liouville operator. We also derive some other versions of uncertainty principle theorems associated with this operator.

1. Introduction and the main result

The uncertainty principle, which plays an important role in harmonic analysis, states that a nonzero function and its Fourier transform cannot simultaneously be very small at infinity. This principle has been researched on various aspects and has several versions named after Hardy, Morgan, Cowling and Price, Gelfand, Beurling and others. The Beurling theorem is the most general case since it implies the other uncertainty principles.

The classical Beurling theorem was proved by Hörmander [1991] and generalized to d dimensions by Bonami et al. [2003]. Here we record the general case:

Lemma 1.1. *For $f \in L^2(\mathbb{R}^d)$ and $N \geqq 0$, if*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)|\,|\widehat{f}(y)|\,e^{\|x\|\|y\|}}{(1+\|x\|+\|y\|)^N}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

then $f(x) = P(x)\,e^{-a\langle Ax,x\rangle}$, $a > 0$, where A is a real positive definite symmetric matrix and $P(x)$ is a polynomial of degree $< (N-d)/2$. In particular, $f = 0$ when $N \leq d$.

In the lemma and the rest of the paper, $\widehat{f}$ is the classic Fourier transform of f in $\mathbb{R}^d$, defined by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}^d} f(x)\,e^{-i\,\lambda\,x}\,\mathrm{d}x, \qquad \lambda \in \mathbb{R}^d.$$

The Beurling theorem has been generalized to different settings. L. Bouattour established an analogue in the framework of Chébli–Trimèche hypergroups $(\mathbb{R}_+, *(A))$ (see [Bouattour and Trimèche 2005]). J. Z. Huang and H. P. Liu [2007a;

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2007b] gave analogues for the Laguerre hypergroup and the Heisenberg group. R. P. Sarkar and J. Sengupta [2007b] established the analogue of the Beurling theorem on the full group SL(2, $\mathbb{R}$). As for the noncompact semisimple Lie group case, S. Thangavelu [2004] first gave the analogue on rank 1 symmetric spaces with an additional condition like the one required in the Cowling–Price theorem, so he called it the Cowbeurling Theorem; then R. P. Sarkar and J. Sengupta [2007a] removed this additional condition and gave the analogue in rank 1 symmetric spaces; recently, L. Bouattour [2008] generalized this result and gave the analogue for real symmetric spaces of rank d. For more Beurling theorems in different settings, refer to [Kamoun and Trimèche 2005; Parui and Sarkar 2008].

In this paper, for $\alpha \geq 0$ we consider the singular partial differential operators

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial x}, \\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}, \quad (r, x) \in (0, +\infty) \times \mathbb{R},\ \alpha \geqq 0, \end{cases}$$

originally studied in [Baccar et al. 2006; Omri and Rachdi 2008]. The latter authors have proved an uncertainty principle that generalized the Heisenberg–Pauli–Weyl inequality for the classical Fourier transform:

Proposition [Omri and Rachdi 2008]. *For all $f \in L^2(d\nu_\alpha)$, we have*

$$\| \, |(r, x)| f \|_{2,\nu_\alpha} \, \|(\mu^2 + 2\lambda^2)^{1/2} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} \geqq \frac{2\alpha+3}{2} \|f\|^2_{2,\nu_\alpha}$$

with equality if and only if

$$f(r, x) = Ce^{-(r^2+x^2)/2t_0^2} \quad \text{for } (r, x) \in \mathbb{R}_+ \times \mathbb{R},\ t_0 > 0,\ C \in \mathbb{C},$$

where $d\nu_\alpha$ is a measure defined on $\mathbb{R}_+ \times \mathbb{R}$ by

$$d\nu_\alpha(r, x) = \mathrm{d}c(r) \otimes \mathrm{d}x \quad \text{with } \mathrm{d}c(r) \overset{\text{def}}{=} \frac{r^{2\alpha+1}}{2^\alpha\, \Gamma(\alpha+1)\sqrt{2\pi}}\, \mathrm{d}r; \tag{1}$$

$d\gamma_\alpha(\mu, \lambda)$ is a measure defined on the set Γ_+

$$\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \big\{(it, x) : (t, x) \in R_+ \times R,\ t \leqq |x|\big\};$$

$|(r, x)|$ is the Euclidean norm in $\mathbb{R}^2$, that is, $|(r, x)| = (r^2 + x^2)^{1/2}$; and $\mathscr{F}_\alpha(f)$ is the generalized Fourier transform associated with the Riemann–Liouville operator.

Our main result is an analogue of the Beurling–Hörmander theorem for this generalized Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann–Liouville operator:

Theorem 1.2. *Let* $K = \mathbb{R}_+ \times \mathbb{R}$, *and assume* $N \geqq 0$. *For* $f \in L^2(K, dv_\alpha)$, *if*

$$\int_{\Gamma_+} \int_K \frac{|f(r,x)|\,|\mathscr{F}_\alpha(f)(\mu,\lambda)|\,e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}v_\alpha(r,x)\,\mathrm{d}r_\alpha(\mu,\lambda) < \infty,$$

then

$$f(r,x) = e^{-ax^2}\biggl(\sum_{j=0}^{k} \psi_j(r) x^j\biggr),$$

where $a > 0$, $k < \dfrac{N-1}{2}$, *and* $\psi_j(r) \in L^2\Bigl([0,+\infty), \dfrac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)}\,\mathrm{d}r\Bigr)$. *In particular, when* $N \leqq 3$,

$$f(r,x) = e^{-ax^2}\,\psi(r),$$

where $\psi(r) \in L^2([0,+\infty), r^{2\alpha+1}/(2^\alpha \Gamma(\alpha+1))\,\mathrm{d}r)$, *and when* $N \leqq 1$, *we have* $f = 0$.

Section 2 contains some preliminary facts about the Riemann–Liouville operator and the generalized Fourier transform. In Section 3, we prove Theorem 1.2. In Section 4, we give some other uncertainty principles. In Section 5, we give a stronger result but at the cost of more strictly constraining the function $f(r,x)$ by utilizing the Riemann–Liouville transform and its dual.

2. Preliminaries

In this section, we set some notation and theorems about the generalized Fourier transform associated with Riemann–Liouville operator. For detailed information, refer to [Baccar et al. 2006; Hamadi and Rachdi 2006; Omri and Rachdi 2008].

From this last reference we know that for all $(\mu,\lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) = -i\,\lambda\,u(r,x), \\ \Delta_2 u(r,x) = -\mu^2\,u(r,x), \\ u(0,0) = 1,\ (\partial u/\partial r)(0,x) = 0, \quad x \in \mathbb{R} \end{cases}$$

admits a unique solution $\varphi_{\mu.\lambda}$, given by

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2+\lambda^2})\,e^{-i\lambda x} \quad \text{for } (\mu,\lambda) \in \mathbb{R}^2, \tag{2}$$

where

$$j_\alpha(x) = 2^\alpha\,\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha} = \Gamma(\alpha+1)\sum_0^\infty \frac{(-1)^n}{n!\Gamma(\alpha+n+1)}\Bigl(\frac{x}{2}\Bigr)^{2n}, \tag{3}$$

and $J_\alpha(x)$ is a Bessel function of the first kind of index α. The modified Bessel function j_α has the following integral representation: for all $\mu,\ r \in \mathbb{R}_+$ we have

$$j_\alpha(r\mu) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \displaystyle\int_0^1 (1-t^2)^{\alpha-1/2} \cos(r\mu t)\,\mathrm{d}t & \text{if } \alpha > -1/2, \\ \cos(r\mu) & \text{if } \alpha = -1/2. \end{cases}$$

The Riemann–Liouville integral transform associated with Δ_1, Δ_2 is defined by

$$\mathcal{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi} \displaystyle\int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1}\,\mathrm{d}t\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha > 0, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^1 f(r\sqrt{1-t^2}, x+rt)\,\dfrac{\mathrm{d}t}{\sqrt{1-t^2}} \qquad\qquad\qquad\quad \text{if } \alpha = 0. \end{cases}$$

Now we give some properties of the eigenfunction $\varphi_{\mu,\lambda}$.

(i) The supremum of $\varphi_{\mu,\lambda}$ satisfies

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1$$

if and only if (μ, λ) belongs to the set

$$\Gamma = \mathbb{R}^2 \cup \{(it, x) : (t, x) \in \mathbb{R}^2, |t| \leqq |x|\}.$$

(ii) The eigenfunction $\varphi_{\mu,\lambda}$ has Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi} \displaystyle\int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-1/2}\,(1-s^2)^{\alpha-1}\,\mathrm{d}t\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha > 0, \\ \dfrac{1}{\pi} \displaystyle\int_{-1}^1 f(r\sqrt{1-t^2}, x+rt)\,\dfrac{\mathrm{d}t}{\sqrt{1-t^2}} \qquad\qquad\qquad\quad \text{if } \alpha = 0, \end{cases}$$

where f is a continuous function on $\mathbb{R}^2$.

From our definition, we can see that the transform $\mathcal{R}_\alpha$ generalizes the "mean operator" defined by

$$\mathcal{R}_0(f)(r,x) = \frac{1}{\pi} \int_0^{2\pi} f(r\ \sin(\theta), x + r\ \cos(\theta))\,\mathrm{d}\theta.$$

In the remainder of the paper, we use the following notation:

(i) $L^p(d\upsilon_\alpha)$ denotes the space of measurable functions f on $K=\mathbb{R}_+\times\mathbb{R}$ such that

$$\|f\|_{p,\upsilon_\alpha}=\left(\int_0^\infty\int_{\mathbb{R}}|f(r,x)|^p\,d\upsilon_\alpha(r,x)\right)^{1/p}<\infty\quad\text{if } p\in[1,+\infty),$$
$$\|f\|_{\infty,\upsilon_\alpha}=\operatorname*{ess\,sup}_{(r,x)\in K}|f(r,x)|<+\infty\qquad\text{if } p=+\infty.$$

(ii) $\langle\,,\,\rangle_{\upsilon_\alpha}$ is the inner product defined on $L^2(d\upsilon_\alpha)$ by

$$\langle f,g\rangle_{\upsilon_\alpha}=\int_0^\infty\int_R f(r,x)\overline{g(r,x)}\,d\upsilon_\alpha(r,x).$$

(iii) $\Gamma_+=\mathbb{R}_+\times\mathbb{R}\cup\big\{(it,x):(t,x)\in\mathbb{R}_+\times\mathbb{R},\,t\le|x|\big\}$.

(iv) $\mathcal{B}_{\Gamma_+}$ is a σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+}=\big\{\theta^{-1}(B):B\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R})\big\},$$

where θ is the bijective function defined on the set Γ_+ by

$$\theta(\mu,\lambda)=(\sqrt{\mu^2+\lambda^2},\lambda).$$

(v) Θ is the operator given by $(\Theta\circ f)(\mu,\lambda)=f(\theta(\mu,\lambda))$ for any function f defined on Γ_+.

(vi) $d\gamma_\alpha$ is a measure on $\mathcal{B}_{\Gamma_+}$ given by

$$\gamma_\alpha(A)=\upsilon_\alpha(\theta(A))\quad\text{for } A\in\mathcal{B}_{\Gamma_+}.$$

(vii) Let $L^p(d\gamma_\alpha)$ denote the space of measurable functions f on Γ_+ such that

$$\|f\|_{p,\gamma_\alpha}=\left(\iint_{\Gamma_+}|f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p}<\infty\quad\text{if } p\in[1,+\infty),$$
$$\|f\|_{\infty,\gamma_\alpha}=\operatorname*{ess\,sup}_{(\mu,\lambda)\in\Gamma_+}e|f(\mu,\lambda)|<+\infty\qquad\text{if } p=+\infty.$$

(viii) $\langle\,,\,\rangle_{\gamma_\alpha}$ is the inner product defined on $L^2(d\gamma_\alpha)$ by

$$\langle f,g\rangle_{\gamma_\alpha}=\int_{\Gamma_+}f(\mu,\lambda)\overline{g(\mu,\lambda)}\,d\gamma_\alpha(\mu,\lambda).$$

Proposition 2.1. (i) *For all nonnegative measurable functions g on Γ_+, we have*

$$\int_{\Gamma_+}g(\mu,\lambda)\,\mathrm{d}\gamma_\alpha(\mu,\lambda)=\frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\bigg(\int_{\mathbb{R}}\int_0^\infty g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha\mu\,\mathrm{d}\mu\,\mathrm{d}\lambda+\int_{\mathbb{R}}\int_0^{|\lambda|}g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha\mu\,\mathrm{d}\mu\,\mathrm{d}\lambda\bigg).$$

(ii) *For all measurable functions f on K, the function $\Theta \circ f$ is measurable on Γ_+. Furthermore, if f is a nonnegative or integrable function on K with respect to the measure $d\upsilon_\alpha$, then we have*

$$\int_{\Gamma_+} (\Theta \circ f)(\mu, \lambda)\, \mathrm{d}\gamma_\alpha(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}} f(r, x)\, \mathrm{d}\upsilon_\alpha(r, x). \tag{4}$$

Now we give the definition of the generalized Fourier transform associated with the Riemann–Liouville operator and some relevant properties.

Definition 2.2. For $f \in L^1(d\upsilon_\alpha)$, the Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann–Liouville operator is defined by

$$\mathscr{F}_\alpha(f)(\mu, \lambda) = \int_K f(r, x)\varphi_{\mu,\lambda}(r, x)\, \mathrm{d}\upsilon_\alpha(r, x) \quad \text{for } (\mu, \lambda) \in \Gamma_+.$$

For this generalized Fourier transform, we have an inversion formula and an Plancherel theorem, just as with the classical Fourier transform in Euclidean space.

Theorem 2.3 (inversion formula). *Let $f \in L^1(\mathrm{d}\upsilon_\alpha)$ such that $\mathscr{F}_\alpha(f) \in L^1(\mathrm{d}\gamma_\alpha)$. Then for almost every $(r, x) \in K$, we have*

$$f(r, x) = \int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu, \lambda)\overline{\varphi_{\mu,\lambda}(r, x)}\, \mathrm{d}\gamma_\alpha(\mu, \lambda).$$

Theorem 2.4 (Plancherel). *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isomorphism from $L^2(\mathrm{d}\upsilon_\alpha)$ onto $L^2(\mathrm{d}\gamma_\alpha)$. In particular, for all $f, g \in L^2(\mathrm{d}\upsilon_\alpha)$, we have a version of Parseval's equality*:

$$\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu, \lambda)\, \overline{\mathscr{F}_\alpha(g)(\mu, \lambda)}\, \mathrm{d}\gamma_\alpha(\mu, \lambda) = \int_K f(r, x)\overline{g(r, x)}\, \mathrm{d}\upsilon_\alpha(r, x).$$

The next two important lemmas will be used later in our proof.

Lemma 2.5. *For $m \in \mathbb{N}$, let*

$$\Phi_m(r) = \sqrt{\frac{2^{\alpha+1}\Gamma(\alpha+1)\, m!}{\Gamma(\alpha+m+1)}}\, e^{-r^2/2}\, L_m^\alpha(r^2).$$

The family $\{\Phi_m(r)\}_{m\in N}$ forms an orthonormal basis of the space

$$L^2(\mathbb{R}_+, r^{2\alpha+1}/(2^\alpha\Gamma(\alpha+1))\, \mathrm{d}r)$$

where $L_m^\alpha(x)$ is the Laguerre polynomial of degree m and order α defined by the expansion [Stempak 1988]

$$\sum_{n=0}^\infty t^n L_n^\alpha(x) = \frac{1}{(1-t)^{\alpha+1}} e^{xt/(t-1)}.$$

For the polynomial $L_m^\alpha(x)$, from [Huang and Liu 2007b], we also have the explicit expression for $L_m^\alpha(x)$:

$$L_m^\alpha(x) = \sum_{j=0}^{m} \frac{\Gamma(m+\alpha+1)}{\Gamma(m-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!}.$$

From the explicit expression of the Laguerre polynomial of degree m and order α, we know that there exists a function $M : \mathbb{N} \to \mathbb{R}_+$ such that for each $m \in \mathbb{N}$, we have $|\Phi_m(x)| \leq M(m)$. The essence of this claim is that the polynomial doesn't grow as rapid as the exponential function when r approaches infinity.

Lemma 2.6 [Omri and Rachdi 2008, page 9]. *For all $m \in N$,*

$$\int_0^\infty e^{-r/2} L_m^\alpha(r) J_\alpha(\sqrt{ry}) r^{\alpha/2}\, \mathrm{d}r = (-1)^m 2\, e^{-y/2} y^{\alpha/2} L_m^\alpha(y).$$

We make the variable replacements $r = a^2$, $y = b^2$, but for simplicity we still use r and y instead of a, b. Then

$$\int_0^\infty e^{-r^2/2} L_m^\alpha(r^2) J_\alpha(ry) r^{\alpha+1}\, \mathrm{d}r = (-1)^m\, e^{-y^2/2} y^\alpha L_m^\alpha(y^2),$$

that is,

$$\int_0^\infty J_\alpha(ry)\, r^{\alpha+1}\, \Phi_m(r)\, \mathrm{d}r = (-1)^m y^\alpha\, \Phi_m(y). \tag{5}$$

3. Proof of the main result

In this section, we will prove Theorem 1.2. From the definition of the generalized Fourier transform, we know that

$$\mathcal{F}_\alpha(f)(\mu, \lambda) = \int_K f(r, x) \varphi_{\mu,\lambda}(r, x)\, \mathrm{d}\nu_\alpha(r, x).$$

Replace $\varphi_{\mu,\lambda}(r, x)$ by the expression in (2) to get

$$\mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^\infty \int_R f(r, x) j_\alpha(r\sqrt{\mu^2+\lambda^2})\, e^{-i\lambda x}\, \mathrm{d}x\, \mathrm{d}c(r)$$

If we let

$$\widetilde{\mathcal{F}_\alpha(f)}(\mu, \lambda) = \int_0^\infty \int_R f(r, x) j_\alpha(r\mu)\, e^{-i\lambda x}\, \mathrm{d}x\, \mathrm{d}c(r),$$

then $\mathcal{F}_\alpha(f)(\mu, \lambda) = (\Theta \circ \widetilde{\mathcal{F}_\alpha(f)})(\mu, \lambda)$. Thus our condition,

$$\int_K \int_{\Gamma_+} \frac{|f(r, x)|\, |\mathcal{F}_\alpha(f)(\mu, \lambda)|\, e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\, \mathrm{d}\nu_\alpha(r, x)\ \mathrm{d}r_\alpha(\mu, \lambda) < \infty,$$

is equivalent to

$$\int_K \int_K \frac{|f(r,x)|\,|\widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)|\,e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}v_\alpha(r,x)\,\mathrm{d}v_\alpha(\mu,\lambda)<\infty$$

by (4) (see Proposition 2.1). Defining

$$f^\lambda(r)=\int_{\mathbb{R}} f(r,x)\,e^{-i\lambda x}\,dx \quad \text{and} \quad f_m(x)=\int_0^\infty f(r,x)\Phi_m(r)\,\mathrm{d}c(r),$$

we obtain

$$\widehat{f_m}(\lambda)=\int_0^\infty f^\lambda(r)\Phi_m(r)\,\mathrm{d}c(r).$$

Before we proceed, we first prove the following useful formula:

$$\left|\int_0^\infty \widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)\Phi_m(\mu)\,\mathrm{d}c(\mu)\right|=\frac{1}{\sqrt{2\pi}}\,|\widehat{f_m}(\lambda)|. \tag{6}$$

Indeed,

$$\begin{aligned}(7)\quad &\int_0^\infty \widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)\Phi_m(\mu)\,\mathrm{d}c(\mu)\\ &=\int_0^\infty\int_0^\infty\int_{\mathbb{R}} f(r,x)e^{-i\lambda x}j_\alpha(r\mu)\Phi_m(\mu)\,\mathrm{d}x\,\mathrm{d}c(r)\,\mathrm{d}c(\mu)\\ &=2^\alpha\,\Gamma(\alpha+1)\int_0^\infty\int_0^\infty f^\lambda(r)\frac{J_\alpha(r\mu)}{(r\mu)^\alpha}\frac{\mu^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Phi_m(\mu)\,\mathrm{d}\mu\,\mathrm{d}c(r).\end{aligned}$$

By (5) (see Lemma 2.6), we know that the right-hand side equals

$$\frac{(-1)^m}{\sqrt{2\pi}}\int_0^\infty f^\lambda(r)\Phi_m(r)\,\mathrm{d}c(r)=\frac{(-1)^m}{\sqrt{2\pi}}\,\widehat{f_m}(\lambda),$$

which proves the claim.

We also need to prove the function $f(r,x)$ is in $L^1(\mathrm{d}v_\alpha)$. Since

$$\int_{\Gamma_+}\int_K \frac{|f(r,x)|\,|\mathscr{F}_\alpha(f)(\mu,\lambda)|\,e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}v_\alpha(r,x)\,\mathrm{d}r_\alpha(\mu,\lambda)<\infty, \tag{8}$$

there must exist a $\lambda_0\in\mathbb{R}$ such that

$$\int_K \frac{|f(r,x)|e^{|x||\lambda_0|}}{(1+|x|+|\lambda_0|)^N}\,\mathrm{d}v_\alpha(r,x)<+\infty.$$

Since there exists a constant $C>0$ such that $(1+|x|+|\lambda_0|)^N<Ce^{|x||\lambda_0|}$ for all $x\in\mathbb{R}$, we obtain

$$\int_K |f(r,x)|\,\mathrm{d}v_\alpha(r,x)<\frac{1}{C}\int_K \frac{|f(r,x)|e^{|x||\lambda_0|}}{(1+|x|+|\lambda_0|)^N}\,\mathrm{d}v_\alpha(r,x)<+\infty,$$

that is, $f(r,x)\in L^1(\mathrm{d}v_\alpha(r,x))$.

To proceed, we first prove that for any $m, n \in \mathbb{N}$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)||\widehat{f_n}(\lambda)|e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}x\,\mathrm{d}\lambda < +\infty. \tag{9}$$

Since

$$|f_m(x)| = \left|\int_0^\infty f(r,x)\,\Phi_m(r)\,\mathrm{d}c(r)\right| \leqq M(m)\int_0^\infty |f(r,x)|\,\mathrm{d}c(r)$$

and

$$\begin{aligned}|\widehat{f_n}(\lambda)| &= \sqrt{2\pi}\left|\int_0^\infty \widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)\Phi_m(\mu)\,\mathrm{d}c(\mu)\right|\\ &\leqq \sqrt{2\pi}\,M(n)\int_0^\infty |\widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)|\,\mathrm{d}c(\mu),\end{aligned}$$

we have, for any $m, n \in \mathbb{N}$,

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)||\widehat{f_n}(\lambda)|e^{|x|\,|\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}x\,\mathrm{d}\lambda\\ &\quad\leqq \sqrt{2\pi}M(m)M(n)\int_K\int_K \frac{|f(r,x)||\widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)\,e^{|x|\,|\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}\nu_\alpha(r,x)\,\mathrm{d}\nu_\alpha(\mu,\lambda)\\ &\quad= \sqrt{2\pi}M(m)M(n)\int_K\int_{\Gamma_+} \frac{|f(r,x)||\mathscr{F}_\alpha(f)(\mu,\lambda)|\,e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}\nu_\alpha(r,x)\,\mathrm{d}\gamma_\alpha(\mu,\lambda)\\ &\quad< +\infty.\end{aligned}$$

In particular, setting $m = n$, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)||\widehat{f_m}(\lambda)|\,e^{|x||\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}x\,\mathrm{d}\lambda < +\infty.$$

Then by Lemma 1.1 (in this case $d = 1$), we have

$$f_m(x) = P_m(x)\,e^{-a_m x^2},$$

where a_m is positive and $P_m(x)$ is a polynomial with degree less than $(N-1)/2$. Further we claim that for all $m \in \mathbb{N}$, we have $a_m = a_n = a$. This holds since if there exist $m, n \in \mathbb{N}$ such that $a_m \neq a_n$, then the equation

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)||\widehat{f_n}(\lambda)|e^{|x|\,|\lambda|}}{(1+|x|+|\lambda|)^N}\,\mathrm{d}x\,\mathrm{d}\lambda < +\infty$$

cannot hold, since it is in contradiction with the same equation derived by exchanging subscripts, which must be equally true. So, by Lemma 2.5,

$$f(r,x) = \sum_{j=0}^{\infty} f_m(x)\,\Phi_m(r) = e^{-ax^2}\left(\sum_{i=0}^{k}\psi_i(r)\,x^i\right),$$

where $k < \dfrac{N-1}{2}$ and

$$\psi_i(r) \in L^2\Big([0,+\infty), \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)}\,\mathrm{d}r\Big).$$

Thus when $N < 3$ we have $f(r,x) = e^{-ax^2}\psi(r)$. In particular, when $N < 1$ we know that $f = 0$, since $f_m(x) = 0$ for each $m \in \mathbb{N}$. This finishes the proof of Theorem 1.2.

4. Some other versions of the uncertainty principle

We now derive other versions of the uncertainty principle as corollaries of our theorem. We start with a Gelfand–Shilov type uncertainty principle, which it is relatively straightforward to prove using Hölder's inequality and reduction to the absurd.

Theorem 4.1 (Gelfand–Shilov type). *Let $N \geqq 0$ and assume $f \in L^2(K, \mathrm{d}\upsilon_\alpha(r,x))$ satisfies*

$$\int_K \frac{|f(r,x)|\, e^{(a^p/p)|x|^p}}{(1+|x|)^N}\,\mathrm{d}\upsilon_\alpha(r,x) < +\infty,$$

$$\int_{\Gamma_+} \frac{|\mathscr{F}_\alpha(f)(\mu,\lambda)|\, e^{(b^q/q)|\lambda|^q}}{(1+|\lambda|)^N}\,\mathrm{d}\gamma_\alpha(\mu,\lambda) < +\infty,$$

where $1 < p, q < \infty$ satisfy $1/p + 1/q = 1$, and a, b are positive numbers such that $ab \geqq 1$. Then $f = 0$ unless $p = q = 2$, $ab = 1$ and $N > 0$, and in this case, we have

$$f(r,x) = e^{-ax^2}\Big(\sum_{j=0}^{m} \varphi_j(r)\, x^j\Big),$$

where $\varphi_j(r) \in L^2(\mathbb{R}_+, \mathrm{d}c(r))$ and $m \le N-1$. In particular, when $N \leqq 1$,

$$f(r,x) = e^{-(a^2/2)\,x^2}\,\psi(r),$$

where $\psi(r) \in L^2(\mathbb{R}_+, \mathrm{d}c(r))$, and when $N < 1$, we have $f = 0$.

Proof. Following the same procedure as in the proof of Theorem 1.2, we derive

$$\int_{\mathbb{R}} \frac{|f_m(x)|\, e^{(a^p/p)|x|^p}}{(1+|x|)^N}\,\mathrm{d}x < \infty, \qquad \int_{\mathbb{R}} \frac{|\widehat{f_m}(\lambda)|\, e^{(b^q/q)|\lambda|^q}}{(1+|\lambda|)^N}\,\mathrm{d}\lambda < \infty.$$

From Hölder's inequality, we have

$$a\,|x|\,b\,|\lambda| \le \frac{a^p\,|x|^p}{p} + \frac{b^q\,|\lambda|^q}{q}.$$

Then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)|\,|\widehat{f_m}(\lambda)|\,e^{ab\,|x|\,|\lambda|}}{(1+|x|+|\lambda|)^{2N}}\,\mathrm{d}x\,\mathrm{d}\lambda \le \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)|\,e^{(a^p/p)|x|^p}}{(1+|x|)^N}\,\frac{|\widehat{f_m}(\lambda)|\,e^{(b^q/q)|\lambda|^q}}{(1+|\lambda|)^N}\,\mathrm{d}x\,\mathrm{d}\lambda < \infty.$$

So, when $ab > 1$, we could first derive the exact form of the function $f_m(x)$ from the Beurling theorem. We then know that with this form for $f_m(x)$, the inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_m(x)|\,|\widehat{f_m}(\lambda)|\,e^{ab\,|x|\,|\lambda|}}{(1+|x|+|\lambda|)^{2N}}\,\mathrm{d}x\,\mathrm{d}\lambda < \infty$$

cannot hold if $f_m(x) \neq 0$. When $ab = 1$ and either $p > 2$ or $q > 2$, also from the Beurling theorem, $f_m(x)$ is the product of polynomial and e^{-cx^2}. We deduce that the inequality

$$\int_{\mathbb{R}} \frac{|f_m(x)|\,e^{(a^p/p)|x|^p}}{(1+|x|)^N}\,\mathrm{d}x < \infty$$

cannot hold when $p > 2$ and the inequality

$$\int_{\mathbb{R}} \frac{|\widehat{f_m}(\lambda)|\,e^{(b^q/q)|\lambda|^q}}{(1+|\lambda|)^N}\,\mathrm{d}\lambda < \infty$$

cannot hold when $q > 2$, if $f_m(x) \neq 0$.

The conclusion in the last possible case, when $ab = 1$ and $p = q = 2$, can be derived from the Beurling theorem directly. □

Following the same idea as in Section 3, we can derive a Morgan-type theorem, which also gives a sharp lower bound for the Gelfand–Shilov type uncertainty principle:

Theorem 4.2. *Let $f \in L^2(K, \mathrm{d}\nu_\alpha(r,x))$ and suppose f satisfies*

$$\int_K |f(r,x)|\,e^{a^p|x|^p/p}\,\mathrm{d}\nu_\alpha(r,x) < \infty, \quad \int_{\Gamma_+} |\mathscr{F}_\alpha(f)(\mu,\lambda)|\,e^{b^q|\lambda|^q/q}\,\mathrm{d}\gamma_\alpha(\mu,\lambda) < \infty,$$

where $1 < p < 2$, $1/p + 1/q = 1$, and a, b are positive numbers. Then $f = 0$ if $ab > |\cos(p\pi/2)|^{1/p}$.

Proof. By the same argument as in the proof of our main theorem, we have

$$\int_{\mathbb{R}} |f_m(x)|\,e^{a^p|x|^p/p}\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}} |\widehat{f_m}(\lambda)\,e^{b^q|\lambda|^q/q}\,\mathrm{d}\lambda < \infty.$$

Then [Bonami et al. 2003, Theorem 1.4], under the condition $ab > |\cos(p\pi/2)|^{1/p}$, implies that $f_m(x) = 0$ for each m, so we have $f(r,x) = 0$. □

Theorem 4.3 (Hardy type). *Suppose* $f \in L^2(K, \mathrm{d}\upsilon_\alpha(r,x))$ *satisfies*

$$|f(r,x)| \leqq C_1 e^{-a(r^2+x^2)} \quad \textit{and} \quad |\mathscr{F}_\alpha(f)(\mu,\lambda)| \leqq C_2 e^{-b(\mu^2+\lambda^2)},$$

where C_1, C_2 *are positive constants and* a, b *are positive real numbers such that* $ab \geqq \frac{1}{4}$. *If* $ab > \frac{1}{4}$, *then* $f = 0$. *If* $ab = \frac{1}{4}$, *then*

$$f(r,x) = e^{-ax^2}\psi(r),$$

where $\psi(r) \in L^2(\mathbb{R}_+, \mathrm{d}c(r))$.

Proof. To prove this corollary, we recall the well-known classical Hardy's theorem for the classical Fourier transform on $\mathbb{R}$ which says that if

$$|f(x)| \leqq Ce^{-ax^2} \quad \text{and} \quad \widehat{f}(\lambda) \leqq Ce^{-b\lambda^2},$$

where $\widehat{f}$ is the Fourier transform of f, then

(i) $f = 0$ when $ab > \frac{1}{4}$;

(ii) $f(x) = ce^{-ax^2}$ when $ab = \frac{1}{4}$;

(iii) there are infinitely many linearly independent functions satisfying the above conditions when $ab < \frac{1}{4}$.

From the conditions in the corollary and using the same method used in Section 3, we have

$$|f_m(x)| \leqq C\, e^{-ax^2} \quad \text{and} \quad |\widehat{f_m}(\lambda)| \leqq C\, e^{-b\lambda^2}.$$

So from the classical Hardy's theorem, we have $f_m(x) = c_m\, e^{-ax^2}$ if $ab = \frac{1}{4}$ for each $m \in N$. Then

$$f(r,x) = e^{-ax^2}\left(\sum_{m=0}^{\infty} c_m\, \Phi_m(r)\right) = e^{-ax^2}\,\psi(r),$$

where $\psi(r) \in L^2(\mathbb{R}_+, \mathrm{d}\,c(r))$. When $ab > \frac{1}{4}$, each $f_m(x)$ vanishes, so we have $f(r,x) = 0$. □

Theorem 4.4 (Morgan type). *Suppose* $f \in L^2(K, \mathrm{d}\upsilon_\alpha(r,x))$ *satisfies*

$$\int_0^\infty |f(r,x)|\, r^{2\alpha+1}\,\mathrm{d}r \leqq C_1 e^{-a|x|^p}, \quad \int_0^\infty |\widetilde{\mathscr{F}_\alpha(f)}(\mu,\lambda)|\, \mu^{2\alpha+1}\,\mathrm{d}\mu \leqq C_2 e^{-b|\lambda|^q},$$

where C_1, C_2 *are positive constants*, $1 < p < 2$, $1/p + 1/q = 1$, *and* a, b *are positive numbers. Then* $f = 0$ *if* $(a\,p)^{1/p}(b\,q)^{1/q} > |\cos(p\pi/2)|^{1/p}$.

Proof. First let $a = \alpha^p/p$ and $b = \beta^q/q$. Then

$$\alpha\,\beta > |\cos(p\,\pi/2)|^{1/p}.$$

There exists an $\epsilon > 0$, such that $(\alpha-\epsilon)(\beta-\epsilon) > |\cos(p\pi/2)|^{1/p}$ also holds. Then

$$\int_{\mathbb{R}} |f_m(x)|\, e^{(\alpha-\epsilon)^p|x|^p/p}\, dx < M(m)\int_{\mathbb{R}} e^{-(\alpha^p-(\alpha-\epsilon)^p)/p\,|x|^p}\, dx < \infty,$$

$$\int_{\mathbb{R}} |\widehat{f_m}(\lambda)\, e^{(\beta-\epsilon)^q|\lambda|^q/q}\, d\lambda < M(m)\int_{\mathbb{R}} e^{-(\beta^q-(\beta-\epsilon)^q)/q\,|\lambda|^q}\, d\lambda < \infty.$$

By [Bonami et al. 2003, Theorem 1.4], we have $f_m(x) = 0$ for each $m \in \mathbb{N}$, so $f = 0$. □

5. More on this topic

We now derive a sharper result than the main theorem, requiring an additional constraint on the function $f(r, x)$.

First we introduce some related notation and propositions about the dual of the Riemann–Liouville operator. For more details, refer to [Baccar et al. 2006]. Let $\mathscr{C}_*(\mathbb{R}^2)$ be the function space of continuous functions on $\mathbb{R}^2$ even with respect to the first variable, and $\mathscr{S}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives even with respect to the first variable. The dual Riemann–Liouville operator (or transform) is defined by

$$\int_0^\infty\int_{\mathbb{R}} \mathscr{R}_\alpha(f)(r,x)g(r,x)\, dx\, r^{2\alpha+1}\, dr = \int_0^\infty\int_{\mathbb{R}} f(r,x)\, {}^t\mathscr{R}_\alpha(g)(r,x)\, dx\ r^{2\alpha+1}\, dr,$$

where $f \in \mathscr{C}_*(\mathbb{R}^2)$ and $g \in \mathscr{S}_*(\mathbb{R}^2)$. This is also why ${}^t\mathscr{R}_\alpha$ called the "dual". We also have for $f \in \mathscr{S}_*(\mathbb{R}^2)$,

$${}^t\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{2\alpha}{\pi}\displaystyle\int_r^\infty \int_{-\sqrt{\mu^2-r^2}}^{\sqrt{\mu^2-r^2}} f(u, x+v)(\mu^2-v^2-r^2)^{\alpha-1}\, dv\, \mu\, d\mu & \\ & \text{if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} f\big(\sqrt{r^2+(x-y)^2}, y\big)\, dy & \text{if } \alpha = 0. \end{cases}$$

Some propositions related to the dual Riemann–Liouville transform are needed before going to our main result in this section.

Lemma 5.1 [Baccar et al. 2006, Lemma 3.6, page 9]. *For $f \in \mathscr{S}_*(\mathbb{R}^2)$,*

$$\mathscr{F}_\alpha(f)(\mu.\lambda) = \wedge_\alpha \circ {}^t\mathscr{R}_\alpha(f)\,(\mu,\lambda) \quad \textit{for } (\mu,\lambda) \in \mathbb{R}^2,$$

where $\wedge_\alpha$ is a constant multiple of the classical Fourier transform on $\mathbb{R}^2$ defined by

$$\wedge_\alpha(f)(\mu,\lambda) = \int_0^\infty\int_{\mathbb{R}} f(r,x)\ \cos(r\mu)\exp(-i\lambda x)\, \frac{1}{\sqrt{2\pi}\,2^\alpha\Gamma(\alpha+1)}\, dx\, dr.$$

Lemma 5.2 [Baccar et al. 2006, Proposition 3.7]. (i) *${}^t\mathscr{R}_\alpha$ is not injective when applied to $\mathscr{S}_*(\mathbb{R}^2)$.*

(ii) ${}^t\mathscr{R}_\alpha(\mathscr{S}_*(R^2)) = \mathscr{S}_*(\mathbb{R}^2)$.

To proceed, we still need to define two special subspaces of $\mathscr{S}_*(\mathbb{R}^2)$. Denote by $\mathscr{S}_*^0(\mathbb{R}^2)$ the subspace of $\mathscr{S}_*(\mathbb{R}^2)$ consisting of functions f such that

$$\operatorname{supp} \widetilde{\mathscr{F}_\alpha(f)} \subset \{(\mu, \lambda) \in \mathbb{R}^2 : |\mu| \geqq |\lambda|\}.$$

Denote by $\mathscr{S}_{*,0}(\mathbb{R}^2)$ the subspace of $\mathscr{S}_*(\mathbb{R}^2)$ consisting of functions f such that

$$\int_0^\infty f(r, x)\, r^{2k}\, \mathrm{d}r = 0 \quad \text{for all } k \in N \text{ and } x \in \mathbb{R}.$$

From Lemma 5.2, we know that ${}^t\mathscr{R}_\alpha$ is not a isomorphism between $\mathscr{S}_*(\mathbb{R}^2)$ and $\mathscr{S}_*(\mathbb{R}^2)$. But things are different on the subspace $\mathscr{S}_*^0(\mathbb{R}^2)$. We have the isomorphism lemma as well as inversion formula for the operator ${}^t\mathscr{R}_\alpha$.

Lemma 5.3. *The dual transform ${}^t\mathscr{R}_\alpha$ is an isomorphism from $\mathscr{S}_*^0(\mathbb{R}^2)$ onto $\mathscr{S}_{*,0}(\mathbb{R}^2)$.*

Lemma 5.4 [Baccar et al. 2006, Theorems 4.5 and 4.6]. *For $g \in \mathscr{S}_{*,0}(\mathbb{R}^2)$ the inversion formula*

$$({}^t\mathscr{R}_\alpha)^{-1}(g) = (K_\alpha^2 \circ \mathscr{R}_\alpha)(g)$$

holds for ${}^t\mathscr{R}_\alpha$, where $\mathscr{R}_\alpha$ is the Riemann–Liouville operator defined in Section 1 and the operator K_α^2 is defined by

$$K_\alpha^2(g)(r, x) = \mathscr{F}_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)}(\mu^2+\lambda^2)^\alpha |\mu| \mathscr{F}_\alpha(g)\right)(r, x).$$

Also K_α^2 is an isomorphism from $\mathscr{S}_*^0(\mathbb{R}^2)$ onto itself.

With the help of these lemmas, we derive our new analogue:

Theorem 5.5. *Suppose $f \in \mathscr{S}_*^0(\mathbb{R}^2)$ satisfies*

$$\int_K \int_{\Gamma_+} \frac{|f(r, x)|\, |\mathscr{F}_\alpha(f)(\mu, \lambda)|\, e^{\|(r,x)\| \|(\mu,\lambda)\|}\, \Xi(\mu, \lambda)}{(1 + \|(r, x)\| + \|(\mu, \lambda)\|)^N}\, \mathrm{d}\gamma_\alpha(\mu, \lambda)\, \mathrm{d}v_\alpha(r, x) < \infty.$$

Then

$$f(r, x) = ({}^t\mathscr{R}_\alpha)^{-1}\big(P(y)\, e^{-\langle Ay, y\rangle}\big),$$

where $y = (r, x)$, $P(y)$ is a polynomial with degree less than $(N-2)/2$, A is a real positive definite symmetric 2×2 matrix, $\|\cdot\|$ is the usual norm in $\mathbb{C}^n$, and $\Xi(\mu, \lambda)$ is defined by

$$\Xi(\mu, \lambda) = \frac{1}{(\mu^2+\lambda^2)^\alpha |\mu|}.$$

In particular, when $N \leqq 2$, we have $f = 0$.

Proof. We first prove that for all $(\mu, \lambda) \in \mathbb{R}^2$, there exists $C > 0$ such that

$$\int_K \frac{|{}^t\mathcal{R}_\alpha(f)(r,x)|\,|\mathcal{F}_\alpha(f)(\mu,\lambda)|\,e^{\|(r,x)\|\|(\mu,\lambda\|)}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}r\,\mathrm{d}x \leqq C\int_K \frac{|f(r,x)||\mathcal{F}_\alpha(f)(\mu,\lambda)\,e^{\|(r,x)\|\,\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}\nu_\alpha(r,x).$$

We first consider the case when $\alpha > 0$; then

$${}^t\mathcal{R}_\alpha(f)(r,x) = \frac{2\alpha}{\pi}\int_r^\infty \int_{-\sqrt{\mu^2-r^2}}^{\sqrt{\mu^2-r^2}} f(\mu, x+v)(\mu^2-v^2-r^2)^{\alpha-1}\,\mathrm{d}v\,\mu\,\mathrm{d}\mu.$$

So we have

$$\begin{aligned}\int_K &\frac{|{}^t\mathcal{R}_\alpha(f)(r,x)|e^{\|(r,x)\|\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}r\,\mathrm{d}x\\ &=\frac{2\alpha}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{\left|\int_r^\infty\int_{\sqrt{\mu^2-r^2}}^{\sqrt{\mu^2-r^2}} f(\mu,x+v)(\mu^2-v^2-r^2)^{\alpha-1}\,\mathrm{d}v\,\mu\,\mathrm{d}\mu\right| e^{\|(r,x)\|\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}x\,\mathrm{d}r\\ &=\frac{2\alpha}{\pi}\int_0^\infty\int_{\mathbb{R}}\int_r^\infty\int_{\sqrt{\mu^2-r^2}}^{\sqrt{\mu^2-r^2}} \frac{|f(\mu,x+v)|(\mu^2-v^2-r^2)^{\alpha-1}e^{\|(r,x)\|\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}v\,\mu\,\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}r.\end{aligned}$$

Changing variables, let $\mu = \mu$, $b = x + v$, $r = r$, $x = x$. For simplicity we will still use v instead of b. Then by a change of variables and integration, we see that the right-hand side above is bounded above by

$$\leqq C_1\int_0^\infty\int_{\mathbb{R}} \frac{|f(r,x)|\,e^{\|(r,x)\|\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N} r^{2\alpha+1}\,\mathrm{d}r\,\mathrm{d}x \leqq C_2\,e\int_K \frac{|f(r,x)|\,e^{\|(r,x)\|\|(\mu,\lambda)\|}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}\nu_\alpha(r,x).$$

For the case $\alpha = 0$, our previous claim also holds by using the same method as in the case $\alpha > 0$, using a different variable replacement by letting $a = \sqrt{r^2 + (x-y)^2}$, $y = y$, and for simplicity still using r instead of a. This proves our claim.

By Proposition 2.1(i), and restricting the integral region Γ_+ to K, we derive the inequality

$$\begin{aligned}&\int_K\int_K \frac{|{}^t\mathcal{R}_\alpha(f)(r,x)|\,|\mathcal{F}_\alpha(f)(\mu,\lambda)|\,e^{\|(r,x)\|\|(\mu,\lambda\|)}}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}\mu\,\mathrm{d}\lambda\\ &\leqq C\times\int_K\int_{\Gamma_+} \frac{|f(r,x)|\,|\mathcal{F}_\alpha(f)(\mu,\lambda)|\,e^{\|(r,x)\|\|(\mu,\lambda\|)}\;\Xi(\mu,\lambda)}{(1+\|(r,x)\|+\|(\mu,\lambda)\|)^N}\,\mathrm{d}\nu_\alpha(r,x)\,\mathrm{d}\gamma_\alpha(\mu,\lambda)\\ &<\infty.\end{aligned}$$

By Lemma 5.1 we know that the above inequality satisfies the conditions of the Beurling theorem (Lemma 1.1) in 2-dimensional Euclidean space. So

$$^{t}\mathcal{R}_\alpha(f)(r,x) = P(y)\, e^{-\langle Ay,y\rangle},$$

where $y=(r,x)$, $P(y)$ is a polynomial such that its degree is less than $(N-2)/2$, and A is a positive definite symmetric 2×2 matrix. From $f\in\mathcal{S}_*^0(\mathbb{R}^2)$ and Lemma 5.3 we know that $P(y)\, e^{-\langle Ay,y\rangle}\in\mathcal{S}_{*,0}(\mathbb{R}^2)$ and

$$f(r,x) = ({}^{t}\mathcal{R}_\alpha)^{-1}\big(P(y)\, e^{-\langle Ay,y\rangle}\big).$$

In particular, if $N\leqq 2$, we have

$$^{t}\mathcal{R}_\alpha(f)(r,x) = 0,$$

which implies $f(r,x)=0$ so our proof is finished. □

Remark. In this section, we gave another analogue of the Beurling–Hörmander theorem. When compared with Theorem 1.2, which just gives the precise structure of x but not r since we only know that $\psi_j(r)\in L^2(\mathbb{R}_+, \mathrm{d}c(r))$, the new analogue derived in this section gives the precise structure of both r and x. However, this requires the additional condition that $f\in\mathcal{S}_*^0(\mathbb{R}^2)$ and it's difficult to remove this condition because the dual Riemann–Liouville transform is not injective on the full space $\mathcal{S}_*(\mathbb{R}^2)$. To conquer this difficulty, a different method might be needed.

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