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We study the problem of finding complete conformal metrics determined by some symmetric function of the modified Schouten tensor on compact manifolds with boundary; which reduces to a Dirichlet problem. We prove the existence of the solution under some suitable conditions. In particular, we prove that every smooth compact n-dimensional manifold with boundary, with $n \geq 3$, admits a complete Riemannian metric g whose Ricci curvature Ric_g and scalar curvature R_g satisfy

$$\det(\mathrm{Ric}_g - R_g g) = \text{const.}$$

This result generalizes Aviles and McOwen's in the scalar curvature case.

1. Introduction

Let $(\overline{M}^n, g)$, for $n \geq 3$, be a compact, n-dimensional smooth Riemannian manifold with smooth boundary ∂M. Let $M = \overline{M} \setminus \partial M$ be the interior of $\overline{M}$, and denote the Ricci tensor and the scalar curvature by Ric and R (or Ric_g and R_g to emphasize the metric), respectively. In [2003], Gursky and Viaclovsky introduced the *modified Schouten tensor*

$$A_g^\tau := \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{\tau}{2(n-1)} R_g g\Big),$$

where $\tau \in \mathbb{R}$. We are interested in deforming the metric in the conformal class $[g]$ of a fixed back ground metric g to certain *complete* metric $\bar{g}$ satisfying

$$\det(\bar{g}^{-1} A_{\bar{g}}^\tau) = \text{const in } M.$$

More generally, let Γ^+ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma^+ \subset \Gamma_1^+$, where

$$\Gamma_k^+ = \{\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n \mid \sigma_j(\kappa) > 0, 1 \leq j \leq k\},$$

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and

$$\sigma_k(\kappa) = \sum_{i_1<\cdots<i_k} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth symmetric function that satisfies some structure conditions in Γ^+, to be listed later. We ask, Does there exist a complete metric $\bar{g}$ in the conformal class $[g]$ such that

$$F(\bar{g}^{-1} A^\tau_{\bar{g}}) = f(x) \quad \text{in } M \tag{1-1}$$

for some given smooth function $f \in C^\infty(\overline{M})$? In this paper, we give a partial answer in the case $\tau > n-1$. We remark that, if $F = \sigma_1$, then (1-1) becomes

$$\frac{(2-\tau)n-2}{2(n-1)(n-2)} R_{\bar{g}} = f(x).$$

In the case $\tau > n-1$ and $f(x)$ is positive, some results have appeared in [Aviles and McOwen 1988].

To find a complete conformal metric satisfying (1-1), we need to solve the Dirichlet problem for (1-1) with larger and larger boundary data. We first write this curvature equation as a partial differential equation. Recall the following formula for the transformation of A^τ under a conformal change of metric $\bar{g} = e^{2u} g$:

$$A^\tau_{\bar{g}} = \frac{\tau-1}{n-2}(\Delta u)g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2}|\nabla u|^2 g + A^\tau_g. \tag{1-2}$$

From (1-2) we may write (1-1) as

$$F\Big(\frac{\tau-1}{n-2}(\Delta u)g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2}|\nabla u|^2 g + A^\tau_g\Big) = f(x)e^{2u}.$$

In this paper, we study a more general equation. Let $h(x,z) : \overline{M}^n \times \mathbb{R}$ be some smooth positive function. Let's consider

$$F(\lambda(\Delta u)g - \nabla^2 u + a(x)du \otimes du + b(x)|\nabla u|^2 g + B) = h(x,u), \tag{1-3}$$

where $\lambda > 1$, B is a symmetric 2-tensor, and $a(x)$ and $b(x)$ are smooth functions on $\overline{M}$. Suppose F is homogeneous of degree one, $F = 0$ on $\partial\Gamma^+$, and F satisfies the following in Γ^+:

(C1) F is positive;

(C2) F is concave (that is, $\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j}$ is negative semidefinite);

(C3) F is monotone (that is, $\frac{\partial F}{\partial \kappa_i}$ is positive).

For convenience, we define

$$W[u] := \nabla^2_{\mathrm{conf}} u + B,$$

and

$$\nabla^2_{\mathrm{conf}} u = \lambda(\Delta u)g - \nabla^2 u + a\,du \otimes du + b|\nabla u|^2 g$$

in the sequel. We call u is *admissible* if $g^{-1}W[u] \in \Gamma^+$.

Theorem 1.1. *For $n \geq 3$, let $(\overline{M}^n, g)$ be a smooth, compact Riemannian manifold with boundary ∂M. If*

1. $B \in \Gamma^+$;
2. $h > 0$ *on* $\overline{M} \times \mathbb{R}$, $\partial_z h(x,z) > 0$ *on* $\overline{M} \times \mathbb{R}$, $\lim_{z\to+\infty} h(x,z) \to +\infty$ *and* $\lim_{z\to-\infty} h(x,z) \to 0$ *in* $M \times \mathbb{R}$; *and*
3. $a(x)$ *is positive on* $\overline{M}$ *and* $\lambda a(x) + b(x)$ *is nonnegative in* M,

then there exists a unique admissible function $u \in C^\infty(\overline{M})$ solving the Dirichlet problem

$$\begin{cases} F(W[u]) = h(x,u) & \text{in } M, \\ u = \varphi & \text{on } \partial M, \end{cases} \tag{1-4}$$

where φ is a smooth function defined on a neighborhood of ∂M.

We may apply Theorem 1.1 to the elementary symmetric functions and their quotients $(\sigma_k/\sigma_l)^{1/(k-l)}$ on Γ_k^+, with $0 \leq l < k \leq n$ and $\sigma_0 = 1$:

Corollary 1.2. *For $n \geq 3$, let $(\overline{M}^n, g)$ be a smooth, compact Riemannian manifold with boundary ∂M. Let $f \in C^\infty(\overline{M})$, let $f > 0$, and let S be a Riemannian metric on ∂M that is conformal to $g|_{\partial M}$. If $A_g^\tau \in \Gamma_k^+$ and $\tau > n-1$, then there exists a smooth metric $\hat{g} \in [g]$ on $\overline{M}$ satisfying*

$$\left(\frac{\sigma_k}{\sigma_l}\right)^{1/(k-l)}(A^\tau_{\hat{g}}) = f \quad \text{in } M \quad \text{and} \quad \hat{g}|_{\partial M} = S,$$

where $0 \leq l < k \leq n$.

Recently Gursky, Streets and Warren [2011] proved that any Riemannian manifold with boundary admits a negative Ricci curvature metric; see also Lohkamp [1994] and Guan [2008]. Once $\mathrm{Ric}_g < 0$, we have $A_g^{2(n-1)} = \frac{1}{n-2}(\mathrm{Ric}_g - R_g g) \in \Gamma_k^+$. Therefore:

Corollary 1.3. *For $n \geq 3$, every smooth compact n-dimensional manifold with boundary admits a Riemannian metric g with its Ricci tensor* Ric *and scalar curvature R satisfying*

$$\sigma_k(g^{-1}(\mathrm{Ric} - Rg)) = \mathrm{const} > 0,$$

where $1 \leq k \leq n$. In the case $k = n$, we have

$$\det(\mathrm{Ric} - Rg) = \mathrm{const} > 0.$$

By solving the infinite boundary data Dirichlet problem, we can produce complete metrics with constant σ_k-A_g^τ curvature, where $\tau > n-1$.

Theorem 1.4. *For $n \geq 3$, let $(\overline{M}^n, g)$ be a smooth, compact Riemannian manifold with boundary ∂M. Choose any smooth positive function $f \in C^\infty(\overline{M})$. If $B \in \Gamma^+$, $a(x)$ is positive on $\overline{M}$, and $\lambda a(x) + b(x)$ is nonnegative in M, then there exists an admissible solution $u \in C^\infty(M)$ to the equation*

$$\begin{cases} F(W[u]) = f(x)e^{2u} & \text{in } M, \\ u = +\infty & \text{on } \partial M. \end{cases} \tag{1-5}$$

Moreover, there exist some constants $C > 0$ and $0 < \gamma \leq 1$, depending on

$$n, \quad \lambda, \quad |f|_{C^2(\overline{M})}, \quad |a|_{L^\infty(\overline{M})}, \quad |b|_{L^\infty(\overline{M})}, \quad |B|_{g(\overline{M})}$$

and the geometry of $(\overline{M}, g)$, such that

$$-C - \gamma \log d(x) \leq u(x) \leq -\log d(x) + C \quad \text{near } \partial M,$$

where $d(x)$ denotes the distance to ∂M with respect to the metric g.

We can combine this with the result of [Gursky et al. 2011]:

Corollary 1.5. *For $n \geq 3$, every smooth compact n-dimensional manifold with boundary admits a complete metric g whose Ricci curvature satisfies*

$$\sigma_k(g^{-1}(\mathrm{Ric} - Rg)) = \mathrm{const} > 0,$$

where $1 \leq k \leq n$. In the case $k = n$, we have

$$\det(\mathrm{Ric} - Rg) = \mathrm{const} > 0.$$

When we consider the modified Schouten tensor with $\tau \leq 0$, it seems reasonable to consider the negative cone, by seeking a complete conformal metric $\bar{g}$ in the conformal class $[g]$, such that $\sigma_k(-\bar{g}A_{\bar{g}}^\tau) = \mathrm{const} > 0$. There are some interesting results, and we refer the reader to [Guan 2008] and [Gursky et al. 2011]. In the case $\tau = 1$, A_g^1 is just the classical Schouten tensor. In [2005], Schnürer fixes the metric at the boundary and realizes a prescribed value for the product of the eigenvalues of the Schouten tensor in the interior, provided there exists a subsolution. In [2007], Guan proved the existence of a conformal metric given its value on the boundary as a prescribed metric conformal to the (induced) background metric, with a prescribed curvature function of the Schouten tensor.

For compact manifolds without boundary, the problem of finding conformal metrics in Γ_k^+ of constant σ_k curvature (that is, of finding $g \in [g_0]$ such that $A_g^1 \in \Gamma_k^+$ and $\sigma_k(g^{-1}A_g^1) = \mathrm{const}$) — known as the higher order k-Yamabe problem for $k \geq 2$ — has attracted enormous interest since the work [Viaclovsky 2000] appeared. It can be viewed as a fully nonlinear version of the Yamabe problem, which was solved by Trudinger [1968], Aubin [1976] and Schoen [1984]. The solvability of the higher order k-Yamabe problem was shown for $k = 2$ in [Sheng et al. 2007] (see also [Chang et al. 2002; Ge and Wang 2006]), for $k = n/2$ in [Trudinger and Wang 2010], for $k > n/2$ in [Gursky and Viaclovsky 2007], and for locally conformally flat manifolds in [Guan and Wang 2003a; Li and Li 2003; Sheng et al. 2007]. For results concerning the modified Schouten tensor on closed manifolds, see [Gursky and Viaclovsky 2003; Li and Sheng 2005] for the case $\tau < 1$, and [Sheng and Zhang 2007] for the case $\tau \geq n-1$.

Our primary task is to solve the Dirichlet problem (1-4). The proof goes via the continuity method and a priori estimates. This paper is organized as follows. In Section 2, we show (1-3) is elliptic at any admissible solution. In Section 3, 4 and 5, we establish a priori estimates that are essential in proving the existence result. We then complete the proof of Theorem 1.1 in Section 6 and solve the infinite boundary data Dirichlet problem (1-5) in Section 7.

2. Ellipticity

In order to discuss the ellipticity properties of Equation (1-3), we define

$$\mathcal{A}[u] := F(g^{-1}W[u]) - h(x,u).$$

We then suppose that $u \in C^2(\overline{M})$ satisfies $\mathcal{A}[u] = 0$. Let $u_s = u + s\psi$, then the linearized operator of $\mathcal{A}$ is

$$\begin{aligned}\mathcal{L}\psi &:= \frac{d}{ds}\mathcal{A}[u_s]|_{s=0} \\ &= F(g^{-1}W[u])^{ij}(\lambda(\Delta\psi)g_{ij} - \psi_{ij} + 2au_i\psi_j + 2b\langle\nabla u, \nabla\psi\rangle g_{ij}) \\ &\qquad - h_z(x,u)\psi.\end{aligned}$$

Defining

$$Q^{ij} = \lambda\sum_l (F^{ll})\delta^{ij} - F^{ij}, \tag{2-1}$$

we have

$$\mathcal{L}\psi = Q^{ij}\psi_{ij} + 2F^{ij}(au_i\psi_j + b\langle\nabla u, \nabla\psi\rangle g_{ij}) - h_z(x,u)\psi. \tag{2-2}$$

Proposition 2.1. *Equation (1-3) is elliptic at any admissible solution.*

Proof. Since F^{ij} is positive definite in Γ^+, we have

$$Q^{ij} \geq (\lambda - 1)\sum_l (F^{ll})\delta^{ij} > 0.$$

Therefore, (1-3) is elliptic by (2-2). □

If $\partial_z h(x,z)$ is positive on $\overline{M}\times\mathbb{R}$, then the coefficient of ψ in the zeroth-order term of (2-2) is strictly negative, and we have this:

Corollary 2.2. *If $\partial_z h(x,z)$ is positive on $\overline{M}\times\mathbb{R}$, then at any admissible solution of (1-3), the linearized operator $\mathscr{L}: C^{2,\alpha}(M)\to C^{\alpha}(M)$ is invertible.*

3. The global C^0 estimates

Proposition 3.1. *If $B\in\Gamma^+$ and $\lim_{z\to+\infty} h(x,z)\to+\infty$, $\lim_{z\to-\infty} h(x,z)\to 0$. Then there exists some positive constant C_0, depending only upon h, B and φ, such that for any $C^2(\overline{M})$ admissible solution u of (1-4), we have*

$$|u|_{C^0(\overline{M})}\le C_0.$$

Proof. Since $\overline{M}$ is compact, we may suppose $\tilde{x}$ is a minimum of the function u. If $\tilde{x}\in M$, we have

$$\begin{aligned} h(\tilde{x},u(\tilde{x})) &= F(\lambda(\Delta u)(\tilde{x})g-\nabla^2u(\tilde{x})+B(\tilde{x}))\\ &\ge \min_M F(B)>0. \end{aligned}$$

Using $\lim_{z\to-\infty} h(x,z)\to 0$, we get the lower bound of u. Otherwise $\tilde{x}\in\partial M$, we get $u\ge\min_{\partial M}\varphi$.

The upper bound of u follows by considering a maximum of the function u and using the fact that $\lim_{z\to+\infty} h(x,z)\to+\infty$. □

4. Gradient estimates

We first establish the interior gradient estimates.

Lemma 4.1. *Suppose $B\in\Gamma^+$ and $\lambda a(x)+b(x)$ is nonnegative in M. If $u\in C^3(B_r)$ is an admissible solution of (1-4) in a ball $B_r\subset M$, then there is a constant C depending only on $|a|_{C^1(M)}$, $|b|_{C^1(M)}$, $\max_{M\times[-C_0,C_0]}|h|_{C^1}$, $|g|_{C^2(M)}$, λ, $|B|_{C^1(M)}$ and $|u|_{C^0(B_r)}$, such that*

$$\sup_{B_{r/2}}|\nabla u|\le C.$$

Proof. Consider the auxiliary function

$$H(x)=\zeta(x)ve^{\phi(u)},$$

where $\zeta(x)\in C_0^\infty(B_r)$ is a cutoff function to be chosen later, $v=(1+\frac{1}{2}|\nabla u|_g^2)$, $\phi:\mathbb{R}\longrightarrow\mathbb{R}$ is a function of the form $\phi(s)=\alpha(\beta+s)^p$, and $|s|\le|u|_{C^0(B_r)}$. The constants α, β and p depend only on $|u|_{C^0(B_r)}$ and $|a|_{L^\infty}$, such that the function $\phi(s)$ satisfies $\phi'(s)>0$ and $\phi''(s)-\phi'^2(s)-|a|_{L^\infty}\phi'(s)\ge\varepsilon_1>0$ for some constant ε_1 depending on $|u|_{C^0(B_r)}$ and $|a|_{L^\infty}$. It is proved in [Gursky and Viaclovsky 2003] that such a function ϕ always exists in the case $|a|_{L^\infty} = 1$. With a slight modification, the proof still works for our case.

Suppose the maximum of H occurs at an interior point $\tilde{x} \in B_r$. Take a normal coordinate system $(x^1, \ldots, x^n)$ at $\tilde{x}$ with respect to g such that $W[u]_{ij}(\tilde{x})$ is diagonal. Then at $\tilde{x}$ we have

$$0 = H_i = (v\zeta_i + \zeta u_{li}u_l + v\zeta\phi' u_i)e^{\phi(u)},$$

that is,

$$\zeta u_{li}u_l = -v(\zeta_i + \zeta\phi' u_i), \tag{4-1}$$

and

$$\begin{aligned} 0 \geq H_{ij} = \zeta(u_l u_{lij} + u_{li}u_{lj} + u_l(u_i u_{lj} + u_{li}u_j)\phi')e^{\phi(u)} \\ + v\zeta((\phi'^2 + \phi'')u_i u_j + \phi' u_{ij})e^{\phi(u)} \\ + u_l(u_{lj}\zeta_i + u_{li}\zeta_j)e^{\phi(u)} + v(\zeta_{ij} + \phi'(u_i\zeta_j + \zeta_i u_j))e^{\phi(u)}. \end{aligned} \tag{4-2}$$

Recall that $Q^{ij} = \lambda(\sum_l F^{ll})\delta^{ij} - F^{ij}$. Since F^{ij} is positive definite in Γ^+, one obtains $\lambda(\sum_l F^{ll})\delta^{ij} \geq Q^{ij} \geq \varepsilon_0(\sum_l F^{ll})\delta^{ij} > 0$, where $\varepsilon_0 = \lambda - 1$. Then (4-2) implies

$$\begin{aligned} 0 \geq \zeta Q^{ij}(u_l u_{lij} + u_{li}u_{lj} + 2u_i u_l u_{lj}\phi') \\ + v\zeta Q^{ij}((\phi'^2 + \phi'')u_i u_j + \phi' u_{ij}) \\ + 2u_l Q^{ij}u_{li}\zeta_j + vQ^{ij}(\zeta_{ij} + 2\phi' u_i\zeta_j). \end{aligned}$$

By the Ricci identity, we have $u_{lij} = u_{ijl} + R_{jlip}u_p$, where R_{ijlp} is the Riemannian curvature tensor of (M, g). Then

$$\begin{aligned} 0 \geq \zeta Q^{ij}\big(u_l u_{ijl} + R_{jlip}u_p u_l + 2u_l u_{li}u_j\phi' + v((\phi'^2 + \phi'')u_i u_j + \phi' u_{ij})\big) \\ + 2u_l Q^{ij}u_{li}\zeta_j + vQ^{ij}(\zeta_{ij} + 2\phi' u_i\zeta_j). \end{aligned} \tag{4-3}$$

Using $h(x, u) = F(W[u]) = F^{ij}W[u]_{ij}$ and $h_l + h_z u_l = F^{ij}W[u]_{ij;l}$, we obtain

$$Q^{ij}u_{ij} = -F^{ij}(au_i u_j + b|\nabla u|^2 g_{ij} + B_{ij}) + h(x, u), \tag{4-4}$$

and

$$\begin{aligned} u_l Q^{ij}u_{ijl} \\ = -F^{ij}(a_l u_l u_i u_j + 2au_i u_{jl}u_l + b_l u_l|\nabla u|^2 g_{ij} + 2bu_k u_{lk}u_l g_{ij} + u_l B_{ijl}) \\ + h_l u_l + h_z|\nabla u|^2. \end{aligned} \tag{4-5}$$

Plugging (4-4) and (4-5) into (4-3), we have

$$\begin{aligned}
0 \geq & -\zeta F^{ij}(a_l u_l u_i u_j + 2a u_i u_{jl} u_l + b_l u_l |\nabla u|^2 g_{ij} + 2b u_k u_{lk} u_l g_{ij} + u_l B_{ijl}) \\
& -\zeta v\phi' F^{ij}(a u_i u_j + b|\nabla u|^2 g_{ij} + B_{ij}) \\
& +\zeta Q^{ij}(R_{jlip} u_p u_l + 2u_l u_{li} u_j \phi' + v(\phi'^2 + \phi'') u_i u_j) \\
& +\zeta(h_l u_l + h_z |\nabla u|^2 + v\phi' h(x,u)) \\
& +2u_l Q^{ij} u_{li} \zeta_j + 2v\phi' Q^{ij} u_i \zeta_j + v Q^{ij} \zeta_{ij}.
\end{aligned}$$

Without loss of generality, we may assume $\frac{1}{2}|\nabla u|^2 \leq v \leq |\nabla u|^2$, and using (4-1), we derive

$$\begin{aligned}
0 \geq \quad & \zeta v\phi' F^{ij}(a u_i u_j + b|\nabla u|^2 g_{ij}) + \zeta v(\phi'' - \phi'^2) Q^{ij} u_i u_j \\
& -\zeta F^{ij}(a_l u_l u_i u_j + b_l u_l |\nabla u|^2 g_{ij} + u_l B_{ijl}) \\
& -\zeta v\phi' F^{ij} B_{ij} + \zeta Q^{ij} R_{jlip} u_p u_l \\
& +\zeta(h_l u_l + h_z |\nabla u|^2 + v\phi' h(x,u)) \\
& -2v\phi' Q^{ij} \zeta_i u_j + 2v(a F^{ij} + b(\textstyle\sum F^{ll})\delta^{ij})\zeta_i u_j \\
& +v Q^{ij} \zeta_{ij} - 2(v/\zeta) Q^{ij} \zeta_i \zeta_j \\
\geq \quad & \zeta v(\phi'' - \phi'^2 - a\phi') Q^{ij} u_i u_j \\
& +\zeta v\phi'(\lambda a(x) + b(x))(\textstyle\sum F^{ll})|\nabla u|^2 - C\zeta(\textstyle\sum F^{ll})(v^{3/2} + 1) \\
& -C\zeta(v+1) - C(\textstyle\sum F^{ll})(|\nabla\zeta| v^{3/2} + |\nabla^2\zeta| v + (|\nabla\zeta|^2/\zeta) v),
\end{aligned} \tag{4-6}$$

in the second inequality, we have used the definition of Q^{ij} to get

$$a\zeta v\phi' F^{ij} u_i u_j = \lambda a\zeta\phi'(\textstyle\sum_l F^{ll})|\nabla u|^2 - a\zeta v\phi' Q^{ij} u_i u_j.$$

Now we choose ζ to satisfy, as in [Guan and Wang 2003b],

$$0 \leq \zeta \leq 1, \quad |\nabla\zeta| \leq b_0 \zeta^{1/2}, \quad |\nabla^2\zeta| \leq b_0$$

for some constant $b_0 > 0$ and

$$\zeta(x) = 1 \text{ in } B_{r/2} \quad \text{and} \quad \zeta(x) = 0 \text{ outside } B_r.$$

By virtue of (4-6), we then have

$$0 \geq (\textstyle\sum_l F^{ll})(\varepsilon_0\varepsilon_1 \zeta v^2 - C\zeta v^{3/2} - C\zeta) - C\zeta(v+1) - C(\textstyle\sum_l F^{ll})(\zeta^{1/2} v^{3/2} + v).$$

Multiplying by ζ on both sides and using that $0 \leq \zeta \leq 1$, we have

$$0 \geq (\textstyle\sum_l F^{ll})(\varepsilon_0\varepsilon_1 \zeta^2 v^2 - C\zeta^{3/2} v^{3/2} - C\zeta v - C) - C(\zeta v + 1). \tag{4-7}$$

Note that Euler formula and concavity of F imply

$$(\textstyle\sum_l F^{ll})(\kappa) = F(\kappa) + \sum_i F^{ii}(\kappa)(1-\kappa_i) \geq F(e) > 0 \quad \text{in } \Gamma^+,$$

where $e = (1, \dots, 1)$. From (4-7), if $\varepsilon_0\varepsilon_1\zeta^2 v^2 - C\zeta^{3/2}v^{3/2} - C\zeta v - C \leq 0$, we have $(\zeta v)(\tilde{x}) \leq C$. Otherwise, we have

$$0 \geq F(e)(\varepsilon_0\varepsilon_1\zeta^2 v^2 - C\zeta^{3/2}v^{3/2} - C\zeta v - C) - C(\zeta v + 1).$$

We then obtain $(\zeta v)(\tilde{x}) \leq C$. Hence $H \leq C$ in B_r; therefore $\sup_{B_{r/2}} |\nabla u| \leq C$. □

We now derive a priori bounds for the boundary gradient of solutions to (1-4) with smooth Dirichlet data φ. Without loss of generality, we may assume that $\varphi \in C^\infty(\overline{M})$ in the sequel. The method is to construct barrier functions near ∂M using the boundary distance function. Let $d(x) = \operatorname{dist}_g(x, \partial M)$ for $x \in M$, and set

$$M_\delta = \{x \in M \mid d(x) < \delta\} \quad \text{for } \delta > 0.$$

Since ∂M is smooth and $|\nabla d| = 1$ on ∂M, we choose $\delta > 0$ sufficiently small so that d is smooth and $\frac{1}{2} \leq |\nabla d| \leq 2$ in M_δ.

Consider the locally defined auxiliary function

$$w^- := \varphi + \theta \log \frac{\delta^2}{d+\delta^2},$$

where θ is some small positive constant. We may directly check that

$$\begin{cases} w^-|_{\partial M} = \varphi, \\ \varphi + \theta \log(\delta/2) \leq w^-|_{\{d(x)=\delta\}} \leq \varphi + \theta \log \delta. \end{cases} \tag{4-8}$$

Since

$$\nabla w^- = \nabla\varphi - \frac{\theta}{d+\delta^2}\nabla d,$$
$$\nabla^2 w^- = \nabla^2\varphi - \frac{\theta}{d+\delta^2}\nabla^2 d + \frac{\theta}{(d+\delta^2)^2}\nabla d \otimes \nabla d,$$

we obtain

$$\begin{aligned} W[w^-]_{ij} &= \frac{(\lambda + b\theta)\theta}{(d+\delta^2)^2}|\nabla d|^2 g_{ij} + \frac{a\theta^2}{(d+\delta^2)^2}d_i d_j - \frac{\theta}{(d+\delta^2)^2}d_i d_j \\ &\qquad - \frac{\theta}{d+\delta^2}(\lambda\Delta d g_{ij} - d_{ij} + a(\varphi_j d_i + \varphi_i d_j) + 2b\,\langle\nabla\varphi, \nabla d\rangle\, g_{ij}) \\ &\qquad\qquad + \lambda\Delta\varphi g_{ij} - \varphi_{ij} + a\varphi_i\varphi_j + b|\nabla\varphi|^2 g_{ij} + B_{ij} \\ &\geq \frac{(\varepsilon_0 - (|a|_{L^\infty(\overline{M})} + |b|_{L^\infty(\overline{M})})\theta)\theta}{(d+\delta^2)^2}|\nabla d|^2 g_{ij} - \frac{\theta}{d+\delta^2}C' g_{ij} - C'' g_{ij}, \end{aligned}$$

where C' and C'' are some sufficiently large constants, depending only on $|\varphi|_{C^2(\bar{M})}$, λ, $|a|_{L^\infty(\overline{M})}$, $|b|_{L^\infty(\bar{M})}$, $|B|_{g(\bar{M})}$ and the geometric quantities of $(\overline{M}, g)$, independent of δ. Choosing

$$\theta \le \frac{\varepsilon_0}{2(|a|_{L^\infty(\overline{M})}+|b|_{L^\infty(\overline{M})})} \quad \text{and} \quad \delta \le \min\left\{1, \frac{\varepsilon_0}{16C'}, \frac{\varepsilon_0\theta}{64C''}\right\},$$

by virtue of $|\nabla d| > 1/2$ in M_δ, we derive

$$\begin{aligned} W[w^-]_{ij} &\ge \frac{\varepsilon_0\theta}{8(d+\delta^2)\delta} g_{ij} - \frac{\theta}{d+\delta^2} C' g_{ij} - C'' g_{ij} \\ &= \frac{\theta}{d+\delta^2}\Big(\frac{\varepsilon_0}{16\delta} - C'\Big) g_{ij} - C'' g_{ij} + \frac{\theta\varepsilon_0}{16\delta(d+\delta^2)} g_{ij} \\ &\ge \frac{\theta\varepsilon_0}{32\delta} g_{ij} - C'' g_{ij} \\ &= \frac{\theta\varepsilon_0}{64\delta} g_{ij} + \Big(\frac{\theta\varepsilon_0}{64\delta} - C''\Big) g_{ij} \ge \frac{\theta\varepsilon_0}{64\delta} g_{ij}, \end{aligned} \tag{4-9}$$

in the first inequality we have used the fact $d+\delta^2 \le 2\delta$, while in the second, we have used that $d+\delta^2 \le 2$.

To estimate the boundary gradient, we need the following maximum principle. We first give a standard definition.

Definition 4.2. We say a subsolution w of (1-3) is *admissible* and

$$F(W[w]) \ge h(x, w) \quad \text{in } M.$$

Changing the direction of the inequality, one gets the definition of the supsolution of (1-3).

Lemma 4.3. *Suppose that w_1 and w_2 are smooth sub- and supersolutions* (*respectively*) *of (1-3) with $w_1|_{\partial M} < w_2|_{\partial M}$. If $\partial_z h(x,z)$ is positive in $M\times\mathbb{R}$, then $w_1 \le w_2$ on $\overline{M}$.*

Proof. We argue by contradiction. Set $\tilde{w} = w_2 - w_1$. Suppose $\tilde{w}(\tilde{x}) = \min_{\overline{M}} \tilde{w} < 0$ for some $\tilde{x} \in \overline{M}$; then $\tilde{x}$ must be an interior point. At this point,

$$\nabla w_2(\tilde{x}) = \nabla w_1(\tilde{x}) \quad \text{and} \quad \nabla^2 w_2(\tilde{x}) \ge \nabla^2 w_1(\tilde{x}).$$

Consequently

$$\begin{aligned} F(W[w_2])(\tilde{x}) &= Q^{ij}\nabla^2_{ij} w_2(\tilde{x}) + F^{ij}(a\nabla_i w_2 \nabla_j w_2 + b|\nabla w_2|^2 g_{ij} + B_{ij})(\tilde{x}) \\ &\ge Q^{ij}\nabla^2_{ij} w_1(\tilde{x}) + F^{ij}(a\nabla_i w_1 \nabla_j w_1 + b|\nabla w_1|^2 g_{ij} + B_{ij})(\tilde{x}) \\ &= F(W[w_1])(\tilde{x}). \end{aligned}$$

We therefore have

$$h(\tilde{x}, w_2(\tilde{x})) \ge F(W[w_2])(\tilde{x}) \ge F(W[w_1])(\tilde{x}) \ge h(\tilde{x}, w_1(\tilde{x})),$$

which contradicts that $w_1(\tilde{x}) > w_2(\tilde{x})$ and $\partial_z h(x,z)$ is positive in $M\times\mathbb{R}$. $\square$

Let x_0 be an arbitrary point on ∂M. We pick local coordinates in M_δ so that ∂M is the plane $x_n = 0$, and let $\{e_\gamma, e_n\}_{\gamma=1}^{n-1}$ be the corresponding coordinate vector fields, where $e_n(x_0)$ denotes the interior normal vector and $e_\gamma(x_0)$ the tangential direction.

Lemma 4.4. *Let u be a $C^2(\overline{M})$ admissible solution of (1-4). If $B \in \Gamma^+$ and $\partial_z h(x, z)$ is positive in $M \times \mathbb{R}$, then there exists a constant C depending on*

$$C_0, \quad \lambda, \quad |\varphi|_{C^2(\overline{M})}, \quad |a|_{L^\infty(\overline{M})}, \quad |b|_{L^\infty(\overline{M})}, \quad |B|_{g(\overline{M})}$$

and the geometric quantities of $(\overline{M}, g)$, such that

$$\partial_n u|_{\partial M} > -C.$$

Proof. Recalling (4-8) and (4-9), we have

$$w^-|_{\partial M} = \varphi \quad \text{and} \quad F(W[w^-]) = F^{ij} W[w^-]_{ij} \geq \frac{\varepsilon_0 \theta}{64\delta} F(e) \quad \text{on } M_\delta.$$

We choose δ smaller, so that

$$F(W[w^-]) \geq \max_{\overline{M}\times[\min_{\overline{M}}\varphi, \max_{\overline{M}}\varphi]} h(x, z) \geq h(x, w^-) \quad \text{on } M_\delta.$$

Since $|u|_{C^0}(\overline{M}) < C_0$, we can regard w^- as a local subsolution of (1-3) on $\overline{M}_\delta = \{x \mid d(x) \leq \delta\}$. Applying Lemma 4.3 to $\overline{M}_\delta$, we have

$$\frac{u(x) - u(x_0)}{d(x, x_0)} \geq \frac{w^-(x) - w^-(x_0)}{d(x, x_0)} \quad \text{for any } x_0 \in \partial M.$$

That is, $\partial_n u|_{\partial M} \geq \partial_n w^-|_{\partial M}$, and our lemma follows. □

We next prove that the $\partial_n u$ have an upper bound; the boundary gradient estimates follow.

Lemma 4.5. *Let u be a $C^2(\overline{M})$ admissible solution of (1-4). If $B \in \Gamma^+$ and $\partial_z h(x, z)$ is positive in $M \times \mathbb{R}$, then we have*

$$\partial_n u(x_0) < C \quad \textit{for any point } x_0 \in \partial M,$$

where C is a positive constant depending on C_0, λ, $|\varphi|_{C^2(\overline{M})}$, $|a|_{L^\infty(\overline{M})}$, $|b|_{L^\infty(\overline{M})}$, $|B|_{g(\bar{M})}$ and the geometric quantities of $(\overline{M}, g)$.

Proof. Since u is admissible and $\Gamma^+ \subset \Gamma_1^+$, we have

$$c_1 \Delta u + c_2 |\nabla u|^2 + \operatorname{tr} B \geq (n\lambda - 1)\Delta u + (a + nb)|\nabla u|^2 + \operatorname{tr} B > 0,$$

where $c_1 = n\lambda - 1$ and $c_2 = |a|_{L^\infty} + n|b|_{L^\infty}$. Therefore the proof reduces to constructing a local supbarrier function of the equation

$$c_1 \Delta v + c_2 |\nabla v|^2 + \operatorname{tr} B = 0.$$

Let's consider $w^+ = \varphi + \theta \log((d+\delta^2/\delta^2))$ in M_δ; then

$$w_i^+ = \theta \frac{d_i}{d+\delta^2} + \varphi_i,$$
$$w_{ij}^+ = -\theta \frac{d_i d_j}{(d+\delta^2)^2} + \theta \frac{d_{ij}}{d+\delta^2} + \varphi_{ij}.$$

We therefore have

$$\begin{aligned} c_1\Delta w^+ + c_2|\nabla w^+|^2 + \operatorname{tr} B \\ &= -\theta(c_1 - c_2\theta)\frac{|\nabla d|^2}{(d+\delta^2)^2} + (c_1\Delta d + 2c_2\langle\nabla d, \nabla\varphi\rangle)\frac{\theta}{d+\delta^2} \\ &\qquad + c_1(\Delta\varphi) + c_2|\nabla\varphi|^2 + \operatorname{tr} B. \end{aligned}$$

Now we choose $\theta < c_1/(2c_2)$. Then using $|\nabla d|^2 > \frac{1}{2}$ in M_δ, we derive

$$\begin{aligned} c_1\Delta w^+ + c_2|\nabla w^+|^2 + \operatorname{tr} B &\le -\frac{c_1\theta}{4(d+\delta^2)^2} + C'\frac{\theta}{d+\delta^2} + C'' \\ &\le \left(-\frac{c_1}{4\delta(1+\delta)} + C'\right)\frac{\theta}{d+\delta^2} + C'' \quad \text{in } M_\delta, \end{aligned}$$

where C' and C'' are two positive constants depending on

$$|\varphi|_{C^2(\overline{M})}, \quad \lambda, \quad |a|_{L^\infty(\overline{M})}, \quad |b|_{L^\infty(\overline{M})}, \quad |B|_{g(\overline{M})}$$

and the geometric quantities of $(\overline{M}, g)$, independent of δ. Next we choose

$$\delta < \min\left\{1, \frac{c_1}{8(C'+1)}, \frac{\theta}{2C''}\right\};$$

then $c_1\Delta w^+ + c_2|\nabla w^+|^2 + \operatorname{tr} B < 0$ in M_δ.

Note that

$$\begin{cases} w^+|_{\partial M} = \varphi, \\ w^+|_{\{x\in M \mid d(x)=\delta\}} \ge \varphi + \theta\log(1/\delta). \end{cases}$$

Without loss of generality, we can assume δ is small; then $|u|_{C^0}(\overline{M}) < C_0$ and the maximum principle imply $u \le w^+$ in $\overline{M}_\delta$. Consequently, for any $x_0 \in \partial M$,

$$\frac{u(x) - u(x_0)}{d(x, x_0)} \le \frac{w^+(x) - w^+(x_0)}{d(x, x_0)}.$$

That is, $\partial_n u|_{\partial M} \le \partial_n w^+|_{\partial M}$, and our lemma follows. □

Combining Lemma 4.1, Lemma 4.4 and Lemma 4.5, we obtain this:

Proposition 4.6. *Suppose $B \in \Gamma^+$, $\lambda a(x) + b(x)$ is nonnegative in M and $\partial_z h(x, z)$ is positive in $M \times \mathbb{R}$. Then for any $C^3(\overline{M})$ admissible solution u of (1-4), there is* *a constant C_1 depending only on*

$$C_0, \quad \lambda, \quad |\varphi|_{C^2(\overline{M})}, \quad |a|_{C^1(\overline{M})}, \quad |b|_{C^1(\overline{M})}, \quad \max_{M\times[-C_0,C_0]}|h|_{C^1}, \quad |B|_{C^1(\overline{M})}$$

and the geometric quantities of $(\overline{M}, g)$, such that $|\nabla u| \le C_1$ on $\overline{M}$.

5. Estimates for the second derivative

As in Section 4, we begin by establishing the interior estimates.

Lemma 5.1. *Let $B \in \Gamma^+$ and $a(x)$ be positive on $\overline{M}$. Let $u \in C^4(B_r)$ be an admissible solution of (1-4) in a ball $B_r \subset M$; there is a constant C depending only on*

$$|a|_{C^2(M)}, \quad |b|_{C^2(M)}, \quad \max_{M\times[-C_0,C_0]}|h|_{C^2}, \quad |g|_{C^2(M)}, \quad |B|_{C^2(M)}, \quad \lambda, \quad |u|_{C^1(B_r)}$$

such that $\sup_{B_{r/2}}|\nabla^2 u| \le C$.

Proof. Since $\Gamma^+ \subset \Gamma_1^+$, we obtain

$$0 < \operatorname{tr} W[u] = (n\lambda - 1)(\Delta u) + (a(x) + nb(x))|\nabla u|^2 + \operatorname{tr} B.$$

Consequently $\Delta u \ge -C$. For obtaining the upper bound of Δu, we consider the auxiliary function

$$G(x) = \zeta(x)(\Delta u + \Lambda a(x)|\nabla u|^2)$$

for some large constant $\Lambda > 1$, depending only on $|a|_{L^\infty}$, $|b|_{L^\infty}$ and λ, to be chosen later; here $\zeta(x) \in C_0^\infty(B_r)$ is a cutoff function as in Lemma 4.1.

Suppose G achieves a maximum at an interior point $\tilde{x} \in M$. We take a normal coordinate system $(x^1, \dots, x^n)$ with respect to g such that $W[u]_{ij}(\tilde{x})$ is diagonal. Without loss of generality, we may assume $G(\tilde{x}) \ge 1$ and $\tilde{x} \in B_r$. Then, at $\tilde{x}$, we have

$$0 = G_i = (\Delta u + \Lambda a|\nabla u|^2)\zeta_i + \zeta(u_{lli} + \Lambda a_i|\nabla u|^2 + 2\Lambda a u_l u_{li}),$$

that is,

$$\zeta u_{lli} = -\Lambda a_i \zeta|\nabla u|^2 - 2\Lambda a\zeta u_l u_{li} - (\Delta u + \Lambda a|\nabla u|^2)\zeta_i, \tag{5-1}$$

and

$$\begin{aligned} 0 \ge G_{ij} = \zeta(u_{llij} &+ \Lambda a_{ij}|\nabla u|^2 + 2\Lambda u_l(a_i u_{lj} + a_j u_{li}) + 2\Lambda a(u_{li}u_{lj} + u_l u_{lij})) \\ &+ (u_{lli} + \Lambda a_i|\nabla u|^2 + 2\Lambda a u_l u_{li})\zeta_j \\ &\quad + (u_{llj} + \Lambda a_j|\nabla u|^2 + 2\Lambda a u_l u_{lj})\zeta_i + (\Delta u + \Lambda a|\nabla u|^2)\zeta_{ij}. \end{aligned} \tag{5-2}$$

Recall that $Q^{ij} = \lambda(\sum_l F^{ll})\delta^{ij} - F^{ij}$. Since F^{ij} is positive definite in Γ^+, one obtains $\lambda(\sum_l F^{ll})\delta^{ij} \ge Q^{ij} \ge \varepsilon_0(\sum_l F^{ll})\delta^{ij} > 0$, where $\varepsilon_0 = \lambda - 1$. Notice that the Ricci identity gives $u_{lij}=u_{ijl}+O(|\nabla u|)$ and $u_{llij}=u_{ijll}+O(|\nabla^2 u|+|\nabla u|)$. Then (5-2) implies

$$
\begin{aligned}
0 &\ge Q^{ij}G_{ij}\\
&= \zeta Q^{ij}\big(u_{llij}+\Lambda a_{ij}|\nabla u|^2+4\Lambda u_l a_i u_{lj}+2\Lambda a(u_{li}u_{lj}+u_l u_{lij})\big)\\
&\qquad +2Q^{ij}(u_{lli}+\Lambda a_i|\nabla u|^2+2\Lambda a u_l u_{li})\zeta_j+(\Delta u+\Lambda a|\nabla u|^2)Q^{ij}\zeta_{ij}\\
&\ge \zeta Q^{ij}(u_{ijll}+2\Lambda a(u_{li}u_{lj}+u_l u_{ijl}))+2Q^{ij}u_{lli}\zeta_j\\
&\qquad\qquad -C\Lambda(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1).
\end{aligned}
\tag{5-3}
$$

Using $h_{ll}+2h_{lz}u_l+h_z u_{ll}=F^{ij}W[u]_{ij;ll}+F^{ij,rs}W[u]_{ij;l}W[u]_{rs;l}$ and the concavity of F, we obtain

$$
\begin{aligned}
Q^{ij}u_{ijll} &\ge -2aF^{ij}(u_{il}u_{jl}+u_i u_{jll})-2b(\textstyle\sum_l F^{ll})(|\nabla^2 u|^2+u_{kll}u_k)\\
&\qquad -C(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1)+h_{ll}+2h_{lz}u_l+h_z u_{ll}.
\end{aligned}
\tag{5-4}
$$

On the other hand, (4-5) implies

$$
2\Lambda a u_l Q^{ij}u_{ijl}\ge -C\Lambda(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1)+2\Lambda a h_l u_l+2\Lambda a h_z|\nabla u|^2. \tag{5-5}
$$

Plugging (5-4) and (5-5) into (5-3), and employing (5-1) we have

$$
\begin{aligned}
0 &\ge 2\Lambda a\zeta Q^{ij}u_{li}u_{lj}-2a\zeta F^{ij}(u_{il}u_{jl}+u_i u_{jll})+2Q^{ij}u_{lli}\zeta_j\\
&\qquad -2b\zeta(\textstyle\sum_l F^{ll})(|\nabla^2 u|^2+u_{kll}u_k)\\
&\qquad -C\Lambda(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1)-C\Lambda(|\nabla^2 u|+1)\\
&\ge 2\zeta(\Lambda a\lambda-b)(\textstyle\sum_l F^{ll})|\nabla^2 u|^2-2a\zeta(\Lambda+1)F^{ij}u_{il}u_{jl}\\
&\qquad -C\Lambda(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1)-C\Lambda(|\nabla^2 u|+1)\\
&\ge 2\zeta(\varepsilon_0 a\Lambda-a-b)(\textstyle\sum_l F^{ll})|\nabla^2 u|^2\\
&\qquad -C\Lambda(\textstyle\sum_l F^{ll})(|\nabla^2 u|+1)-C\Lambda(|\nabla^2 u|+1).
\end{aligned}
$$

Since a is positive on $\overline{M}$, we assume $a(x)\ge\varepsilon_2>0$. We now choose $\Lambda>\max\{1, 2(|a|_{L^\infty}+|b|_{L^\infty})/(\varepsilon_0\varepsilon_2)\}$, and multiply ζ on both sides to produce

$$
0\ge \Lambda(\textstyle\sum_l F^{ll})(\varepsilon_0\varepsilon_2\zeta^2|\nabla^2 u|^2-C\zeta|\nabla^2 u|-C)-C\Lambda(\zeta|\nabla^2 u|+1). \tag{5-6}
$$

It follows that $(\zeta|\nabla^2 u|)(\tilde{x})\le C$. Therefore $\sup_{B_{r/2}}\Delta u\le C$.

If $\Gamma^+\subset\Gamma_2^+$, then $\sup_{B_{r/2}}\Delta u\le C$ implies that $\sup_{B_{r/2}}|\nabla^2 u|\le C$. To get the Hessian bounds of u in general, we simply consider the maximum of

$$
\zeta(x)\max_{\xi\in(T_xM\cap\mathbb{S}^n)}(\nabla_\xi\nabla_\xi u+\Lambda a(x)(\nabla_\xi u)^2).
$$

The calculation is similar. $\square$

We next derive a priori bounds for second derivatives of solutions to (1-4). The method we use is similar to that of [Guan 2007; Guan 2008; Gursky et al. 2011]. The notation below is the same as in Section 4.

We use a barrier function

$$v(x) = p(qd^2 - d) \quad \text{in } M_\delta,$$

where p and q are positive constants. Let's define a linear operator

$$\mathcal{P}(\psi) = Q^{ij}\psi_{ij} + 2F^{ij}(a(x)u_i\psi_j + b(x)\langle \nabla u, \nabla\psi\rangle g_{ij}). \tag{5-7}$$

Then

$$\mathcal{P}d = Q^{ij}d_{ij} + 2F^{ij}(au_i d_j + b\langle \nabla u, \nabla d\rangle g_{ij}),$$

and consequently

$$|\mathcal{P}d| \le C_\# \textstyle\sum_l F^{ll} \quad \text{in } M_\delta,$$

where $C_\#$ depends on λ, $|u|_{C^1(\overline{M})}$, $|a|_{L^\infty(\bar{M})}$, $|b|_{L^\infty(\overline{M})}$ and the geometric quantities of $(\overline{M}, g)$. On the other hand, we have in M_δ

$$\begin{aligned}
\mathcal{P}d^2 &= 2Q^{ij}(d_i d_j) + 2d\mathcal{P}d \\
&\ge 2\varepsilon_0\big(\textstyle\sum_l F^{ll}\big)|\nabla d|^2 - 2dC_\# \textstyle\sum_l F^{ll} \\
&\ge (\varepsilon_0 - 2C_\#\delta)\textstyle\sum_l F^{ll},
\end{aligned}$$

where $\varepsilon_0 = \lambda - 1$ as before. After we choose

$$q > 2(1 + C_\#)/\varepsilon_0 \quad \text{and} \quad \delta < \min\{\varepsilon_0/(4C_\#), 1/(2q)\},$$

the function v satisfies

$$\mathcal{P}v \ge p\{q(\varepsilon_0 - 2C_\#\delta) - C_\#\}\textstyle\sum_l F^{ll} \ge p\textstyle\sum_l F^{ll}, \tag{5-8}$$

and

$$v \le -\tfrac{1}{2}pd \quad \text{in } M_\delta. \tag{5-9}$$

Let x_0 be an arbitrary point on ∂M. Let $r(x) = \operatorname{dist}_g(x, x_0)$ to denote the distance from x to x_0 with respect to the background metric. Let $\Omega_\delta(x_0) = B_\delta(x_0) \cap M_\delta$, where $B_\delta(x_0) = \{x \in \overline{M} \mid r(x) < \delta\}$. Since δ is small, we assume r^2 is smooth in $\Omega_\delta(x_0)$. A similar calculation implies

$$\tfrac{1}{2}\varepsilon_0\textstyle\sum_l F^{ll} \le \mathcal{P}r^2 \le (2\lambda + \tfrac{1}{2}\varepsilon_0)\textstyle\sum_l F^{ll} \quad \text{in } \Omega_\delta(x_0). \tag{5-10}$$

Now we pick a local coordinates in M_δ so that ∂M is the plane $x_n = 0$, and we let $\{e_\gamma, e_n\}_{\gamma=1}^{n-1}$ be the corresponding coordinate vector fields, where $e_n(x_0)$ denotes the interior normal vector and $e_\gamma(x_0)$ the tangential direction. Fix some γ and consider the locally defined function $\phi = e_\gamma(u - \varphi)$, where u is a $C^3(\overline{M})$ admissible solution of (1-4). In order to derive the boundary estimates for second derivatives, we need the following lemma.

Lemma 5.2. *In the notation above, there exists a constant C, depending only on C_0, C_1, $|a|_{C^1(\bar{M})}$, $|b|_{C^1(\overline{M})}$, $|h|_{C^1(\overline{M}\times[-C_0,C_0])}$ and $|\varphi|_{C^3(M_\delta)}$, such that*

$$|\mathcal{P}\phi| \le C(1+\textstyle\sum_l F^{ll}).$$

Proof. Differentiating Equation (1-3) with respect to e_γ yields

$$\begin{aligned} Q^{ij}u_{ij\gamma} + 2F^{ij}(au_{i\gamma}u_j + bu_l u_{l\gamma}g_{ij}) \\ = -F^{ij}(a_\gamma u_i u_j + b_\gamma |\nabla u|^2 g_{ij} + B_{ij\gamma}) + h_z u_\gamma + h_\gamma. \end{aligned}$$

Exchanging derivatives implies

$$u_{ij\gamma} = u_{\gamma ij} + (Rm * \nabla u)_{ij\gamma}.$$

Combining these calculations yields

$$\begin{aligned} \mathcal{P}\phi &= Q^{ij}u_{\gamma ij} + 2F^{ij}(au_i u_{\gamma j} + bu_k u_{\gamma k}g_{ij}) \\ &\qquad - Q^{ij}\varphi_{\gamma ij} - 2F^{ij}(au_i\varphi_{\gamma j} + bu_k\varphi_{\gamma k}g_{ij}) \\ &= -F^{ij}(a_\gamma u_i u_j + b_\gamma|\nabla u|^2 g_{ij} + B_{ij\gamma}) + h_z u_\gamma + h_\gamma \\ &\qquad - Q^{ij}\varphi_{\gamma ij} - 2F^{ij}(au_i\varphi_{\gamma j} + bu_k\varphi_{\gamma k}g_{ij}) - Q^{ij}(Rm*\nabla u)_{ij\gamma} \end{aligned}$$

Therefore

$$|\mathcal{P}\phi| \le C(\textstyle\sum_l F^{ll}) + C. \qquad \square$$

We are now ready to prove the boundary estimates for second derivatives.

Lemma 5.3. *Let $u \in C^3(\overline{M})$ be an admissible solution of (1-4). Then*

$$|\nabla^2 u| \le C \quad \text{on } \partial M,$$

where the constant $C > 0$ depends on

$$C_0, \quad C_1, \quad |a|_{C^1(\overline{M})}, \quad |b|_{C^1(\overline{M})}, \quad |h|_{C^1(\overline{M}\times[-C_0,C_0])}, \quad |\varphi|_{C^3(M_\delta)}, \quad |B|_{C^1(\overline{M})}$$

and the geometric quantities of $(\overline{M}, g)$.

Proof. We require separate proofs for the different types $\nabla_\gamma\nabla_\eta u$, $\nabla_\gamma\nabla_n u$ and $\nabla_n\nabla_n u$ of boundary second derivatives.

Let x_0 be an arbitrary point on ∂M. Using that $u-\varphi = 0$ on ∂M, we obtain

$$\nabla_\gamma\nabla_\eta(u-\varphi)(x_0) = -\nabla_n(u-\varphi)\Pi(e_\gamma, e_\eta)(x_0),$$

where $1 \le \gamma, \eta \le n-1$ and Π denotes the second fundamental form of ∂M. We therefore have the estimates for the pure tangential second order derivatives.

Combining (5-8), (5-10) and Lemma 5.2, we have for any positive constant μ

$$\mathcal{P}(\phi - v + \mu r^2) \le (C - p + \mu(2\lambda + \tfrac{1}{2}\varepsilon_0)) \textstyle\sum_l F^{ll} + C.$$

Picking μ large enough and $p > \mu^2$, we get

$$\mathcal{P}(\phi - v + \mu r^2) \le -\tfrac{1}{2} p F(e) + C < 0.$$

Thus by the maximum principle, we conclude that the minimum of $\phi - v + \mu r^2$ occurs on the boundary of $\Omega_\delta(x_0)$. It remains to check these boundary values. There are two components of $\partial\Omega_\delta(x_0)$ to check. Firstly, since $\phi \equiv 0$ and $v \equiv 0$ on $\partial\Omega_\delta(x_0) \cap \partial M$, we get $\phi - v + \mu r^2 \ge 0$ on $\partial\Omega_\delta(x_0) \cap \partial M$ and $(\phi - v + \mu r^2)(x_0) = 0$. Since μ is large, (5-9) implies $\phi - v + \mu r^2 > \phi + (p/2)d + \mu r^2 > 0$ on $\partial\Omega_\delta(x_0) \setminus \partial M$. It follows that the normal derivative of $\phi - v + \mu r^2$ is nonnegative, and therefore we conclude

$$\begin{aligned}\nabla_n \nabla_\gamma u(x_0) &> \nabla_n(\nabla_\gamma \varphi + v - \mu r^2)(x_0)\\ &= \nabla_n \nabla_\gamma \varphi(x_0) - p > -C.\end{aligned}$$

However, using Lemma 5.2 again, it is clear that the same argument applies to $-\phi$, and one deduces the mixed second derivative estimates

$$|\nabla_n \nabla_\gamma u| < C.$$

Once we bound $\nabla_\gamma \nabla_\eta u$ and $\nabla_\gamma \nabla_n u$, to estimate the double normal second derivative $\nabla_n \nabla_n u$ we only need to bound Δu. Note that $W[u]_{ij} \in \Gamma_1^+$, that is,

$$(n\lambda - 1)(\Delta u) + (a(x) + nb(x))|\nabla u|^2 + \operatorname{tr} B > 0.$$

Consequently Δu is bounded from below and we have to establish an upper bound

$$u_{nn} \le C \quad \text{on } \partial M.$$

Without loss of generality, one can assume $u_{nn} \ge 0$ on ∂M (otherwise we are done). Orthogonally decompose the matrix W at $x_0 \in \partial M$ in terms of e_γ and e_n. Using the known bounds, we find

$$\begin{aligned}W[u]_{ij}(x_0) &= (\lambda \Delta u g_{ij} - u_{ij} + a u_i u_j + b|\nabla u|^2 g_{ij} + B_{ij})(x_0)\\ &\ge \begin{pmatrix}\lambda u_{nn} I_{n-1} & 0\\ 0 & (\lambda - 1)u_{nn}\end{pmatrix}(x_0) - C\delta_{ij}\\ &\ge (\varepsilon_0 u_{nn}(x_0) - C)\delta_{ij},\end{aligned}$$

where C depends on $|u|_{C^1(\bar{M})}$, $|a|_{C^0(\bar{M})}$, $|b|_{C^0(\bar{M})}$, $|B|_{C^0(\bar{M})}$, $|\nabla_\gamma \nabla_\eta u|$ and $|\nabla_\gamma \nabla_n u|$. It is clear that

$$\begin{aligned} C &> \max_{M\times[-|u|_{C^0(\bar{M})},|u|_{C^0(\bar{M})}]} |h| \\ &\geq F^{ij}(x_0) W[u]_{ij}(x_0) \\ &\geq (\varepsilon_0 u_{nn}(x_0) - C) \textstyle\sum_l F^{ll}(x_0) \\ &\geq (\varepsilon_0 u_{nn}(x_0) - C) F(e). \end{aligned}$$

Thus we obtain the upper bound as desired. □

Combining Lemma 5.1 and Lemma 5.3, we have the global estimates for the second derivative.

Proposition 5.4. *Suppose $B \in \Gamma^+$ and $a(x)$ is positive on $\bar{M}$. Then for any $C^4(\bar{M})$ admissible solution u of (1-4), there is a constant C_2 depending only on C_0, C_1, λ, $|a|_{C^2(\bar{M})}$, $|b|_{C^2(\bar{M})}$, $|h|_{C^2(M\times[-C_0,C_0])}$, $|\varphi|_{C^3(\bar{M})}$, $|B|_{C^2(\bar{M})}$ and the geometric quantities of $(\bar{M}, g)$ such that*

$$|\nabla^2 u| \leq C_2 \quad \text{on } \bar{M}.$$

6. Proof of Theorem 1.1

The proof of Theorem 1.1 is standard. We only sketch it here. For $t \in [0, 1]$, we consider the equations

$$(\star_t) \qquad \begin{cases} F(\nabla^2_{\mathrm{conf}} u + B^t) = h^t, \\ \qquad\qquad u|_{\partial M} = \varphi^t, \end{cases}$$

where

$$B^t = tB + \frac{1-t}{F(e)} g, \quad h^t = (1-t)e^{2u} + th(x,u), \quad \varphi^t = t\varphi.$$

For $t = 0$, the admissible solution is $u \equiv 0$ on $\bar{M}$; for $t = 1$, it is our desired Equation (1-4). It is direct to check that

- $B^t \in \Gamma^+$.
- $h^t > 0$ on $\bar{M} \times \mathbb{R}$, $\partial_z h^t(x,z) > 0$ on $\bar{M} \times \mathbb{R}$, $\lim_{z\to+\infty} h^t(x,z) \to +\infty$ and $\lim_{z\to-\infty} h^t(x,z) \to 0$ in $M \times \mathbb{R}$.
- There exists a uniform constant $C > 0$, independent of $t \in [0, 1]$, such that $|B^t|_{C^2(\bar{M})} < C$, $|h^t|_{C^2(\bar{M}\times[-C,C])} < C$ and $|\varphi^t|_{C^3(\bar{M})} < C$.

Applying our a priori estimates Proposition 3.1, 4.6 and 5.4 to $(\star_t)$ and noting that F is concave, we obtain, by Evans–Krylov estimates,

$$|u_t|_{C^{2,\alpha}(\bar{M})} \leq C \quad \text{for all } t \in [0, 1].$$

Combining this with Corollary 2.2, we see by standard degree theory that $(\star_t)$ is solvable for $t = 1$. Uniqueness follows by Lemma 4.3.

7. Proof of Theorem 1.4

To solve the Dirichlet problem for large boundary conditions, we need to control the behavior of the solution near the boundary. We can do this by constructing barrier functions for some suitable equation.

Recall that F is concave, then

$$F(\kappa) \le \omega \sum \kappa_i \quad \text{in } \Gamma^+$$

for some uniform constant $\omega > 0$. For any $C^2(\overline{M})$ admissible function u satisfying

$$F(W[u]) = f(x)e^{2u} \quad \text{in } M,$$

u is a subsolution of the equation

$$b_1 \Delta u + b_2 |\nabla u|^2 + b_3 = e^{2u}, \tag{7-1}$$

where

$$b_1 = \frac{\omega(n\lambda - 1)}{\min_{\overline{M}} f}, \quad b_2 = \frac{\omega(|a|_{L^\infty} + n|b|_{L\infty})}{\min_{\overline{M}} f} \quad b_3 = \frac{\omega|\operatorname{tr} B|_{L^\infty}}{\min_{\overline{M}} f}.$$

Before constructing a local supsolution of (7-1), we give some notation. Take a point $y_0 \in M_{\delta/4}$ near the boundary ∂M. Suppose $x_0 \in \partial M$ is the point that satisfies $d(y_0) = \operatorname{dist}_g(x_0, y_0)$. Consider a geodesic running from x_0, passing through y_0, and going out a small distance to a point z_0 with $\operatorname{dist}_g(z_0, x_0) = \eta$. We use $r(x)$ to denote the distance from z_0 to x with respect to the background metric g. We assume that δ and η are small enough that $r^2(x) = (\operatorname{dist}_g(x, z_0))^2$ is smooth in the ball $B_\eta(z_0)$. We may choose normal coordinates $\{e_k\}$. Then we have

$$\Delta r^2(z_0) = 2n.$$

We now assume

$$1 \le \Delta r^2 \le 3n \quad \text{in } B_\eta(z_0).$$

Consider the following auxiliary function defined in $B_\eta(z_0)$:

$$\overline{w}(x) = -\log(\eta^2 - r^2) + \theta \log \frac{\eta^2 - r^2 + \epsilon}{\epsilon} + \log 2 + \tfrac{1}{2}\log(nb_1 + b_2) + \log \eta,$$

where θ and ϵ are constants to be chosen later. It is easy to check that

$$\overline{w}_i = \frac{2rr_i}{\eta^2 - r^2} - \theta \frac{2rr_i}{\eta^2 - r^2 + \epsilon},$$

and

$$\overline{w}_{ij} = \frac{\nabla^2_{ij} r^2}{\eta^2 - r^2} + \frac{4r^2 r_i r_j}{(\eta^2 - r^2)^2} - \theta \frac{\nabla^2_{ij} r^2}{\eta^2 - r^2 + \epsilon} - \theta \frac{4r^2 r_i r_j}{(\eta^2 - r^2 + \epsilon)^2}.$$

Consequently, using $|\nabla r| = 1$ and $1 \le \Delta r^2 \le 3n$ in $B_\eta(z_0)$, we derive

$$\begin{aligned}
&b_1 \Delta \overline{w} + b_2 |\nabla \overline{w}|^2 + b_3 \\
&\qquad = b_1 \frac{\Delta r^2}{\eta^2 - r^2} + \frac{4(b_1 + b_2) r^2}{(\eta^2 - r^2)^2} - \frac{b_1 \theta \Delta r^2}{\eta^2 - r^2 + \epsilon} - \frac{4(b_1 - b_2\theta)\theta r^2}{(\eta^2 - r^2 + \epsilon)^2} \\
&\qquad\qquad - \frac{8 b_2 \theta r^2}{(\eta^2 - r^2)(\eta^2 - r^2 + \epsilon)} + b_3 \\
&\qquad \le \frac{3n b_1 \eta^2 + (3b_1 + 4b_2) r^2}{(\eta^2 - r^2)^2} - \frac{b_1 \theta}{\eta^2 - r^2 + \epsilon} - \frac{4(b_1 - b_2\theta)\theta r^2}{(\eta^2 - r^2 + \epsilon)^2} + b_3.
\end{aligned}$$

Now choosing $\theta < b_1/(2b_2)$, $\eta < \sqrt{b_1\theta/(2b_3)}$, $\epsilon < \eta^2$, and using $r \le \eta$, one obtains

$$b_1 \Delta \overline{w} + b_2 |\nabla \overline{w}|^2 + b_3 \le \frac{4(nb_1 + b_2)\eta^2}{(\eta^2 - r^2)^2} \le e^{2\overline{w}}.$$

Since $\overline{w}|_{\partial B_\eta(z_0)} = +\infty$, maximum principle implies

$$u \le \overline{w} \quad \text{ in } B_\eta(z_0);$$

hence

$$u(y_0) \le -\log d(y_0) + \theta \log \frac{2\eta d(y_0) + \epsilon}{\epsilon} + \log 2 + \tfrac{1}{2} \log(nb_1 + b_2). \tag{7-2}$$

Now we complete the proof as follows.

Proof of Theorem 2. We use the notation of Section 4. The argument here is similar to that in [Guan 2008]. Let's consider the locally defined auxiliary functions

$$v_m^\gamma := \gamma \log \frac{m\delta^2}{md + \delta^2} \quad \text{ in } M_\delta,$$

where γ is some small positive constant to be chosen later and $m = 1, 2, 3, \ldots$. It is direct to check that

$$\begin{gathered}
v_m^\gamma|_{\partial M} = \gamma \log m, \\
\gamma \log \tfrac{1}{2}\delta \le v_m^\gamma|_{\{d(x)=\delta\}} \le \gamma \log \delta.
\end{gathered} \tag{7-3}$$

By a direct computation, we obtain

$$
\begin{aligned}
W[v_m^\gamma]_{ij} &= \frac{(\lambda+b\gamma)\gamma m^2}{(md+\delta^2)^2}|\nabla d|^2 g_{ij} + \frac{a\gamma^2 m^2}{(md+\delta^2)^2} d_i d_j - \frac{\gamma m^2}{(md+\delta^2)^2} d_i d_j \\
&\qquad - \frac{\gamma m}{md+\delta^2}(\lambda \Delta d g_{ij} - d_{ij}) + B_{ij} \\
&\geq \frac{(\varepsilon_0 - (|a|_{L^\infty(\overline{M})} + |b|_{L^\infty(\overline{M})})\gamma)\gamma m^2}{(md+\delta^2)^2}|\nabla d|^2 g_{ij} \\
&\qquad - \frac{\gamma m}{md+\delta^2} C' g_{ij} - C'' g_{ij},
\end{aligned}
$$

where C' and C'' are some large constants depending only on λ, $|B|_{g(\overline{M})}$ and the geometric quantities of $(\overline{M}, g)$, independent of δ. Choosing

$$
\gamma \leq \frac{\varepsilon_0}{2(|a|_{L^\infty(\overline{M})} + |b|_{L^\infty(\overline{M})})} \quad \text{and} \quad \delta \leq \min\left\{1, \frac{\varepsilon_0}{16C'}, \frac{\varepsilon_0\gamma}{64C''}\right\},
$$

and observing that $|\nabla d| > 1/2$ in M_δ, we derive

$$
\begin{aligned}
W[v_m^\gamma]_{ij} &\geq \left(\frac{\varepsilon_0 m}{4(md+\delta^2)} - C'\right)\frac{\gamma m}{md+\delta^2} g_{ij} - C'' g_{ij} \\
&\geq \frac{\varepsilon_0 \gamma m^2}{8(md+\delta^2)^2} g_{ij} - C'' g_{ij} \\
&\geq \frac{\varepsilon_0 \gamma m^2}{16(md+\delta^2)^2} g_{ij}.
\end{aligned}
$$

Consequently, if $\gamma \leq \min\{1, \frac{1}{2}\varepsilon_0/(|a|_{L^\infty(\overline{M})} + |b|_{L^\infty(\overline{M})})\}$ and δ is small enough, then

$$
\begin{aligned}
F(W[v_m^\gamma]) &\geq \frac{\varepsilon_0 \gamma m^2}{16(md+\delta^2)^2} F(e) \\
&= \frac{\varepsilon_0 \gamma F(e)}{16\delta^4} \exp(2v_m^\gamma/\gamma) \\
&\geq f(x) e^{2v_m^\gamma}
\end{aligned}
\tag{7-4}
$$

in M_δ. For any integer $m \geq 1$, let $u_m \in C^\infty(\overline{M})$ be the admissible solution of the Dirichlet problem

$$
\begin{cases}
F(W[u]) = f(x)e^{2u} & \text{in } M, \\
u = \gamma \log m & \text{on } \partial M,
\end{cases}
$$

where γ is the constant has been fixed. Then (7-3), (7-4) and Lemma 4.3 imply

$$
u_m \geq v_m^\gamma = \gamma \log \frac{m\delta^2}{md+\delta^2}. \tag{7-5}
$$

Recalling (7-2), we obtain for any $m \geq 1$

$$u_m \leq -\log d + C. \tag{7-6}$$

Since $u_m \leq u_{m+1}$ for $m \geq 1$, and the u_m have the boundary control (7-5) and (7-6), the limit

$$u(x) := \lim_{m\to\infty} u_m(x)$$

exists for all $x \in M$ and satisfies

$$-C - \gamma \log d \leq u(x) \leq -\log d + C$$

near ∂M.

For any compact subset $K \subset M$, by the boundary control above and the a priori estimates of Proposition 3.1, Lemma 4.1 and Lemma 5.1, we obtain

$$|u_m|_{C^{2,\alpha}(K)} \leq C,$$

where $0 < \alpha < 1$, $C = C(K)$ is independent of m. Thus u is a solution of (1-5). □

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