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The zero-divisor graph of a ring R is the graph whose vertices consist of the nonzero zero-divisors of R in which two distinct vertices a and b are adjacent if and only if either $ab = 0$ or $ba = 0$. In this paper, we investigate some properties of zero-divisor graphs of Boolean rings. Among other results, we prove that for any two rings R and S with $\Gamma(R) \simeq \Gamma(S)$, if R is Boolean and $|R| > 4$, then $R \simeq S$.

1. Introduction

Throughout the paper, R denotes a ring, not necessarily with identity, and $\mathcal{D}(R)$ denotes the set of all zero-divisors of R . If X is either an element or a subset of R , then the *left annihilator* of X is $\text{Ann}_\ell(X) = \{a \in R \mid aX = 0\}$ and the right annihilator of X , denoted by $\text{Ann}_r(X)$, is similarly defined. For any subset Y of R , we let $Y^* = Y \setminus \{0\}$. The *zero-divisor graph* of R , denoted by $\Gamma(R)$, is a graph with the vertex set $\mathcal{D}(R)^*$ such that two vertices x and y are joined by an undirected edge if and only if $x \neq y$ and either $xy = 0$ or $yx = 0$. Notice that a ring R is a domain if and only if $\Gamma(R)$ is the null graph. For a commutative ring R with identity, the definition of a zero-divisor graph of R that was first introduced in [Beck 1988] coincides with the above definition of $\Gamma(R)$. The zero-divisor graph concept for noncommutative rings was first defined in [Redmond 2002]. The zero-divisor graphs offer a graphical representation of rings so that we may discover some new algebraic properties of rings that are hidden from the viewpoint of classical ring theorists. For an instance, using the notion of a zero-divisor graph, it has been proven in [Redmond 2004] that for any finite ring R , $\sum_{x \in R} |\text{Ann}_\ell(x) \setminus \text{Ann}_r(x)|$ is even. A simple proof of this result is given in [Akbari and Mohammadian 2007].

Let us recall some definitions regarding graph theory and ring theory. For a vertex v of a graph G , $\mathcal{N}(v)$ denotes the set of all vertices of G adjacent to v , and the *degree* of v is defined by $|\mathcal{N}(v)|$. A graph G is called a *star* if G contains at least two vertices and there exists a vertex that is joined to all other vertices

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and G has no other edges. A *path* \mathcal{P} in a graph G is a sequence of distinct vertices v_1, v_2, \dots, v_{k+1} in which every two consecutive vertices are adjacent. The number k is called the *length* of \mathcal{P} . For two vertices u and v in a graph G , the *distance* between u and v , denoted by $d(u, v)$, is the length of the shortest path between u and v , if such a path exists; otherwise, we define $d(u, v) = \infty$. The *diameter* of a graph G is defined by $\text{diam } G = \sup\{d(u, v) \mid u \text{ and } v \text{ are distinct vertices of } G\}$. In [Redmond 2002] it was shown that for any ring R , $\text{diam } \Gamma(R) \leq 3$. Furthermore, two graphs G_1 and G_2 are said to be *isomorphic* if there is a bijective map φ between the vertex set of G_1 and the vertex set of G_2 such that the adjacency relation is preserved. Finally, we recall that a ring is called *reduced* if it has no nonzero nilpotent elements. A ring whose elements are all idempotent is called *Boolean*. We denote by \mathbb{Z}_n the ring of integers modulo n and by \mathbb{F}_q the field with q elements.

In this article we study the zero-divisor graphs of Boolean rings. We show that for any reduced ring R that is not a domain, $\Gamma(R)$ is isomorphic to the zero-divisor graph of a nonreduced ring, provided that $\Gamma(R)$ is a star. As a consequence, we prove that Boolean rings with more than four elements are determined by their zero-divisor graphs.

2. The results

In [Akbari and Mohammadian 2006, Theorem 17], it is proven that for any finite ring R that is not a field, if $\Gamma(R)$ is isomorphic to the zero-divisor graph of a reduced ring S , then $R \simeq S$, unless $S \simeq \mathbb{Z}_2 \times \mathbb{F}_q$, where either $q = 2$ or $(q + 1)/2$ is a prime power. Since for any finite field F , $\Gamma(\mathbb{Z}_2 \times F)$ is a star, the following theorem presents an analogue of this result for the general case.

Remark 1. Let $\{\mathcal{A}_i\}_{i \in I}$ and $\{\mathcal{B}_j\}_{j \in J}$ be two families of commutative domains with identity, where $|I| \geq 2$. In [Anderson et al. 2003, Theorem 2.1], it is shown that $\Gamma(\prod_{i \in I} \mathcal{A}_i) \simeq \Gamma(\prod_{j \in J} \mathcal{B}_j)$ if and only if there is a bijective map $\pi : I \rightarrow J$ such that $|\mathcal{A}_i| = |\mathcal{B}_{\pi(i)}|$ for all $i \in I$. Hence there are many examples of nonisomorphic pairs of infinite reduced commutative rings whose zero-divisor graphs are isomorphic.

Theorem 2. *Let S be a reduced ring such that S is not a domain and $\Gamma(S)$ is not a star. If R is a ring such that $\Gamma(R) \simeq \Gamma(S)$, then R is also a reduced ring.*

Proof. We recall a well-known fact about reduced rings: for all elements x and y of a reduced ring T , $xy = 0$ if and only if $yx = 0$. For this, note that if $xy = 0$ for some elements $x, y \in T$, then $(yx)^2 = 0$ and since T is reduced, we find that $yx = 0$. This fact implies that if two vertices u and v of $\Gamma(S)$ are adjacent, then $uv = vu = 0$. We use this property frequently in what follows. We also state two properties of $\Gamma(S)$:

- (i) For every two adjacent vertices u and v of $\Gamma(S)$ with at least one common neighbor, $u + v$ is a vertex of $\Gamma(S)$ and $\mathcal{N}(u + v) = \mathcal{N}(u) \cap \mathcal{N}(v)$. For this, note that if $x \in \mathcal{N}(u) \cap \mathcal{N}(v)$, then $xu = xv = 0$, and hence $x(u + v) = 0$. Also, $u + v \neq 0$ since $uv = 0$ and S is reduced. Therefore, $x \in \mathcal{N}(u + v)$. Conversely, if $x \in \mathcal{N}(u + v)$, then $(xu)u = x(u + v)u = 0$ and thus $u(xu) = 0$. Therefore $(xu)^2 = 0$ and so $xu = 0$. This means that $x \in \mathcal{N}(u)$ and with a similar argument, we find that $x \in \mathcal{N}(v)$, as required.
- (ii) For every three mutually adjacent vertices u, v and w of $\Gamma(S)$, we have $\mathcal{N}(u) \not\subseteq \mathcal{N}(v) \cup \mathcal{N}(w)$. Indeed, it easily seen that $v + w \in \mathcal{N}(u) \setminus (\mathcal{N}(v) \cup \mathcal{N}(w))$.

Suppose that R is a ring with $\Gamma(R) \simeq \Gamma(S)$. So properties (i) and (ii) also hold for $\Gamma(R)$. To the contrary, assume that $a^2 = 0$ for some element $a \in R^*$.

Since S is reduced, [Akbari and Mohammadian 2006, Corollary 4] yields that $\Gamma(R)$ has at least two vertices. Note that a is not adjacent to all other vertices of $\Gamma(R)$. To prove this, suppose otherwise. Since $\Gamma(R)$ is not a star, there exist two adjacent vertices $x, y \in \mathcal{N}(a)$. So $\mathcal{N}(x) \subseteq \mathcal{N}(a) \cup \mathcal{N}(y)$, which contradicts (ii). Moreover, we have $|\mathcal{N}(a)| \geq 2$. For this, suppose otherwise. Since $\Gamma(R)$ is a connected graph [Redmond 2002] with at least two vertices, we may assume that $\mathcal{N}(a) = \{b\}$ for some vertex b of $\Gamma(R)$. From $a + b \in \text{Ann}_\ell(a) \cup \text{Ann}_r(a)$, we conclude that $a + b = 0$. Hence $b = -a$ and therefore $\Gamma(R)$ is a star on two vertices, a contradiction.

We claim that either $Ra = \{0, a\}$ or $aR = \{0, a\}$. Suppose that there exist two elements $b \in Ra \setminus \{0, a\}$ and $c \in aR \setminus \{0, a\}$. If $b \neq c$, then a, b and c are three mutually adjacent vertices and $\mathcal{N}(a) \subseteq \mathcal{N}(b) \cup \mathcal{N}(c)$, which contradicts (ii). Hence $b = c$. For some vertex $d \in \mathcal{N}(a) \setminus \{b\}$, the vertices a, b and d are mutually adjacent and $\mathcal{N}(a) \subseteq \mathcal{N}(b) \cup \mathcal{N}(d)$, which again contradicts (ii). Since $Ra \neq \{0\}$ and $aR \neq \{0\}$, the claim is proved.

We assume that $Ra = aR = \{0, a\}$. For any two vertices $x, y \notin \mathcal{N}(a)$, we have $xa = ya = a$. Thus $(xy)a = a$ and so $xy \neq 0$. This means that every edge of $\Gamma(R)$ has at least one endpoint in $\mathcal{N}(a)$. Working towards a contradiction, assume that no two vertices in $\mathcal{N}(a)$ are adjacent. This means that $\Gamma(R)$, and so $\Gamma(S)$ is a bipartite graph, and using [Akbari et al. 2003, Theorem 2.4], $\Gamma(S)$ and thus $\Gamma(R)$ is a complete bipartite graph. Let $r \notin \mathcal{N}(a)$ and $s \in \mathcal{N}(a) \cap \mathcal{N}(r)$. Since $\Gamma(R)$ is a complete bipartite graph and $a + s \in \mathcal{N}(a)$, r is adjacent to $a + s$. Therefore $a = r(a + s)r = 0$, a contradiction. Hence there are two adjacent vertices $b, c \in \mathcal{N}(a)$. We now consider the two following cases.

Case I. Suppose that a together with one of the elements b, c are contained in one of the one-sided annihilators of the third element. Without loss of generality, assume that $\{a, c\} \in \text{Ann}_\ell(b)$. By (i), there exists a vertex d in $\Gamma(R)$ such that $d \notin \{a, b\}$ and $\mathcal{N}(d) = \mathcal{N}(a) \cap \mathcal{N}(b)$. If $b \neq a + c$, then $a + c \in \mathcal{N}(a) \cap \mathcal{N}(b)$, and

hence $a = d(a + c)d = 0$, a contradiction. Thus $b = a + c$, and it follows from $ab = 0$ that $ac = 0$. Moreover, if $ca \neq 0$, then $a = ca = cb - c^2 = -c^2$, which contradicts $d \in \mathcal{N}(c) \setminus \mathcal{N}(a)$. Therefore $ca = 0$ and so $c^2 = cb - ca = 0$. Since $b = a + c$, we find that the product of any two elements of $\{a, b, c\}$ is zero.

Suppose towards a contradiction that there is a vertex $r \in (\mathcal{N}(b) \cap \mathcal{N}(c)) \setminus \{a\}$. We have $rar = r(b - c)r = 0$ and by $Ra = aR = \{0, a\}$, we deduce that $r \in \mathcal{N}(a)$. By (i), there exists a vertex s in $\Gamma(R)$ such that $\mathcal{N}(s) = \mathcal{N}(a) \cap \mathcal{N}(r)$. This implies that $sas = s(b - c)s = 0$, since $\{b, c\} \subseteq \mathcal{N}(a) \cap \mathcal{N}(r)$. On the other hand, $s \notin \mathcal{N}(a)$ and $Ra = aR = \{0, a\}$ yields that $sas = a$, a contradiction. This establishes that $\mathcal{N}(b) \cap \mathcal{N}(c) = \{a\}$.

For convenience and without loss of generality, assume that $cd = 0$. From $\{b, c\} \subseteq \text{Ann}_r(c)$, $d \notin \mathcal{N}(a) \cup \mathcal{N}(b)$ and $\mathcal{N}(b) \cap \mathcal{N}(c) = \{a\}$, we have $Rc = \{0, c\}$. Therefore $[R : \text{Ann}_\ell(a) \cap \text{Ann}_\ell(c)] \leq [R : \text{Ann}_\ell(a)][R : \text{Ann}_\ell(c)] = |Ra||Rc| = 4$. Since $\mathcal{N}(b) \cap \mathcal{N}(c) = \{a\}$ and the product of any two elements of $\{a, b, c\}$ is zero, we find that $\text{Ann}_\ell(a) \cap \text{Ann}_\ell(c) = \{0, a, b, c\}$. This yields that $|R| \leq 16$. Using (i), let e be a vertex of $\Gamma(R)$ in which $\mathcal{N}(e) = \mathcal{N}(a) \cap \mathcal{N}(c)$. It is not hard to see that

$$R = \{0, a, b, c\} \cup (d + \{0, a, b, c\}) \cup (e + \{0, a, b, c\}) \cup (d + e + \{0, a, b, c\}).$$

Therefore $\text{Ann}_\ell(a) = \text{Ann}_r(a) = \{0, a, b, c\} \cup (d + e + \{0, a, b, c\})$. Because $e \notin \mathcal{N}(a) \cup \mathcal{N}(c)$, $Ra = \{0, a\}$ and $Rc = \{0, c\}$, we conclude that $ea = a$ and $ec = c$. Therefore $eb = b$ and by $b \in \mathcal{N}(a) \cap \mathcal{N}(c)$, we obtain that $be = 0$. Furthermore, $e \notin \mathcal{N}(a) \cup \mathcal{N}(c)$ and $\mathcal{N}(b) \cap \mathcal{N}(c) = \{a\}$ yield that $Rb = \{0, b\}$. It follows from $d \notin \mathcal{N}(b)$ that $d + e \in \text{Ann}_\ell(b)$, and so $\mathcal{N}(a) \subseteq \mathcal{N}(b) \cup \mathcal{N}(c)$, which contradicts (ii).

Case II. When **Case I** does not occur, by replacing b with c if necessary, we may assume that $ab = bc = ca = 0$ and none of ba, cb and ac is zero. We have $\{a, b\} \in \text{Ann}_\ell(cb)$, and so, applying the argument in the first paragraph of **Case I** for cb and b instead of b and c , respectively, we obtain in particular that $ba = 0$, which is a contradiction.

Next, with no loss of generality, assume that $aR = \{0, a\}$ and there exists an element $g \in Ra \setminus \{0, a\}$. Since $aR = \{0, a\}$, $-a = a$ and so $-g = g$. Also, from $g \in Ra$ and $aR = \{0, a\}$, we easily obtain that $ag = ga = 0$. By (i), there exists a vertex h in $\Gamma(R)$ such that $\mathcal{N}(h) = \mathcal{N}(a) \cap \mathcal{N}(g)$. We claim that $\text{Ann}_r(a) \subseteq \text{Ann}_r(h) \cup \{0, a, g, a + g\}$. Suppose $x \in \text{Ann}_r(a) \setminus (\text{Ann}_r(h) \cup \{a, g\})$. Since $g \in Ra$ and $\mathcal{N}(h) = \mathcal{N}(a) \cap \mathcal{N}(g)$, we conclude that $x \in \text{Ann}_\ell(h)$. Moreover, $h \notin \mathcal{N}(a)$ and $aR = \{0, a\}$, so $ah = a$. We have $a + g \in \mathcal{N}(a) \cap \mathcal{N}(g)$ and $(a + g)h = a + g$, and hence $h(a + g) = 0$. These equalities yield that $h(a + g + x) = hx \neq 0$ and $(a + g + x)h = a + g \neq 0$. On the other hand, $a(a + g + x) = 0$, so it follows from $g \in Ra$ and $\mathcal{N}(h) = \mathcal{N}(a) \cap \mathcal{N}(g)$ that $a + g + x = 0$. Therefore, $x = a + g$, and the claim is proved. Since $\text{Ann}_r(a)$, $\text{Ann}_r(h)$ and $\{0, a, g, a + g\}$

are three additive subgroups of R in which $\text{Ann}_r(a) \subseteq \text{Ann}_r(h) \cup \{0, a, g, a + g\}$ and $a \in \text{Ann}_r(a) \setminus \text{Ann}_r(h)$, we deduce that $\text{Ann}_r(a) = \{0, a, g, a + g\}$. Applying (ii), there exists a vertex $y \in \mathcal{N}(a) \setminus (\mathcal{N}(g) \cup \mathcal{N}(a + g))$. We have $ya = 0$ and $ay \neq 0$. By $[R : \text{Ann}_r(a)] = |aR|$, we conclude that $R = \text{Ann}_r(a) \cup y + \text{Ann}_r(a)$. It follows from $\text{Ann}_r(a) = \{0, a, g, a + g\}$ that $Ra = \{0\}$, a contradiction. Now the proof is complete. \square

Example 3. The condition on $\Gamma(S)$ in Theorem 2 is necessary. For examples involving infinite rings, let \mathcal{S} be an arbitrary infinite domain, \mathcal{R} be the polynomial ring in the set of variables $\{x\} \cup \{x_\alpha \mid \alpha \in \mathcal{S}\}$ with coefficients in \mathbb{Z}_2 , and \mathcal{I} be the ideal of \mathcal{R} generated by $\{x^2\} \cup \{xx_\alpha - x \mid \alpha \in \mathcal{S}\} \cup \{x_\alpha x - x \mid \alpha \in \mathcal{S}\}$. It is easy to verify that $\Gamma(\mathcal{R}/\mathcal{I})$ is a star on $|\mathcal{S}|$ vertices and $x + \mathcal{I}$ is that vertex which is adjacent to all other vertices of the graph. Therefore $\Gamma(\mathcal{R}/\mathcal{I}) \simeq \Gamma(\mathbb{Z}_2 \times \mathcal{S})$, while \mathcal{R}/\mathcal{I} is not reduced.

Remark 4. It is easy to establish that every reduced ring whose zero-divisor graph is a star is isomorphic to the direct product of \mathbb{Z}_2 and a domain. For this, let R be a reduced ring with $\Gamma(R)$ a star and let e be that vertex which is adjacent to all other vertices of $\Gamma(R)$. Obviously, e is idempotent, and using the fact that all idempotent elements of a reduced ring are central, we may write $R \simeq eR \times (1 - e)R$. Since $\Gamma(R)$ is a star, we clearly conclude that $eR = \{0, e\}$ and $(1 - e)R$ is a domain, as required. From this, Theorem 2, Example 3, and [Akbari and Mohammadian 2006, Theorem 17], we imply that for every reduced ring R that is not a domain, $\Gamma(R)$ is isomorphic to the zero-divisor graph of a nonreduced ring if and only if $\Gamma(R)$ is either an infinite star or a star with q vertices, where either $q = 2$ or both q and $(q + 1)/2$ are prime powers.

In [LaGrange 2007, Theorem 4.1], it is shown that if R and S are two commutative rings with identity such that S is a Boolean ring with more than four elements and $\Gamma(R) \simeq \Gamma(S)$, then $R \simeq S$. In what follows, we generalize this result to every arbitrary ring R . We need the following easy lemmas.

Lemma 5. *Let R be a ring such that all elements in $\mathcal{D}(R)$ are idempotent. Then R is either a domain or a Boolean ring.*

Proof. Suppose that R is not a domain. By the hypotheses, R is reduced. Using the fact that all idempotent elements of a reduced ring are central, $\mathcal{D}(R)$ is contained in the center of R . Therefore, for every two elements $a \in R$ and $z \in \mathcal{D}(R)^*$, we have $az \in \mathcal{D}(R)$. Hence $(az)^2 = az$, and so $(a^2 - a)z = 0$. The latter equality shows that $a^2 - a \in \mathcal{D}(R)$ and also $\text{Ann}_\ell(a^2 - a) = \mathcal{D}(R)$. Thus $a^2 - a = (a^2 - a)^2 = 0$ for each element $a \in R$, as desired. \square

Lemma 6. *Let R be a Boolean ring with $|R| > 4$. Then $\Gamma(R)$ contains no vertex adjacent to all other vertices of the graph.*

Proof by contradiction. Suppose that a vertex r is adjacent to all other vertices of $\Gamma(R)$. Let $z \in \mathcal{D}(R) \setminus \{0, r\}$. We have $r(r+z) = r \neq 0$ and so $r+z$ is a nonzero-divisor idempotent of R . Thus $1 = r+z$ is the identity of R and so $R = \{0, 1, r, 1-r\}$, which contradicts $|R| > 4$. \square

Theorem 7. *Let S be a Boolean ring with $|S| > 4$. Suppose that R is a ring and $\varphi : \Gamma(R) \rightarrow \Gamma(S)$ is a graph isomorphism. Then φ is extendable to a ring isomorphism from R to S . In particular, $R \simeq S$.*

Proof. Recall that the characteristic of every Boolean ring is 2. We first state the following properties of $\Gamma(S)$.

- (i) For every two vertices u and v of $\Gamma(S)$, if $\mathcal{N}(u) = \mathcal{N}(v)$, then $u = v$. For this, note that if $u \neq uv$, then $u + uv \in \mathcal{N}(v) \setminus \mathcal{N}(u)$, which is impossible. So we conclude that $u = uv$, and similarly $v = uv$, which yield that $u = v$, as desired.
- (ii) For every two adjacent vertices u and v of $\Gamma(S)$, using (i) together with an easy argument, we find that $u + v$ is the unique vertex of $\Gamma(S)$ such that $\mathcal{N}(u + v) = \mathcal{N}(u) \cap \mathcal{N}(v)$, if $\mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset$; and otherwise, $1 = u + v$ is the identity of S , because in this case $u + v$ is a nonzero-divisor idempotent of S . Moreover, if S has identity, then $v = 1 + u$ is the unique neighbor of u in $\Gamma(S)$ such that $\mathcal{N}(u) \cap \mathcal{N}(v) = \emptyset$. For uniqueness, note that for any vertex $x \in \mathcal{N}(u)$, if $x \neq 1 + u$, then $1 + u + x \in \mathcal{N}(u) \cap \mathcal{N}(x)$.
- (iii) For every two nonadjacent vertices u and v of $\Gamma(S)$, $\mathcal{N}(u) \cup \mathcal{N}(v) \subseteq \mathcal{N}(uv)$; and if $\mathcal{N}(u) \cup \mathcal{N}(v) \subseteq \mathcal{N}(w)$ for some vertex w of $\Gamma(S)$, then $\mathcal{N}(uv) \subseteq \mathcal{N}(w)$. For the second statement, let $x \in \text{Ann}_\ell(uv)$. We have $vx \in \text{Ann}_\ell(u)$. Since $\mathcal{N}(u) \subseteq \mathcal{N}(w)$, $w(vx) = 0$ and so $wx \in \text{Ann}_\ell(v)$. It follows from $\mathcal{N}(v) \subseteq \mathcal{N}(w)$ that $w(wx) = 0$ and thus $x \in \text{Ann}_\ell(w)$, as required.

Since $\Gamma(R) \simeq \Gamma(S)$, the above properties also hold for $\Gamma(R)$. Using [Theorem 2](#) and [Lemma 6](#), R is reduced. It is easily checked that $\mathcal{N}(z^2) = \mathcal{N}(z)$ for each vertex z of $\Gamma(R)$. By (i), we have $z^2 = z$ for every element $z \in \mathcal{D}(R)$. Applying [Lemma 5](#), R is a Boolean ring. Define $\varphi(0) = 0$. By (ii), $\Gamma(R)$ (respectively, $\Gamma(S)$) contains two adjacent vertices with no common neighbors if and only if R (respectively, S) has identity. Since $\Gamma(R) \simeq \Gamma(S)$, either both R and S have identity or neither of them has identity. When the first case occurs, we define $\varphi(1) = 1$. Furthermore, the properties (ii) and (iii) imply that for every two vertices u and v of $\Gamma(S)$, the elements

- $1 + u$, if S has identity;
- $u + v$, if u and v are adjacent and $u + v \neq 1$; and
- uv , if u and v are not adjacent,

can be determined by $\Gamma(S)$. We claim that for every two distinct nonadjacent vertices u and v of $\Gamma(S)$, the element $u + v$ can also be determined by $\Gamma(S)$. First assume that $uv \notin \{u, v\}$. Using (iii), we obtain that the element uv is determined by $\Gamma(S)$. By (i) and (ii), $u + uv$ is the unique vertex of $\Gamma(S)$ such that $\mathcal{N}(u) = \mathcal{N}(uv) \cap \mathcal{N}(u + uv)$. This and a similar argument establish that the elements $u + uv$ and $v + uv$ are determined by $\Gamma(S)$. Since the vertices $u + uv$ and $v + uv$ are adjacent, we are done using (ii). Next, with no loss of generality, suppose that $uv = u$. In this case, the vertices u and $u + v$ are adjacent, and so applying (i) and (ii), we find that $u + v$ is the unique vertex of $\Gamma(S)$ such that $\mathcal{N}(v) = \mathcal{N}(u) \cap \mathcal{N}(u + v)$. This proves the claim. Now, by $\Gamma(R) \simeq \Gamma(S)$ and the above reasonings, it is not hard to verify that $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$, as desired. \square

As an interesting fact, it is well-known that every isomorphism between multiplicative semigroups of two Boolean rings is a ring isomorphism. Obviously, Theorem 7 generalizes this fact. The following theorem asserts that the zero-divisor graph of a Boolean ring R determines whether R has identity or not.

Theorem 8. *Let R be a Boolean ring and $|R| > 4$. Then $\text{diam } \Gamma(R) = 3$ if R has identity, and otherwise $\text{diam } \Gamma(R) = 2$.*

Proof. We know from [Redmond 2002] that for any ring T , $\text{diam } \Gamma(T) \leq 3$. First suppose that R has identity. Since $|R| > 4$, we can take an element $e \notin \{0, 1\}$. We have $R = eR \oplus (1 - e)R$, so either $|eR| > 2$ or $|(1 - e)R| > 2$. With no loss of generality, let $f \in eR \setminus \{0, e\}$. Since e and $1 + e + f$ are two nonadjacent vertices with no common neighbors and $\text{diam } \Gamma(R) \leq 3$, the result follows.

Next suppose that R has no identity. Applying Lemma 6, we find $\text{diam } \Gamma(R) \geq 2$. Now, let a and b be two nonadjacent vertices of $\Gamma(R)$. Since R has no identity, there exists an element c such that $(a + b + ab)c \neq c$. We have $c + ac + bc + abc \in \mathcal{N}(a) \cap \mathcal{N}(b)$, which clearly completes the proof. \square

It is well-known that every finite Boolean ring has identity. We generalize this fact in the following theorem.

Theorem 9. *Let R be a Boolean ring such that $\Gamma(R)$ has a vertex of finite degree. Then R has identity.*

Proof. Recall that the adjoint multiplication \circ of an arbitrary ring T is defined by $x \circ y = x + y + xy$ for any two elements $x, y \in T$. Suppose that a is a vertex of finite degree of $\Gamma(R)$ and $\mathcal{N}(a) = \{a_1, \dots, a_n\}$ for some integer $n \geq 1$. Let $b = a_1 \circ \dots \circ a_n$. Clearly, $ab = 0$ and $a_i b = a_i$ for all i . We show that $a + b$ is the identity of R . Indeed, it is enough to prove that $a + b$ is a nonzero-divisor. Toward a contradiction, assume that $(a + b)z = 0$ for some element $z \in R^*$. Multiplying this equality by a , we find that $az = 0$, and hence $z = a_j$ for some $j \in \{1, \dots, n\}$.

Also, multiplying the equality $(a + b)z = 0$ by a_j yields that $a_jz = 0$, which is impossible. This completes the proof. \square

Remark 10. The converse of [Theorem 9](#) is not true. Let \mathcal{R} be the set consisting of the empty set together with all finite unions of all left-closed right-open intervals and all left-unbounded right-open intervals of real numbers. Clearly, \mathcal{R} is a Boolean ring with identity with respect to symmetric difference as the addition operation and intersection as the multiplication operation, while obviously every vertex of $\Gamma(\mathcal{R})$ has infinite degree.

We conclude the paper with the following theorem on the polynomial rings over Boolean rings.

Theorem 11. *Let R and S be two Boolean rings such that $\Gamma(R[x]) \simeq \Gamma(S[x])$. Then $R \simeq S$.*

Proof. Let T be an arbitrary Boolean ring. $\Gamma(T[x])$ is the null graph if and only if $T \simeq \mathbb{Z}_2$. Hence we may assume that $\mathcal{D}(R)^*$ and $\mathcal{D}(S)^*$ are both nonempty. Using [Theorem 7](#), it suffices to establish that $\Gamma(R) \simeq \Gamma(S)$. Since finitely generated one-sided ideals of von Neumann regular rings, including Boolean rings, are principal [[Lam 2001](#), (4.23)], for each finitely generated ideal I of T , there exists a unique element e such that $I = (e)$. For a polynomial $f(x) = a_nx^n + \dots + a_0 \in T[x]$, let $\widehat{f(x)}$ be the unique element of T such that $(a_0, \dots, a_n) = (\widehat{f(x)})$. From [[Armendariz 1974](#), Lemma 1], every reduced ring is Armendariz, and hence it is not hard to see that for any polynomial $f(x) \in \mathcal{D}(T[x])^*$, $\widehat{f(x)}$ is the unique element of T such that $\mathcal{N}(f(x)) = \mathcal{N}(\widehat{f(x)})$.

Now, assume that $\phi : \Gamma(R[x]) \rightarrow \Gamma(S[x])$ is a graph isomorphism. We define $\psi : \Gamma(R) \rightarrow \Gamma(S)$ by $\psi(a) = \widehat{\phi(a)}$ for all $a \in \mathcal{D}(R)^*$, and we claim that ψ is a graph isomorphism. If a and b are two adjacent vertices of $\Gamma(R)$, then $\phi(a) \in \mathcal{N}(\phi(b)) = \mathcal{N}(\psi(b))$. This yields that $\psi(b) \in \mathcal{N}(\phi(a)) = \mathcal{N}(\psi(a))$ and therefore $\psi(a)$ and $\psi(b)$ are adjacent in $\Gamma(S)$. The converse is clearly true, and so ψ preserves the adjacency relation. Moreover, if $\psi(a) = \psi(b)$ for two vertices a and b of $\Gamma(R)$, then $\mathcal{N}(\phi(a)) = \mathcal{N}(\phi(b))$ and thus $\mathcal{N}(a) = \mathcal{N}(b)$. In particular, $\mathcal{N}(a) \cap R = \mathcal{N}(b) \cap R$. Using the property (i) of the zero-divisor graphs of Boolean rings given in the proof of [Theorem 7](#), we deduce that $a = b$. This concludes the injectivity of ψ . Finally, we prove that ψ is surjective. Suppose $s \in \mathcal{D}(S)^*$ and let

$$r = \widehat{\phi^{-1}(s)}.$$

Since $\mathcal{N}(\phi^{-1}(s)) = \mathcal{N}(r)$, we find that $\mathcal{N}(s) = \mathcal{N}(\phi(r)) = \mathcal{N}(\psi(r))$ and hence $s = \psi(r)$. This establishes the claim and completes the proof. \square

Remark 12. Let $n \geq 2$ and \mathcal{R} and \mathcal{S} be two rings which each of them is the direct product of n arbitrary finite fields. Using the result mentioned in [Remark 1](#), it is

easily checked that $\Gamma(\mathcal{R}[x]) \simeq \Gamma(\mathcal{S}[x])$. Therefore the conclusion of [Theorem 11](#) is not true if one of R and S is not Boolean.

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