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UNIQUENESS OF THE FOLIATION OF CONSTANT MEAN CURVATURE SPHERES IN ASYMPTOTICALLY FLAT 3-MANIFOLDS

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This paper studies the constant mean curvature surface in asymptotically flat 3-manifolds with general asymptotics. Under some weak conditions, the foliation of stable spheres of constant mean curvature is shown to be unique outside some compact set in the asymptotically flat 3-manifold with positive mass.

1. Introduction

A three-manifold M with a Riemannian metric g and a two-tensor K is called an initial data set (M, g, K) if g and K satisfy the constraint equations

$$R_g - |K|_g^2 + (\mathrm{tr}_g(K))^2 = 16\pi\rho \quad \text{and} \quad \mathrm{div}_g(K) - d(\mathrm{tr}_g(K)) = 8\pi J, \tag{1-1}$$

where R_g is the scalar curvature of the metric g, $\mathrm{tr}_g(K)$ denotes $g^{ij}K_{ij}$, ρ is the observed energy density, and J is the observed momentum density.

Definition 1.1. Let $q \in \left(\frac{1}{2}, 1\right]$. An initial data set (M, g, K) is called an *asymptotically flat* (AF) manifold if there is a compact subset $\tilde{K} \subset M$ and coordinates $\{x^i\}$ with the following properties: $M \setminus \tilde{K}$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$; ρ and J from (1-1) satisfy $\rho(x) = O(|x|^{-2-2q})$ and $J(x) = O(|x|^{-2-2q})$; and

$$g_{ij}(x) = \delta_{ij} + h_{ij}(x) \quad \text{with} \quad h_{ij}(x) = O_5(|x|^{-q})K_{ij}(x) = O_1(|x|^{-1-q}),$$

where $f = O_k(|x|^{-q})$ means $\partial^l f = O(|x|^{-l-q})$ for $l = 0, \dots, k$. We call $M \setminus \tilde{K}$ an *end* of the AF manifold (M, g, K); we will only consider AF manifolds with one end. The *mass* of this end is defined as

$$m = \lim_{r\to\infty} \frac{1}{16\pi} \int_{|x|=r} (h_{ij,j} - h_{jj,i}) v_g^i \, d\mu_g,$$

where v_g and $d\mu_g$ are the unit normal vector and volume form with respect to the metric g. From [Bartnik 1986], we know the mass is well defined when $q > \frac{1}{2}$.

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Given a function f, let $f^{\text{odd}}(x) = f(x) - f(-x)$ and $f^{\text{even}}(x) = f(x) + f(-x)$.

Definition 1.2. An AF manifold (M, g, K) is said to satisfy the Regge–Teitelboim condition, and is called an AF-RT manifold, if ρ and J satisfy

$$\rho^{\text{odd}}(x) = O(|x|^{-3-2q}), \quad J^{\text{odd}}(x) = O(|x|^{-3-2q})$$

and g, K satisfy the asymptotically even/odd conditions

$$h_{ij}^{\text{odd}}(x) = O_2(|x|^{-1-q}), \quad K_{ij}^{\text{even}}(x) = O_1(|x|^{-2-q}).$$

For AF-RT manifolds, the *center of mass* C is defined by

$$C^\alpha = \frac{1}{16\pi m} \lim_{r\to\infty} \biggl(\int_{|x|=r} x^\alpha (h_{ij,i} - h_{ii,j}) v_g^j \, d\mu_g - \int_{|x|=r} (h_{i\alpha} v_g^i - h_{ii} v_g^\alpha) \, d\mu_g \biggr).$$

From [Huang 2009], we know the center of mass is well defined.

Let Σ be a surface of constant mean curvature (CMC). We say that Σ is *stable* if the second variation operator has only nonnegative eigenvalues when restricted to the functions with 0 mean value, that is,

$$\int_\Sigma (|A|^2 + \text{Ric}(v_g, v_g)) f^2 \, d\mu \le \int_\Sigma |\nabla f|^2 \, d\mu$$

for f a function with $\int_\Sigma f \, d\mu = 0$, where A is the second fundamental form and $\text{Ric}(v_g, v_g)$ is the Ricci curvature in the normal direction with respect to the metric g.

We discuss the existence and uniqueness of CMC spheres that separate the compact part from infinity in AF-RT manifolds. The following two theorems are due to Lan-Hsuan Huang [2010]:

Theorem 1.3 (existence). *If (M, g, K) is AF-RT with $q \in \bigl(\frac{1}{2}, 1\bigr]$ and $m > 0$, there exists a foliation by spheres $\{\Sigma_R\}$ with constant mean curvature $H(\Sigma_R) = 2/R + O(R^{-1-q})$ in the exterior region of M. Each leaf Σ_R is a $(c_0 R^{1-q})$-graph over $S_R(C)$ and is strictly stable.*

Set $r(x) = \bigl(\sum (x^i)^2\bigr)^{1/2}$. For a CMC sphere Σ separating infinity from $\tilde{K}$, define

$$r_0(\Sigma) = \inf\{r(x) \mid x \in \Sigma\}, \quad r_1(\Sigma) = \sup\{r(x) \mid x \in \Sigma\}.$$

Theorem 1.4 (uniqueness). *Assume that (M, g, K) is AF-RT with $q \in \bigl(\frac{1}{2}, 1\bigr]$ and $m > 0$. There exist σ_1 and C_1 such that, if Σ is a topological sphere of constant mean curvature $H = H(\Sigma_R)$ for some $R \ge \sigma_1$, and moreover Σ is stable and satisfies $r_1(\Sigma) \le C_1 r_0^{1/a}$ for some $a \in \bigl(\frac{5-q}{2(2+q)}, 1\bigr]$, then $\Sigma = \Sigma_R$.*

Uniqueness is the harder problem. In Theorem 1.4, Huang needs the assumption $r_1 \le C_1 r_0^{1/a}$ for the radius of the surface. To get a sharper uniqueness result as in [Qing and Tian 2007], we consider metrics of the following form:

Definition 1.5. An AF-RT manifold (M, g, K) with mass m is called an (m, k, ε)-*AF-RT manifold*, where $k > 2$ and $\varepsilon > 0$, if the metric g can be expressed as

$$g_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + h^1_{ij}(\theta)/r + Q, \tag{1-2}$$

with $h^1_{ij}(\theta) \in C^5(S^2)$, $Q = O_5(|x|^{-2})$, and

$$\|h^1_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \le \varepsilon. \tag{1-3}$$

Here $\theta = (\theta_1, \theta_2)$ is the coordinate on $S^2 \subset \mathbb{R}^3$.

Remark 1.6. From (1-3), we know that the mass m has positive two-side bounds. We can certainly consider the case $\|h^1_{ij}(\theta) - C\delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \le \varepsilon$ for any positive C, but here we only assume $C = 1$ without lost of generality.

Our main uniqueness result is this:

Theorem 1.7. *For any $k > 2$ there exists $\varepsilon > 0$, depending only on k, with the following property. For any (m, k, ε)-AF-RT manifold (M, g, K), there exists a compact $\tilde{K}$ and a constant $C > 0$ such that, for any constant $H > 0$ sufficiently small, there is a unique stable CMC sphere Σ separating $\tilde{K}$ from infinity and such that $H(\Sigma) = H$ and $\log r_1(\Sigma) \le C r_0(\Sigma)^{1/4}$.*

Remark 1.8. This is an improvement on Huang's result, as can be seen by comparing $r_1 \le C_1 r_0^{1/a}$ with $\log r_1(\Sigma) \le C r_0(\Sigma)^{1/4}$ (since $\log r_1$ grows much more slowly than any positive power of r_1).

Remark 1.9. In the proof of Theorem 1.7, the RT condition is needed only for the existence theorem in [Huang 2010].

Remark 1.10. Here I can only deal with the case when $q = 1$. When $q \in \left(\frac{1}{2}, 1\right)$ it seems that $\|h_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{k,2}(S^2)} \le \varepsilon$ is not a proper condition.

The paper is organized much like [Qing and Tian 2007]: In Section 2 we give an a priori estimate on stable CMC spheres based on Simon's identity. In Section 3, we introduce blow-down analysis in three different scales. In Section 4 we recall the asymptotic analysis from [Qing and Tian 1997] and prove a technical lemma. In Section 5 we introduce asymptotically harmonic coordinates. In Section 6 we introduce the notion of the center of mass and prove Theorem 1.7.

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2. Curvature estimates

In this section and the next we assume that (M, g, K) is AF-RT with $q \in \left(\frac{1}{2}, 1\right]$. Let Σ be a CMC sphere in the asymptotically flat end $(M \setminus \tilde{K}, g)$, and assume Σ separates the compact part from infinity. First we have the following estimate, similar to [Huisken and Yau 1996, Lemma 5.2].

Lemma 2.1. *Let $X = x^i(\partial/\partial x^i)$ be the Euclidean coordinate vector field and*

$$r = \left(\sum (x^i)^2\right)^{1/2}.$$

With respect to the metric g, let v be the outward normal vector field, $d\mu$ be the volume form of Σ. Then we have the estimate

$$\int_\Sigma \langle X, v\rangle^2 r^{-4}\, d\mu \le H^2|\Sigma|.$$

Moreover for each $a \ge a_0 > 2$ and r_0 sufficiently large,

$$\int_\Sigma r^{-a}\, d\mu \le C(a_0) r_0^{2-a} H^2 |\Sigma|.$$

Proof. Because the mean curvature H is constant, for some smooth vector field Y on Σ, we have the divergence formula

$$\int_\Sigma \operatorname{div}_\Sigma Y\, d\mu = H \int_\Sigma \langle Y, v\rangle\, d\mu.$$

Choose $Y = Xr^{-a}$, $a \ge 2$ and let e_α be the orthonormal basis on Σ, $\alpha = 1, 2$. Supposing $e_\alpha = a_\alpha^i(\partial/\partial x^i)$, it is obvious that a_α^i is bounded because the manifold is asymptotically flat. Then

$$\begin{aligned}
\operatorname{div}_\Sigma Y = \operatorname{div}_\Sigma(Xr^{-a}) &= \langle \nabla_{e_\alpha}(Xr^{-a}), e_\alpha\rangle \\
&= r^{-a} \operatorname{div}_\Sigma X - ar^{-a-2} a_\alpha^i a_\alpha^j x^i x^j + O(r^{-a-q}) \\
&= r^{-a} \operatorname{div}_\Sigma X - \alpha r^{-a-2} |X^\tau|^2 + O(r^{-a-q}),
\end{aligned}$$

where X^τ is the tangent projection of X. Also,

$$|\operatorname{div}_\Sigma X - 2| = O(r^{-q}).$$

Note that $|X^\tau|^2 = r^2 - \langle X, \nu\rangle^2 + O(r^{2-q})$. Combining all of these,

$$\left|(2-a)\int_\Sigma r^{-a}\,d\mu + a\int_\Sigma \langle X,\nu\rangle^2 r^{-a-2}\,d\mu - H\int_\Sigma \langle X,\nu\rangle r^{-a}\,d\mu\right| \le C\int_\Sigma r^{-a-q}\,d\mu. \tag{2-1}$$

Choosing $a = 2$, from Hölder's inequality, we have

$$\int_\Sigma \langle X,\nu\rangle^2 r^{-4}\,d\mu \le \tfrac{1}{4}H^2|\Sigma| + C\int_\Sigma r^{-2-q}\,d\mu. \tag{2-2}$$

Then choosing $a = 2+q$ gives

$$\int_\Sigma r^{-2-q}\,d\mu \le 4r_0^{-q}\left(\int_\Sigma \langle X,\nu\rangle^2 r^{-4}\,d\mu + H^2|\Sigma| + C\int_\Sigma r^{-2-q}\,d\mu\right).$$

This combined with (2-2) implies

$$\int_\Sigma \langle X,\nu\rangle^2 r^{-4}\,d\mu \le H^2|\Sigma|.$$

Again from (2-1), we have for $a \ge a_0 > 2$,

$$\int_\Sigma r^{-a} \le C(a_0-2)^{-1} r_0^{2-a} H^2|\Sigma|. \qquad \square$$

Now we can derive the integral estimate for $|\mathring{A}|$ from the stability of the surface as in [Huisken and Yau 1996, Proposition 5.3]:

Lemma 2.2. *Suppose Σ is a stable CMC sphere in an asymptotically flat manifold. For r_0 sufficiently large*,

$$\int_\Sigma |\mathring{A}|^2\,d\mu \le C r_0^{-q},$$
$$H^2|\Sigma| \le C,$$
$$\int_\Sigma H^2\,d\mu = 16\pi + O(r_0^{-q}).$$

Proof. Since Σ is stable,

$$\int_\Sigma |\nabla f|^2\,d\mu \ge \int_\Sigma (|A|^2 + \mathrm{Ric}(\nu,\nu)) f^2\,d\mu$$

for any function f with $\int_\Sigma f\,d\mu = 0$, where A is the second fundamental form of Σ and Ric is the Ricci curvature of M.

Choose ψ to be a conformal map of degree 1 from Σ to the standard S^2 in $\mathbb{R}^3$. Each component ψ_i of ψ can be chosen such that $\int \psi_i\,d\mu = 0$ [Li and Yau 1982].

For each ψ_i,

$$\int_\Sigma |\nabla\psi_i|^2\,d\mu = \frac{8\pi}{3}.$$

Since $\sum \psi_i^2 \equiv 1$ we conclude that

$$\int_\Sigma |A|^2 + \mathrm{Ric}(v, v)\,d\mu \le 8\pi.$$

From the Gauss equation

$$\tfrac{1}{2}|A|^2 + \mathrm{Ric}(v, v) - \tfrac{1}{2}R + K = \tfrac{1}{2}H^2, \tag{2-3}$$

we have

$$|A|^2 + \mathrm{Ric}(v, v) = \tfrac{1}{2}|\mathring{A}|^2 + \tfrac{3}{4}H^2 + \tfrac{1}{2}R - K,$$

where K is the Gauss curvature of Σ and $\mathring{A}$ is defined as $\mathring{A}_{ij} = A_{ij} - (H/2)g_{ij}$.

Then

$$\int_\Sigma \tfrac{1}{2}|\mathring{A}|^2 + \tfrac{3}{4}H^2|\Sigma| \le 12\pi + r_0^{-q}H^2|\Sigma|$$

because $R = O(r^{-2-2q})$, from the constraint equation (1-1). So $H^2|\Sigma| \le 16\pi$.

Using the Gauss equation in a different way, we have

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2\,d\mu &= \int_\Sigma |A|^2 - \frac{H^2}{2}\,d\mu\\ &= \frac{1}{2}\int_\Sigma |A|^2 + \mathrm{Ric}(v, v)\,d\mu + \frac{1}{2}\int_\Sigma R - 3\,\mathrm{Ric}(v, v) - 2K\,d\mu\\ &\le \int_\Sigma r^{-2-q}\,d\mu\\ &= O(r_0^{-q}).\end{aligned}$$

Then again from the Gauss equation (2-3),

$$\int_\Sigma H^2\,d\mu = 4\int_\Sigma K\,d\mu + O(r_0^{-q}) = 16\pi + O(r_0^{-q}).$$ □

Lemma 2.3. *Suppose M is a constant mean curvature surface in an asymptotically flat end* $(\mathbb{R}^3 \setminus B_1(0), g)$. *Then*

$$\int_\Sigma H_e^2\,d\mu_e = 16\pi + O(r_0^{-q}).$$

Proof. We follow the calculation of Huisken and Ilmanen [2001] to obtain

$$g_{ij} = \delta_{ij} + h_{ij}.$$

Suppose that

$$g_{ij}|_\Sigma = f_{ij}, \quad \delta_{ij}|_\Sigma = \varepsilon_{ij},$$

where f^{ij} and ϵ^{ij} are the corresponding inverse matrices. Let $v, \omega, A, H, d\mu$ represent the normal vector, the dual form of v, the second fundamental form, the mean curvature and the volume form of Σ in the metric g and $v_e, \omega_e, A_e, H_e, \mu_e$ represent the corresponding ones in Euclidean metric. Easy calculation gives

$$f^{ij} - \varepsilon^{ij} = -f^{ik}h_{kl}f^{lj} \pm C|h|^2, \tag{2-4}$$

$$g^{ij} - \delta^{ij} = -g^{ik}h_{kl}g^{lj} \pm C|h|^2, \tag{2-5}$$

$$\omega = \frac{\omega_e}{|\omega_e|}, \quad v^i = g^{ij}\omega_j, \tag{2-6}$$

$$(\omega_e)_i = \omega_i \pm C|P|, \quad v_e^i = v^i + C|h|, \quad 1 - |\omega_e| = \tfrac{1}{2}h_{ij}v^iv^j, \tag{2-7}$$

$$\Gamma_{ij}^k = \tfrac{1}{2}g^{kl}(\overline{\nabla}_i h_{jl} + \overline{\nabla}_j h_{il} - \overline{\nabla}_l h_{ij}) \pm C|h| \pm C|\overline{\nabla} h|, \tag{2-8}$$

where Γ_{ij}^k is the Christoffel symbol for $\overline{\nabla} - \overline{\nabla}_e$ and we denote the gradient for the metrics g and δ by $\overline{\nabla}$ and $\overline{\nabla}_e$.

We have the formula

$$|\omega_e|_g A_{ij} = (A_e)_{ij} - (\omega_e)_k \Gamma_{ij}^k. \tag{2-9}$$

This implies

$$\begin{aligned} H - H_e &= f^{ij}A_{ij} - \varepsilon^{ij}(A_e)_{ij} \\ &= (f^{ij} - \varepsilon^{ij})A_{ij} + \varepsilon^{ij}A_{ij}(1 - |\omega_e|_g) + \varepsilon^{ij}(|\omega_e|_g A_{ij} - (A_e)_{ij}). \end{aligned}$$

From (2-4), (2-5), (2-7),

$$\varepsilon^{ij}A_{ij}(1 - |\omega_e|_g) = \tfrac{1}{2}Hv^iv^jh_{ij} \pm C|h|^2|A|.$$

Using (2-4)–(2-9), we obtain

$$\begin{aligned} \varepsilon^{ij}(|\omega_e|A_{ij} - (A_e)_{ij}) &= -\varepsilon^{ij}(\omega_e)_k\Gamma_{ij}^k \\ &= -\tfrac{1}{2}f^{ij}\omega_k g^{kl}(\overline{\nabla}_i h_{jl} + \overline{\nabla}_j h_{il} - \overline{\nabla}_l h_{ij}) \pm C|h|\,|\overline{\nabla} h| \\ &= -f^{ij}v^l\overline{\nabla}_i h_{jl} + \tfrac{1}{2}f^{ij}v^l\overline{\nabla}_l h_{ij} \pm C|h|\,|\overline{\nabla} h|. \end{aligned}$$

At last,

$$\begin{aligned} H - H_e = -f^{ik}h_{kl}f^{lj}A_{ij} + \tfrac{1}{2}Hv^iv^jh_{ij} - f^{ij}v^l\overline{\nabla}_i h_{jl} \\ + \tfrac{1}{2}f^{ij}v^l\overline{\nabla}_l h_{ij} \pm C|h|\,|\overline{\nabla} h| \pm C|h|^2|A|, \end{aligned} \tag{2-10}$$

and

$$\begin{aligned}\int_\Sigma H_e^2\, d\mu_e &= (1+O(r_0^{-q}))\int_\Sigma H_e^2\, d\mu\\ &\le (1+O(r_0^{-q}))\Big(\int_\Sigma H^2\, d\mu + \int_\Sigma (H_e-H)^2 + 2|H(H_e-H)|\, d\mu\Big)\\ &\le (1+O(r_0^{-q}))\Big(16\pi + O(r_0^{-q}) + \int_\Sigma (H_e-H)^2\\ &\qquad\qquad + \Big(\int_\Sigma H^2\, d\mu\Big)^{1/2}\Big(\int_\Sigma (H_e-H)^2\, d\mu\Big)^{1/2}\Big)\end{aligned}$$

$$\begin{aligned}\int (H_e-H)^2\, d\mu &\le \int O(|x|^{-2q})|A|^2 + H^2 O(|x|^{-2q}) + O(|x|^{-2-2q})\, d\mu\\ &\le \int O(|x|^{-2q})H^2 + O(|x|^{-2q})|\mathring{A}|^2 + O(|x|^{-2-2q})\, d\mu\\ &= O(r_0^{-2q}),\end{aligned}$$

so we have

$$\int_\Sigma H_e^2\, d\mu_e \le 16\pi + O(r_0^{-q}).$$

On the other hand, by Euler's formula,

$$K_e = \tfrac{1}{4}H_e^2 - \tfrac{1}{2}|\mathring{A}_e|^2.$$

So we have

$$\int H_e^2\, d\mu_e \ge 16\pi$$

which implies

$$\int_\Sigma H_e^2\, d\mu_e = 16\pi + O(r_0^{-q}). \qquad \square$$

Based on [Michael and Simon 1973] we have the following Sobolev inequality.

Lemma 2.4. *Suppose Σ is a constant mean curvature surface in an asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ with $r_0(\Sigma)$ sufficiently large and that $\int_\Sigma H^2 \le C$. Then*

$$\Big(\int_\Sigma f^2\, d\mu\Big)^{1/2} \le C\Big(\int_\Sigma |\nabla f|\, d\mu + \int_\Sigma H|f|\, d\mu\Big). \tag{2-11}$$

Proof. Note that this is valid for a surface in Euclidean space. So by the uniform equivalence of the metrics g and δ, we have

$$\Big(\int |f|^2\, d\mu\Big)^{1/2} \le C\Big(\int |f|^2\, d\mu_e\Big)^{1/2} \le C\Big(\int |\nabla f| + H|f| + |H-H_e|\,|f|\, d\mu\Big).$$

To bound the last term on the right, we use

$$\int |H - H_e|\,|f|\,d\mu \le \int O(|x|^{-q})|A|\,|f| + O(|x|^{-q})H|f| + O(|x|^{-1-q})|f|\,d\mu$$
$$\le O(r_0^{-q})\int H|f| + \Big(\int |\mathring{A}|^2\,d\mu\Big)^{1/2} O(r_0^{-q})\|f\|_{L^2} + O(r_0^{-q})\|f\|_{L^2}.$$

So we can choose r_0 sufficiently large to get the desired result. □

Lemma 2.5. *Suppose Σ is a constant mean curvature surface in an asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ with $r_0(\Sigma)$ sufficiently large. Then*

$$C_1 H^{-1} \le \operatorname{diam}(\Sigma) \le C_2 H^{-1}.$$

where $\operatorname{diam}(\Sigma)$ *denotes the diameter of Σ in the Euclidean space $\mathbb{R}^3$. In particular, if the surface Σ separates infinity from the compact part, then*

$$C_1 H^{-1} \le r_1(\Sigma) \le C_2 H^{-1}.$$

Proof. We already know that

$$\int_\Sigma H_e^2\,d\mu_e = 16\pi + O(r_0^{-q}).$$

Then from [Simon 1993, Lemma 1.1],

$$\sqrt{\frac{2|\Sigma|_e}{F(\Sigma)}} \le \operatorname{diam}(\Sigma) \le C\sqrt{|\Sigma|_e F(\Sigma)},$$

where $F(\Sigma) = \frac{1}{2}\int_\Sigma H_e^2$ is the Willmore functional and $|\Sigma|_e$ is the volume of Σ with respect to the Euclidean metric. Since the Euclidean metric is uniformly equivalent to g, we get the result. □

To get the pointwise estimate for $\mathring{A}$, we use Simon's identity (2-12) below and Moser's iteration argument.

Lemma 2.6 [Schoen et al. 1975]. *Suppose N is a hypersurface in a Riemannian manifold (M, g). Then the second fundamental form satisfies the identity*

$$\Delta A_{ij} = \nabla_i\nabla_j H + HA_{ik}A_{jk} - |A|^2 A_{ij} + HR_{3i3j} - A_{ij}R_{3k3k} + A_{jk}R_{klil} + A_{ik}R_{kljl} - 2A_{lk}R_{iljk} + \overline{\nabla}_j R_{3kik} + \overline{\nabla}_k R_{3ijk}, \tag{2-12}$$

where R_{ijkl} and $\overline{\nabla}$ are the curvature and gradient operator of (M, g).

From this we easily deduce for CMC surfaces the inequality

$$-|\mathring{A}|\Delta|\mathring{A}| \le |\mathring{A}|^4 + CH|\mathring{A}|^3 + CH^2|\mathring{A}|^2 + C|\mathring{A}|^2|x|^{-2-q} + CH|\mathring{A}|\,|x|^{-2-q} + C|\mathring{A}|\,|x|^{-3-q}.$$

We also need an inequality for $\nabla\mathring{A}$ because we also want to estimate the higher derivative:

$$-|\nabla\mathring{A}|\Delta|\nabla\mathring{A}| \le C|\nabla\mathring{A}|^2\big(|\mathring{A}|^2 + H|\mathring{A}| + H^2 + O(|x|^{-2-q})\big) + |\nabla\mathring{A}|\big((|\mathring{A}|^2 + H|\mathring{A}| + H^2)O(|x|^{-2-q}) + (|\mathring{A}| + H)O(|x|^{-3-q}) + O(|x|^{-4-q})\big).$$

Then we can get the pointwise estimates for $\mathring{A}$ and $\nabla\mathring{A}$.

Theorem 2.7 [Qing and Tian 2007]. *Suppose that* $(\mathbb{R}^3 \setminus B_1(0), g)$ *is an asymptotically flat end. Then there exist positive numbers* σ_0, δ_0 *such that for any constant mean curvature surface in the end which separates infinity from the compact part, we have*

$$|\mathring{A}|^2(x) \le C|x|^{-2}\int_{B_{\delta_0|x|}(x)} |\mathring{A}|^2\,d\mu + C|x|^{-2-2q} \le C|x|^{-2}r_0^{-q} \tag{2-13}$$

and

$$|\nabla\mathring{A}|^2(x) \le C|x|^{-2}\int_{B_{\delta_0|x|}(x)} |\nabla\mathring{A}|^2\,d\mu + C|x|^{-4-2q} \le C|x|^{-4}r_0^{-q/2}, \tag{2-14}$$

provided that $r_0 \ge \sigma_0$.

Proof. In the Sobolev inequality (2-11), take $f = u^2$. Then

$$\Big(\int_\Sigma u^4\,d\mu\Big)^{1/2} \le C\Big(2\int_\Sigma |u||\nabla u|\,d\mu + \int_\Sigma Hu^2\,d\mu\Big) \le C\Big(\int_\Sigma u^2\Big)^{1/2}\Big(\int_\Sigma |\nabla u|^2\,d\mu\Big)^{1/2} + C\Big(\int_{\mathrm{supp}(u)} H^2\,d\mu\Big)^{1/2}\Big(\int_\Sigma u^4\,d\mu\Big)^{1/2}.$$

To proceed, we need some auxiliary results.

Lemma 2.8. *For any* $\varepsilon > 0$, *we can find a uniform* δ_0 *sufficiently small such that*

$$\int_{B_{\delta_0|x|}(x)} H^2\,d\mu \le \varepsilon \quad \textit{for any } x \in \Sigma.$$

Proof. The metric g is equivalent to Euclidean metric δ. Thus we need only to prove that there exists C such that

$$|B_{\delta_0|x|}(x)|_e \le C\delta_0^2|x|^2,$$

because then

$$H^2|B_{\delta_0|x|}(x)|_e \le C\delta_0^2|x|^2H^2 \le C\delta_0^2.$$

From the proof of Lemma 1.1 in [Simon 1993] we know that, for any $x \in \Sigma$, if $B_\sigma(x)$ denotes the Euclidean ball of radius σ with center x in $\mathbb{R}^3$ and $\Sigma_\sigma = \Sigma \cap B_\sigma(x)$, then there exists C such that for $0 < \sigma \le \rho < \infty$,

$$\sigma^{-2}|\Sigma_\sigma|_e \le C(\rho^{-2}|\Sigma_\rho| + F(\Sigma_\rho)),$$

where $F(\Sigma_\rho)$ is the Willmore functional. The constant C does not depend on Σ, σ, ρ.

Letting $\rho \to \infty$, $\rho^{-2}|\Sigma_\rho| \to 0$ gives

$$\sigma^{-2}|\Sigma_\sigma|_e \le CF(\Sigma) \le C.$$

This proves the lemma. □

So if $\operatorname{supp}(u) \subset B_{\delta_0|x|}(x)$, we have the scaling invariant Sobolev inequality

$$\left(\int_\Sigma u^4\,d\mu\right)^{1/2} \le C\left(\int_\Sigma u^2\right)^{1/2}\left(\int_\Sigma |\nabla u|^2\,d\mu\right)^{1/2}.$$

Lemma 2.9 [Qing and Tian 2007, Lemma 2.6]. *Suppose a nonnegative function $v \in L^2$ solves*

$$-\Delta v \le fv + h$$

on $B_{2R}(x_0)$, where

$$\int_{B_{2R}(x_0)} f^2\,d\mu \le CR^{-2}$$

and $h \in L^2(B_{2R}(x_0))$. Also, suppose that

$$\left(\int_\Sigma u^4\,d\mu\right)^{1/2} \le C\left(\int_\Sigma u^2\right)^{1/2}\left(\int_\Sigma |\nabla u|^2\,d\mu\right)^{1/2}$$

holds for all u with support inside $B_{2R}(x_0)$. Then

$$\sup_{B_R(x_0)} v \le CR^{-1}\|v\|_{L^2(B_{2R}(x_0))} + CR\|h\|_{L^2(B_{2R(x_0)})}.$$

Then we find that

$$\begin{aligned}-\Delta|\mathring{A}| &\le (|\mathring{A}|^2 + H^2 + H|\mathring{A}| + C|x|^{-2-q})|\mathring{A}| + CH|x|^{-2-q} + C|x|^{-3-q}\\ &= f_1|\mathring{A}| + h_1\end{aligned}$$

and

$$-\Delta|\nabla\mathring{A}| \le C|\nabla\mathring{A}|(|\mathring{A}|^2 + H|\mathring{A}| + H^2 + O(|x|^{-3})) \\ + \big((|\mathring{A}|^2 + H|\mathring{A}| + H^2)O(|x|^{-3}) + (|\mathring{A}| + H)O(|x|^{-4}) + O(|x|^{-5})\big) \\ = f_2|\nabla\mathring{A}| + h_2.$$

As in Theorem 2.5 of [Qing and Tian 2007] we have $\|f_i\|^2_{L^2(B_{2\delta_0|x|}(x))} \le C|x|^{-2}$ for $i = 1, 2$. Further, it is easy to show that

$$\|h_1\|^2_{L^2(B_{2\delta_0|x|}(x))} = O(|x|^{-4-2q}) \quad \text{and} \quad \|h_2\|^2_{L^2(B_{2\delta_0|x|}(x))} = O(|x|^{-6-2q}).$$

At last we know that

$$\int_{B_{\delta_0|x|}(x)} |\mathring{A}|^2 d\mu \le C|x|^{-2} r_0^{-q}, \tag{2-15}$$

and

$$\int_{B_{\delta_0|x|}(x)} |\nabla\mathring{A}|^2 d\mu \le |x|^{-2}\bigg(\int_{B_{\delta_0|x|}(x)} |\mathring{A}|^2 d\mu\bigg)^{1/2} \le |x|^{-2} r_0^{-q/2}. \tag{2-16}$$

The first inequality follows from Lemma 2.2 and the second one from Simon's identity (2-12). This concludes the proof of Theorem 2.7. □

3. Blow down analysis

Now like [Qing and Tian 2007], we blow down the surface in three different scales. First we consider

$$\tilde{N} = \tfrac{1}{2}HN = \big\{\tfrac{1}{2}Hx \,\big|\, x \in N\big\}.$$

Suppose there is a sequence of constant mean curvature surfaces $\{N_i\}$ such that

$$\lim_{i\to\infty} r_0(N_i) = \infty.$$

We know that

$$\lim_{i\to\infty} \int_{N_i} H_e^2 d\mu_e = 16\pi.$$

Hence, by the curvature estimates established in the previous section combined with the proof of [Simon 1993, Theorem 3.1], we have:

Lemma 3.1. *Suppose that $\{N_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty.$$

Also, suppose that N_i separates infinity from the compact part. Then, there is a subsequence of $\{\tilde{N}_i\}$ which converges in Gromov–Hausdorff distance to a round *sphere* $S_1^2(a)$ *of radius* 1 *and centered at* $a \in \mathbb{R}^3$. *Moreover, the convergence is* $C^{2,\alpha}$ *away from the origin.*

Then, we use a smaller scale r_0 to blow down the surface

$$\hat{N} = r_0(N)^{-1} N = \{r_0^{-1} x \mid x \in N\}.$$

Lemma 3.2. *Suppose that* $\{N_i\}$ *is a sequence of constant mean curvature surfaces in a given asymptotically flat end* $(\mathbb{R}^3 \setminus B_1(0), g)$ *and that* $\lim_{i\to\infty} r_0(N_i) = \infty$. *Also, suppose that*

$$\lim_{i\to\infty} r_0(N_i) H(N_i) = 0.$$

Then there is a subsequence of $\{\hat{N}_i\}$ *converging to a* 2*-plane at distance* 1 *from the origin. Moreover the convergence is in* $C^{2,\alpha}$ *in any compact set of* $\mathbb{R}^3$.

We must understand the behavior of the surfaces N_i in the scales between $r_0(N_i)$ and $H^{-1}(N_i)$. We consider the scale r_i such that

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0, \qquad \lim_{i\to\infty} r_i H(N_i) = 0$$

and blow down the surfaces

$$\bar{N}_i = r_i^{-1} N = \{r_i^{-1} x \mid x \in N\}.$$

Lemma 3.3. *Suppose that* $\{N_i\}$ *is a sequence of constant mean curvature surfaces in a given asymptotically flat end* $(\mathbb{R}^3 \setminus B_1(0), g)$ *and that*

$$\lim_{i\to\infty} r_0(N_i) = \infty.$$

Also, suppose that r_i *are such that*

$$\lim_{i\to\infty} \frac{r_0(N_i)}{r_i} = 0, \qquad \lim_{i\to\infty} r_i H(N_i) = 0.$$

Then there is a subsequence of $\{\bar{N}_i\}$ *converging to a* 2*-plane at the origin in Gromov–Hausdorff distance. Moreover the convergence is* $C^{2,\alpha}$ *in any compact subset away from the origin.*

4. Asymptotic analysis

In this section and the next two we assume that (M, g, K) is an (m, k, ε)-AF-RT manifold, with $q = 1$. First we revise [Qing and Tian 1997, Proposition 2.1], proving a different version. Set

$$\|u\|_{1,i}^2 = \int_{[(i-1)L, iL]\times S^1} |u|^2 + |\nabla u|^2 \, dt \, d\theta,$$

where (t, θ) is the standard column coordinate.

Lemma 4.1. *Suppose $u \in W^{1,2}(\Sigma, \mathbb{R}^k)$ satisfies*

$$\Delta u + A \cdot \nabla u + B \cdot u = h$$

in Σ, where $\Sigma = [0, 3L] \times S^1$ for L large. Then there exists a positive number δ_0 such that if

$$\|h\|_{L^2(\Sigma)} \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i} \quad \text{and} \quad \|A\|_{L^\infty(\Sigma)} \leq \delta_0 \|B\|_{L^\infty(\Sigma)} \leq \delta_0,$$

the following conditions are satisfied:

(a) $\|u\|_{1,3} \leq e^{-(1/2)L} \|u\|_{1,2}$ *implies* $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1}$.

(b) $\|u\|_{1,1} \leq e^{-(1/2)L} \|u\|_{1,2}$ *implies* $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3}$.

(c) *If both*

$$\int_{L \times S^1} u \, d\theta \quad \text{and} \quad \int_{2L \times S^1} u \, d\theta \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i},$$

then either $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1}$ *or* $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3}$.

Proof. Supposing $u \in W^{1,2}(\Sigma)$ and u is harmonic, we can deduce that u satisfies (a), (b) and this variant condition:

(c′) If both

$$\int_{L \times S^1} u \, d\theta \quad \text{and} \quad \int_{2L \times S^1} u \, d\theta = 0,$$

then either $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,1}$ or $\|u\|_{1,2} < e^{-(1/2)L} \|u\|_{1,3}$.

A harmonic function u can be written as

$$u = a_0 + b_0 t + \sum_{n=1}^{\infty} \big(e^{nt} (a_n \cos n\theta + b_n \sin n\theta) + e^{-nt} (a_{-n} \cos n\theta + b_{-n} \sin n\theta) \big).$$

Then it follows that for $i = 1, 2, 3$,

$$\begin{aligned} \|u\|_{1,i}^2 = 2\pi \big((a_0^2 + b_0^2) L + a_0 b_0 L^2 (2i - 1) + \tfrac{1}{3} b_0^2 L^3 (3i^2 - 3i + 1) \big) \\ + \frac{\pi}{2} \sum_{n=1}^{\infty} \Big(\frac{e^{2nL-1}}{n} \big(e^{2(i-1)nL} (a_n^2 + b_n^2) + e^{-2niL} (a_{-n}^2 + b_{-n}^2) \big) \\ + 4L (a_n a_{-n} + b_n b_{-n}) \Big) \\ + \pi \sum_{n=1}^{\infty} \Big(\frac{e^{2nL-1}}{n} \big(e^{2(i-1)nL} (n^2 a_n^2 + n^2 b_n^2) + e^{-2niL} (n^2 a_{-n}^2 + n^2 b_{-n}^2) \big) \\ + 4L (n^2 a_n a_{-n} + n^2 b_n b_{-n}) \Big). \end{aligned}$$

If L is fixed and sufficiently large, then

$$\|u\|_{1,2}^2 < \tfrac{1}{2} (e^L \|u\|_{1,3}^2 + e^{-L} \|u\|_{1,1}^2),$$

which implies (a). We get (b) in the same way. For (c′), we have $a_0 = b_0 = 0$, so then

$$\|u\|_{1,2}^2 < \tfrac{1}{2}e^{-L}(\|u\|_{1,3}^2 + \|u\|_{1,1}^2)$$

which implies (c′).

The second step is to pass to limits. If the proposition were false, then one would have a sequence $\delta_k \to 0$ and a sequence of solutions u_k, each violating (a), (b), or (c), with $\|h_k\|_{L^2} \le \delta_k \max_{1\le i\le 3} \|u_k\|_{1,i}$, $\|A_k\|_\infty \le \delta_k$ and $\|B_k\|_\infty \le \delta_k$ solving

$$\Delta u_k + A_k \cdot \nabla u_k + B_k \cdot u_k = h_k.$$

We may assume $\max_{1\le i\le 3} \|u_k\|_{1,i} = 1$; otherwise we can normalize them. So we know $\|u_k\|_{1,2} > C > 0$, because u_k violates (a), (b), or (c). We know that there is a subsequence that converges to some harmonic function $u \in W^{1,2}(\Sigma)$ weakly. From the interior $W^{2,p}$ estimate we know the convergence is strongly $W^{1,2}$ in I_2, which implies that u is not trivially zero.

Because $u_i \rightharpoonup u$ weakly in $W^{1,2}(\Sigma)$ sense, we know $u_i \rightharpoonup u$ in the $W^{1,2}(I_1)$ and $W^{1,2}(I_3)$ senses. Then

$$\liminf_{i\to\infty} \|u_i\|_{1,1} \ge \|u\|_{1,1}, \qquad \lim_{i\to\infty} \|u_i\|_{1,2} = \|u\|_{1,2}, \qquad \liminf_{i\to\infty} \|u_i\|_{1,3} \ge \|u\|_{1,3}.$$

Then u_i converges to some nontrivial harmonic function u which violates one of (a), (b), or (c′), proving the lemma. □

Given a surface N in $\mathbb{R}^3$, recall from, for example, [Kenmotsu 2003, (8.5)], that

$$\Delta_e v + |\nabla_e v|^2 v = \nabla_e H_e,$$

where v is the Gauss map from $N \to S^2$.

Lemma 4.2. *For the constant mean curvature surfaces in the asymptotically flat end* $(\mathbb{R}^3 \setminus B_1(0), g)$, *we have*

$$|\nabla_e H_e|(x) \le C|x|^{-2} r_0^{-1}.$$

Proof. Because the metric g and the Euclidean metric are uniformly equivalent, we must prove that

$$|\nabla H_e|(x) \le C|x|^{-2} r_0^{-1}.$$

From (2-10), (2-13), (2-14), and Lemma 2.5 (now $q = 1$), we know that

$$\begin{aligned} |\nabla H_e| \le |\overline{\nabla} h_{ij}|\,|A| + |h_{ij}|\,|A|^2 + |h_{ij}|\,|\nabla \mathring{A}_{ij}| + H|A|\,|h_{ij}| + H|\overline{\nabla} h_{ij}| \\ + |A|\,|\overline{\nabla} h_{ij}| + |\overline{\nabla}^2 h| \\ \le |x|^{-2} r_0^{-1}, \end{aligned}$$

which completes the proof. □

Suppose Σ is a constant mean curvature surface in the asymptotically flat end. Set

$$A_{r_1,r_2} = \{x \in \Sigma \mid r_1 \le |x| \le r_2\},$$

and let $A^0_{r_1,r_2}$ be the standard annulus in $\mathbb{R}^2$. We are concerned with the behavior of v on $A_{Kr_0(\Sigma),sH^{-1}(\Sigma)}$ of Σ where K is fixed large and s is fixed small. The lemma below gives us good coordinates on the surface.

Lemma 4.3. *Suppose Σ is a constant mean curvature surface in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$. Then, for any $\varepsilon > 0$ and L large, there are M, s and K such that, if $r_0 \ge M$ and $Kr_0(\Sigma) < r < sH^{-1}(\Sigma)$, then $(r^{-1}A_{r,e^L r}, r^{-2}g_e)$ may be represented as $(A^0_{1,e^L}, \bar{g})$ and*

$$\|\bar{g} - |dx|^2\|_{C^1(A^0_{1,e^L})} \le \varepsilon.$$

In other words, in the cylindrical coordinates $(S^1 \times [\log r, L + \log r, \bar{g}_c])$,

$$\|\bar{g}_c - (dt^2 + d\theta^2)\|_{C^1(S^1\times[\log r, L+\log r])} \le \varepsilon.$$

Proof. Suppose this is not true. Then we can assume that such K (or such s) cannot be found. Then by Lemma 3.2, for some $\varepsilon_0 > 0$, there is a sequence Σ_n with $r_0(\Sigma_n) \to \infty$ and $\tilde{l}_n \to \infty$ such that

$$\left((Kr_0 e^{\tilde{l}_n L})^{-1} A_{Kr_0 e^{\tilde{l}_n L}, Kr_0 e^{(\tilde{l}_n+1)L}}, (Kr_0 e^{\tilde{l}_n L})^{-2} g_e\right)$$

is not ε_0 close to $(A^0_{1,e^L}, \bar{g})$.

By Lemma 3.1,

$$\frac{Kr_0 e^{\tilde{l}_n L}}{sH^{-1}(\Sigma_n)} \to 0$$

must hold because we have chosen s sufficiently small.

So if we assume $r_n = Kr_0 e^{\tilde{l}_n L}$, then

$$\lim_{n\to\infty} \frac{r_n}{Kr_0} = \infty, \quad \lim_{n\to\infty} \frac{r_n}{sH^{-1}} = 0.$$

Blowing down the surface using r_n gives a contradiction with Lemma 3.3. □

Now consider the cylindrical coordinates (t, θ) on $(S^1 \times [\log Kr_0, \log sH^{-1}])$. The tension field satisfies

$$|\tau(v)| = r^2 |\nabla_e H_e| \le C r_0^{-1}$$

for $t \in [\log Kr_0, \log sH^{-1}]$. Thus,

$$\int_{S^1\times[t,t+L]} |\tau(v)|^2 \, dt \, d\theta \le C r_0^{-2}.$$

Let I_i be $S^1 \times [\log Kr_0 + (i-1)L, \log Kr_0 + iL]$, and N_i be $I_{i-1} \cup I_i \cup I_{i+1}$. On Σ_n, assume $\log(sH^{-1}) - \log(Kr_0) = l_n L$. As in [Qing and Tian 1997], first we prove:

Lemma 4.4. *For each $i \in [3, l_n - 2]$, there exists a geodesic γ such that*

$$\int_{I_i} |\widetilde{\nabla}(v-\gamma)|^2 \, dt \, d\theta \le C(e^{-iL} + e^{-(l_n - i)L})s^2 + Cr_0^{-1}, \tag{4-1}$$

where $\widetilde{\nabla}$ is the gradient on $S^1 \times [\log(Kr_0), \log(sH^{-1})]$.

Lemma 4.5. *By Theorem 2.7,*

$$[v]_{C^\alpha(I_i)} \le \|\widetilde{\nabla} v\|_{L^\infty} \le C(r_0^{-1/2} + s).$$

Thus if r_0 sufficiently large and s sufficiently small, then $[v]_{C^\alpha(N_i)}$ is very small.

Proof of Lemma 4.4. To apply the Lemma 4.1 to prove this lemma we choose points P and Q on S^2 (the image of Gauss map) satisfying

$$\left| P - \frac{1}{2\pi} \int_{(i-1)L \times S^1} v \, d\theta \right| \le C \max_{(i-1)L \times S^1} |v - P|^2,$$
$$\left| Q - \frac{1}{2\pi} \int_{iL \times S^1} v \, d\theta \right| \le C \max_{iL \times S^1} |v - Q|^2.$$

Note that S^2 is compact and smooth, so by Lemma 4.5 we can always find such P and Q that are very close. So there is a unique geodesic γ_i connecting P and Q whose velocity is sufficiently small.

If we write down the equation satisfied by $v - \gamma_i$ on $S^1 \times [\log(Kr_0), \log(sH^{-1})]$,

$$\tilde{\Delta} u + A \cdot \widetilde{\nabla} u + B \cdot u = \tau,$$

where $u = v - \gamma_i$, we have

$$\begin{aligned} |A| &\le C(|\widetilde{\nabla} v| + |\widetilde{\nabla} \gamma_i|) \le \delta_0, \\ |B| &\le C \min\{|\widetilde{\nabla} v|^2, |\widetilde{\nabla} \gamma_i|^2\} \le \delta_0. \end{aligned} \tag{4-2}$$

If Lemma 4.1(c) cannot be used, the only reason is that

$$\|v - \gamma_i\|_{1,i} \le C \|\tau\|_{L^2(N_i)},$$

and so

$$\int_{I_i} |\widetilde{\nabla}(v - \gamma_i)|^2 \, dt \, d\theta \le Cr_0^{-2},$$

which implies (4-1).

If Lemma 4.1(c) can be used, then applying it for $u = v - \gamma_i$ over N_i, we have one of the following:

$$\|u\|_{1,i} < e^{-(1/2)L} \|u\|_{1,i-1},$$
$$\|u\|_{1,i} < e^{-(1/2)L} \|u\|_{1,i+1}.$$

Suppose the first one happens (without loss of generality). Then we may push this relation to the left because (4-2) holds regardless of t's position. If the theorem can be used on N_{j+1} but not on N_j for some $j \geq 2$, then

$$\|u\|_{1,i} < e^{-(1/2)(i-j)L}\|u\|_{1,j} \leq Ce^{-(1/2)(i-j)L} r_0^{-1} \leq Cr_0^{-1}.$$

If the theorem can be used until I_2, then

$$\begin{aligned} e^{L/2}\|u\|_{1,2} &\leq \|u\|_{1,1} \\ &= \left(\int_{I_1} u^2\,dt\,d\theta\right)^{1/2} + \left(\int_{I_1} |\widetilde{\nabla} u|^2\,dt\,d\theta\right)^{1/2} \\ &\leq \left(\int_{I_2} u^2\,dt\,d\theta\right)^{1/2} + \left(\int_{I_1} \big(u(t,\theta)-u(t+L,\theta)\big)^2\,dt\,d\theta\right)^{1/2} \\ &\qquad + \left(\int_{I_1} |\widetilde{\nabla} u|^2\,dt\,d\theta\right)^{1/2}. \end{aligned}$$

So we have

$$\begin{aligned} (e^{L/2}-1)\|u\|_{1,2} &\leq \left(\int_{I_1}\left(\int_0^L \left|\frac{\partial u}{\partial t}(t+s,\theta)\right| ds\right)^2 dt\,d\theta\right)^{1/2} + \left(\int_{I_1} |\widetilde{\nabla} u|^2\,dt\,d\theta\right)^{1/2} \\ &\leq \int_0^L \left(\int_{I_1}\left|\frac{\partial u}{\partial t}(t+s,\theta)\right|^2 dt\,d\theta\right)^{1/2} ds + \left(\int_{I_1} |\widetilde{\nabla} u|^2\,dt\,d\theta\right)^{1/2} \\ &\leq C\left(\int_{I_1\cup I_2} |\widetilde{\nabla} u|^2\,dt\,d\theta\right)^{1/2} \\ &\leq C\left(\int_{I_1\cup I_2} |\widetilde{\nabla} v|^2\,dt\,d\theta\right)^{1/2} + C\left(\int_{I_1\cup I_2} |\widetilde{\nabla} \gamma_i|^2\,dt\,d\theta\right)^{1/2} \\ &\leq C(r_0^{-1/2}+s). \end{aligned}$$

This implies the estimate

$$\|u\|_{1,i} \leq Ce^{(-(i-2)/2)L}\|u\|_{1,2} \leq Ce^{-(i/2)L}(r_0^{-1/2}+s).$$

If $\|u\|_{1,i} < e^{-(1/2)L}\|u\|_{1,i+1}$ happens, we will have similarly

$$\|u\|_{1,i} \leq Ce^{-((l_n-i)/2)L}(r_0^{-1/2}+s).$$

Finally we get

$$\|u\|_{1,i} \leq C(e^{-(i/2)L} + e^{-((l_n-i)/2)L})s + Cr_0^{-1/2},$$

which implies (4-1). □

Then to get the energy decay, we use the Hopf differential

$$\Phi = |\partial_t v|^2 - |\partial_\theta v|^2 - 2\sqrt{-1}\,\partial_t v\cdot\partial_\theta v.$$

We know that the L^1 norm of Φ is invariant under conformal change of the coordinates. Now (t,θ) is the coordinate of $A_{Kr_0e^{(i-2)L},Kr_0e^{(i+1)L}}$. We find another coordinate for it: Set $r_i = Kr_0e^{iL}$. Then

$$(r_i^{-1}A_{Kr_0e^{(i-2)L},Kr_0e^{(i+1)L}}, r_i^{-2}g_e)$$

can be represented as $(A^0_{e^{-2L},e^L}, \bar{g})$, where

$$\|\bar{g}-|dx|^2\|_{C^1(A^0_{e^{-2L},e^L})} \le \varepsilon.$$

Assume this Euclidean coordinate is (x, y), so

$$\int_{S^1\times[\log Kr_0+(i-1)L,\log Kr_0+iL]} |\Phi|\,dt\,d\theta = \int_{A^0_{e^{-L},1}} |\Phi|\,dx\,dy.$$

To estimate the right hand side, we use the Cauchy integral formula on $\Omega = A^0_{e^{-2L},e^L}$, and set $\Omega' = A^0_{e^{-L},1}$. Then for any $z\in\Omega'$,

$$\Phi(v)(z) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{\Phi(w)}{w-z}\,dw + \frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\partial\Phi(w)}{\partial\overline{w}}\frac{dw\wedge d\overline{w}}{w-z}.$$

We know

$$\begin{aligned}|\partial_x v|, |\partial_y v| \le CKr_0e^{iL}|A| &\le CKr_0e^{iL}(|x|^{-1}r_0^{-1/2}+r_1^{-1})\\ &\le C(r_0^{-1/2}+se^{-(l_n-i)L}),\end{aligned}$$

so we have

$$\left|\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Omega}\frac{\Phi(w)}{w-z}\,dw\right| \le C(r_0^{-1}+s^2e^{-2(l_n-i)L}).$$

For the second term, notice that by easy calculation

$$\frac{\partial\Phi(w)}{\partial\overline{w}} = \partial v\cdot\bar{\tau}(v),$$

where $\bar{\tau}(v)$ is the tension field under this coordinate. Now,

$$|\bar{\tau}(v)| \le (Kr_0e^{il})^2|\nabla_e H_e| \le Cr_0^{-1}$$

so we have

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Omega}\frac{\partial\Phi(w)}{\partial\overline{w}}\frac{dw\wedge d\overline{w}}{w-z} \le Cr_0^{-1}.$$

Then we get

$$\int_{\Omega'}|\Phi| \le C(r_0^{-1}+s^2e^{-2(l_n-i)L}).$$

By direct calculation,

$$\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_t v|^2\,dt\,d\theta \le \int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\Phi|\,dt\,d\theta+\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_\theta v|^2\,dt\,d\theta,$$

and we can get the estimate of

$$\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\partial_\theta v|^2\,dt\,d\theta$$

directly by (4-1). So we get the estimate

$$\int_{S^1\times[Kr_0e^{(i-1)L},Kr_0e^{iL}]}|\widetilde{\nabla} v|^2\,dt\,d\theta\le C(e^{-iL}+e^{-(l_n-i)L})s^2+Cr_0^{-1}.$$

Proposition 4.6. *Suppose that* $\{\Sigma_n\}$ *is a sequence of constant mean curvature surfaces in a given asymptotically flat end* $(\mathbb{R}^3\setminus B_1(0),g)$ *and that*

$$\lim_{i\to\infty} r_0(\Sigma_n)=\infty.$$

Also, suppose that

$$\lim_{n\to\infty} r_0(\Sigma_n)H(\Sigma_n)=0.$$

Then there exists a large number K, *a small number* s *and a number* n_0 *such that, when* $n\ge n_0$,

$$\max_{I_i}|\widetilde{\nabla} v|\le C(e^{-(i/2)L}+e^{-((l_n-i)/2)L})s+Cr_0^{-1/2},$$

where

$$I_i=S^1\times[\log(Kr_0(\Sigma_n))+(i-1)L,\log(Kr_0(\Sigma_n))+iL],$$
$$i\in[0,l_n]\log(Kr_0(\Sigma_n))+l_nL=\log(sH^{-1}(\Sigma_n)).$$

Proof. We use the interior estimate of the elliptic equation

$$\tilde{\Delta}v+|\widetilde{\nabla} v|^2v=\tau.$$

We know $\|\widetilde{\nabla} v\|_\infty\le C(r_0^{-1/2}+s)$, and $\|\tau\|_\infty\le Cr_0^{-1}$. Assume that

$$I_i\subset\subset\tilde{I}_i\subset\subset N_i.$$

Then for some $p>2$,

$$\begin{aligned}\sup_{I_i}|\widetilde{\nabla} v|&\le C\|\widetilde{\nabla} v\|_{W^{1,p}(I_i)}\le C(\|v\|_{L^p(\tilde{I}_i)}+r_0^{-1})\le C(\|v\|_{L^2(N_i)}+r_0^{-1})\\&\le C(e^{-(i/2)L}+e^{-((l_n-i)/2)L})s+Cr_0^{-1/2}.\end{aligned}$$

□

This analysis improves our understanding of the blowdowns that we discussed in the previous section. Namely,

Corollary 4.7. *Assume the same conditions as in Proposition 4.6 and, in addition,*

$$\lim_{r_0\to\infty}\frac{\log r_1}{r_0^{1/4}}=0.$$

Then the limit plane in Lemmas 3.2 and 3.3 are all orthogonal to the same vector a. In fact, we may choose s small and i large enough so that

$$|v(x)+a|\le\varepsilon$$

for all $x\in\Sigma_n$ and $|x|\le sH^{-1}(\Sigma_n)$.

Proof. We want to prove that

$$\operatorname{Osc}_{B_{sH^{-1}}\cap\Sigma_n} v$$

is sufficiently small if $r_0(\Sigma_n)$ is large and s is small. We already know that

$$\operatorname{Osc}_{B_{Kr_0}\cap\Sigma_n} v$$

is very small from Lemma 3.2, so we need only to prove that

$$\operatorname{Osc}_{(B_{sH^{-1}}\setminus B_{Kr_0})\cap\Sigma_n} v$$

is small.

From Proposition 4.6 above we find that

$$\begin{aligned}\operatorname{Osc}_{(B_{sH^{-1}}\setminus B_{Kr_0})\cap\Sigma_n} v &\le \sum_{i=1}^{l_n}\operatorname{Osc}_{I_i} v\le C\sum_{i=1}^{l_n}\sup_{I_i}|\widetilde{\nabla}v|\\ &\le C\sum_{i=1}^{l_n}((e^{-(i/2)L}+e^{-((l_n-i)/2)L})s+r_0^{-1/2})\le Cs+l_nr_0^{-1/2}.\end{aligned}$$

From the inequalities $C^{-1}r_1\le H^{-1}\le Cr_1$, we have

$$l_nr_0^{-1/2}=L^{-1}(\log(sH^{-1})-\log(Kr_0))r_0^{-1/2}\le C\frac{\log r_1}{r_0^{1/2}}\to 0$$

as $r_0\to\infty$, which proves the corollary. □

Corollary 4.8. *Assume the same conditions as in Proposition 4.6. Let $v_n = v(p_n)$ for some $p_n \in I_{l_n/2}$. Then for $i \in [0, \frac{1}{2}l_n]$,*

$$\sup_{I_i} |v - v_n| \le C(e^{-(1/2)iL} + e^{-(1/4)l_n L})s + l_n r_0^{-1/2},$$

and for $i \in [\frac{1}{2}l_n, l_n]$,

$$\sup_{I_i} |v - v_n| \le C(e^{-(1/4)l_n L} + e^{-(1/2)(l_n - i)L})s + l_n r_0^{-1/2}.$$

5. Harmonic coordinates

We assume that the metric g can be expanded in the coordinate $\{x^i\}$ as

$$g_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + h^1_{ij}(\theta)/r + Q,$$

where θ is the coordinate on the unit sphere S^2, $h^1_{ij}(\theta)$ is a function extended constantly along the radial direction, and Q satisfies

$$\sup r^{2+k} |\partial^k Q| \le C$$

for $k = 0, 1, \ldots, 5$.

First, note that

$$\begin{aligned}\Delta_g x^k &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\Big(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} x^k\Big) \\ &= \frac{\partial}{\partial x^i} g^{ik} + \tfrac{1}{2} g^{ik} g^{mn} g_{mn,i} = -g^{mn}\Gamma^k_{mn} = O(|x|^{-2}).\end{aligned}$$

Our aim is to find an asymptotically harmonic coordinate, that is, a coordinate y^i such that $\Delta_g y^k = O(|x|^{-3})$:

$$\begin{aligned}\Delta_g x^k &= -g^{jl} g^{ik} \frac{1}{2}\Big(\frac{\partial}{\partial x^j} h_{li} + \frac{\partial}{\partial x^l} h_{ji} - \frac{\partial}{\partial x^i} h_{jl}\Big) \\ &= -g^{jl} g^{ik} \frac{1}{2}\bigg(r^{-2}\bigg(\Big(h^1_{li,j}(\theta) - h^1_{li}(\theta)\frac{x^j}{r}\Big) + \Big(h^1_{ji,l}(\theta) - h^1_{ji}(\theta)\frac{x^l}{r}\Big) \\ &\qquad - \Big(h^1_{jl,i}(\theta) - h^1_{jl}(\theta)\frac{x^i}{r}\Big)\bigg)\bigg) + \partial Q \\ &= -g^{jl} g^{ik} \tfrac{1}{2} r^{-2} f^1_{lij}(\theta) + O(|x|^{-3}).\end{aligned}$$

We also know that $g^{ij} = \delta^{ij} - h^1_{ij}(\theta)/r + O(r^{-2})$.

Then

$$\Delta_g x^k = -\tfrac{1}{2} r^{-2} f^1_{jkj}(\theta) + O(r^{-3}).$$

Suppose $0 = \xi_0 > \xi_1 \ge \xi_2 \ge \cdots$ are the eigenvalues of $\Delta|_{S^2}$, and $A_n(\theta)$ are the corresponding orthonormal eigenvectors.

Set

$$y^k = x^k + \sum_{n=0}^{\infty} f_n^k(r) A_n(\theta).$$

We have

$$\Delta_g y^k = \Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3}(f_n^k(r) A_n(\theta)) + \sum_{n=0}^{\infty}(\Delta_g - \Delta_{\mathbb{R}^3})(f_n^k(r) A_n(\theta)).$$

Solve the equation

$$\Delta_g x^k + \sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3}(f_n^k(r) A_n(\theta)) = O(|x|^{-3}).$$

Assume

$$\tfrac{1}{2} f_{jkj}^1(\theta) = \sum_{n=0}^{\infty} \lambda_n^k A_n(\theta).$$

So we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta_{\mathbb{R}^3}(f_n^k(r) A_n(\theta)) &= r^{-2} \sum_{n=0}^{\infty} \lambda_n^k A_n(\theta), \\
\tfrac{1}{r^2}(2r f_n^{k\prime} + r^2 f_n^{k\prime\prime} + f_n^k(r)\xi_n) &= \lambda_n^k && n = 0, \dots, \infty, \\
f_0^k &= \lambda_0^k \log r, \\
f_n^k &= \frac{\lambda_n^k}{\xi_n} && n > 0,
\end{aligned}$$

and this solution satisfies

$$\sum_{n=0}^{\infty}(\Delta_g - \Delta_{\mathbb{R}^3})(f_n^k(r) A_n(\theta)) = O(|x|^{-3}).$$

So if

$$y^k = x^k + \frac{1}{2\sqrt{\pi}} \lambda_0^k \log r + \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n} A_n(\theta), \tag{5-1}$$

then we must have

$$\Delta y^k = O(|x|^{-3}).$$

Note that

$$\Delta|_{S^2} \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n} A_n(\theta) = \sum_{n=1}^{\infty} \lambda_n^k A_n(\theta) = \tfrac{1}{2} f_{jkj}^1(\theta) - \tfrac{1}{2} \overline{f_{jkj}^1(\theta)}$$

where $\overline{f_{jkj}^1(\theta)}$ is its mean value on the unit sphere.

Set

$$g_k^1(\theta) = \sum_{n=1}^{\infty} \frac{\lambda_n^k}{\xi_n} A_n(\theta) = \Delta^{-1}\Big(\tfrac{1}{2} f_{jkj}^1(\theta) - \tfrac{1}{2}\overline{f_{jkj}^1(\theta)}\Big),$$

$$\text{(5-2)} \qquad \frac{\partial y^k}{\partial x^i} = \delta_{ik} + \frac{\lambda_0^k}{2\sqrt{\pi}} \frac{1}{r}\frac{x^i}{r} + g_k^1(\theta)_i \frac{1}{r},$$

$$\frac{\partial x^i}{\partial y^k} = \delta_{ik} + O(|x|^{-1}).$$

So we get

$$\tilde{g}_{ij} = g\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = \delta_{ij} + O(|x|^{-1}).$$

We now define $\tilde{h}_{ij}$ by

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{h}_{ij}$$

and discuss its ellipticity. We have

$$\tilde{h}_{ij} = h_{ij} - \frac{1}{2r\sqrt{\pi}}\Big(\lambda_0^i \frac{x^j}{r} + \lambda_0^j \frac{x^i}{r}\Big) - \frac{(g_{i,j}^1(\theta) + g_{j,i}^1(\theta))}{r},$$

where $g_{i,j}^1(\theta)$ denotes the constant extension along the radial direction of function $(\partial g_i^1(\theta)/\partial x^j)|_{S^2}$.

Example 5.1. For the metric $g_{ij} = \delta_{ij} + \delta_{ij}/r$, we have

$$\Delta_g x^k = -\frac{1}{2}\frac{x^k}{r^3} + O(|x|^{-3}).$$

On S^2, we have $\Delta|_{S^2} x^k = -2x^k$. So if we let

$$y^k = x^k - \frac{1}{4}\frac{x^k}{r},$$

then $\Delta_g y^k = O(|x|^{-3})$. Thus,

$$\frac{\partial y^k}{\partial x^i} = \delta_{ki} - \frac{1}{4}\Big(\frac{\delta_{ki}}{r} - \frac{x^k x^i}{r^3}\Big),$$

$$\tilde{h}_{ij} = \frac{3\delta_{ij}}{2r} - \frac{x^i x^j}{2r^3} + O(r^{-2}).$$

Lemma 5.2. *Suppose in some coordinate $\{x^i\}$ that $g_{ij} = \delta_{ij} + h_{ij}^1(\theta)/r + Q$. Then for any $m > 2$ there exists $\varepsilon > 0$ such that if $\|h_{ij}^1(\theta) - \delta_{ij}(\theta)\|_{W^{m,2}(S^2)} \le \varepsilon$, then in the asymptotically harmonic coordinate $\{y^i\}$ from above, we have*

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{h}_{ij},$$

where $\tilde{h}_{ij} = O(|y|^{-1})$ s and $|y|\tilde{h}_{ij}$ is uniformly elliptic.

Proof. We know easily from (5-2) that $\tilde{h}_{ij} = O(|x|^{-1})$ and that $\lim_{|x|\to\infty} |y|/|x| = 1$. Then $\tilde{h}_{ij} = O(|y|^{-1})$. So we need only to prove that $|y|\tilde{h}_{ij}$ is uniformly elliptic.

First, from $\|h^1_{ij}(\theta) - \delta_{ij}(\theta)\|_{W^{m,2}(S^2)} \le \varepsilon$,

$$\Big\| \tfrac{1}{2} f^1_{jkj}(\theta) - \frac{1}{2}\frac{x^k}{r} \Big\|_{W^{m-1,2}(S^2)} \le C\varepsilon.$$

Note that $\frac{1}{2} f^1_{jkj}(\theta) = \sum_{n=0}^{\infty} \lambda^k_n A_n(\theta)$ and x^k is an eigenvector of Δ_{S^2}, so we can assume that $A_1(\theta) = C_k x^k|_{S^2}$ without loss of generality. Now,

$$\|\lambda^k_0 A_0(\theta) + \big(\lambda^k_1 C_k - \tfrac{1}{2}\big)x^k + \sum_{n=2}^{\infty} \lambda^k_n A_n(\theta)\|_{W^{m-1,2}(S^2)} \le \varepsilon,$$

so we get

$$|\lambda^k_0| \le \varepsilon, \quad \lambda^k_1 C_k - \tfrac{1}{2} \le \varepsilon, \quad \sum_{n=2}^{\infty} (|\xi_n|^{(m-1)/2} \lambda^k_n)^2 \le \varepsilon.$$

By (5-1),

$$\frac{\partial y^k}{\partial x^i} = \delta_{ik} + \frac{\lambda^k_0}{2\sqrt{\pi}} \frac{1}{r}\frac{x^i}{r} - \frac{1}{2}\big(\tfrac{1}{2} \pm \varepsilon\big)\Big(\frac{\delta_{ik}}{r} - \frac{x^i x^k}{r^3}\Big) + \sum_{n=2}^{\infty} \frac{\lambda^k_n}{\xi_n} \frac{\partial A_n(\theta)}{\partial x^i},$$

where the last term on the right can be estimated, for some $p > 0$, as

$$\begin{aligned}
\Big| \sum_{n=2}^{\infty} \frac{\lambda^k_n}{\xi_n} \frac{\partial A_n(\theta)}{\partial x^i} \Big| &\le \sum_{n=2}^{\infty} \frac{|\lambda^k_n|}{|\xi_n|} \frac{|\nabla_{S^2} A_n(\theta)|}{r} \\
&\le \sum_{n=2}^{\infty} \frac{|\lambda^k_n|}{|\xi_n|} \frac{\|A_n(\theta)\|_{W^{2+p,2}}}{r} \\
&\le \sum_{n=2}^{\infty} \frac{|\lambda^k_n|}{|\xi_n|} \frac{|\xi_n|^{1+p/2}\|A_n(\theta)\|_{L^2}}{r} \\
&\le \frac{1}{r} \sum_{n=2}^{\infty} |\lambda^k_n|\, |\xi_n|^{(m-1)/2} |\xi_n|^{(p-m+1)/2} \\
&\le \frac{1}{r} \Big(\sum_{n=2}^{\infty} (|\lambda^k_n|\, |\xi_n|^{(m-1)/2})^2 \Big)^{1/2} \Big(\sum_{n=2}^{\infty} |\xi_n|^{p-m+1} \Big)^{1/2}.
\end{aligned}$$

Let $p = (m-2)/2$. Then from $\xi_n = O(n)$ we have

$$\sum_{n=2}^{\infty} |\xi_n|^{p-m+1} \le C,$$

so

$$\left|\sum_{n=2}^{\infty} \frac{\lambda_n^k}{\xi_n} \frac{\partial A_n(\theta)}{\partial x^i}\right| \leq \frac{C\varepsilon}{r}.$$

Then we have

$$\frac{\partial y^k}{\partial x^i} = \delta_{ik} - \frac{1}{4}\left(\frac{\delta_{ik}}{r} - \frac{x^i x^k}{r^3}\right) + \frac{C\varepsilon}{r},$$

so we can deduce that

$$\tilde{h}_{ij} = h_{ij} + \frac{\delta_{ij}}{2r} - \frac{x^i x^j}{2r^3} + \frac{C\varepsilon}{r}.$$

It follows from $\left\|h_{ij}^1(\theta) - \delta_{ij}(\theta)\right\|_{W^{m,2}(S^2)} \leq \varepsilon$ that rh_{ij} is uniformly elliptic. The eigenvalues of $(x^i x^j)/r^2$ are between 0 and 1, so $|y|\tilde{h}_{ij}$ is uniformly elliptic, from the fact that $\lim_{r\to\infty} |y|/r = 1$ and ε is sufficiently small. □

So all the analysis in Sections 2–4 can be done in the asymptotically harmonic coordinate $\{y^i\}$.

Lemma 5.3. *In the asymptotically harmonic coordinate $\{y^i\}$, we have*

$$-\tfrac{1}{2}\Delta_g \log|\tilde{g}| = R(g) + O(|y|^{-4}).$$

Proof. From direct calculation we have

$$R(g) = \tilde{g}^{jk}\tilde{g}^{il}\tilde{g}_{ml}\left(\frac{\partial\widetilde{\Gamma}_{jk}^m}{\partial y^i} - \frac{\partial\widetilde{\Gamma}_{ik}^m}{\partial y^j}\right) + O(|y|^{-4}),$$

$$\begin{aligned}
\tilde{g}^{jk}\tilde{g}^{il}\tilde{g}_{ml}\frac{\partial\widetilde{\Gamma}_{jk}^m}{\partial y^i} &= \tilde{g}^{il}\tilde{g}_{ml}\frac{\partial(\tilde{g}^{jk}\widetilde{\Gamma}_{jk}^m)}{\partial y^i} + O(|y|^{-4}) \\
&= -\tilde{g}^{il}\tilde{g}_{ml}\frac{\partial\Delta_g y^m}{\partial y^i} + O(|y|^{-4}) = O(|y|^{-4}), \\
-\tilde{g}^{jk}\tilde{g}^{il}\tilde{g}_{ml}\frac{\partial\widetilde{\Gamma}_{ik}^m}{\partial y^j} &= -\tfrac{1}{2}\tilde{g}^{jk}\tilde{g}^{ip}\frac{\partial^2\tilde{g}_{ip}}{\partial y^j\partial y^k} + O(|y|^{-4}) \\
&= -\tfrac{1}{2}\Delta_g \log|\tilde{g}| + O(|y|^{-4}),
\end{aligned}$$

which proves the lemma. □

Corollary 5.4. *If in addition $R = O(|x|^{-3-\tau})$ for some $\tau > 0$, then in the asymptotically harmonic coordinate $\{y^i\}$, we have*

$$\sum_{i=1}^{3}\tilde{h}_{ii} = 8m(g)/|y| + o(|y|^{-1-\tau/2}).$$

Proof. First we know that

$$\lim_{|x|\to\infty}\frac{|y|}{|x|} = 1.$$

Then from Lemma 5.3, in the coordinate $\{y^i\}$ we have

$$\Delta_g \log |\tilde{g}| = O(|y|^{-3-\tau}).$$

We know that

$$\log |\tilde{g}| = O(|y|^{-1}).$$

From the theory of harmonic functions in $\mathbb{R}^n$, there exists a constant C such that

$$\log |\tilde{g}| = \frac{C}{|y|} + o(|y|^{-1-\tau/2}).$$

From Bartnik's result, we know the mass is invariant under the change of coordinates because $R(g) \in L^1$:

$$m(g) = \lim_{R\to\infty} \frac{1}{16\pi} \int_{S_R} (\tilde{h}_{ij,j} - \tilde{h}_{jj,i}) v_g^i \, d\mu.$$

Now,

$$\begin{aligned}\tilde{g}_{ik,k} - \tfrac{1}{2}\tilde{g}_{kk,i} &= \tilde{g}^{ij}\tilde{g}^{kl}\Big(\tilde{g}_{jk,l} - \tfrac{1}{2}\tilde{g}_{kl,j}\Big) + O(|y|^{-3}) \\ &= -\Delta_g y^i + O(|y|^{-3}) = O(|y|^{-3}),\end{aligned}$$

therefore

$$\begin{aligned}m(g) &= \lim_{R\to\infty} \frac{1}{16\pi} \int_{S_R} \Big(-\tfrac{1}{2}\tilde{h}_{jj,i}\Big) v_g^i \, d\mu \\ &= -\lim_{R\to\infty} \frac{1}{32\pi} \int_{S_R} \frac{\partial \log |\tilde{g}|}{\partial y^i} v_g^i \, d\mu \\ &= \lim_{R\to\infty} \frac{1}{32\pi} \int_{S_R} \frac{C y^i}{|y|^3} v_g^i \, d\mu \\ &= \frac{C}{8}.\end{aligned}$$

So we get the result by easy calculation. □

Remark 5.5. We can replace the constraint equation by the condition

$$R = O(|x|^{-3-\tau}) \quad \text{for some } \tau > 0.$$

6. Proof of Theorem 1.7

Now let's prove Theorem 1.7. First note that, if it were false, we could find a sequence Σ_n of stable constant mean curvature surfaces with $r_0(\Sigma_n) \to \infty$ and $\log r_1(\Sigma_n) \leq \frac{1}{n} r_0(\Sigma_n)^{1/4}$; but the Σ_n do not belong to the foliation constructed by the standard method. So $r_1 \leq C r_0$ cannot hold with a uniform C, by Huang's uniqueness theorem.

Recall that, for any surface Σ embedded in $\mathbb{R}^3$ and any given vector $b \in \mathbb{R}^3$,

$$\int_\Sigma H_e \langle v_e \cdot b \rangle_e \, d\mu_e = 0,$$

where H_e and v_e denote the mean curvature and normal vector field with respect to the Euclidean metric.

On the other hand, if Σ is a constant mean curvature surface in the asymptotically flat end, then

$$\int_\Sigma H \langle v_e \cdot b \rangle_e \, d\mu_e = 0.$$

So we have

$$\int_\Sigma (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e = 0.$$

Now we want to prove that, for a sequence of stable constant mean curvature spheres Σ_n with $r_0(\Sigma_n) \to \infty$ and $\log r_1(\Sigma_n) \leq \frac{1}{n} r_0(\Sigma_n)^{1/4}$, if there does not exist a uniform constant C such that $r_1 \leq C r_0$ for every Σ_n, then there exists a subsequence (also denoted by Σ_n) and a constant vector b, such that

$$\limsup_{n \to \infty} \int_{\Sigma_n} (H - H_e) \langle v_e, b \rangle_e d\mu_e < 0. \tag{6-1}$$

Then Theorem 1.7 follows from this contradiction.

From now on, our calculation is in the asymptotically harmonic coordinate $\{x^i\}$. We have calculated $H - H_e$, so

$$\begin{aligned} \int_\Sigma (H - H_e) \langle v_e \cdot b \rangle_e \, d\mu_e &= \int_\Sigma \Big(-f^{ik} h_{kl} f^{lj} A_{ij} + \tfrac{1}{2} H v^i v^j h_{ij} - f^{ij} v^l \overline{\nabla}_i h_{jl} \\ &\quad + \tfrac{1}{2} f^{ij} v^l \overline{\nabla}_l h_{ij} \pm C |h| \, |\overline{\nabla} h| \pm C |h|^2 |A| \Big) \langle v_e \cdot b \rangle_e \, d\mu_e. \end{aligned} \tag{6-2}$$

For the sequence of constant mean curvature surfaces Σ_n chosen above, we have

$$\lim_{n \to \infty} r_0(\Sigma_n) = \infty, \qquad \lim_{n \to \infty} H(\Sigma_n) r_0(\Sigma_n) = 0,$$

and

$$\lim_{n \to \infty} \frac{\log r_1(\Sigma_n)}{r_0(\Sigma_n)^{1/4}} = 0 \tag{6-3}$$

because all the radius conditions are preserved when the coordinates turn into asymptotically harmonic coordinates.

So we can choose s sufficiently small and K sufficiently large with $s H^{-1} > K r_0$ for r_0 sufficiently large.

We know that

$$|h| = O(|x|^{-1}), \quad |\overline{\nabla} h| = O(|x|^{-2}), \quad |A| \le CH + C|\mathring{A}|.$$

From the estimate

$$|\mathring{A}| \le r_0^{-1/2} O(|x|^{-1}),$$

we have

$$\left| \int_\Sigma \bigl(\pm C|h|\,|\overline{\nabla} h| \pm C|h|^2|A|\bigr)\langle v_e \cdot b\rangle_e \, d\mu_e \right| \le C\int_\Sigma (H|x|^{-2} + |x|^{-3}) = O(r_0^{-1})$$

by the estimates in Section 2.

Now we calculate other terms in (6-2):

$$\begin{aligned}
&\int_{\Sigma_n} -f^{ij} v^l (\overline{\nabla}_i h_{jl}) v^m b^m \, d\mu_e \\
&= \frac{1}{2}\int_{\Sigma_n} (f^{ij} h_{jk} f^{kl} A_{li} - H v^j v^l h_{jl}) v^m b^m \, d\mu_e + \frac{1}{2}\int_{\Sigma_n} f^{ij} v^l h_{jl} A_{ik} f^{km} b^m \, d\mu_e \\
&\qquad - \frac{1}{2}\int_{\Sigma_n} f^{ij} v^l (\overline{\nabla}_i h_{jl}) v^m b^m \, d\mu_e
\end{aligned}$$

because $d\mu_e = (1+O(r^{-1}))\, d\mu$, $v_e = (1+O(r^{-1}))v$ and $\langle v_e \cdot b\rangle_e = \langle v \cdot b\rangle_g + O(r^{-1})$.

So we have

$$\begin{aligned}
&\int_{\Sigma_n} (H - H_e)\langle v_e \cdot b\rangle_e \, d\mu_e \\
&\qquad = \int_{\Sigma_n} -\tfrac{1}{2} f^{ik} h_{kl} f^{lj} A_{ij} v^m b^m + f^{ij} v^l h_{jl} A_{ik} f^{km} b^m \\
&\qquad\qquad - \tfrac{1}{2} f^{ij} v^l \overline{\nabla}_i h_{jl} v^m b^m + \tfrac{1}{2} f^{ij} v^l \overline{\nabla}_l h_{ij} v^m b^m + O(r_0^{-1}) d\bar{\mu}.
\end{aligned}$$

Note that

$$A_{ij} = \mathring{A}_{ij} + \frac{f_{ij}}{2} H, \quad \sup|\mathring{A}| \le r_0^{-1/2} O(|x|^{-1}),$$

hence

$$\begin{aligned}
\int_{\Sigma_n} (H - H_e)\langle v_e \cdot b\rangle_e \, d\mu_e &= \int_{\Sigma_n} -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m \\
&\qquad + \tfrac{1}{2} f^{ij} (\overline{\nabla}_l h_{ij}) v^l v^m b^m - \tfrac{1}{2} f^{ij} (\overline{\nabla}_i h_{jl}) v^l v^m b^m \\
&\qquad \pm C \int_{\Sigma_n} |x|^{-2} r_0^{-1/2} + O(r_0^{-1}).
\end{aligned}$$

In this case we calculate

$$\int_{\Sigma_n} |x|^{-2} r_0^{-1/2} \, d\mu_e.$$

We divide the integral into three parts:

$$\int_{\Sigma_n} |x|^{-2} r_0^{-1/2} = \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} + \int_{\Sigma_n \cap B_{Kr_0}(0)} + \int_{\Sigma_n \cap (B_{sH^{-1}} \setminus B_{Kr_0})} |x|^{-2} r_0^{-1/2}.$$

Then by the blowdown results in Section 3,

$$\int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} |x|^{-2} r_0^{-1/2}\, d\mu_e = \int_{\tilde{\Sigma}_n \cap B^c_s(0)} |\tilde{x}|^{-2} r_0^{-1/2} d\tilde{\mu} \le C r_0^{-1/2}$$

$$\int_{\Sigma_n \cap B_{Kr_0}(0)} |x|^{-2} r_0^{-1/2}\, d\mu_e = \int_{\hat{\Sigma}_n \cap B_k(0)} |\hat{x}|^{-2} r_0^{-1/2} d\hat{\mu} \le C r_0^{-1/2}$$

$$\begin{aligned}\int_{\Sigma_n \cap (B_{sH^{-1}} \setminus B_{Kr_0})} |x|^{-2} r_0^{-1/2}\, d\mu_e &= \sum_{i=0}^{n} \int_{\Sigma_n \cap (B_{Kr_0 e^{4iL}} \setminus B_{Kr_0 e^{4(i-1)L}})} |x|^{-2} r_0^{-1/2}\, d\mu_e \\ &\le C \sum_{i=0}^{n} \int_{B_{e^{4L}} \setminus B_1} |\bar{x}|^{-2} r_0^{-1/2} d\bar{\mu} \le C r_0^{-1/2} l_n L,\end{aligned}$$

where $e^{l_n L} K r_0 = s H^{-1}$.

So if

$$\lim_{r_0 \to 0} \frac{|\log H|}{r_0^{1/2}} = 0,$$

in other words,

$$\lim_{r_0 \to 0} \frac{|\log r_1|}{r_0^{1/2}} = 0,$$

we have

$$\int_{\Sigma} |x|^{-2} r_0^{-1/2} d\bar{\mu} \to 0$$

as $r_0 \to \infty$.

From the property of the asymptotically harmonic coordinate,

$$g^{ij} h_{ij} = \frac{8m(g)}{r} + o(r^{-1-\tau/2}),$$

$$g^{kl}\Big(g_{ik,l} - \tfrac{1}{2} g_{kl,i}\Big) = O(|x|^{-3}),$$

$$\begin{aligned}&\int_{\Sigma_n} -\frac{H}{4} f^{kl} h_{kl} v^m b^m + \frac{H}{4} f^{jm} h_{jl} v^l b^m + \tfrac{1}{2} f^{ij} (\overline{\nabla}_l h_{ij} - \overline{\nabla}_i h_{jl}) v^l v^m b^m \\ &\quad = \int_{\Sigma_n} -\frac{H}{4} g^{kl} h_{kl} v^m b^m + \frac{H}{4} g^{jm} h_{jl} v^l b^m \\ &\qquad\qquad + \tfrac{1}{2} g^{ij} (\overline{\nabla}_l h_{ij} - \overline{\nabla}_i h_{jl}) v^l v^m b^m + O(|r_0|^{-1}) \\ &\quad = -2m(g) \int_{\Sigma_n} \Big(\frac{H}{r} \langle v_e \cdot b_e \rangle_e + \frac{\langle x \cdot v_e \rangle_e \langle v_e \cdot b \rangle_e}{r^3}\Big) + \int_{\Sigma_n} \frac{H}{4} h_{ml} v^l b^m + o(1).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\Big(-2m(g)\int_{\Sigma_n}\Big(\frac{H}{r}\langle v_e\cdot b\rangle_e+\frac{\langle x\cdot v_e\rangle_e\langle v_e\cdot b\rangle_e}{r^3}\Big)+\int_{\Sigma_n}\frac{H}{4}\,h_{ml}v^l b^m\Big)=0.$$

Note that

$$h_{ml}v^l=(h_{ml}-\frac{\operatorname{tr}(h)}{2}\delta_{ml})v^l+\frac{\operatorname{tr}(h)}{2}v^m,$$

where $\operatorname{tr}(h)=g^{ij}h_{ij}$.

Assume that the three eigenvalues of h_{ml} are

$$\lambda_1\ge\lambda_2\ge\lambda_3\ge 0.$$

For $p\in\Sigma$ fixed, choose coordinates properly so

$$h_{ml}-\frac{\operatorname{tr}(h)}{2}\delta_{ml}$$

can be written as

$$\begin{pmatrix}\lambda_1-\frac{\operatorname{tr}(h)}{2} & 0 & 0\\ 0 & \lambda_2-\frac{\operatorname{tr}(h)}{2} & 0\\ 0 & 0 & \lambda_3-\frac{\operatorname{tr}(h)}{2}\end{pmatrix}.$$

Assume $v=(\tilde{v}^1,\tilde{v}^2,\tilde{v}^3)$ and $(\tilde{v}^1)^2+(\tilde{v}^2)^2+(\tilde{v}^3)^2=1$. Then

$$\sum_{i=1}^{3}\Big(\Big(\lambda_i-\frac{\operatorname{tr}(h)}{2}\Big)\tilde{v}^i\Big)^2=\frac{(\operatorname{tr}(h))^2}{4}-\sum_{i=1}^{3}\lambda_i(\operatorname{tr}(h)-\lambda_i)(\tilde{v}^i)^2.$$

Because of uniform ellipticity, there exists $C>0$ such that

$$\frac{\operatorname{tr}(h)}{C}\le\lambda_3\le\lambda_2\le\lambda_1\le\Big(1-\frac{1}{C}\Big)\operatorname{tr}(h),$$

so

$$\lambda_i(\operatorname{tr}(h)-\lambda_i)\ge\frac{1}{C}\Big(1-\frac{1}{C}\Big)(\operatorname{tr}(h))^2.$$

Hence

$$\sum_{i=1}^{3}\Big(\Big(\lambda_i-\frac{\operatorname{tr}(h)}{2}\Big)\tilde{v}^i\Big)^2\le\Big(\frac{1}{4}-\frac{1}{C}\Big(1-\frac{1}{C}\Big)\Big)(\operatorname{tr}(h))^2$$

$$\begin{aligned}\int_{\Sigma_n}\frac{H}{4}h_{ml}v^l b^m&=\int_{\Sigma_n}\frac{H}{4}\Big(\frac{\operatorname{tr}(h)}{2}(v\cdot b)+\Big(h_{ml}-\frac{\operatorname{tr}(h)}{2}\delta_{ml}\Big)v^l b^m\Big)\\&\le\int_{\Sigma_n}\frac{H\operatorname{tr}(h)}{4}\Big(\frac{1}{2}(v\cdot b)+\sqrt{\frac{1}{4}-\frac{1}{C}\Big(1-\frac{1}{C}\Big)}\Big)\\&=\int_{\Sigma_n}\frac{H\,m(g)}{r}\Big((v\cdot b)+1-\frac{2}{C}\Big).\end{aligned}$$

Thus, as $n \to \infty$,

$$\int_{\Sigma_n} (H - H_e)\langle \nu_e \cdot b\rangle_e \\ \le -m \int_{\Sigma_n} \frac{H}{r}\langle \nu_e \cdot b\rangle_e + \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e + \Big(1 - \frac{2}{C}\Big) m(g) \int_{\Sigma_n} \frac{H}{r}\, d\mu_e + o(1).$$

From Lemma 3.1, we have that $(H/2)\Sigma_n$ subconverges to some sphere $S_1^2(a)$ with $|a| = 1$. Now we choose $b = -a$. Then by the calculation in [Qing and Tian 2007],

$$-m(g) \int_{\Sigma_n} \frac{H}{r}\langle \nu_e \cdot b\rangle_e \to -\frac{8}{3}\pi m(g)$$
$$-m(g) \int_{\Sigma_n} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \to -\tfrac{16}{3}\pi m(g)$$
$$(1 - \frac{2}{C}) m(g) \int_{\Sigma_n} \frac{H}{r} \to \Big(1 - \frac{2}{C}\Big) 8\pi m(g)$$

as $n \to \infty$.

Since there is a small difference from [Qing and Tian 2007], we prove these convergences again. Notice from Lemma 3.1 that $(H/2)\Sigma_n$ subconverges to some sphere $S_1(a)$ with $|a| = 1$, and the first and third integral converge, respectively, to

$$-m(g) \int_{S_1(a)} \frac{2}{r}\langle \nu_e \cdot b\rangle_e = -\tfrac{8}{3}\pi m(g) \text{ and } \Big(1 - \frac{2}{C}\Big) m(g) \int_{S_1(a)} \frac{2}{r} = \Big(1 - \frac{2}{C}\Big) 8\pi m(g).$$

To deal with (6-4), first notice that

$$\int_{S^2(a)} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e = \tfrac{4}{3}\pi.$$

Then we break up the integral (6-4) into three parts:

$$\int_{\Sigma_n} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e \\ = \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} + \int_{\Sigma_n \cap B_{Kr_0}(0)} + \int_{\Sigma_n \cap B_{sH^{-1}} \setminus B_{Kr_0}} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e.$$

Then

$$\lim_{n \to \infty} \int_{\Sigma_n \cap B^c_{sH^{-1}}(0)} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e = \int_{S^2(a) \cap B^c_s} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e,$$
$$\lim_{n \to \infty} \int_{\Sigma_n \cap B_{Kr_0}(0)} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e = \int_{P \cap B_K(0)} \frac{2}{r^3}\langle x \cdot \nu_e\rangle_e \langle \nu_e \cdot b\rangle_e \, d\mu_e,$$

where P is the limit plane in Lemma 3.2. From Corollary 4.7, the normal vector of P is v_e. Due to an easy calculation,

$$\int_P \frac{2}{r^3}\langle x\cdot v_e\rangle_e \langle v_e\cdot b\rangle_e \, d\mu_e = 4\pi.$$

From the divergence theorem,

$$\int_{\Sigma_n} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e = 8\pi$$

for any n and

$$\int_{S^2(a)} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e = 4\pi,$$

because the origin is on the sphere $S^2(a)$. Since

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B^c_{sH^{-1}}(0)} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e = \int_{S^2(a)\cap B^c_s(0)} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e,$$

$$\lim_{n\to\infty}\int_{\Sigma_n\cap B_{Kr_0}(0)} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e = \int_{P\cap B_K(0)} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e,$$

$$\int_P \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e = 4\pi,$$

we have

$$\lim_{s\to 0, K\to\infty}\ \limsup_{n\to\infty}\left|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \, d\mu_e\right| = 0. \tag{6-4}$$

Now we want to prove that

$$\lim_{s\to 0, K\to\infty}\ \limsup_{n\to\infty}\left|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \langle v_e\cdot b\rangle_e \, d\mu_e\right| = 0. \tag{6-5}$$

Use Corollary 4.8 to get (6-5) from (6-4), but there is a small difference from [Qing and Tian 2007]:

$$\begin{aligned}
&\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \langle v_e\cdot b\rangle_e \, d\mu_e \\
&\qquad = \langle v_n\cdot b\rangle_e \int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} \frac{2}{r^3}\langle v_e\cdot b\rangle_e \, d\mu_e \\
&\qquad\qquad + \int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})} \frac{2}{r^3}\langle x\cdot v_e\rangle_e \langle (v_e - v_n)\cdot b\rangle_e \, d\mu_e.
\end{aligned}$$

The first term will converge to 0. We deal with the second term using the cylindrical coordinates in Section 4:

$$\begin{aligned}
&\left|\int_{\Sigma_n\cap(B_{sH^{-1}}\setminus B_{Kr_0})}\frac{2}{r^3}\langle x\cdot\nu_e\rangle_e\langle(\nu_e-\nu_n)\cdot b\rangle_e\,d\mu_e\right| \\
&\qquad\qquad= \left|\sum_{j=1}^{l_n}\int_{A_{Kr_0e^{(j-1)L},Kr_0e^{jL}}}\frac{2}{r^3}\langle x\cdot\nu_e\rangle_e\langle(\nu_e-\nu_n)\cdot b\rangle_e\,d\mu_e\right| \\
&\qquad\qquad\le C\sum_{j=1}^{l_n}L\max_{I_j}|\nu_e-\nu_n| \\
&\qquad\qquad= C\sum_{j=1}^{l_n/2}L\max_{I_j}|\nu_e-\nu_n| + C\sum_{j=l_n/2+1}^{l_n}L\max_{I_j}|\nu_e-\nu_n|.
\end{aligned}$$

From Corollary 4.8,

$$CL\sum_{i=1}^{l_n/2}\sup_{I_i}|\nu-\nu_n| + CL\sum_{i=l_n/2+1}^{l_n}\sup_{I_i}|\nu-\nu_n| \le C(l_ne^{-(1/4)l_nL}+C)s + l_n^2r_0^{-1/2}.$$

But from the condition

$$\lim_{n\to\infty}\frac{\log r_1(\Sigma_n)}{r_0(\Sigma_n)^{1/4}}=0,$$

we know

$$\lim_{n\to\infty} l_n^2r_0^{-1/2} = \lim_{n\to\infty}\left(\frac{L^{-1}(\log sH^{-1}-\log Kr_0)}{r_0^{1/4}}\right)^2=0,$$

so (6-5) holds.

Then

$$0\le -\tfrac{8}{3}\pi m(g) - \tfrac{16}{3}\pi m(g) + \left(1-\frac{2}{C}\right)8\pi m(g) = -\frac{16}{C}\pi m(g).$$

But $m(g)>0$, so

$$\limsup_{n\to\infty}\int_{\Sigma_n}(H-H_e)\langle\nu_e,b\rangle_e d\mu_e<0. \tag{6-6}$$

This proves Theorem 1.7.

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