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A CLASS OF IRREDUCIBLE INTEGRABLE MODULES FOR THE EXTENDED BABY TKK ALGEBRA

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The baby TKK algebra is a core of the extended affine Lie algebra of type A_1 over a semilattice in $\mathbb{R}^2$. In this paper, we classify the irreducible integrable weight modules for the extended baby TKK algebra under the assumption that its center acts nontrivially.

1. Introduction

Extended affine Lie algebras (EALAs) were first introduced in [Høegh-Krohn and Torrésani 1990] and studied systematically in [Allison et al. 1997; Berman et al. 1996]. They are natural generalizations of finite-dimensional simple Lie algebras and affine Kac–Moody algebras. There are many examples of EALAs, such as toroidal algebras and TKK algebras [Moody et al. 1990; Mao and Tan 2007a; 2007b; Eswara Rao 2004; Tan 1999]. In [Eswara Rao 2004], the author studied the irreducible integrable weight modules of toroidal algebras.

The baby TKK algebra $\hat{\mathscr{G}}(\mathscr{J}(S))$ is the universal central extension of $\mathscr{G}(\mathscr{J}(S))$ obtained by the Tits–Kantor–Koecher construction. Its vertex operator representation and quantum analogue were studied in [Tan 1999; Gao and Jing 2010].

We recall this construction [Allison et al. 1997; Tan 1999]: Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the unit elements in the lattice $\mathbb{Z}^2$. Let S_i for $0 \le i \le 3$ be the cosets of $2\mathbb{Z}^2$ in $\mathbb{Z}^2$ defined by

$$S_0 = 2\mathbb{Z}^2, \quad S_1 = e_1 + 2\mathbb{Z}^2, \quad S_2 = e_2 + 2\mathbb{Z}^2, \quad S_3 = e_1 + e_2 + 2\mathbb{Z}^2. \tag{1-1}$$

Let $S = S_0 \cup S_1 \cup S_2$. For $\sigma \in S$, let x^σ be a symbol. Then we obtain a Jordan algebra $\mathscr{J}(S) = \bigoplus_{\sigma \in S} \mathbb{C}x^\sigma$ with multiplication

$$x^r x^s = \begin{cases} x^{r+s} & \text{if } r, s \in S_0 \cup S_i \text{ and } 0 \le i \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1-2}$$

Let $L_{\mathscr{J}(S)}$ be the set of multiplication operators of $\mathscr{J}(S)$ and

$$\operatorname{Inder}(\mathscr{J}(S)) = [L_{\mathscr{J}(S)}, L_{\mathscr{J}(S)}] = \operatorname{span}_{\mathbb{C}}\{[L_a, L_b] : a, b \in \mathscr{J}(S)\}$$

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where $[L_a, L_b]$ is an inner derivation of the Jordan algebra $\mathcal{J}(S)$. Let $\mathfrak{sl}_2(\mathbb{C})$ be the 3-dimensional simple Lie algebra. We use x_+, x_- and $\alpha^\vee$ to denote the Chevalley basis of $\mathfrak{sl}_2(\mathbb{C})$ with relations

$$[x_+, x_-] = \alpha^\vee \quad \text{and} \quad [\alpha^\vee, x_\pm] = \pm 2x_\pm. \tag{1-3}$$

Define a Lie algebra $\mathcal{G}(\mathcal{J}(S)) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{J}(S)\big) \oplus \mathrm{Inder}(\mathcal{J}(S))$ with multiplication

$$\begin{aligned}
&[A \otimes x^r, B \otimes x^s] = [A, B] \otimes x^r x^s + 2\,\mathrm{tr}(AB)\,[L_{x^r}, L_{x^s}],\\
&[D, A \otimes x^r] = A \otimes Dx^r,\\
&\big[D, [L_{x^r}, L_{x^s}]\big] = [L_{Dx^r}, L_{x^s}] + [L_{x^r}, L_{Dx^s}],
\end{aligned}$$

for $A, B \in \mathfrak{sl}_2(\mathbb{C})$, $x^r, x^s \in \mathcal{J}(S)$, and $D \in \mathrm{Inder}(\mathcal{J}(S))$. The Lie algebra $\mathcal{G}(\mathcal{J}(S))$ is a perfect Lie algebra. Its universal central extension $\hat{\mathcal{G}}(\mathcal{J}(S))$ is called the *baby TKK algebra*.

Let $\langle \mathcal{J}(S), \mathcal{J}(S)\rangle$ be the quotient space $(\mathcal{J}(S) \otimes \mathcal{J}(S))/I$, where I is the subspace of $\mathcal{J}(S) \otimes \mathcal{J}(S)$ spanned by all vectors of the form

$$a \otimes b + b \otimes a \quad \text{or} \quad ab \otimes c + bc \otimes a + ca \otimes b$$

for $a, b, c \in \mathcal{J}(S)$. We will use $\langle a, b\rangle$ to denote the element $a \otimes b + I$ in $(\mathcal{J}(S) \otimes \mathcal{J}(S))/I$. In [Tan 1999], the baby TKK algebra $\hat{\mathcal{G}}(\mathcal{J}(S))$ is realized as the vector space

$$\hat{\mathcal{G}}(\mathcal{J}(S)) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{J}(S)\big) \oplus \langle \mathcal{J}(S), \mathcal{J}(S)\rangle, \tag{1-4}$$

with the Lie bracket given by

$$\begin{aligned}
&[A \otimes a, B \otimes b] = [A, B] \otimes ab + 2\,\mathrm{tr}(AB)\,\langle a, b\rangle,\\
&[\langle a, b\rangle, A \otimes c] = A \otimes [L_a, L_b]c,\\
&[\langle a, b\rangle, \langle c, d\rangle] = \big\langle [L_a, L_b]c, d\big\rangle + \big\langle c, [L_a, L_b]d\big\rangle,
\end{aligned} \tag{1-5}$$

for $a, b, c, d \in \mathcal{J}(S)$ and $A, B \in \mathfrak{sl}_2(\mathbb{C})$. A vertex operator representation of $\hat{\mathcal{G}}(\mathcal{J}(S))$ was given in [Tan 1999] on a mixed bosonic-fermionic Fock space.

Let d_1, d_2 be the derivations on the baby TKK algebra $\hat{\mathcal{G}}(\mathcal{J}(S))$ given by

$$\begin{aligned}
&[d_i, A \otimes x^\sigma] = (\sigma \cdot e_i) A \otimes x^\sigma,\\
&[d_i, \langle x^\sigma, x^\tau\rangle] = ((\sigma + \tau) \cdot e_i)\,\langle x^\sigma, x^\tau\rangle,
\end{aligned} \tag{1-6}$$

for $\sigma, \tau \in S$, $A \in \mathfrak{sl}_2(\mathbb{C})$, $i, j = 1, 2$, where $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^2$.

The *extended baby TKK algebra* $\mathcal{L}$ is defined to be

$$\mathcal{L} = \hat{\mathcal{G}}(\mathcal{J}(S)) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2. \tag{1-7}$$

The center of $\mathscr{L}$ is two-dimensional, denoted by $\mathbb{C}C_1 \oplus \mathbb{C}C_2$, where $C_1 = \langle x^{e_1}, x^{-e_1}\rangle$ and $C_2 = \langle x^{e_2}, x^{-e_2}\rangle$.

In this paper, we study the irreducible integrable weight modules of the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts nonzero while C_2 acts as zero. We identify $\mathfrak{sl}_2(\mathbb{C})$ with the subalgebra $\mathfrak{sl}_2(\mathbb{C}) \otimes 1$ of $\mathscr{L}$. Then, $\mathscr{L}$ has a five-dimensional Cartan subalgebra $\mathbb{C}\alpha^\vee \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. Let Δ be the root system of $\mathscr{L}$ with respect to this Cartan subalgebra. In Section 2, we will decompose Δ into $\Delta = \Delta_- \cup \Delta_0 \cup \Delta_+$ and, correspondingly, have a "triangular decomposition" of the extended baby TKK algebra $\mathscr{L}$,

$$\mathscr{L} = \mathscr{L}(\Delta_-) \oplus \mathscr{L}(\Delta_0) \oplus \mathscr{L}(\Delta_+), \tag{1-8}$$

where $\mathscr{L}(\Delta_\pm) = \bigoplus_{\beta\in\Delta_\pm} \mathscr{L}_\beta$ and $\mathscr{L}(\Delta_0) = \bigoplus_{\beta\in\Delta_0} \mathscr{L}_\beta$, where $\mathscr{L}_\beta$ denotes the root space for $\beta \in \Delta$. By a highest-weight module we mean a weight module generated by a weight vector that is annihilated by $\mathscr{L}(\Delta_+)$. We show that any irreducible integrable module V for $\mathscr{L}$ with the actions of $C_1 > 0$ and $C_2 = 0$ is a highest-weight module, and we also determine the conditions for a highest weight module to be integrable.

The paper is organized as follows: In Section 2, we recall some results on the structure of the extended baby TKK algebra $\mathscr{L}$, and give the definition of integrable modules of $\mathscr{L}$. We close the section with a lemma about the properties of irreducible integrable modules of $\mathscr{L}$. In Section 3, we study the highest-weight modules of $\mathscr{L}$. Let $\mathscr{K} = \hat{\mathscr{G}}(\mathscr{J}(S)) \oplus \mathbb{C}d_1$ be a subalgebra of $\mathscr{L}$. We define irreducible highest-weight modules, denoted by $V(\bar{\psi})$ and $L(\psi)$, for the Lie algebras $\mathscr{L}$ and $\mathscr{K}$, respectively. We show that the integrability of the $\mathscr{L}$-module $V(\bar{\psi})$ is equivalent to the integrability of the $\mathscr{K}$-module $L(\psi)$. Then, we investigate the conditions for the $\mathscr{K}$-module $L(\psi)$ to be integrable. In Section 4, we prove that every irreducible integrable module of $\mathscr{L}$ with the actions of $C_1 > 0$ and $C_2 = 0$ is isomorphic to a highest-weight module $V(\bar{\psi})$ constructed in Section 3.

We denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{C}$ the sets of integers, nonnegative integers, positive integers, real numbers, and complex numbers, respectively. $U(\mathfrak{g})$ stands for the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. All algebras are over $\mathbb{C}$.

2. Basic concepts

We recall the structure of $\mathscr{L}$ and its root system. Following [Tan 1999], we define

$$x_\pm(\sigma) = x_\pm(m,n) := \begin{cases} x_\pm \otimes x^\sigma & \text{if } \sigma \in S, \\ 0 & \text{if } \sigma \in S_3, \end{cases}$$

$$\alpha^\vee(\sigma) = \alpha^\vee(m,n) := \begin{cases} \alpha^\vee \otimes x^\sigma & \text{if } \sigma \in S, \\ 2\langle x^{e_1}, x^{\sigma - e_1}\rangle & \text{if } \sigma \in S_3, \end{cases}$$

and

$$C_i(\sigma)=C_i(m,n):=\begin{cases}\langle x^{e_i},x^{\sigma-e_i}\rangle & \text{if } \sigma\in S_0,\\ 0 & \text{if } \sigma\notin S_0,\end{cases}$$

where $i=1,2,\ m,n\in\mathbb{Z}$ and $\sigma=(m,n)$. We also define

$$\Omega(\tau):=\begin{cases}0 & \text{if } \tau\in S_0,\\ -1 & \text{if } \tau\in S_1,\\ 1 & \text{if } \tau\in S_2,\end{cases}$$

for $\tau\in S$. The sets S_0, S_1, S_2, S_3 and S were defined in (1-1).

Proposition 2.1 [Tan 1999]. *The universal central extension* $\hat{\mathscr{G}}(\mathscr{J}(S))$ *of* $\mathscr{G}(\mathscr{J}(S))$ *is spanned by the elements* $\{x_\pm(\sigma),\alpha^\vee(\tau),C_i(\rho)\}$, *for* $i=1,2,\ \sigma\in S,\ \tau\in\mathbb{Z}^2=\mathbb{Z}e_1+\mathbb{Z}e_2$, *and* $\rho\in S_0$, *and satisfies the following relations*:

(R1) *For* $\sigma,\tau\in S$,

$$[x_\pm(\sigma),x_\pm(\tau)]=0,$$

$$[x_+(\sigma),x_-(\tau)]=\begin{cases}\Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \textit{if } \sigma+\tau\notin S,\\ \alpha^\vee(\sigma+\tau)+2\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \textit{if } \sigma+\tau\in S.\end{cases}$$

(R2) *For* $\sigma\in\mathbb{Z}^2,\ \tau\in S$,

$$[\alpha^\vee(\sigma),x_\pm(\tau)]=\begin{cases}\pm 2x_\pm(\sigma+\tau) & \textit{if } \sigma\in S,\\ 2\Omega(\tau)\,x_\pm(\sigma+\tau) & \textit{if } \sigma\notin S.\end{cases}$$

(R3) *For* $\sigma,\tau\in\mathbb{Z}^2$,

$$[\alpha^\vee(\sigma),\alpha^\vee(\tau)]=\begin{cases}2\Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \textit{if } \sigma\notin S,\tau\in S,\\ -4\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \textit{if } \sigma,\tau\notin S,\\ 4\sum_{i=1,2}(\sigma\cdot e_i)\,C_i(\sigma+\tau) & \textit{if } \sigma,\tau\in S \textit{ and } \sigma+\tau\in S,\\ 2\Omega(\tau)\,\alpha^\vee(\sigma+\tau) & \textit{if } \sigma,\tau\in S \textit{ and } \sigma+\tau\notin S.\end{cases}$$

(R4) $C_i(\sigma)$ *are central for* $\sigma\in S_0$ *and* $i=1,2$, *and satisfy*

$$(\sigma\cdot e_1)\,C_1(\sigma)+(\sigma\cdot e_2)\,C_2(\sigma)=0. \qquad \square$$

Remark 2.2. We set $\mathfrak{h}_0=\mathbb{C}\alpha^\vee(0,0)=\mathbb{C}\alpha^\vee$ and the Cartan subalgebra

$$\mathfrak{h}=\mathfrak{h}_0\oplus\mathbb{C}C_1\oplus\mathbb{C}C_2\oplus\mathbb{C}d_1\oplus\mathbb{C}d_2$$

of the baby TKK algebra $\mathscr{L}=\hat{\mathscr{G}}(\mathscr{J}(S))\oplus\mathbb{C}d_1\oplus\mathbb{C}d_2$. The center $\mathfrak{Z}(\mathscr{L})$ of $\mathscr{L}$ is $\mathbb{C}C_1\oplus\mathbb{C}C_2$.

Remark 2.3. $\mathscr{L}$ contains as a subalgebra the affine Kac–Moody algebra

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \Big(\mathfrak{sl}_2(\mathbb{C}) \otimes \Big(\sum_{n\in\mathbb{Z}} \mathbb{C}x^{ne_1}\Big)\Big) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1.$$

Definition 2.4. A module M over $\mathscr{L}$ is called a *weight module* if

$$M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda,$$

where $M_\lambda = \{v \in M : h\,.\,v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. The set $P(M) = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}$ is called the *weight set* of M. For $\lambda \in P(M)$, M_λ is called a *weight space* associated to λ.

Lemma 2.5. *If M is any irreducible weight module over $\mathscr{L}$, then the actions of C_1 and C_2 are constant.* □

From this lemma, we see that, for any irreducible weight module M over $\mathscr{L}$, the actions of C_1 and C_2 are always linearly dependent. Due to this, in this paper we will consider modules with the actions of C_1 nonzero and $C_2 = 0$.

Define the elements α, δ_i and w_i in $\mathfrak{h}^*$ $(i = 1, 2)$ by

$$\begin{aligned}
&\alpha(\alpha^\vee) = 2, \quad \alpha(d_j) = \alpha(C_j) = 0,\\
&\delta_i(\alpha^\vee) = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad \delta_i(C_j) = 0,\\
&w_i(\alpha^\vee) = 0, \quad w_i(d_j) = 0, \quad w_i(C_j) = \delta_{ij},
\end{aligned}$$

for $j = 1, 2$. Define also

$$\begin{aligned}
\Delta^{\mathrm{Re}} &= \{\pm\alpha + n_1\delta_1 + n_2\delta_2 : (n_1, n_2) \in S\},\\
\Delta^{\mathrm{Im}} &= \{n_1\delta_1 + n_2\delta_2 : (n_1, n_2) \in \mathbb{Z}^2\},\\
\Delta &= \Delta^{\mathrm{Re}} \cup \Delta^{\mathrm{Im}}.
\end{aligned}$$

The elements in Δ^{Re} and Δ^{Im} are called real and imaginary (or isotropic) roots, respectively. Then, $\mathscr{L}$ has a root space decomposition

$$\mathscr{L} = \bigoplus_{\beta\in\Delta} \mathscr{L}_\beta,$$

where $\mathscr{L}_\beta = \{x \in \mathscr{L} : [h, x] = \beta(h)x \text{ for all } h \in \mathfrak{h}\}$ and $\mathscr{L}_0 = \mathfrak{h}$.

Define the coroot $\gamma^\vee = \pm\alpha^\vee + 2n_1C_1 + 2n_2C_2$ for $\gamma = \pm\alpha + n_1\delta_1 + n_2\delta_2 \in \Delta^{\mathrm{Re}}$, and define the reflection r_γ on $\mathfrak{h}^*$ by setting

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma.$$

Let $\mathscr{W}$ be the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by $\{r_\gamma : \gamma \in \Delta^{\mathrm{Re}}\}$. We call $\mathscr{W}$ the *Weyl group* of $\mathscr{L}$. One can read more about the structure of $\mathscr{W}$ in [Azam 1999].

Set

$$\begin{aligned}\Delta_+ &= \big((\alpha+\mathbb{N}\delta_1+\mathbb{Z}\delta_2)\cup(-\alpha+\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\cup(\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\big)\cap\Delta,\\ \Delta_- &= \big((\alpha-\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\cup(-\alpha-\mathbb{N}\delta_1+\mathbb{Z}\delta_2)\cup(-\mathbb{Z}_+\delta_1+\mathbb{Z}\delta_2)\big)\cap\Delta,\\ \Delta_0 &= \mathbb{Z}\delta_2.\end{aligned}$$

Correspondingly, set

$$\mathscr{L}(\Delta_+)=\bigoplus_{\beta\in\Delta_+}\mathscr{L}_\beta,\quad \mathscr{L}(\Delta_-)=\bigoplus_{\beta\in\Delta_-}\mathscr{L}_\beta,\quad \mathscr{L}(\Delta_0)=\bigoplus_{\beta\in\Delta_0}\mathscr{L}_\beta.$$

Then, one has $\Delta=\Delta_-\cup\Delta_0\cup\Delta_+$ and $\mathscr{L}=\mathscr{L}(\Delta_-)\oplus\mathscr{L}(\Delta_0)\oplus\mathscr{L}(\Delta_+)$.

Remark 2.6. The three subspaces $\mathscr{L}(\Delta_\pm)$ and $\mathscr{L}(\Delta_0)$ are all Lie subalgebras of $\mathscr{L}$.

Definition 2.7. A module M for $\mathscr{L}$ is said to be integrable if

(1) M is a weight module,

(2) each weight space of M is finite-dimensional,

(3) for any $\beta\in\Delta^{\mathrm{Re}}$, $x\in\mathscr{L}_\beta$ and $v\in M$, there exists some $k\in\mathbb{Z}_+$ such that $x^k\,.\,v=0$; that is, x acts locally nilpotent on M.

Lemma 2.8. *If M is an irreducible integrable module for $\mathscr{L}$, then*

(1) *the weight set $P(M)$ is $\mathscr{W}$-invariant*;

(2) $\dim M_\lambda=\dim M_{\omega\lambda}$, *for all $\lambda\in P(M)$ and $\omega\in\mathscr{W}$*;

(3) *for any real root γ and weight $\lambda\in P(M)$, $\lambda(\gamma^\vee)\in\mathbb{Z}$*;

(4) *if γ is real, $\lambda\in P(M)$ and $\lambda(\gamma^\vee)>0$, then $\lambda-\gamma\in P(M)$*;

(5) *for $i=1,2$, the action of $2C_i$ on M is a constant integer.*

Proof. Without loss of generality, we take a real root $\gamma=\alpha+n_1\delta_1+n_2\delta_2$ and set $\sigma=n_1e_1+n_2e_2$. Let $\mathfrak{sl}_2(\gamma)=\mathrm{span}_{\mathbb{C}}\{x_+(\sigma),\ x_-(-\sigma),\ \gamma^\vee=\alpha^\vee+2n_1C_1+2n_2C_2\}$, which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Set $s_\gamma=\exp(x_-(-\sigma))\cdot\exp(-x_+(\sigma))\cdot\exp(x_-(-\sigma))$. Then, s_γ is well-defined on M. It is easy to check that $s_\gamma M_\lambda\subset M_{r_\gamma\lambda}$ and, hence, $s_\gamma M_\lambda=M_{r_\gamma\lambda}$. Statements (1) and (2) follow from these observations.

Statement (3): Since $x_+(\sigma)$ and $x_-(-\sigma)$ are nilpotent on any nonzero vector $v\in M_\lambda$, by the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ one sees that $\lambda(\gamma^\vee)$ is an integer.

Statement (4): For any $v_\lambda\in M_\lambda$, $W=U(\mathfrak{sl}_2(\gamma))v_\lambda$ is finite dimensional. As a $(\mathfrak{sl}_2(\gamma)+\mathfrak{h})$-module, the weights of W are $\lambda-p\gamma,\dots,\lambda+q\gamma$, where p,q are nonnegative integers, and $p-q=\lambda(\gamma^\vee)$. Now, if $\lambda(\gamma^\vee)>0$, then $p>0$ and, hence, $\lambda-\gamma\in P(M)$.

Statement (5) follows from (3) and Lemma 2.5. □

3. The highest- and lowest-weight modules

We define highest-weight and lowest-weight modules over $\mathscr{L}$, and construct a class of irreducible highest-weight modules $V(\bar{\psi})$ for $\mathscr{L}$ so that $2C_1$ acts as a positive integer and C_2 acts as zero. Then, we investigate sufficient conditions for $V(\bar{\psi})$ to be integrable.

Definition 3.1. A module M over $\mathscr{L}$ is called a *highest-* (respectively, *lowest-*) *weight module*, if there exists some $0 \neq v \in M$ such that

(1) v is a weight vector; that is, for all $h \in \mathfrak{h}$, we have $h \,.\, v = \lambda(h) v$ for some $\lambda \in \mathfrak{h}^*$;

(2) $\mathscr{L}(\Delta_+) \,.\, v = 0$ (respectively, $\mathscr{L}(\Delta_-) \,.\, v = 0$);

(3) $U(\mathscr{L}) \,.\, v = M$.

Let $H = \mathrm{span}_{\mathbb{C}}\{\alpha^\vee(\sigma), C_1(2\sigma), C_2, d_1 : \sigma \in \mathbb{Z}e_2\}$ and ψ be a linear functional on H satisfying $\psi(C_1) \neq 0$ and $\psi(C_2) = 0$. Note that $\mathscr{L}(\Delta_0) = H \oplus \mathbb{C}d_2$ and that $H/\mathbb{C}C_2$ is abelian. Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial ring. Define an associative algebra homomorphism $\bar{\psi}$ by

$$\begin{aligned} \bar{\psi} :\ & U(H) \to \mathbb{C}[t, t^{-1}], \\ & X_1 \ldots X_k \mapsto \psi(X_1) \ldots \psi(X_k) t^{m_1+\cdots+m_k}, \end{aligned} \tag{3-1}$$

where X_i is homogeneous in H and $[d_2, X_i] = m_i X_i$ for $1 \leq i \leq k$.

Denote by $A_{\bar{\psi}}$ the image of $\bar{\psi}$ in $\mathbb{C}[t, t^{-1}]$. Since $\mathscr{L}(\Delta_0)$ is $\mathbb{Z}$-graded with respect to d_2, we have a $\mathscr{L}(\Delta_0)$-module structure on $A_{\bar{\psi}}$ defined, for $X \in H$, by

$$X \,.\, t^n = \bar{\psi}(X) t^n \quad \text{and} \quad d_2 \,.\, t^n = n t^n.$$

Lemma 3.2 [Rao 1995]. *The $\mathscr{L}(\Delta_0)$-module $A_{\bar{\psi}}$ defined by (3-1) is irreducible if and only if each homogeneous element of $A_{\bar{\psi}}$ is invertible in $A_{\bar{\psi}}$.* □

Let $\bar{\psi}$ be given by (3-1) such that $A_{\bar{\psi}}$ is irreducible as an $\mathscr{L}(\Delta_0)$-module, and let $\mathscr{L}(\Delta_+)$ act trivially on $A_{\bar{\psi}}$. Consider the following induced module for $\mathscr{L}$:

$$M(\bar{\psi}) = U(\mathscr{L}) \otimes_{U(\mathscr{L}(\Delta_0) \oplus \mathscr{L}(\Delta_+))} A_{\bar{\psi}}.$$

Let ψ_0 be the restriction of ψ on $\mathfrak{h}_1 = \mathfrak{h}_0 \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}d_1$. We extend ψ_0 to a linear functional (still denoted by ψ_0) on $\mathfrak{h}$ by setting $\psi_0(d_2) = 0$.

Proposition 3.3. (1) *$M(\bar{\psi})$ is a highest-weight module over $\mathscr{L}$.*

(2) *The weight set $P(M(\bar{\psi}))$ is a subset of $\psi_0 + \mathbb{Z}\delta_2 - \mathrm{span}_{\mathbb{N}}\Delta_-$. Moreover, $x \in M(\bar{\psi})$ has a weight of form $\psi_0 + n\delta_2$ if and only if $x \in A_{\bar{\psi}}$.*

(3) *$M(\bar{\psi})$ has a unique irreducible quotient $V(\bar{\psi})$.*

Proof. (1) Applying the Poincaré–Birkhoff–Witt (PBW) theorem, we have $M(\bar{\psi}) = U(\mathscr{L}(\Delta_-))\,A_{\bar{\psi}}$. Noting that $1 = t^0 \in A_{\bar{\psi}}$ and $A_{\bar{\psi}}$ is irreducible as $\mathscr{L}(\Delta_0)$-module, we see that $A_{\bar{\psi}} = U(\mathscr{L}(\Delta_0))\,t^0$. Hence, $M(\bar{\psi}) = U(\mathscr{L}(\Delta_-))\,U(\mathscr{L}(\Delta_0))\,(1 \otimes t^0) = U(\mathscr{L})\,(1 \otimes t^0)$. It follows that $M(\bar{\psi})$ is a highest-weight module over $\mathscr{L}$.

(2) This is clear.

(3) Let W_1 and W_2 be two nonzero proper submodules of $M(\bar{\psi})$. Since $A_{\bar{\psi}}$ is irreducible as $\mathscr{L}(\Delta_0)$-module, it follows that $A_{\bar{\psi}} \cap W_i = 0$ for $i = 1, 2$. Now, we check that $(W_1 + W_2) \cap A_{\bar{\psi}} = \{0\}$, that is, $W_1 + W_2$ is still a proper submodule of $M(\bar{\psi})$. If $(W_1 + W_2) \cap A_{\bar{\psi}} \neq \{0\}$, we may write a weight vector $x \in A_{\bar{\psi}}$ as $x = y_1 + y_2$ for some $y_i \in W_i$ for $i = 1, 2$. By (2), we can assume that the weight of x is $\psi_0 + n\delta_2$ for some $n \in \mathbb{Z}$. Then, in at least one of W_1 and W_2, there exists a weight vector of weight $\psi_0 + n\delta_2$, which is again impossible by (2). If M is the sum of all proper submodules of $M(\bar{\psi})$, then $V(\bar{\psi}) = M(\bar{\psi})/M$ is the unique irreducible quotient. □

In the rest of this section, we investigate the conditions for $V(\bar{\psi})$ to be integrable. We will show in next section that any irreducible integrable module of $\mathscr{L}$ with the actions $C_1 > 0$ and $C_2 = 0$ is isomorphic to $V(\bar{\psi})$ for some $\bar{\psi}$.

Let $\mathscr{K} = \hat{\mathscr{G}}(\mathscr{J}(S)) \oplus \mathbb{C}d_1$ be a subalgebra of $\mathscr{L}$. Then, $\mathscr{K} = \mathscr{L}(\Delta_-) \oplus H \oplus \mathscr{L}(\Delta_+)$.

Definition 3.4. A $\mathscr{K}$-module W is called a *highest-weight module* if there exists a nonzero vector $v \in W$ such that

(1) $\mathscr{L}(\Delta_+)\,.\,v = 0$,

(2) $U(\mathscr{K})\,.\,v = W$,

(3) there exists some $\psi \in H^*$ with $\psi(C_2) = 0$ such that $h.v = \psi(h)\,v$ for all h in H.

Let ψ be in H^* with $\psi(C_2) = 0$. We view $\mathbb{C}$ as a one-dimensional $H \oplus \mathscr{L}(\Delta_+)$-module, on which h acts as the scalar $\psi(h)$ for $h \in H$, and $\mathscr{L}(\Delta_+)$ acts trivially. Consider the induced module for $\mathscr{K}$,

$$W(\psi) = U(\mathscr{K}) \otimes_{U(H \oplus \mathscr{L}(\Delta_+))} \mathbb{C}.$$

Clearly, $W(\psi)$ has a unique irreducible quotient denoted by $L(\psi)$, with the highest weight vector $v = 1 \otimes 1$.

Consider any $\bar{\psi}$ defined by (3-1) such that $A_{\bar{\psi}}$ is an irreducible $\mathscr{L}(\Delta_0)$-module. Define a linear map $\mathfrak{X} : A_{\bar{\psi}} \to \mathbb{C}$ by evaluating the polynomials at 1. In other words, $\mathfrak{X}(f(t)) = f(1)$ for all $f(t) \in A_{\bar{\psi}}$. If $\psi = \mathfrak{X} \circ (\bar{\psi}|_H)$, then we get the $L(\psi)$ defined above. One can easily check that the following action gives an $\mathscr{L}$-module structure on the vector space $L(\psi) \otimes \mathbb{C}[t, t^{-1}]$:

$$X\,.\,(a \otimes t^m) = (X\,.\,a) \otimes t^{m+n} \quad \text{and} \quad d_2\,.\,(a \otimes t^m) = ma \otimes t^m \tag{3-2}$$

for $X \in \mathscr{K}$ satisfying $[d_2, X] = nX$, $a \in L(\psi)$, and $m \in \mathbb{Z}$.

Theorem 3.5. *If $A_{\bar{\psi}}$ is irreducible as an $\mathscr{L}(\Delta_0)$-module, then $L(\psi) \otimes \mathbb{C}[t, t^{-1}]$ is completely reducible as an $\mathscr{L}$-module, and the component containing $v \otimes 1$ is isomorphic to $V(\bar{\psi})$ as an $\mathscr{L}$-module.*

Proof. First, note that $A_{\bar{\psi}} = \mathbb{C}[t^N, t^{-N}]$ for some nonnegative integer N. Take $G = \{0, 1, \dots, N-1\}$ if $N \geq 1$, or $G = \mathbb{Z}$ if $N = 0$. We will show that

$$L(\psi) \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{n \in G} U(\mathscr{L})(v \otimes t^n),$$

and that each $U(\mathscr{L})(v \otimes t^n)$ is irreducible as an $\mathscr{L}$-module.

If $w \otimes t^m \in L(\psi) \otimes \mathbb{C}[t, t^{-1}]$, then there exists some $X \in U(\mathscr{H})$ such that $Xv = w$ in $L(\psi)$. Write $X = \sum_n X_n$, where $[d_2, X_n] = nX_n$. We have $\sum_n X_n . (v \otimes t^{m-n}) = w \otimes t^m$, which implies that $L(\psi) \otimes \mathbb{C}[t, t^{-1}] = \sum_{n \in \mathbb{Z}} U(\mathscr{L})(v \otimes t^n)$.

For $t^r \in A_{\bar{\psi}}$, we have $\bar{\psi}(X') = t^r$ for some $X' \in U(H)$, and then $X' . (v \otimes t^m) = v \otimes t^{m+r}$. Hence, $U(\mathscr{L})(v \otimes t^m) = U(\mathscr{L})(v \otimes t^{m+r})$ and

$$L(\psi) \otimes \mathbb{C}[t, t^{-1}] = \sum_{n \in G} U(\mathscr{L})(v \otimes t^n). \tag{3-3}$$

Next, we prove that $U(\mathscr{L})(v \otimes t^m)$ is irreducible as an $\mathscr{L}$-module when $m \in G$. Let W be a nonzero $\mathscr{L}$-submodule of $U(\mathscr{L})(v \otimes t^m)$. Consider the linear map

$$\pi : W \to L(\psi), \quad w \otimes t^m \mapsto w.$$

It is clear that π is a homomorphism of $\mathscr{H}$-modules. Since $L(\psi)$ is irreducible as a $\mathscr{H}$-module, π has to be surjective. Using the fact that W is $\mathbb{Z}$-graded with respect to d_2, it follows that W contains $v \otimes t^n$ for some integer n. Clearly, $v \otimes t^n \in U(\mathscr{L})(v \otimes t^m)$ implies that $v \otimes t^n \in U(\mathscr{L}(\Delta_0))(v \otimes t^m)$. Then, there exists some $Y \in U(H)$ such that $Y(v \otimes t^m) = v \otimes t^n$, which means that $\bar{\psi}(Y) = t^{n-m} \in A_{\bar{\psi}}$. Choose $Z \in U(H)$ such that $\bar{\psi}(Z) = t^{m-n}$. Then, $v \otimes t^m = Z(v \otimes t^n) \in W$ and hence $W = U(\mathscr{L})(v \otimes t^m)$, as required. From the above, we see that $v \otimes t^m \in U(\mathscr{L})(v \otimes t^n)$ if and only if $m - n \in G \pmod N$. Therefore,

$$L(\psi) \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{n \in G} U(\mathscr{L})(v \otimes t^n). \tag{3-4}$$

Finally, the assertion that "the component containing $v \otimes 1$ is isomorphic to $V(\bar{\psi})$ as an $\mathscr{L}$-module" is clear. □

Proposition 3.6. *If $\bar{\psi}$ is defined by (3-1) and such that* $\dim A_{\bar{\psi}} = 1$, *then at least one of the weight spaces of $V(\bar{\psi})$ is infinite-dimensional.*

Proof. Since $\dim A_{\bar{\psi}} = 1$, we have that $C_1 v \neq 0$ and $C_1(0, 2m) v = 0$ for all $m \neq 0$. First, we show that $\alpha^\vee(-2, 2m) v \neq 0$ in $L(\psi)$ for all $m \in \mathbb{Z}$. Otherwise, we assume that $\alpha^\vee(-2, 2m)\,v = 0$ for some $n \in \mathbb{Z}$. Then,

$$0 = \alpha^\vee(2, -2m)\,\alpha^\vee(-2, 2m)\,v = [\alpha^\vee(2, -2m), \alpha^\vee(-2, 2m)]\,v = 8C_1 v,$$

which is a contradiction.

We complete the proof by showing that the set

$$\{\alpha^\vee(-2, 2m)\,\alpha^\vee(-2, -2m)\,(v \otimes 1) : m > 0\}$$

is linearly independent in $V(\bar{\psi})$. Otherwise, we may assume that we have a relation

$$\sum_m b_m\,\alpha^\vee(-2, 2m)\,\alpha^\vee(-2, -2m)\,(v \otimes 1) = 0$$

with some $b_m \neq 0$. Under the action of $\alpha^\vee(2, 2s)$, we obtain

$$\sum_m b_m \big(\alpha^\vee(-2, -2m)\,C_1(0, 2(m+s)) + \alpha^\vee(-2, 2m)\,C_1(0, 2(-m+s))\big)\,(v \otimes 1) = 0.$$

For any element $s \in \{m : b_m \neq 0\}$, we deduce that $b_s = 0$ — a contradiction. □

Proposition 3.7. *Let $\bar{\psi}$ be defined by (3-1) and such that $A_{\bar{\psi}}$ is an irreducible $\mathscr{L}(\Delta_0)$-module with* $\dim A_{\bar{\psi}} > 1$. *Then, $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$ if and only if $L(\psi)$ has finite-dimensional weight spaces with respect to $\mathfrak{h}_1$.*

Proof. Suppose that $V(\psi)$ has finite-dimensional weight spaces with respect to $\mathfrak{h}_1$. Then, $L(\psi) \otimes \mathbb{C}[t, t^{-1}]$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$. By Theorem 3.5, we see that $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$.

Suppose now that $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\mathfrak{h}$, and consider the $\mathscr{K}$-module homomorphism

$$\begin{aligned} \zeta : L(\psi) \otimes \mathbb{C}[t, t^{-1}] &\to L(\psi), \\ w \otimes t^n &\mapsto w, \end{aligned} \tag{3-5}$$

where $w \in L(\psi)$ and $n \in \mathbb{Z}$. For $k \in \mathbb{Z}$, let ζ_k be the restriction of ζ to $L(\psi) \otimes t^k$. Then, ζ_k is a $\mathscr{K}$-module isomorphism. If $L(\psi)$ has a weight space $L(\psi)_\nu$ satisfying $\dim L(\psi)_\nu = \infty$, then $\zeta_k^{-1}(L(\psi)_\nu) = (L(\psi) \otimes t^k)_\nu$ is infinite-dimensional. Note that G is a finite set. Therefore, there is at least one $n \in G$ such that the weight space $(U(\mathscr{L})(v \otimes t^n))_{\nu'}$ of $U(\mathscr{L})(v \otimes t^n)$ is infinite dimensional, where $\nu'|_{\mathfrak{h}_1} = \nu$ and $\nu'(d_2) = k$. This is a contradiction. □

Now, we investigate the conditions for $L(\psi)$ to be integrable.

Theorem 3.8. *Let $\lambda_1, \ldots, \lambda_k; -\mu_1, \ldots, -\mu_l$ be nonnegative integers, and take two sets of nonzero distinct complex numbers, $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_l\}$.*

If $\psi : H \to \mathbb{C}$ *is a linear map such that*

$$\psi(\alpha^\vee(0,m)) = \sum_{i=1}^{k} \lambda_i a_i^m, \tag{3-6}$$

$$\psi\big(\alpha^\vee(0,2m) - 2C_1(0,2m)\big) = \sum_{i=1}^{l} \mu_i b_i^m, \tag{3-7}$$

$$\psi(C_2) = 0, \tag{3-8}$$

then $L(\psi)$ *is an integrable module for* $\mathcal{K}$.

Conversely, if $L(\psi)$ *is integrable (with* $\psi(C_2) = 0$*) for* $\mathcal{K}$, *then* ψ *has to be defined as above.*

Before proving Theorem 3.8, we present several results which we will use later.

Lemma 3.9. *The Lie subalgebra* $\mathcal{L}(\Delta_+)$ *is generated by the set*

$$\{x_+(0,n),\ x_-(1,2n),\ x_-(2,2n+1) : n \in \mathbb{Z}\}. \tag{3-9}$$

Proof. It is straightforward to check. □

For $n \in \mathbb{Z}$, we define

$$X_{1,n} = x_+(0,n), \qquad X_{2,n} = x_-(1,2n), \qquad X_{3,n} = x_-(2,2n+1). \tag{3-10}$$

Recall that an element $X \in \mathcal{K}$ is said to be *locally nilpotent* on $L(\psi)$ if, for any element $w \in L(\psi)$, one has $X^m w = 0$ when $m \gg 0$. For an arbitrary Lie algebra $\mathfrak{g}$, we have the following results:

Proposition 3.10 [Kac 1990]. *Let* $v_1, v_2, \ldots$ *be a system of generators of a* $\mathfrak{g}$-*module* V, *and let* $x \in \mathfrak{g}$ *be such that* $\operatorname{ad} x$ *is locally nilpotent on* $\mathfrak{g}$ *and* $x^{N_i}(v_i) = 0$ *for some positive integers* N_i, $i = 1, 2, \ldots$. *Then* x *is locally nilpotent on* V. □

Proposition 3.11 [Moody and Pianzola 1995]. *Let* $\pi : \mathfrak{g} \to \mathrm{gl}(V)$ *be a representation of* $\mathfrak{g}$ *on a vector space* V. *If* $x \in \mathfrak{g}$ *is such that both* $\operatorname{ad} x$ *and* $\pi(x)$ *are locally nilpotent, then, for all* $y \in \mathfrak{g}$,

$$\pi((\exp \operatorname{ad} x)(y)) = (\exp \pi(x))\pi(y)(\exp \pi(x))^{-1}. \qquad \square$$

Let $\alpha_0 = -\alpha + \delta_1$. Then, $\{\alpha,\ \alpha_0\}$ is a set of simple roots of the affine Kac–Moody algebra $\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \big(\mathfrak{sl}_2(\mathbb{C}) \otimes (\sum_{k\in\mathbb{Z}} \mathbb{C}x^{ke_1})\big) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1$ (see Remark 2.3). Let $\mathcal{W}_{\mathrm{aff}}$ be the subgroup of $\mathcal{W}$ generated by the reflections associated to α and α_0. Then, $\mathcal{W}_{\mathrm{aff}}$ is the Weyl group of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$.

Lemma 3.12. *If* $\gamma = \pm\alpha + n_1\delta_1 + n_2\delta_2 \in \Delta^{\mathrm{Re}}$ *is a real root, then there exists some* $\omega \in \mathcal{W}_{\mathrm{aff}}$ *such that* $\omega(\gamma) = \alpha + n_2\delta_2$ *or* $\omega(\gamma) = \alpha_0 + n_2\delta_2$. *In any case,* $\omega(\gamma)$ *is still a root in* Δ^{Re}.

Proof. Denote $\gamma' = \gamma - n_2\delta_2$. Since γ' is a real root of the affine Kac–Moody algebra $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, there exists $\omega \in \mathcal{W}_{\text{aff}}$ such that $\omega(\gamma') = \alpha$ or $\omega(\gamma') = \alpha_0$. We see that $\omega(\gamma') = \alpha$ (respectively, α_0) if n_1 is even (respectively, odd). Thus, $\omega(\gamma) = \alpha + n_2\delta_2$ or $\omega(\gamma) = \alpha_0 + n_2\delta_2$. In either case, $\omega(\gamma)$ is a root in Δ^{Re}. □

Lemma 3.13. *Suppose that, for all $m \in \mathbb{Z}$, both $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on the highest-weight vector v in $L(\psi)$, where $\sigma_m = -e_1 + 2me_2$ and $\tau_m = me_2$. Then, $x_\pm(\sigma)$ are locally nilpotent on $L(\psi)$ for all $\sigma = k_1e_1 + k_2e_2 \in S$.*

Proof. Since $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on v and locally nilpotent on $\mathcal{L}$ under the adjoint action, they are locally nilpotent on $L(\psi)$ by Proposition 3.10. Thus, $L(\psi)$ is an integrable module (without the finite-dimensional weight-spaces condition) for the $\mathfrak{sl}_2(\mathbb{C})$-copies $\{x_+(-\tau_m), x_-(\tau_m), \alpha^\vee\}$ and $\{x_+(\sigma_m), x_-(-\sigma_m), \alpha^\vee - 2C_1\}$ (we are assuming $C_2 = 0$).

Let $\gamma = \pm\alpha + k_1\delta_1 + k_2\delta_2$ be the root of $x_\pm(\sigma)$ for $\sigma = k_1e_1 + k_2e_2$. By Lemma 3.12, there exists some $\omega \in \mathcal{W}_{\text{aff}}$ such that $\omega(\gamma) = \beta + k_2\delta_2$ for $\beta \in \{\alpha, \alpha_0\}$. Let s_ω be the inner automorphism of $\mathcal{L}$ associated to ω, and take $Y \in \mathcal{L}_{\beta+k_2\delta_2}$ to be a nonzero root vector. Up to a nonzero constant multiple, we have $s_\omega(x_\pm(\sigma)) = Y$. By Proposition 3.11, we know that $x_\pm(\sigma)$ are locally nilpotent on $L(\psi)$. □

Consider the loop algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. Let $u_1, \dots, u_n$ be nonzero complex numbers and $\xi_1, \dots, \xi_n$ (with $n > 0$) be nonnegative integers. Let B be the $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by an element w subject to the relations

$$(x_+ \otimes \mathbb{C}[t, t^{-1}]) \,.\, w = 0, \quad (\alpha^\vee \otimes t^m) \,.\, w = \sum_{j=1}^{n} \xi_j u_j^m w, \quad (x_- \otimes 1)^{\sum_j \xi_j + 1} \,.\, w = 0,$$

with $m \in \mathbb{Z}$. We have:

Theorem 3.14 [Chari and Pressley 2001]. (1) *The $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module B (associated with $u_1, \dots, u_n$ and $\xi_1, \dots, \xi_n$ with $n > 0$) is finite-dimensional.*

(2) *If B' is any finite-dimensional $\widehat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by an element w' such that* $\dim U(\alpha^\vee \otimes \mathbb{C}[t, t^{-1}])\, w' = 1$, *then B' is a quotient of some module B constructed as above.* □

Lemma 3.15. *If ψ is as in Theorem 3.8, then, for all $m \in \mathbb{Z}$, both $x_+(\sigma_m)$ and $x_-(\tau_m)$ are nilpotent on the generator v of $L(\psi)$, where $\sigma_m = -e_1 + 2me_2$ and $\tau_m = me_2$.*

Proof. As $L(\psi)$ is irreducible, it is enough to show that

$$\mathcal{L}(\Delta_+) \,.\, (x_+(\sigma_m))^N v = 0 \quad \text{and} \quad \mathcal{L}(\Delta_+) \,.\, (x_-(\tau_m))^N v = 0 \tag{3-11}$$

for some $N \gg 0$. By Lemma 3.9, $\mathscr{L}(\Delta_+) . (x_+(\sigma_m))^N v = 0$ is equivalent to

$$X_{1,n}(x_+(\sigma_m))^N v = 0, \tag{3-12}$$

$$X_{2,n}(x_+(\sigma_m))^N v = 0, \tag{3-13}$$

$$X_{3,n}(x_+(\sigma_m))^N v = 0. \tag{3-14}$$

It is easy to see that (3-12) and (3-14) hold for $N \geq 0$. To show (3-13), we set

$$x_n = x_+(\sigma_n), \quad y_n = x_-(-\sigma_{-n}), \quad h_n = \alpha^\vee(0, 2n) - 2C_1(0, 2n), \tag{3-15}$$

for $n \in \mathbb{Z}$. Noting that $C_2 = 0$ on $L(\psi)$, these vectors satisfy

$$[x_a, y_b] = h_{a+b}, \quad [h_c, x_a] = 2x_{c+a}, \quad [h_c, y_b] = -2y_{b+c}.$$

Hence, they form a basis for a loop algebra of type A_1. Denote this subalgebra by $\mathfrak{S}$. In $W(\psi)$, we consider the $\mathfrak{S}$-submodule generated by v. From Theorem 3.14, we know that $(x_+(\sigma_m))^N v$ belongs to a proper submodule of $U(\mathfrak{S})v$ for some $N \gg 0$. Applying the PBW Theorem to $W(\psi)$, we see that (3-13) holds. The proof that $\mathscr{L}(\Delta_+) . (x_-(\tau_m))^N v = 0$ is similar and is omitted. □

The following proposition gives the first part of Theorem 3.8.

Proposition 3.16. *For ψ as in Theorem 3.8, $L(\psi)$ is integrable as a $\mathscr{K}$-module.*

Proof. By applying Lemmas 3.13 and 3.15, we show that, with respect to $\mathfrak{h}_1$, the weight spaces of $L(\psi)$ are finite-dimensional.

Let ψ_1 be the restriction of ψ on $\mathfrak{h}_1$. Then, the weight set $P(L(\psi))$ is a subset of $\psi_1 - (\mathbb{Z}_+\alpha_0 + \mathbb{Z}_+\alpha)$. Consider any weight space $L(\psi)_{\psi_1 - \eta}$ with $\eta \in \mathbb{Z}_+\alpha_0 + \mathbb{Z}_+\alpha$. From applying the PBW Theorem to $L(\psi)$, the vector space $L(\psi)_{\psi_1 - \eta}$ is spanned by some vectors of the form

$$X(\beta_1, n_1) X(\beta_2, n_2) \dots X(\beta_k, n_k) v, \tag{3-16}$$

where $X(\beta_i, n_i)$ is a root vector of $\mathscr{L}(\Delta_-)$ with root $\beta_i + n_i\delta_2$, and the β_i are negative affine roots satisfying $\sum \beta_i = -\eta$. For a fixed η, only finitely many β_i will appear. It suffices to show that, for fixed $\beta_1, \dots, \beta_k$, the vectors of the form (3-16) span a finite-dimensional vector space.

As a subalgebra of $\mathscr{J}(S)$, the subspace $\mathscr{T} = \bigoplus_{s \in \mathbb{Z}} \mathbb{C}x^{se_2}$ is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Define

$$p = \sum_{i=0}^{k} \epsilon_i x^{ie_2} = \prod_{j=1}^{k} (x^{e_2} - a_j) \quad \text{and} \quad q = \sum_{i=0}^{l} \epsilon'_i x^{2ie_2} = \prod_{j=1}^{l} (x^{2e_2} - b_j).$$

Let $s = pq$. We use P, Q and S to denote the ideals $p\mathscr{T}$, $q\mathscr{T}$ and $s\mathscr{T}$ of $\mathscr{T}$, respectively. Write $s = \sum_i \epsilon''_i x^{ie_2}$. By using the definition of ψ, it is straightforward to check the following two identities:

$$\psi(\alpha^\vee \otimes S) = 0. \tag{3-17}$$

$$\psi\Big(\sum_{m=0}^{l} \epsilon'_m h_{m+n}\Big) = 0 \quad \text{for all } n \in \mathbb{Z} \text{ (see (3-15))}. \tag{3-18}$$

First, we show that, for any negative affine root β and all $m \in \mathbb{Z}$, we have $\sum_i \epsilon''_i X(\beta, m+i)v = 0$, where $X(\beta, m+i)$ is a root vector of $\mathcal{L}(\Delta_-)$ with root $\beta + (m+i)\delta_2$. We prove this by induction on the height of $-\beta$. When the height of $-\beta$ is 1, we need

$$\sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big)\,.\,v = 0. \tag{3-19}$$

$$\sum_i \epsilon''_i \big(x_+ \otimes x^{-e_1+(m+i)e_2}\big)\,.\,v = 0. \tag{3-20}$$

Since $L(\psi)$ is irreducible, this is equivalent to both $\sum_i \epsilon''_i(x_- \otimes x^{(m+i)e_2})\,.\,v$ and $\sum_i \epsilon''_i(x_+ \otimes x^{-e_1+(m+i)e_2})\,.\,v$ being annihilated by $\mathcal{L}(\Delta_+)$. By Lemma 3.9, it is enough to check that they are annihilated by $X_{1,n}$, $X_{2,n}$ and $X_{3,n}$ for $n \in \mathbb{Z}$. Now, it is clear that

$$X_{2,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big)\,.\,v = 0 \quad \text{and} \quad X_{3,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big)\,.\,v = 0.$$

But, by (3-17) and using that $C_2 = 0$ on $L(\psi)$,

$$X_{1,n} \sum_i \epsilon''_i \big(x_- \otimes x^{(m+i)e_2}\big)\,.\,v = \alpha^\vee \otimes (x^{ne_2} s)\,.\,v = 0.$$

Similarly, we can prove (3-20). If the height of $-\beta$ is 2, then $\sum_i \epsilon''_i X(\beta, m+i)v$ is 0, as it is annihilated by $X_{i,n}$ for $i = 1, 2, 3$. Now, we assume that the height of $-\beta$ is 3. Then, $\beta = -\alpha - \delta_1$ or $\alpha - 2\delta_1$. In case $\beta = -\alpha - \delta_1$, one can easily see that

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0 \quad \text{for } j = 1, 2, 3.$$

So, $\sum_i \epsilon''_i X(\beta, m+i)v = 0$. In case $\beta = \alpha - 2\delta_1$,

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0 \quad \text{for } j = 1, 2.$$

Thus, $X_{3,n} \sum_i \epsilon''_i X(\beta, m+i)v = 0$ by (3-17) and (3-18). When the height of $-\beta$ is greater than 3, consider

$$X_{j,n} \sum_i \epsilon''_i X(\beta, m+i)v = \sum_i \epsilon''_i [X_{j,n}, X(\beta, m+i)]\,.\,v.$$

Clearly, the negative of the height decreases and hence it is zero by induction, as required.

For the fixed negative affine roots $\gamma_1, \dots, \gamma_l$ $(1 \le j \le l)$, we show that

$$\sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_j, n+i) X(\gamma_{j+1}, n_{j+1}) \dots X(\gamma_l, n_l) \,.\, v = 0,$$

for all integers $n, n_1, \dots, n_l$, using induction on the height of $-(\gamma_{j+1} + \cdots + \gamma_l)$. It is clear when $\beta_{j+1}, \dots, \beta_l$ are 0. Now, since

$$\begin{aligned}\sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_j, n+i) X(\gamma_{j+1}, n_{j+1}) \dots X(\gamma_l, n_l) \,.\, v \\ = \sum_i \epsilon_i'' X(\gamma_1, n_1) \dots [X(\gamma_j, n+i), X(\gamma_{j+1}, n_{j+1})] \dots X(\gamma_l, n_l) \,.\, v \\ + \sum_i \epsilon_i'' X(\gamma_1, n_1) \dots X(\gamma_{j+1}, n_{j+1}) X(\gamma_j, n+i) \dots X(\gamma_l, n_l) \,.\, v,\end{aligned}$$

the terms on the right hand side are zero by induction.

Since $\dim(\mathcal{T}/S) < \infty$, for fixed $\beta_1, \dots, \beta_k$, the vectors of the form (3-16) span a finite-dimensional vector space. Therefore, we know that the weight spaces of $L(\psi)$ are finite-dimensional. This completes the proof of this proposition. □

The second part of Theorem 3.8 follows from the next proposition.

Proposition 3.17. *If $L(\psi)$ is integrable as a $\mathcal{K}$-module, with the action $C_2 = 0$, then ψ satisfies the conditions of Theorem 3.8.*

Proof. We consider the affine algebra $\mathfrak{T} = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{T} \oplus \mathbb{C}C_2$. Denote by V the irreducible quotient of $U(\mathfrak{T})v$ of $\mathfrak{T}$. We claim that $\dim V < \infty$. From the integrability of $L(\psi)$, the set

$$\{x_-(0, n) \,.\, v : n \in \mathbb{Z}\}$$

is linearly dependent. So, there exists some nonzero polynomial $f = \sum_i f_i x^{ie_2}$ such that $(x_- \otimes f)v = 0$. Set $F = f\mathcal{T}$. We have $(x_- \otimes F).v = 0$ and $(\alpha^\vee \otimes F).v = 0$. The first identity follows since

$$0 = \alpha^\vee(0, m)(x_- \otimes f)v = (x_- \otimes f)\alpha^\vee(0, m)v - 2(x_- \otimes x^{me_2} f)v$$

and $\alpha^\vee(0, m)$ acts on v as a constant. The second identity follows from the first.

It follows that $(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2).v = 0$, and we show that $(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2).V = 0$. In fact, if we define $W = \{w \in V : (\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2) \,.\, w = 0\}$, then W is a nonzero submodule. Hence $V = W$, since V is irreducible. We deduce that V is an irreducible integrable module for $(\mathfrak{sl}_2(\mathbb{C}) \otimes \mathcal{T} \oplus \mathbb{C}C_2)/(\mathfrak{sl}_2(\mathbb{C}) \otimes F \oplus \mathbb{C}C_2)$. This implies that $\dim V < \infty$. Using Theorem 3.14, we can see that ψ satisfies the condition (3-6) of Theorem 3.8. Similarly, we can prove that ψ satisfies (3-7). □

4. The classification theorem

We classify the irreducible integrable modules for the extended baby TKK algebra $\mathcal{L}$ with actions $C_1 \neq 0$ and $C_2 = 0$.

Proposition 4.1. *If V is an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts as a positive number and C_2 acts as zero, then V is a highest-weight module.*

Proof. By Lemma 2.8, we may assume that $2C_1$ acts on V as a positive integer, say $2c_1$.

First, we show that, for any fixed $\lambda \in P(V)$, there exists some $\lambda' \in P(V)$ such that $\lambda' + n\alpha$ is not a weight for any positive integer n, and that $\lambda'(d_i) = \lambda(d_i)$ for $i = 1, 2$.

Let $W = \{w \in V : d_i w = \lambda(d_i) w, i = 1, 2\}$. Write $P_1 = \{\mu \in P(V) : V_\mu \subset W\}$. Then, for any $\mu \in P_1$, we can write μ in the form

$$\mu = \bar{\mu} + \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1,$$

where $\bar{\mu} = \mu|_{\mathfrak{h}_0}$. Set $\bar{P}_1 = \{\bar{\mu} : \mu \in P_1\}$. Since W is an integrable module for the Lie subalgebra $\operatorname{span}_{\mathbb{C}}\{x_\pm, \alpha^\vee\}$, with finite-dimensional weight spaces with respect to $\mathfrak{h}_0 = \mathbb{C}\alpha^\vee$, it follows from Weyl's theorem that W can be decomposed as

$$W = \bigoplus_{\bar{\mu} \in \mathfrak{h}_0^*} V(\bar{\mu}),$$

where each $V(\bar{\mu})$ is an irreducible finite-dimensional module for $\operatorname{span}_{\mathbb{C}}\{x_\pm, \alpha^\vee\}$ with highest weight $\bar{\mu}$. Since V is irreducible, for any two weights μ, ν in P_1, we have $\mu - \nu = n\alpha$ for some integer n. Thus, $\bar{P}_1$ belongs to either $\mathbb{Z}\alpha$ or $\frac{1}{2}\alpha + \mathbb{Z}\alpha$. Set

$$\mu = \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1 \quad \text{if } \bar{P}_1 \subset \mathbb{Z}\alpha, \quad \text{or}$$
$$\mu = \tfrac{1}{2}\alpha + \lambda(d_1)\delta_1 + \lambda(d_2)\delta_2 + c_1 w_1 \quad \text{if } \bar{P}_1 \subset (1/2)\alpha + \mathbb{Z}\alpha.$$

By $\mathfrak{sl}_2(\mathbb{C})$-theory, we know that $\bar{\mu}$ is a common weight of the $V(\bar{\nu})$-terms that occur in $W = \bigoplus_{\bar{\nu} \in \mathfrak{h}_0^*} V(\bar{\nu})$. Since V_μ is finite-dimensional, P_1 is a finite set. Take $\lambda' \in P_1$ so that $\bar{\lambda'}(\alpha^\vee)$ is maximal. Then, λ' is the required weight.

Recall that $\{\alpha_0 = -\alpha + \delta_1, \alpha\}$ is a set of simple roots of the affine Kac–Moody Lie algebra

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \left(\mathfrak{sl}_2(\mathbb{C}) \otimes \left(\sum_{j \in \mathbb{Z}} \mathbb{C}x^{je_1}\right)\right) \oplus \mathbb{C}C_1 \oplus \mathbb{C}d_1.$$

Define a partial order $\preceq$ on $\mathfrak{h}^*$ by setting

$$\lambda \preceq \mu \quad \text{if and only if} \quad \lambda - \mu = n_1\alpha_0 + n_2\alpha \ \text{ for some } n_1, n_2 \in -\mathbb{N}.$$

If λ' is as above and such that $\lambda' + n\alpha$ is not a weight for any positive integer n, then $\lambda'(\alpha^\vee) \geq 0$ by Lemma 2.8. Let $\Pi = \{\alpha + m\delta_1 : m \geq 0\} \cup \{-\alpha + m\delta_1 : m > 0\}$ be the set of positive real roots of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, and $\Pi_{\lambda'} = \{\gamma \in \Pi : \lambda'(\gamma^\vee) \leq 0\}$. Since $\lambda'(C_1) > 0$, it follows that $\Pi_{\lambda'}$ is a finite set. Using a similar technique as in the proof of [Chari 1986, Thm 2.4], we get a nonzero weight vector $v \in V_{\lambda' + p\delta_1}$, $p \geq 0$, such that $\mathscr{L}_{r\delta_1} v = 0$ for all $r > 0$, and $\mathscr{L}_\beta v = 0$ for all but finitely many roots $\beta \in \Pi$.

Using an argument similar to the first paragraph of the proof of [Eswara Rao 2004, Prop 2.8], we obtain a weight $\mu \in P(V)$ such that

$$\mu + \eta \notin P(V) \quad \text{for all } \eta \not\leq 0. \tag{4-1}$$

In particular, $\mu + \beta \notin P(V)$ for all $\beta \in \Pi$.

By Lemma 2.8, we have $\mu(\beta^\vee) \geq 0$ for all $\beta \in \Pi$. In particular, $\mu(\alpha) \geq 0$. To prove that the module V has a highest-weight vector, we divide the argument into two cases: case 1, for $\mu(\alpha) > 0$, and case 2, for $\mu(\alpha) = 0$.

Case 1: Suppose that $\mu(\alpha) > 0$. If $\mu + \beta + m\delta_2 \notin P(V)$ for all integers m such that $\beta + m\delta_2 \in \Delta_+$, then it is clear that $\mathscr{L}(\Delta_+) \,.\, v = 0$ for any $0 \neq v \in V_\mu$, and we are done. On the other hand, assume that there exist some $\beta \in \Pi$ and $m_0 \in \mathbb{Z}$ such that $\beta + m_0\delta_2 \in \Delta_+$ and $V_{\mu+\beta+m_0\delta_2} \neq 0$. Let $\nu = \mu + \beta + m_0\delta_2$. We show that ν is a highest weight. That is, $V_{\nu+\gamma+k\delta_2} = 0$ for all $\gamma \in \Pi$ and all $k \in \mathbb{Z}$ such that $\gamma + k\delta_2 \in \Delta_+$. Suppose this is false. Then, $V_{\nu+\gamma+k_0\delta_2} \neq 0$ for some $\gamma \in \Pi$ and $k_0 \in \mathbb{Z}$ such that $\gamma + k_0\delta_2 \in \Delta_+$. Let $\gamma_1 = \beta + (m_0 + k_0)\delta_2$. We divide the argument into three subcases. In each subcase, we will get a contradiction with (4-1).

Subcase 1.1: Suppose $\beta, \gamma \in \{\alpha + m\delta_1 : m \geq 0\}$ or $\beta, \gamma \in \{-\alpha + m\delta_1 : m > 0\}$. We have $(\beta + \gamma)(\beta^\vee) > 0$ and $(\beta + \gamma)(\gamma^\vee) > 0$. If γ_1 is a root in Δ_+, then $(\nu + \gamma + k_0\delta_2)(\gamma_1^\vee) = (\mu + \beta + \gamma)(\beta^\vee) > 0$, which implies that

$$\mu + \gamma = (\nu + \gamma + k_0\delta_2) - \gamma_1 \in P(V),$$

which contradicts (4-1). If γ_1 is not a root, then we take $\gamma_1 - \delta_1$, which is obviously a root in Δ. Similar arguments show that $\mu + \gamma + \delta_1 \in P(V)$, contradicting (4-1) again.

Subcase 1.2: Suppose $\beta = \alpha + m\delta_1$ and $\gamma = -\alpha + n\delta_1$ for some $m \geq 0$ and $n > 0$. If $\gamma_1 \in \Delta_+$, then we have $(\mu + \beta + \gamma + (m_0 + k_0)\delta_2)(\gamma_1^\vee) = \mu(\beta^\vee) > 0$, which implies that

$$\mu + \gamma = (\mu + \beta + \gamma + (m_0 + k_0)\delta_2) - (\beta + (m_0 + k_0)\delta_2) \in P(V).$$

This contradicts (4-1). If $\gamma_1 \notin \Delta_+$, then $(\mu + \beta + \gamma + (m_0 + k_0)\delta_2)((\gamma_1 - \delta_1)^\vee) > 0$, which gives

$$\mu + \gamma + \delta_1 = (\mu + \beta + \gamma + (m_0 + k_0)\delta_2) - (\beta - \delta_1 + (m_0 + k_0)\delta_2) \in P(V).$$

This contradicts (4-1) again.

Subcase 1.3: Suppose $\beta = -\alpha + m\delta_1$ and $\gamma = \alpha + n\delta_1$ for some $m > 0$ and $n \geq 0$. This can be dealt with similarly to Subcase 1.2. This completes the proof of Case 1.

Case 2: Suppose now that $\mu(\alpha^\vee) = 0$. We assume that there exist some $\beta_0 \in \Pi$ and $t \in \mathbb{Z}$ such that $\beta_0 + t\delta_2 \in \Delta_+$ and $V_{\mu+\beta_0+t\delta_2} \neq 0$. Let $\mu_1 = \mu + \beta_0 + t\delta_2$.

If $\mu_1+\beta+m\delta_2 \notin P(V)$ for all integers m such that $\beta+m\delta_2 \in \Delta_+$, then, for any $0 \neq v \in V_{\mu_1}$, we have $\mathscr{L}(\Delta_+)\,.\,v=0$ and we are done. On the other hand, we assume that there exist some $\beta' \in \Pi$ and $m_1 \in \mathbb{Z}$ such that $\beta'+m_1\delta_2 \in \Delta_+$ and $V_{\mu_1+\beta'+m_1\delta_2} \neq 0$. Let $\nu_1=\mu_1+\beta'+m_1\delta_2$. We prove that ν_1 is a highest weight. That is, $V_{\nu_1+\gamma+k\delta_2}=0$ for all $\gamma \in \Pi$ and all $k \in \mathbb{Z}$ such that $\gamma+k\delta_2 \in \Delta_+$. Suppose this is false. Then, $V_{\nu_1+\gamma'+k_1\delta_2} \neq 0$ for some $\gamma' \in \Pi$ and $k_1 \in \mathbb{Z}$ such that $\gamma'+k_1\delta_2 \in \Delta_+$. Let $\gamma_2=\beta'+(t+m_1+k_1)\delta_2$. We divide the arguments into four subcases. In each subcase, we will get a contradiction with (4-1).

<u>Subcase 2.1</u>: Suppose $\beta',\gamma' \in \{\alpha+m\delta_1 : m \geq 0\}$. In this case, $(\beta'+\gamma')(\beta'^\vee)>0$ and $(\beta'+\gamma')(\gamma'^\vee)>0$. If γ_2 is a root in Δ_+, then

$$(\nu_1+\gamma'+k_1\delta_2)(\gamma_2^\vee)=(\mu+\beta_0+\beta'+\gamma')(\beta'^\vee)>0,$$

which implies that

$$\mu+\beta_0+\gamma'=(\nu_1+\gamma'+k_1\delta_2)-\gamma_2 \in P(V).$$

If $\beta_0 \in \{-\alpha+m\delta_1 : m>0\}$, then we arrive at a contradiction with (4-1). If $\beta_0 \in \{\alpha+m\delta_1 : m \geq 0\}$, then $(\mu+\beta_0+\gamma')(\gamma'^\vee)>0$, which means that $\mu+\beta_0 \in P(V)$ — a contradiction again. If γ_2 is not a root, then we take $\gamma_2-\delta_1$, which is a root in Δ. Similar arguments give a contradiction with (4-1).

<u>Subcase 2.2</u>: Suppose $\beta',\gamma' \in \{-\alpha+m\delta_1 : m\rangle 0\}$. This is very similar to the arguments for Subcase 2.1.

<u>Subcase 2.3</u>: Suppose $\beta'=\alpha+m'\delta_1$ and $\gamma'=-\alpha+n'\delta_1$ for some $m' \geq 0$ and $n'>0$. We have these two subcases:

<u>Subcase 2.3.1</u>: Suppose $\beta_0 \in \{\alpha+m\delta_1 : m \geq 0\}$. If $\gamma_2 \in \Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)(\gamma_2^\vee)=(\mu+\beta_0+\beta'+\gamma')(\beta'^\vee)>0.$$

This implies that $\mu+\beta_0+\gamma' \in P(V)$, which is impossible by (4-1). If $\gamma_2 \notin \Delta_+$, we consider $\gamma_2-\delta_1 \in \Delta_+$. Then,

$$\begin{aligned}(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)((\gamma_2-\delta_1)^\vee)\\ =(\mu+\beta_0+\beta'+\gamma')((\beta'-\delta_1)^\vee)>0.\end{aligned}$$

This implies that $\mu+\beta_0+\gamma'+\delta_1 \in P(V)$, which is also impossible.

<u>Subcase 2.3.2</u>: Suppose $\beta_0 \in \{-\alpha+m\delta_1 : m>0\}$. We denote $\gamma_3=\gamma'+(t+m_1+k_1)\delta_2$. If $\gamma_3 \in \Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)(\gamma_3^\vee)=(\mu+\beta_0+\beta'+\gamma')(\gamma'^\vee)>0.$$

So we have $\mu+\beta_0+\beta'\in P(V)$, which is impossible. If $\gamma_3\notin\Delta_+$, then

$$(\mu+\beta_0+\beta'+\gamma'+(t+m_1+k_1)\delta_2)((\gamma_3-\delta_1)^\vee) = (\mu+\beta_0+\beta'+\gamma')((-\alpha+(n'-1)\delta_1)^\vee) > 0.$$

We get $\mu+\beta_0+\beta'+\delta_1\in P(V)$, which is a contradiction.

Subcase 2.4: Finally, suppose $\beta'=-\alpha+m'\delta_1$ and $\gamma'=\alpha+n'\delta_1$ for some $m'>0$ and $n'\geq 0$. This can be discussed similarly to Subcase 2.3, and thus completes the proof of Case 2.

In every case, there exists some weight vector, say $v\in V$, such that $\mathscr{L}(\Delta_+).v=0$. Therefore, V is a highest-weight module for $\mathscr{L}$. □

Lemma 4.2 [Eswara Rao 2001]. *Any $\mathbb{Z}$-graded simple commutative and associative algebra, with all its homogeneous subspaces finite-dimensional, is isomorphic to a subalgebra $A_{\bar\psi}$ of $\mathbb{C}[t,t^{-1}]$ for some $\bar\psi$ (as defined by (3-1)). Furthermore, every nonzero homogeneous element in $A_{\bar\psi}$ is invertible in $A_{\bar\psi}$.* □

Theorem 4.3. *Let V be an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ such that C_1 acts as a positive number and C_2 acts as zero. Then, V is isomorphic to $V(\bar\psi)$, for some $\bar\psi$ given in Section 3, such that $A_{\bar\psi}$ is an irreducible $\mathscr{L}(\Delta_0)$-module.*

Proof. By Proposition 4.1, there exists some nonzero weight vector $v\in V$ such that $\mathscr{L}(\Delta_+)\,.\,v=0$. Let M be the $\mathscr{L}(\Delta_0)$-module generated by v. In fact,

$$M=\{w\in V:\mathscr{L}(\Delta_+)\,.\,w=0\}$$

and M is irreducible as an $\mathscr{L}(\Delta_0)$-module by the irreducibility of V. Let $I=\{X\in U(H):X\,.\,v=0\}$. It is clear that $M\cong U(H)/I$ as $\mathscr{L}(\Delta_0)$-modules. Since $U(H)/(U(H)C_2)$ is commutative and I is an ideal of $U(H)$, we see that $U(H)/I$ is a $\mathbb{Z}$-graded simple commutative and associative algebra. By Lemma 4.2, M is isomorphic to some $A_{\bar\psi}$. It is now clear that V is isomorphic to $V(\bar\psi)$. □

In view of Proposition 4.1, we have:

Corollary 4.4. *If V is an irreducible integrable module for the extended baby TKK algebra $\mathscr{L}$ with $C_1<0$ and $C_2=0$, then V is a lowest-weight module.* □

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