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Some duality properties for induced representations of enveloping algebras involve the character $\mathrm{Trad}_{\mathfrak{g}}$. We extend them to deformation Hopf algebras A_h of a noetherian Hopf k-algebra A_0 satisfying $\mathrm{Ext}^i_{A_0}(k, A_0) = \{0\}$ except for $i = d$ where it is isomorphic to k. These duality properties involve the character of A_h defined by right multiplication on the one-dimensional free $k[[h]]$-module $\mathrm{Ext}^d_{A_h}(k[[h]], A_h)$. In the case of quantized enveloping algebras, this character lifts the character $\mathrm{Trad}_{\mathfrak{g}}$. We also prove Poincaré duality for such deformation Hopf algebras in the case where $k[[h]]$ is an A_h-module of finite projective dimension. We explain the relation of our construction with quantum duality.

1. Introduction

Let k be a field of characteristic 0 and set $K = k[[h]]$. Let A_0 be a noetherian algebra. Assume k has a left A_0-module structure such that, for some integer d,

$$\begin{cases} \mathrm{Ext}^i_{A_0}(k, A_0) = \{0\} & \text{if } i \neq d, \\ \mathrm{Ext}^d_{A_0}(k, A_0) \simeq k. \end{cases}$$

It follows from Poincaré duality that any finite-dimensional Lie algebra $\mathfrak{g}$ verifies these assumptions. In this case, $d = \dim \mathfrak{g}$ and the character defined by the right representation of $U(\mathfrak{g})$ on $\mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\mathrm{Trad}_{\mathfrak{g}}$ [Chemla 1994]. The algebra of regular functions on an affine algebraic Poisson group and the algebra of formal power series also satisfy these hypothesis. Let A_h be a deformation algebra of A_0. Assume that there exists an A_h-module structure on K that reduces modulo h to the A_0-module structure we started with. The main theorem of the paper constructs a new character of A_h that will be denoted by θ_{A_h}.

Theorem 4.1. *With the assumptions made above*:

(a) $\mathrm{Ext}^i_{A_h}(K, A_h) = \{0\}$ *if* $i \neq d$.

(b) $\mathrm{Ext}^d_{A_h}(K, A_h)$ *is a free* K*-module of dimension one. The right* A_h*-module structure given by right multiplication lifts that of* A_0 *on* $\mathrm{Ext}^d_{A_0}(k, A_0)$.

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The right A_h-module $\mathrm{Ext}^d_{A_h}(K, A_h)$ will be denoted by Ω_{A_h}. If there is an ambiguity, the integer d will be written d_{A_h}.

Theorem 4.1 applies to universal quantum enveloping algebras, quantization of affine algebraic Poisson groups and quantum formal series Hopf algebras.

Let $\mathfrak{g}$ be a Lie bialgebra. Denote by $F[\mathfrak{g}]$ the formal series Poisson algebra $U(\mathfrak{g})^*$. If $F_h[\mathfrak{g}]$ is a quantum formal series algebra such that $F_h[\mathfrak{g}]/hF_h[\mathfrak{g}]$ is isomorphic to $F[\mathfrak{g}]$ as a Poisson Hopf algebra, we construct a resolution of the trivial $F_h[\mathfrak{g}]$-module that lifts the Koszul resolution of the trivial $F[\mathfrak{g}]$-module k and that behaves well with respect to quantum duality [Drinfeld 1987, Gavarini 2002]. This construction is not explicit, but it allows us to show that if $F_h[\mathfrak{g}]$ and $U_h(\mathfrak{g}^*)$ are linked by quantum duality, the relation $\theta_{F_h[\mathfrak{g}]} = h\theta_{U_h(\mathfrak{g}^*)}$ holds.

As an application of Theorem 4.1, we show Poincaré duality:

Theorem 7.1. *We make the same assumptions as above. Let M be an A_h-module. Assume that K is an A_h-module of finite projective dimension. For all integers i, the K-modules $\mathrm{Ext}^i_{A_h}(K, M)$ and $\mathrm{Tor}^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M)$ are isomorphic.*

Convention. From now on, we assume that A_h is a deformation Hopf algebra.

Brown and Levasseur [1985] and Kempf [1991] showed that, in the semisimple context, the Ext-dual of a Verma module is a Verma module. In [Chemla 1994] we extended this result to the Ext-dual of an induced representation of any Lie superalgebra. In this article, we show that this result can be generalized to quantum groups provided that the quantization is functorial. Such a quantization has been constructed in [Etingof and Kazhdan 1996, 1998a, 1998b, Etingof and Schiffmann 2002]. As the result holds for quantized universal enveloping algebras, for quantized functions algebras and for quantum formal series Hopf algebras, we state it in the more general setting of Hopf algebras.

Corollary 7.3. *Let A_h and B_h be topological Hopf deformations of A_0 and B_0, respectively. We assume that there exists a morphism of Hopf algebras from B_h to A_h such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of the Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then, if S_h denotes the antipode of B_h, one has:*

(a) $\mathrm{Ext}^i_{A_h}(A_h \underset{B_h}{\otimes} V, A_h)$ *is* $\{0\}$ *if i is different from d_{B_h}.*

(b) *The right A_h-module $\mathrm{Ext}^{d_{B_h}}_{A_h}(A_h \otimes_{B_h} V, A_h)$ is isomorphic to $(\Omega_{B_h} \otimes V^*) \otimes_{B_h} A_h$, where $\Omega_{B_h} \otimes V^*$ is endowed with the right B_h-module structure given by*

$$(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{B_h}(u'_{j,n})\, \omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n\to+\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, *for all $u \in B_h$, all $f \in V^*$, and all $\omega \in \Omega_{B_h}$.*

Proposition 7.4. *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfies the hypothesis of Theorem 4.1. Let V (respectively W) be a B_h-module (respectively C_h-module) which is a free finite dimensional K-module. Then, for all integers n, there is an isomorphism*

$$\operatorname{Ext}_{A_h}^{n+d_{B_h}}\Big(A_h \underset{B_h}{\otimes} V,\, A_h \underset{C_h}{\otimes} W\Big) \simeq \operatorname{Ext}_{A_h}^{n+d_{C_h}}\Big((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h,\, (\Omega_{B_h} \otimes V^*) \underset{B_h}{\otimes} A_h\Big).$$

The right B_h-module structure on $\Omega_{B_h} \otimes V^*$ and the C_h-module structure on $\Omega_{C_h} \otimes W^*$ are as in Corollary 7.3.

Remarks. Proposition 7.4 was already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, and A_h, B_h, C_h are the corresponding enveloping algebras. In this case, $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, Boe and Collingwood [1985] and Gyoja [2000], generalizing a result of G. Zuckerman, proved a part of this theorem (the case where $\mathfrak{h} = \mathfrak{g}$ and $n = \dim \mathfrak{h} = \dim \mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semisimple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. In [Collingwood and Shelton 1990], such a duality is also proved in a slightly different context (but still under the semisimple hypothesis).

M. Duflo [1987] proved Proposition 7.4 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one-dimensional representations.

Proposition 7.4 is proved in full generality in the context of Lie superalgebras in [Chemla 1994].

Wet set $A_h^e = A_h \otimes A_h^{op}$. Using the properties of a Hopf algebra [Chemla 2004], we show that all the $\operatorname{Ext}^i_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes_{k[[h]]}} A_h)$'s are zero except one. More precisely:

Proposition 7.5. *Assume that A_h satisfies the conditions of Theorem 4.1. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \widehat{\otimes_{k[[h]]}} A_h$ with the following $\widehat{A_h^e}$-module structure: for any α, β, x, y in A_h, $\alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta$.*

(a) *$HH^i_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is zero if $i \neq d_{A_h}$.*

(b) *The $\widehat{A_h^e}$-module $HH^{d_{A_h}}_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the following $\widehat{A_h^e}$-module structure: for any α, β, x in A_h,*

$$\alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta_i') \otimes S(\beta_i'') x S^{-1}(\alpha), \quad \text{where } \beta = \sum_i \beta_i' \otimes \beta_i''.$$

This result has already been obtained in [Dolgushev and Etingof 2005] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. From Proposition 7.5, as in [van den Bergh 1998], we deduce a duality between Hochschild homology and Hochschild cohomology.

Organization of the paper. In Section 2, we gather all the necessary results about decreasing filtrations, and in Section 3, we recall some basic facts about deformation algebras. The main theorem of the paper, Theorem 4.1, is stated, proved and illustrated by examples in Section 4. In Section 5, we study the behavior of the character θ_{F_h} with respect to quantum duality. Section 6 is devoted to the study of an example. In Section 7, we give applications of our main theorem.

Our study of algebras endowed with a decreasing filtration and filtered modules over such algebras relies on the use of the associated graded algebra and graded module, and on topological arguments. We apply this study to deformation algebras endowed with the h-adic filtration and filtered modules over such algebras. In [Kashiwara and Schapira 2008], a study of the derived category of A_h-modules is carried out using the right derived functor of the functor $M \mapsto M/(hM)$.

2. Decreasing filtrations

In this section, we give results about decreasing filtrations. These results are proved in [Schneiders 1994] in the framework of increasing filtrations. Most of our proofs are obtained by adjusting those of Schneiders.

Let $GA = \bigoplus_{t\in\mathbb{Z}} G_t A$ be a $\mathbb{Z}$-graded algebra. Let $GM = \bigoplus_{t\in\mathbb{Z}} G_t M$ and $GN = \bigoplus_{t\in\mathbb{Z}} G_t N$ be two graded GA-modules. A morphism of graded GA-modules from GM to GN is a morphism of GA-modules $f : GM \to GN$, such that $f(G_t M) \subset G_t N$. The group of morphisms of graded GA-modules from GM to GN will be denoted by $\mathrm{Hom}_{GA}(GM, GN)$.

For $r \in \mathbb{Z}$ and a graded GA-module GM, define the shifted graded GA-module $GM(r)$ to be the GA-module GM with the grading defined by $G_t M(r) = G_{t+r} M$. Denote by $\underline{\mathrm{Hom}}_{GA}(GM, GN)$ the graded group defined by setting

$$G_t \underline{\mathrm{Hom}}_{GA}(GM, GN) = \mathrm{Hom}_{GA}(GM, GN(t)).$$

The i-th right derived functor of the functor $\underline{\mathrm{Hom}}_{GA}(-, N)$ will be denoted by $\underline{\mathrm{Ext}}^i_{GA}(-, N)$.

A graded GA-module GL is finite free if there are integers $d_1, \dots, d_n$ such that

$$GL \simeq \bigoplus_{i=1}^{n} GA(-d_i).$$

A graded GA-module GM is of finite type if there exists a finite free graded GA-module GL and an exact sequence in the category of graded GA-modules $GL \to GM \to 0$.

A graded ring GA is noetherian if any graded GA-submodule of a graded GA-module of finite type is of finite type.

Henceforth, all the GA-modules we consider will be graded, so we refer to graded GA-modules simply as GA-modules.

We are now going to consider a k-algebra endowed with a decreasing filtration $\cdots \subset F_{t+1}A \subset F_tA \subset \cdots \subset F_1A \subset F_0A = A$. The order of an element a, $o(a)$, is the biggest t such that $a \in F_tA$. The principal symbol of a is the image of a in $F_{o(a)}/F_{o(a)+1}$. It will be denoted by $[a]$.

A filtered module over FA is the data of an A-module M and a family $(F_tM)_{t\in\mathbb{Z}}$ of k-subspaces, such that

$$\bigcup_{t\in\mathbb{Z}} F_tM = M, \quad F_{t+1}M \subset F_tM, \quad F_tA \cdot F_lM \subset F_{t+l}M.$$

We will assume that $F_tM = M$ for $t << 0$. The principal symbol of an element of M is defined. We endow such a module with the topology for which a basis of neighborhoods is $(F_tM)_{t\in\mathbb{Z}}$. The topological space M is Hausdorff if and only if $\bigcap_{t\in\mathbb{Z}} F_tM = \{0\}$. If M is Hausdorff, the topology defined by the filtration is that of the metric given by

$$d(x, y) = \|x - y\| = 2^{-\sup\{j\in\mathbb{Z} \mid x-y\in F_jM\}} \quad \text{for all } (x, y) \in FM.$$

Example. Let k be a field and set $K = k[[h]]$. If V is a K-module, it is endowed with the following decreasing filtration $\cdots \subset h^nV \subset h^{n-1}V \subset \cdots \subset hV \subset V$. The topology induced by this filtration is the h-adic topology.

Lemma 2.1 [Schwartz 1986, page 245]. *Let N be a Hausdorff filtered module. Let P be a submodule of N which is closed in N. Let p be the canonical projection from N to N/P.*

(a) *The topology defined by the filtration $p(F_tN)$ on N/P is the quotient topology. N/P is Hausdorff and its topology is defined by the distance*

$$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\|, \quad \textit{where } \|\bar{x}\| = \inf\{\|a\|, a \in \bar{x}\}.$$

(b) *If N is complete, then N/P is complete for the quotient topology.*

Let FM and FN be two filtered FA-modules. $Fu : FM \to FN$, a filtered morphism, is a morphism $u : M \to N$ of the underlying A-modules, such that $u(F_tM) \subset F_tN$. It is continuous if we endow M and N with the topology defined by the filtrations. Denote the morphism $u_{|F_tM} : F_tM \to F_tN$ by F_tu. Denote the group of filtered morphisms from FM to FN by $\mathrm{Hom}_{FA}(FM, FN)$. The kernel of Fu is the kernel of u filtered by the family $\mathrm{Ker}\, Fu \cap F_tM$. If M is complete and N is Hausdorff, then $\mathrm{Ker}\, Fu$, endowed with the induced topology is complete.

A graded ring $GA = \bigoplus_{t\in\mathbb{N}} F_tA/F_{t+1}A$ is associated to a filtered ring FA. A graded GA-module $GM = \bigoplus_{t\in\mathbb{Z}} F_tM/F_{t+1}M$ is associated to a filtered FA-module FM. If x is in F_tM, we will write $\sigma_t(x)$ for the class of x in $F_tM/F_{t+1}M$. We will denote by $Gu : GM \to GN$ the morphism of GA-modules induced by Fu.

An arrow $Fu : FM \to FN$ is strict if it satisfies

$$u(F_t M) = u(M) \cap F_t N.$$

An exact sequence of FA-modules is a sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$, such that Ker $F_t v =$ Im $F_t u$. It follows from this definition that Fu is strict. If, moreover, Fv is strict, we say that it is a strict exact sequence.

Proposition 2.2. (a) *Consider $Fu : FM \to FN$ and $Fv : FN \to FP$ two filtered FA-morphisms such that $Fv \circ Fu = 0$. If the sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$ is strict exact, then $GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$ is exact.*

(b) *Conversely, assume that FM is complete for the topology defined by the filtration and FN is Hausdorff for the topology defined by the filtration. If the sequence $GM \xrightarrow{Gu} GN \xrightarrow{Gv} GP$ is exact, then the sequence $FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP$ is strict exact.*

Corollary 2.3. *Let FA be a filtered k-algebra and let FM and FN be two FA-modules. Let $Fu : FM \to FN$ be a morphism of FA-modules. Then it follows that $G \operatorname{Ker} Fu \subset \operatorname{Ker} GFu$ and $\operatorname{Im} GFu \subset G \operatorname{Im} Fu$. Assume moreover that FM is complete and FN is Hausdorff. Then the following conditions are equivalent*:

(a) *Fu is strict.*

(b) $G \operatorname{Ker} Fu = \operatorname{Ker} GFu$.

(c) $\operatorname{Im} GFu = G \operatorname{Im} Fu$.

Proposition 2.4. *Let $(M^\bullet, d^\bullet)$ be a complex of complete FA-modules. $H^i(M^\bullet)$ is filtered as follows*:

$$F_t H^i(M^\bullet) = \frac{\operatorname{Ker} d_i \cap F_t M^i + \operatorname{Im} d_{i-1}}{\operatorname{Im} d_{i-1}} \simeq \frac{\operatorname{Ker} d_i \cap F_t M^i}{\operatorname{Im} d_{i-1} \cap F_t M^{i-1}}.$$

If d_i and d_{i-1} are strict, then $GH^i(M^\bullet)$ is isomorphic to $H^i(GM^\bullet)$

Remark. The isomorphism from $G_t H^i(M^\bullet)$ to $H^i(G_t M^\bullet)$ associates $\mathrm{cl}(\sigma_t(x))$ to $\sigma_t\, \mathrm{cl}(x)$.

For any $r \in \mathbb{Z}$ and for any FA-module FM, we define the shifted module $FM(r)$ as the module M endowed with the filtration $(F_{t+r} M)_{t \in \mathbb{Z}}$.

An FA-module module is finite free if it is isomorphic to an FA-module of the type $\bigoplus_{i=1}^p FA(-d_i)$, where $d_1, \dots, d_p$ are integers. An FA-module FM is of finite type if there exists a strict epimorphism $FL \to FM$, where FL is a finite free FA-module. This means that we can find $m_1 \in F_{d_1} M, \dots, m_p \in F_{d_p} M$, such that any $m \in F_d M$ may be written as

$$m = \sum_{i=1}^p a_{d-d_i} m_i, \quad \text{where } a_{d-d_i} \in F_{d-d_i} A.$$

Proposition 2.5. *Let FA be a filtered k-algebra and FM be an FA-module.*

(a) *If FM is an FA-module of finite type generated by $(s_1, \dots, s_r)$, then GM is a GA-module of finite type generated by $([s_1], \dots, [s_r])$. Conversely, assume that FA is complete for the topology given by the filtration, and FM is an FA-module which is Hausdorff for the topology defined by the filtration. If GM is a GA-module of finite type generated by $([s_1], \dots, [s_r])$, then FM is an FA-module of finite type generated by $(s_1, \dots, s_r)$.*

(b) *If FM is a finite free FA-module, then GM is a finite free GA-module. Conversely, assume that FA is complete for the topology given by the filtration, and FM is an FA-module that is Hausdorff for the topology defined by the filtration. If GM is a finite free GA-module, then FM is a finite free FA-module.*

Definition 2.6. A filtered k-algebra is said to be (filtered) noetherian if it satisfies one of the following equivalent conditions:

- Any filtered submodule (not necessarily a strict submodule) of a finite-type FA-module is of finite type.
- Any filtered ideal (not necessarily a strict ideal) of FA is of finite type.

Proposition 2.7. *Let FA be a filtered complete k-algebra and GA its associated graded algebra. If GA is graded noetherian, then FA is filtered noetherian.*

Proof of Proposition 2.7. We assume that GA is a noetherian algebra. We need to prove that a filtered submodule FM' of a finitely generated FA-module FM is finitely generated.

First we assume that FM is Hausdorff. For this case, the proof is identical to that of [Schneiders 1994].

We no longer assume that FM is Hausdorff. As FM is a finite-type FA-module, there exists a strict exact sequence

$$FL = \bigoplus_{i=1}^{n} FA(-d_i) \xrightarrow{p} FM \to 0.$$

We may apply the first case to the submodule of FL, $p^{-1}(FM')$, endowed with the filtration

$$F_t[p^{-1}(M')] = p_t^{-1}(F_t M') = p^{-1}(F_t M') \cap F_t L.$$

The general case follows easily. □

Proposition 2.8. *Assume that FA is noetherian for the topology given by the filtration. Any FA-module of finite type has an infinite resolution by finite free FA-modules.*

Remark. The sequence $\cdots \to GL_s \to GL_{s-1} \to \cdots \to GL_0 \to GM \to 0$ is a resolution of the GA-module GM for such a resolution of FM.

Proposition 2.9. *Assume FA is noetherian and complete. If FN is a finite-type FA-module, then it is complete.*

Proof of Proposition 2.9. Assume that FN is Hausdorff. Let FN be a finite-type Hausdorff FA-module. We have $FL = \bigoplus_{i=1}^{n} FA(-d_i) \xrightarrow{p} FN \to 0$, a strict exact sequence. The filtration on FN is given by $p(F_t L)$. Let us endow the kernel K of p with the induced topology. We have $0 \to FK \to FL \to FN \to 0$, a strict exact sequence. As N is Hausdorff, $K = p^{-1}(\{0\})$ is closed in FL. The filtered FA-module FN is isomorphic to FL/K, endowed with the quotient topology. Hence, FN is complete (see Lemma 2.1).

We no longer assume that FN is Hausdorff. From the first case, FK, endowed with the induced topology, is complete and therefore closed in FL. We have $FN \simeq FL/K$, so the FA-module FN is Hausdorff. □

Remark. Proposition 2.9 is proved in [Kashiwara and Schapira 2008] in the case of an A_h-module (A_h being a deformation algebra) endowed with the h-adic filtration.

3. Deformation algebras

In this section k will be a field of characteristic 0 and we will set $K = k[[h]]$.

Definition 3.1. A topologically free K-algebra A_h is a topologically free K-module together with a K-bilinear (multiplication) map $A_h \times A_h \to A_h$, making A_h into an associative algebra.

Let A_0 be an associative k-algebra. A deformation of A_0 is a topologically free K-algebra A_h such that $A_0 \simeq A_h/hA_h$ as algebras.

Remark. If A_h is a deformation algebra of A_0, we may endow it with the h-adic filtration. We then have

$$GA_h = \bigoplus_{i \in \mathbb{N}} \frac{h^i A_h}{h^{i+1} A_h} \simeq A_0[h]$$

as $k[h]$-algebras. From Proposition 2.7, we deduce that a deformation algebra of a noetherian algebra is noetherian.

Definition 3.2. A deformation of a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over a field k is a topological Hopf algebra $(A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over the ring $k[[h]]$, such that

(i) A_h is isomorphic to $A_0[[h]]$ as a $k[[h]]$-module, and

(ii) A_h/hA_h is isomorphic to A_0 as a Hopf algebra.

Example 3.3 (QUEA: quantized universal enveloping algebras). Let $\mathfrak{g}$ be a Lie bialgebra. A Hopf algebra deformation of $U(\mathfrak{g})$, $U_h(\mathfrak{g})$, such that $U_h(\mathfrak{g})/(hU_h(\mathfrak{g}))$ is isomorphic to $U(\mathfrak{g})$ as a coPoisson Hopf algebra, is called a quantization of $U(\mathfrak{g})$.

Quantizations of Lie bialgebras have been constructed in [Etingof and Kazhdan 1996].

Example 3.4 (quantization of affine algebraic Poisson groups). A quantization of an affine algebraic Poisson group $(G, \{,\})$ is a Hopf algebra deformation $\mathscr{F}_h(G)$ of the Hopf algebra $\mathscr{F}(G)$ of regular functions on G, such that $\mathscr{F}_h(G)/(h\mathscr{F}_h(G))$ is isomorphic to $(\mathscr{F}(G), \{\,,\})$ as a Poisson Hopf algebra.

Etingof and Kazhdan [1998b] have constructed quantizations of affine algebraic Poisson groups. (See also [Chari and Pressley 1994] for the case of G simple.)

Example 3.5 (QFSHA: quantum formal series Hopf algebras). The vector space dual $U(\mathfrak{g})^*$ of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra can be identified with an algebra of formal power series and has a natural Hopf algebra structure, provided we interpret the tensor product $U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$ in a suitable, completed sense. If $\mathfrak{g}$ is a Lie bialgebra, then $U(\mathfrak{g})^*$ is a Hopf Poisson algebra.

A quantum formal series Hopf algebra is a topological Hopf algebra B_h over $k[\![h]\!]$, such that $B_h/(hB_h)$ is isomorphic to $U(\mathfrak{g})^*$ as a topological Poisson Hopf algebra, for some finite-dimensional Lie bialgebra.

Proposition 3.6 [Kashiwara and Schapira 2008, Theorem 2.6]. *Let A_h be a deformation algebra of A_0 and let M be an A_h-module. If*

(i) *M has no h-torsion*,

(ii) *$M/(hM)$ is a flat A_0-module*, *and*

(iii) $M = \varprojlim_n M/(h^n M)$,

then M is a flat A_h-module.

4. A quantization of the character trad

Theorem 4.1. *Let A_0 be a noetherian k-algebra and let A_h be a deformation of A_0. Assume that k has a left A_0-module structure such that there exists an integer d*, *such that*

$$\begin{cases} \mathrm{Ext}^i_{A_0}(k, A_0) = \{0\} & \text{if } i \neq d, \\ \mathrm{Ext}^d_{A_0}(k, A_0) \simeq k. \end{cases}$$

Assume that K is endowed with an A_h-module structure, *which reduces modulo h to the A_0-module structure on k that we started with. Then*:

(a) $\mathrm{Ext}^i_{A_h}(K, A_h)$ *is zero if $i \neq d$.*

(b) $\mathrm{Ext}^d_{A_h}(K, A_h)$ *is a free K-module of dimension* 1, *and a right A_h-module under right multiplication. It is a lift of the right A_0-module structure* (*given by right multiplication*) *on* $\mathrm{Ext}^d_{A_0}(k, A_0)$.

Notation. We denote by Ω_{A_h} the right A_h-module $\mathrm{Ext}^d_{A_h}(k, A_h)$,i and by and the character defined by this action θ_{A_h}.

Remark. Kashiwara and Schapira [2008, Section 6] make a similar construction in the setup of DQ-algebroids. In [Chemla 2004], it is shown that a result similar to Theorem 4.1 holds for $U_q(\mathfrak{g})$ ($\mathfrak{g}$ semisimple).

Example 4.2. Poincaré duality gives us the following result for any finite dimensional Lie algebra.

$$\begin{cases} \mathrm{Ext}^i_{U(\mathfrak{g})}(k, U(\mathfrak{g})) = \{0\} & \text{if } i \neq 0, \\ \mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g})) \simeq \Lambda^{\dim \mathfrak{g}}(\mathfrak{g}^*). \end{cases}$$

The character defined by the right action of $U(\mathfrak{g})$ on $\mathrm{Ext}^{\dim \mathfrak{g}}_{U(\mathfrak{g})}(k, U(\mathfrak{g}))$ is $\mathrm{trad}_{\mathfrak{g}}$ [Chemla 1994]. Thus, the character defined by Theorem 4.1 is a quantization of the character $\mathrm{trad}_{\mathfrak{g}}$.

- If $\mathfrak{g}$ is a complex semisimple algebra, as $H^1(\mathfrak{g}, k) = \{0\}$ [Hilton and Stammbach 1997, page 247], there exists a unique lift of the trivial representation of $U_h(\mathfrak{g})$, hence the representation $\Omega_{U_h(\mathfrak{g})}$ is the trivial representation.
- Let $\mathfrak{a}$ be a k-Lie algebra. Denote by $\mathfrak{a}_h$ the Lie algebra obtained from $\mathfrak{a}$ by multiplying the bracket of $\mathfrak{a}$ by h. Thus, it is true that for any elements X and Y of $\mathfrak{a}_h \simeq \mathfrak{a}$, one has $[X, Y]_{\mathfrak{a}_h} = h[X, Y]_{\mathfrak{a}}$. Denote by $\widehat{U(\mathfrak{a}_h)}$ the h-adic completion of $U(\mathfrak{a}_h)$. Then $\widehat{U(\mathfrak{a}_h)}$ is a Hopf deformation of $(\mathfrak{a}^{ab}, \delta = 0)$. The character $\theta_{\widehat{U(\mathfrak{a}_h)}}$ defined by the theorem in this case is given by
$$\theta_{\widehat{U(\mathfrak{a}_h)}}(X) = h\, \mathrm{trad}_{\mathfrak{a}}(X) \quad \text{for all } X \in \mathfrak{a}.$$
Thus, even if $\mathfrak{g}$ is unimodular, the character defined by the right action of $U_h(\mathfrak{g})$ on $\Omega_{U_h(\mathfrak{g})} \simeq \bigwedge^{\dim \mathfrak{g}}(\mathfrak{g}^*)[[h]]$ might not be trivial.
- We consider the following Lie algebra: $\mathfrak{a} = \bigoplus_{i=1}^5 ke_i$ with nonzero bracket $[e_2, e_4] = e_1$. Consider $k[[h]]$-Lie algebra structure on $\mathfrak{a}[[h]]$ defined by the nonzero brackets $[e_3, e_5] = he_3$ and $[e_2, e_4] = 2e_1$. Then $\widehat{U(\mathfrak{a}[[h]])}$ is a quantization of $U(\mathfrak{a})$. It is easy to see that
$$\theta_{\widehat{U(\mathfrak{a}[[h]])}}(e_i) = \begin{cases} 0 & \text{if } i \neq 5, \\ -h & \text{if } i = 5. \end{cases}$$

Example 4.3. Theorem 4.1 also applies to quantization of affine algebraic Poisson groups. If G is an affine algebraic Poisson group with neutral element e, we take

k to be given by the counit of the Hopf algebra $\mathcal{F}(G)$. By [Altman and Kleiman 1970], we have $\operatorname{Ext}^i_{\mathcal{F}(G)}(k, \mathcal{F}(G)) = \{0\}$ if $i \neq \dim G$, while

$$\operatorname{Ext}^{\dim G}_{\mathcal{F}(G)}(k, \mathcal{F}(G)) \simeq \bigwedge^{\dim G}(\mathcal{M}_e/\mathcal{M}_e^2)^*, \quad \text{where } \mathcal{M}_e = \{f \in \mathcal{F}(G) \mid f(e) = 0\}.$$

Let $\mathfrak{g}$ be a real Lie algebra. The algebra of regular functions on $\mathfrak{g}^*$, $\mathcal{F}(\mathfrak{g}^*)$, is isomorphic to $S(\mathfrak{g})$ and is naturally equipped with a Poisson structure given by the following: if X and Y are in $\mathfrak{g}$, then $\{X, Y\} = [X, Y]$. In the example above, $\widehat{U(\mathfrak{g}_h)}$ is a quantization of the Poisson algebra $\mathcal{F}(\mathfrak{g}^*)$. $\mathcal{F}(\mathfrak{g}^*)$ acts trivially on $\operatorname{Ext}^{\dim \mathfrak{g}}_{\mathcal{F}(\mathfrak{g}^*)}(k, \mathcal{F}(\mathfrak{g}^*))$, whereas the action of $\mathcal{F}_h(\mathfrak{g}^*) \simeq \widehat{U(\mathfrak{g}_h)}$ on $\operatorname{Ext}^{\dim \mathfrak{g}}_{\mathcal{F}_h(\mathfrak{g}^*)}(k, \mathcal{F}_h(\mathfrak{g}^*))$ is not trivial.

Example 4.4. Theorem 4.1 also applies to quantum formal series Hopf algebras.

Proof of Theorem 4.1. Let us consider a resolution of the A_h-module K by filtered finite free A_h-modules

$$\cdots \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \to K \to \{0\},$$

with $FL^i = \bigoplus_{k=1}^{d_i} FA_h(-m_{j,i})$, so that the graded complex

$$\dots GL^i \xrightarrow{G\partial_i} \cdots \to GL^1 \xrightarrow{G\partial_1} GL^0 \to k[h] \to \{0\}$$

is a resolution of the $A_0[h]$-module $k[h]$. Consider the complex

$$M^\bullet = (\operatorname{Hom}_{A_h}(L^\bullet, A_h), {}^t\partial_\bullet).$$

Recall that there is a natural filtration on $\operatorname{Hom}_{A_h}(L^i, A_h)$ defined by

$$F_t \operatorname{Hom}_{A_h}(L^i, A_h) = \{\lambda \in \operatorname{Hom}_{A_h}(L^i, A_h) \mid \lambda(F_p L^i) \subset F_{t+p} A_h\}.$$

One has an isomorphism of right FA-modules $F \operatorname{Hom}_{A_h}(L^i, A_h) = \bigoplus_{j=1}^{d_i} FA(m_{j,i})$. Hence,

$$GF \operatorname{Hom}_{A_h}(L^i, A_h) \simeq \underline{\operatorname{Hom}}_{GA_h}(GL^i, GA_h),$$

and the complex $\underline{\operatorname{Hom}}_{GA_h}(GL^i, GA_h)$ computes $\underline{\operatorname{Ext}}^i_{GA_h}(k[h], GA_h)$. We have the following isomorphisms of right $A_0[h]$-modules.

$$\underline{\operatorname{Ext}}^i_{GA_h}(k[h], GA_h) \simeq \underline{\operatorname{Ext}}^i_{A_0[h]}(k[h], A_0[h]) \simeq \operatorname{Ext}^i_{A_0}(k, A_0)[h].$$

If $i \neq d$, then $\underline{\operatorname{Ext}}^i_{GA_h}(k[h], GA_h) = \{0\}$. This means that the sequence

$$\underline{\operatorname{Hom}}_{GA}(GL_{i-1}, GA_h) \xrightarrow{{}^tG\partial_i} \underline{\operatorname{Hom}}_{GA}(GL_i, GA_h) \xrightarrow{{}^tG\partial_{i+1}} \underline{\operatorname{Hom}}_{GA}(GL_{i+1}, GA_h)$$

is an exact sequence of GA_h-modules. Applying Proposition 2.2, the sequence

$$F \operatorname{Hom}_{FA}(FL_{i-1}, FN) \xrightarrow{{}^t\partial_i} F \operatorname{Hom}_{FA}(FL_i, FN) \xrightarrow{{}^t\partial_{i+1}} F \operatorname{Hom}_{FA}(FL_{i+1}, FN)$$

is strict exact. As FL_i is finite free, the underlying module of $F\operatorname{Hom}_{FA}(FL_i, FN)$ is $\operatorname{Hom}_A(L_i, N)$. Hence, we have proved that $\operatorname{Ext}^i_{A_h}(K, A_h) = \{0\}$ if $i \neq d$.

We have also proved that all the maps ${}^t\partial_i$ are strict. Hence, by Proposition 2.4, we have

$$G\operatorname{Ext}^i_{A_h}(k[\![h]\!], A_h) \simeq \underline{\operatorname{Ext}}^i_{GA_h}(k[h], A_0[h]) \simeq \operatorname{Ext}^i_{A_0}(k, A_0)[h],$$

for all integers i. The FA_h-modules $\operatorname{Ext}^i_{A_h}(K, A_h)$ are finite-type FA-modules. They are therefore Hausdorff, in fact, they are even complete (Proposition 2.9). As $\operatorname{Ext}^d_{A_h}(K, A_h)$ is Hausdorff and $G\operatorname{Ext}^d_{A_h}(k[\![h]\!], A_h) \simeq \operatorname{Ext}^d_{A_0}(k, A_0)[h]$, the $k[\![h]\!]$-module $\operatorname{Ext}^d_{A_h}(K, A_h)$ is one-dimensional. This finishes the proof. □

From now on, we assume that A_h is a topological Hopf algebra and that its action on K is given by the counit. The antipode of A_h will be denoted by S_h.

If V is a left A_h-module, we define the right A_h-module V^r by

$$v \cdot_{S_h} a = S_h(a) \cdot v \quad \text{for all } a \in A_h \text{ and } v \in V,$$

and the right A_h-module V^ρ by

$$v \cdot_{S_h^{-1}} a = S_h^{-1}(a) \cdot v \quad \text{for all } a \in A_h \text{ and } v \in V.$$

Similarly, if W is a right A_h-module, we define the left A_h-module W^l by

$$a \cdot_{S_h} w = w \cdot S_h(a) \quad \text{for all } a \in A_h \text{ and } w \in W,$$

and the left A_h-module W^λ by

$$a \cdot_{S_h^{-1}} w = w \cdot S_h^{-1}(a) \quad \text{for all } a \in A_h \text{ and } w \in W.$$

One has $(V^r)^\lambda = V$, $(V^\rho)^l = V$, $(W^l)^\rho = W$ and $(W^\lambda)^r = W$. Thus, we have defined two (in the case where $S_h^2 \neq \mathrm{id}$) equivalences of categories between the category of left A_h-modules and the category of right A_h-modules, that is, left A_h^{op}-modules.

Let $\operatorname{Mod}(A_h)$ be the abelian category of left A_h-modules and $D(\operatorname{Mod}(A_h))$ be the derived category of the abelian category $\operatorname{Mod}(A_h)$. We may consider A_h as an $A_h \otimes A_h^{op}$-module. Introduce a functor D_{A_h} from $D(\operatorname{Mod}(A_h))$ to $D(Mod(A_h^{op}))$ by setting

$$D_{A_h}(M^\bullet) = R\operatorname{Hom}_{A_h}(M^\bullet, A_h) \quad \text{for all } M^\bullet \in D(A_h).$$

If M is a finitely generated module, the canonical arrow $M \to D_{A_h^{op}} \circ D_{A_h}(M)$ is an isomorphism.

Let V be a left A_h-module. Then, by transposition, $V^* = \operatorname{Hom}_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see it as a left module structure. Thus, one has

$$u \cdot f = f \cdot S_h(u) \quad \text{for all } u \in A_h \text{ and } f \in V^*.$$

We endow $\Omega_{A_h} \otimes V^*$ with the right A_h-module structure given by

$$(\omega \otimes f) \cdot u = \lim_{n\to+\infty} \sum_j \theta_{A_h}(u'_{j,n})\omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n\to+\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, for all $u \in A_h$, all $f \in V^*$, and all $\omega \in \Omega_{A_h}$.

Theorem 4.5. *Let V be an A_h-module free of finite type as a $k[\![h]\!]$-module. Then $D_{A_h}(V)$ and $\Omega_{A_h} \otimes V^*$ are isomorphic in $D\left(A_h^{op}\right)$.*

To prove the theorem, we need the following lemma [Duflo 1982; Chemla 1994]:

Lemma 4.6. *Let W be a left A_h-module. $A_h \widehat{\otimes} W$ is endowed with two different $(A_h \otimes A_h^{op})$-module structures, as follows. Set*

$$\Delta(a) = \lim_{n\to+\infty} \sum_i a'_{i,n} \otimes a''_{i,n} \quad \textit{for } a \in A_h. \tag{4-1}$$

The first structure, denoted by $(A_h \widehat{\otimes} W)_1$, is given by

$$(u \otimes w) \cdot a = ua \otimes w \quad \text{and} \quad a \cdot (u \otimes w) = \lim_{n\to+\infty} \sum_i a'_{i,n} u \otimes a''_{i,n} \cdot w,$$

where $w \in W$ and $u, a \in A_h$. The second structure, denoted by $(A_h \widehat{\otimes} W)_2$, is given by

$$a \cdot (u \otimes w) = au \otimes w \quad \text{and} \quad (u \otimes w) \cdot a = \lim_{n\to+\infty} \sum_i ua'_{i,n} \otimes S_h(a''_{i,n}) \cdot w.$$

The $A_h \otimes A_h^{op}$-modules $(A_h \widehat{\otimes} W)_1$ and $(A_h \widehat{\otimes} W)_2$ are isomorphic.

Proof of Lemma 4.6. The map $\Psi : (A_h \widehat{\otimes} W)_2 \to (A_h \widehat{\otimes} W)_1$ given by

$$u \otimes w \mapsto \lim_{n\to+\infty} \sum_i u'_{i,n} \otimes u''_{i,n} \cdot w,$$

with Δ as in (4-1), is an isomorphism of $A_h \otimes A_h^{op}$-modules from $(A_h \widehat{\otimes} W)_2$ to $(A_h \widehat{\otimes} W)_1$. Moreover, $\Psi^{-1}(u \otimes w) = \sum u'_{i,n} \otimes S_h(u''_{i,n}) \cdot w$. □

Proof of Theorem 4.5. Let $L^\bullet$ be a resolution of K by free A_h-modules. We endow $L^i \otimes V$ with the following left A_h-module structure:

$$a \cdot (l \otimes v) = \lim_{n\to+\infty} \sum_i a'_{i,n} \cdot l \otimes a''_{i,n} \cdot v.$$

Then $L^\bullet \otimes V$ is a resolution of V by free A_h-modules. Using the relation

$$a \cdot l \otimes v = \lim_{n\to+\infty} \sum_i a'_{i,n} \left(l \otimes S_h(a''_{i,n}) \cdot v\right),$$

one shows the sequence of A_h-isomorphisms

$$\begin{aligned}D_{A_h}(V) &\simeq \operatorname{Hom}_{A_h}(L\otimes V, A_h) \simeq \operatorname{Hom}_{A_h}\big(L, (A_h\otimes V^*)_1\big)\\ &\simeq \operatorname{Hom}_{A_h}\big(L, (A_h\otimes V^*)_2\big) \simeq R\operatorname{Hom}_{A_h}(K, A_h)\otimes V^*.\end{aligned}$$ □

5. Link with quantum duality

Review of the quantum dual principle [Drinfeld 1987, Gavarini 2002]. There are two functors,

$$(\ )' : \mathrm{QUEA} \to \mathrm{QFSA} \quad \text{and} \quad (\ )^\vee : \mathrm{QFSA} \to \mathrm{QUEA},$$

which are inverse to each other. If $U_h(\mathfrak{g})$ is a quantization of $U(\mathfrak{g})$ and $F_h[\![\mathfrak{g}]\!]$ is a quantization of $F[\![\mathfrak{g}]\!] = U(\mathfrak{g})^*$, then $U_h(\mathfrak{g})'$ is a quantization of $F[\![\mathfrak{g}^*]\!]$ and $F_h[\![\mathfrak{g}]\!]^\vee$ is a quantization of $U(\mathfrak{g}^*)$. We recall the construction of the functor $(\ )^\vee$, which is the one we will need. Let $\mathfrak{g}$ be a Lie bialgebra and $F_h[\![\mathfrak{g}]\!]$ a quantization of $F[\![\mathfrak{g}]\!] = U(\mathfrak{g})^*$. For simplicity we will write F_h instead of $F_h[\![\mathfrak{g}]\!]$. If ϵ_h denotes the counit of F_h, set $I := \epsilon_h^{-1}(hk[\![h]\!])$ and $J = \operatorname{Ker}\epsilon_h$. Let

$$F_h^\times := \sum_{n\geq 0} h^{-n} I^n = \sum_{n\geq 0} (h^{-1}I)^n = \bigcup_{n\geq 0} (h^{-1}I)^n$$

be the $k[\![h]\!]$-subalgebra of $k((h))\otimes_{k[\![h]\!]} F_h$ generated by $h^{-1}I$. As $I = J + hF_h$, one has

$$F_h^\times = \sum_{n\geq 0} h^{-n} J^n.$$

Define $F_h^\vee$ to be the h-adic completion of the $k[\![h]\!]$-module $F_h^\times$. The Hopf algebra structure on F_h induces a Hopf algebra structure on $F_h^\vee$. A precise description of $F_h^\vee$ is given in [Gavarini 2002]. The algebras F_h/hF_h and $k[\![\bar{x}_1, \dots, \bar{x}_n]\!]$ are isomorphic. We denote $\pi : F_h \to F_h/hF_h$ be the natural projection. We may choose $x_j \in \pi^{-1}(\bar{x}_j)$ for any j, such that $\epsilon_h(x_j) = 0$. Then F_h and $k[\![x_1, \dots, x_n, h]\!]$ are isomorphic as $k[\![h]\!]$-topological modules and J is the set of formal series f whose degree in the x_j, $\partial_X(f)$ (that is, the degree of the lowest-degree monomials occurring in the series with nonzero coefficients) is strictly positive. As F_h/hF_h is commutative, one has $x_i x_j - x_j x_i = h\chi_{i,j}$ with $\chi_{i,j} \in F_h$. Since $\chi_{i,j}$ is in J, it can be written as

$$\chi_{i,j} = \sum_{a=1}^{n} c_a(h) x_a + f_{i,j}(x_1, \dots, x_n, h), \quad \text{with } \partial_X(f_{i,j}) > 1.$$

If $\check{x}_i = h^{-1}x_j$, then

$$F_h^\vee = \left\{ f = \sum_{r\in\mathbb{N}} P_r(\check{x}_1, \dots, \check{x}_n) h^r \;\middle|\; P_r(X_1, \dots, X_n) \in k[X_1, \dots, X_n] \right\}.$$

The topological $k[\![h]\!]$-modules $F_h^\vee$ and $k[\check{x}_1, \dots, \check{x}_n][\![h]\!]$ are isomorphic. One has

$$\check{x}_i \check{x}_j - \check{x}_j \check{x}_i = \sum_{a=1}^{n} c_a(h) \check{x}_a + h^{-1} \check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h),$$

where $\check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h)$ is obtained from $f_{i,j}(x_1, \dots, x_n)$ by writing $x_j = h\check{x}_j$. The element $h^{-1}\check{f}_{i,j}(\check{x}_1, \dots, \check{x}_n, h)$ is in $hk[\check{x}_1, \dots, \check{x}_n][\![h]\!]$ (as $\partial_X(f_{i,j}) > 1$). The k-span of the set of cosets $\{e_i = \check{x}_i \bmod hF_h^\vee\}$ is a Lie algebra isomorphic to $\mathfrak{g}^*$, and the map $\Psi : F_h^\vee \to U(\mathfrak{g}^*)[\![h]\!]$ defined by

$$\Psi\left(\sum_{r\in\mathbb{N}} P_r(\check{x}_1, \dots, \check{x}_n) h^r\right) = \sum_{r\in\mathbb{N}} P_r(e_1, \dots, e_n) h^r$$

is an isomorphism of topological $k[\![h]\!]$-modules. Denote by $\cdot_h$ multiplication on F_h and its transposition to $U(\mathfrak{g}^*)[\![h]\!]$ by Ψ. If u and v are in $U(\mathfrak{g}^*)$, one writes $u \cdot_h v = \sum_{r\in\mathbb{N}} h^r \mu_r(u, v)$. One knows that the first nonzero μ_r is a 1-cocycle of the Hochschild cohomology.

If P in $k[X_1, \dots, X_n]$ can be written $P = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} X_1^{i_1} \dots X_n^{i_n}$ and $g \in k[X_1, \dots, X_n][\![h]\!]$ can be written $g = \sum_{i=1}^{r} P_r(X_1, \dots, X_r) h^r$, then one sets

$$P^{\otimes}(e_1, \dots, e_n) = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} e_1^{\otimes i_1} \dots e_n^{\otimes i_n} \in T_k\left(\bigoplus_{i=1}^{n} ke_i\right),$$

$$g^{\otimes}(e_1, \dots, e_n) = \sum_{i=1}^{r} P_r^{\otimes}(e_1, \dots, e_r) h^r.$$

$(F_h)^\vee$ is isomorphic as an algebra to

$$U_h(\mathfrak{g}^*) \simeq \frac{T_{k[\![h]\!]}\left(\bigoplus_{i=1}^{n} k[\![h]\!] e_i\right)}{I},$$

where I is the closure (in the h-adic topology) of the two sided ideal generated by the relations

$$e_i \otimes e_j - e_j \otimes e_i = \sum_{k=1}^{n} c_k(h) e_k + h^{-1} \check{f}_{i,j}^{\otimes}(e_1, \dots, e_n, h).$$

Quantum duality and deformation of the Koszul complex. We may construct resolutions of the trivial $F_h[\mathfrak{g}]$ and $F_h[\mathfrak{g}]^\vee$-modules that respect the quantum duality.

Theorem 5.1. *Let $\mathfrak{g}$ be a Lie bialgebra, $F_h[\mathfrak{g}]$ a QFSHA such that $F_h[\mathfrak{g}]/(hF_h[\mathfrak{g}])$ is isomorphic to $F[\mathfrak{g}]$ as a topological Poisson Hopf algebra and $F_h[\mathfrak{g}]^\vee = U_h(\mathfrak{g}^*)$, the quantization of $U(\mathfrak{g}^*)$ constructed from $F_h[\mathfrak{g}]$ by the quantum duality principle. Let $\bar{x}_1, \dots, \bar{x}_n$ be elements of $F[\mathfrak{g}]$ such that $F[\mathfrak{g}] \simeq k[\![\bar{x}_1, \dots, \bar{x}_n]\!]$. Choose $x_1, \dots, x_n$, elements of $F_h[\mathfrak{g}]$, such that $x_i = \bar{x}_i \bmod h$ and $\epsilon_h(x_i) = 0$. Then*

$U_h(\mathfrak{g}^*) \simeq k[\check{x}_1, \dots, \check{x}_n][[h]]$ with $\check{x}_i = h^{-1}x_i$. Let $(\epsilon_1, \dots, \epsilon_n)$ be a basis of $\mathfrak{g}^*$ and $C^a_{i,j}$ the structural constants of $\mathfrak{g}^*$ with respect to this basis. We can construct a resolution of the trivial $F_h[\mathfrak{g}]$-module $K^h_\bullet = \big(F_h[\mathfrak{g}] \otimes \bigwedge \mathfrak{g}^*, \partial^h_q\big)$ of the form

$$\begin{aligned}
&\partial^h_q(1 \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_q}) \\
&\quad = \sum_{i=1}^{q} (-1)^{i-1} x_i \otimes \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_i}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{r<s} \sum_a (-1)^{r+s} h C^a_{p_r,p_s} 1 \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_r}} \wedge \cdots \wedge \widehat{\epsilon_{p_s}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{t_1,\dots,t_{q-1}} h\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \otimes \epsilon_{t_1} \wedge \cdots \wedge \epsilon_{t_{q-1}},
\end{aligned}$$

such that $\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \in I = \epsilon_h^{-1}(hk[[h]])$. Set

$$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check{x}_1, \dots, \check{x}_n) = \alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1, \dots, x_n).$$

$\check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}$ is in $hk[\check{x}_1, \dots, \check{x}_n][[h]]$. Now define the morphism of $U_h(\mathfrak{g}^*)$-modules $\check{\partial}^h_q : U_h(\mathfrak{g}^*) \otimes \bigwedge^q(\mathfrak{g}^*) \to U_h(\mathfrak{g}^*) \otimes \bigwedge^{q-1}(\mathfrak{g}^*)$ by

$$\begin{aligned}
&\check{\partial}^h_q(1 \otimes \epsilon_{p_1} \wedge \cdots \wedge \epsilon_{p_q}) \\
&\quad = \sum_{i=1}^{n} (-1)^{i-1} \check{x}_i \otimes \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_i}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{r<s} \sum_a (-1)^{r+s} C^a_{p_r,p_s} 1 \otimes \epsilon_a \wedge \epsilon_{p_1} \wedge \cdots \wedge \widehat{\epsilon_{p_r}} \wedge \cdots \wedge \widehat{\epsilon_{p_s}} \wedge \cdots \wedge \epsilon_{p_q} \\
&\qquad + \sum_{t_1,\dots,t_{q-1}} \check{\alpha}^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q} \otimes \epsilon_{t_1} \wedge \cdots \wedge \epsilon_{t_{q-1}}.
\end{aligned}$$

Then $\check{K}^\bullet_h = (U_h(\mathfrak{g}^*) \otimes \bigwedge^\bullet \mathfrak{g}^*, \check{\partial}^h_q)$ is a resolution of the trivial $U_h(\mathfrak{g}^*)$-module $k[[h]]$.

Proof of Theorem 5.1. One sets $x_i x_j - x_j x_i = \sum_{a=1}^n hC^a_{i,j} x_a + hu^a_{i,j} x_a$. We know that $u^a_{i,j}$ is in I. Take $\partial^h_0 = \epsilon_h$, $\partial^h_1(1 \otimes \epsilon_i) = x_i$. Set

$$\partial^h_2(1 \otimes \epsilon_i \wedge \epsilon_j) = x_i \otimes \epsilon_j - x_j \otimes \epsilon_i - \sum_a hC^a_{i,j} \otimes \epsilon_a - h \sum_a u^a_{i,j} \otimes \epsilon_a.$$

We have $\partial^h_1 \circ \partial^h_2 = 0$ and we may choose $\alpha^a_{i,j} = u^a_{i,j}$.

Assume that $\partial^h_0, \partial^h_1, \dots, \partial^h_q$ have been constructed such that

- $\partial^h_{r-1}\partial^h_r = 0$ for all $r \in [1, q]$;
- $\operatorname{Im} \partial^h_r = \operatorname{Ker} \partial^h_{r-1}$ for all $r \in [1, q]$ (and the required relations are satisfied);
- $\alpha^{q_1,\dots,q_{r-1}}_{p_1,p_2,\dots,p_r} \in I$.

Let us show that we can construct ∂^h_{q+1} satisfying these three conditions.

A computation [Knapp 1988, page 173] shows that

$$\partial^h_q\Big(\sum_{i=1}^{q+1}(-1)^{i-1}x_{p_i}\otimes\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\Big)$$
$$+\partial^h_q\Big(\sum_{k<l}\sum_a(-1)^{k+l}hC^a_{p_k,p_l}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\widehat{\epsilon_{p_k}}\wedge\cdots\wedge\widehat{\epsilon_{p_l}}\wedge\cdots\wedge\epsilon_{p_{q+1}}\Big)$$
$$=\sum_{j<i}(-1)^{i+j}\Big(x_{p_i}x_{p_j}-x_{p_j}x_{p_i}-\sum_a hC^a_{p_i,p_j}x_a\Big)\otimes\epsilon_1\wedge\cdots\wedge\widehat{\epsilon_{p_j}}\wedge\cdots\wedge\widehat{\epsilon_{p_i}}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_i(-1)^{i-1}hx_{p_i}\alpha_{p_1,\dots,\widehat{p_i},\dots,p_{q+1}}+\sum_{r<s}(-1)^{r+s}h^2C^a_{p_r,p_s}\alpha_{a,p_1,\dots,\widehat{p_r},\dots,\widehat{p_l},\dots,p_{q+1}}.$$

Modulo h, this expression is zero. Since $\partial^h_{q-1}\partial^h_q$, vanishes, this same expression is in $h\operatorname{Ker}\partial^h_{q-1}=h\operatorname{Im}\partial^h_q$. Hence it equals $-\partial^h_q(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})$, for of an appropriate choice of $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ in $F_h[\mathfrak{g}]$.

We prove that $\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}$ is in I. Clearly, $-\partial^h_q(h\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q})$ is an element of $I^3\otimes\bigwedge^q\mathfrak{g}^*$. Note that ∂^h_q sends $I^r\otimes\bigwedge^q\mathfrak{g}^*$ to $I^{r+1}\otimes\bigwedge^q\mathfrak{g}^*$. Let us write

$$\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}=\sum_{i_1,\dots,i_n}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}x_1^{i_1}\dots x_n^{i_n},$$

with $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{i_1,\dots,i_n}$ in $k[\![h]\!]$. From the remarks just made, we see that

$$\partial^h_q\Big(h\sum_{t_1,\dots,t_q}(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}\Big)\in I^3\otimes{\textstyle\bigwedge^q}\mathfrak{g}^*.$$

Hence, $(\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}})_{0,\dots,0}$ is in $hk[\![h]\!]$.

Since $\operatorname{Im}G\partial^h_{q+1}=\operatorname{Ker}G\partial^h_q$, one has $\operatorname{Im}\partial^h_{q+1}=\operatorname{Ker}\partial^h_q$.

Set $\check\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(\check x_1,\dots,\check x_n)=\alpha^{t_1,\dots,t_{q-1}}_{p_1,\dots,p_q}(x_1,\dots,x_n)$. Then $\check\partial_0=\epsilon$, $\check\partial_1(1\otimes\epsilon_i)=\check x_i$, $\check\partial_2(1\otimes\epsilon_i\wedge\epsilon_j)=\check x_i\otimes\epsilon_j-\check x_j\otimes\epsilon_j\sum_a C^a_{i,j}\otimes\epsilon_a-\sum_a\check u^a_{i,j}\otimes\epsilon_a$, and

$$\check\partial^h_{q+1}(1\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_{q+1}})$$
$$=\sum_{i=1}^{q+1}(-1)^{i-1}\check x_i\otimes\epsilon_{p_1}\wedge\cdots\wedge\hat\epsilon_{p_i}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_{r<s}\sum_a(-1)^{r+s}C^a_{p_r,p_s}1\otimes\epsilon_a\wedge\epsilon_{p_1}\wedge\cdots\wedge\hat\epsilon_{p_r}\wedge\cdots\wedge\hat\epsilon_{p_s}\wedge\cdots\wedge\epsilon_{p_{q+1}}$$
$$+\sum_{t_1,\dots,t_{q-1}}\check\alpha^{t_1,\dots,t_q}_{p_1,\dots,p_{q+1}}\otimes\epsilon_{t_1}\wedge\cdots\wedge\epsilon_{t_q}.$$

If P is in F_h, one has $\partial_q(P\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q})=h\check{\partial}(\check{P}\otimes\epsilon_{p_1}\wedge\cdots\wedge\epsilon_{p_q})$. The relation $\check{\partial}_q\check{\partial}_{q+1}=0$ is obtained by multiplying the relation $\partial_q^h\partial_{q+1}^h=0$ by h^{-2}. As $G\check{\partial}_q^h$ is the differential of the Koszul complex of the trivial $U(\mathfrak{g}^*)[h]$-module, the complex $\check{K}_h^\bullet=\big(U_h(\mathfrak{g}^*)\otimes\bigwedge^\bullet\mathfrak{g}^*,\check{\partial}_n^h\big)$ is a resolution of the trivial $U_h(\mathfrak{g}^*)$-module. □

A link between θ_{F_h} and $\theta_{F_h^\vee}$. The remainder of this section is devoted to the proof of this equality:

Theorem 5.2. $$\theta_{F_h}=h\theta_{F_h^\vee}.$$

Proof. We keep the notation of the previous proposition and we will use the proof of Theorem 4.1.

The complex $\big(\bigwedge^\bullet\mathfrak{g}\otimes F_h,{}^t\,\partial_n^h\big)$ computes the $k[[h]]$-modules $\operatorname{Ext}^i_{F_h}(k[[h]],F_h)$. The cohomology class $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of

$$\underline{\operatorname{Ext}}^n_{F[\mathfrak{g}][h]}(k[h],F[\mathfrak{g}][h])\simeq G\operatorname{Ext}^n_{F_h}(k[[h]],F_h).$$

Hence, there exists $\sigma=1+h\sigma_1+\cdots\in\operatorname{Ker}^t\,\partial_n^h$ such that $[\operatorname{cl}(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)]$ is a basis of $G\operatorname{Ext}^n_{F_h}(k[[h]],F_h)$. As the filtration on $\operatorname{Ext}^n_{F_h}(k[[h]],F_h)$ is Hausdorff, the cohomology class $\operatorname{cl}(\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\operatorname{Ext}^n_{F_h}(k[[h]],F_h)$.

Define $\check{\sigma}$ by $\check{\sigma}(\check{x}_1,\ldots,\check{x}_n)=\sigma(x_1,\ldots,x_n)$. One has ${}^t\partial_n=h\,{}^t\check{\partial}_n$, and it is easy to check that $\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*$ is in $\operatorname{Ker}^t\,\check{\partial}_{n-1}^h$. If we had

$$\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*={}^t\,\check{\partial}_{n-1}^h\bigg(\sum_{i=1}^n\check{\sigma}_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\bigg),$$

then, reducing modulo h, we would get

$$\overline{\check{\sigma}}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\overline{{}^t\check{\partial}_{n-1}^h}\bigg(\sum_{i=1}^n\overline{\check{\sigma}_i}\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*\bigg).$$

This would imply that $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is 0 in $\operatorname{Ext}^n_{U(\mathfrak{g}^*)}(k,U(\mathfrak{g}^*))$, which is impossible because $\operatorname{cl}(1\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a basis of $\operatorname{Ext}^n_{U(\mathfrak{g}^*)}(k,U(\mathfrak{g}^*))$. Thus, $\operatorname{cl}(\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*)$ is a nonzero element of $\operatorname{Ext}^{\dim\mathfrak{g}^*}_{U_h(\mathfrak{g}^*)}(k[[h]],U_h(\mathfrak{g}^*))$. For all i in $[1,n]$, one has the relation

$$\sigma x_i\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\theta_{F_h}(x_i)\sigma\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*+{}^t\,\partial_n^h(\mu)$$

Let us write $\mu=\sum_i\mu_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*$ with $\mu_i\in F_h[\mathfrak{g}]$. We set

$$\check{\mu}_i(\check{x}_1,\ldots,\check{x}_n)=\mu_i(x_1,\ldots,x_n)\quad\text{and}\quad\check{\mu}=\sum_i\check{\mu}_i\otimes\epsilon_1^*\wedge\cdots\wedge\widehat{\epsilon_i^*}\wedge\cdots\wedge\epsilon_n^*.$$

Then we have $h\check{\sigma}\check{x}_i\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*=\theta_{F_h}(x_i)\,\check{\sigma}\otimes\epsilon_1^*\wedge\cdots\wedge\epsilon_n^*+h\,{}^t\check{\partial}_n^h(\check{\mu})$. □

6. Study of an example

We will now explicitly study an example suggested by B. Enriquez. Chloup [1997] introduced the triangular Lie bialgebra

$$(\mathfrak{g} = kX_1 \oplus kX_2 \oplus kX_3 \oplus kX_4 \oplus kX_5, r = 4(X_2 \wedge X_3)),$$

where the nonzero brackets are given by $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ and $[X_1, X_4] = X_5$, and the cobracket $\delta_{\mathfrak{g}}$ is the following:

$$\text{if } X \in \mathfrak{g}, \quad \text{then } \delta(X) = X \cdot 4(X_2 \wedge X_3).$$

The dual Lie bialgebra of $\mathfrak{g}$ will be denoted by $\big(\mathfrak{a} = \bigoplus_{i=1}^{5} ke_i, \delta\big)$. The only nonzero Lie bracket of $\mathfrak{a}$ is $[e_2, e_4] = 2e_1$ and its cobracket δ is nonzero on the basis vectors e_3, e_4, e_5:

$$\delta(e_3) = e_1 \otimes e_2 - e_2 \otimes e_1 = 2e_1 \wedge e_2, \quad \delta(e_4) = 2e_1 \wedge e_3, \quad \delta(e_5) = 2e_1 \wedge e_4.$$

We may twist the trivial deformation of $\big(U(\mathfrak{g})[\![h]\!], \mu_0, \Delta_0, \iota_0, \epsilon_0, S_0\big)$ by the invertible element

$$R = \exp\big(h(X_2 \otimes X_3 - X_3 \otimes X_2)\big)$$

of $U(\mathfrak{g})[\![h]\!] \widehat{\otimes} U(\mathfrak{g})[\![h]\!]$ (see [Chari and Pressley 1994, page 130]). The topological Hopf algebra obtained has the same multiplication, antipode, unit and counit. However, its coproduct is $\Delta^R = R^{-1}\Delta_0 R$. It is a quantization of $(\mathfrak{g}, r)$. We will denote it by $U_h(\mathfrak{g})$. The Hopf algebra $U_h(\mathfrak{g})^*$ is a QFSHA and $(U_h(\mathfrak{g})^*)^\vee$ is a quantization of $(\mathfrak{a}, \delta_{\mathfrak{a}})$. We will compute it explicitly.

Proposition 6.1. (a) $(U(\mathfrak{g})^*)^\vee$ *is isomorphic as a topological Hopf algebra to the topological k[[h]]-algebra*

$$\frac{T_{k[\![h]\!]}(k[\![h]\!]e_1 \oplus k[\![h]\!]e_2 \oplus k[\![h]\!]e_3 \oplus k[\![h]\!]e_4 \oplus k[\![h]\!]e_5)}{I},$$

where I is the closure of the two-sided ideal generated by

$$e_2 \otimes e_4 - e_4 \otimes e_2 - 2e_1,$$
$$e_3 \otimes e_5 - e_5 \otimes e_3 - \tfrac{2}{3}h^2 e_1 \otimes e_1 \otimes e_1,$$
$$e_4 \otimes e_5 - e_5 \otimes e_4 - \tfrac{1}{6}h^3 e_1 \otimes e_1 \otimes e_1 \otimes e_1,$$
$$e_2 \otimes e_5 - e_5 \otimes e_2 + he_1 \otimes e_1,$$
$$e_3 \otimes e_4 - e_4 \otimes e_3 + he_1 \otimes e_1,$$
$$e_i \otimes e_j - e_j \otimes e_i, \quad if\ \{i, j\} \neq \{2, 4\}, \{3, 5\}, \{4, 5\}, \{2, 5\}, \{3, 4\},$$

with the coproduct Δ_h, counit ϵ_h and antipode S defined as follows:

$$\begin{aligned}
&\Delta_h(e_1) = e_1 \otimes 1 + 1 \otimes e_1,\\
&\Delta_h(e_2) = e_2 \otimes 1 + 1 \otimes e_2,\\
&\Delta_h(e_3) = e_3 \otimes 1 + 1 \otimes e_3 - he_2 \otimes e_1,\\
&\Delta_h(e_4) = e_4 \otimes 1 + 1 \otimes e_4 - he_3 \otimes e_1 + \tfrac{1}{2}h^2 e_2 \otimes e_1^2,\\
&\Delta_h(e_5) = e_5 \otimes 1 + 1 \otimes e_5 - he_4 \otimes e_1 + \tfrac{1}{2}h^2 e_3 \otimes e_1^2 - \tfrac{1}{6}h^3 e_2 \otimes e_1^3,\\
&\epsilon_h(e_i) = 0 \quad \textit{and} \quad S(e_i) = -e_i \quad \textit{for } i \in [1,5].
\end{aligned}$$

(b) $(U(\mathfrak{g})^*)^\vee$ *is not isomorphic to the trivial deformation of* $U(\mathfrak{a})$ *as an algebra.*

Proof of Proposition 6.1. Let ξ_i be the element of $U(\mathfrak{g})^*$ defined by

$$\langle \xi_i, X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5}\rangle = \delta_{a_1,0} \dots \delta_{a_i,1} \dots \delta_{a_5,0}.$$

The algebras $U(\mathfrak{g})^*$ and $k[\![\xi_1, \dots, \xi_n]\!]$ are isomorphic. The topological Hopf algebra $(U_h(\mathfrak{g})^*, {}^t\Delta_0^R = \cdot_h, {}^t\mu_0 = \Delta_h, {}^t\epsilon_0, {}^t\iota_0 = \epsilon_h, {}^tS_0)$ is a QFSHA. $U_h(\mathfrak{g})^*$ and $k[\![\xi_1, \dots, \xi_n, h]\!]$ are isomorphic as $k[\![h]\!]$-modules. The elements $\xi_1, \dots, \xi_n$ generate topologically the $k[\![h]\!]$- algebra $U_h(\mathfrak{g})^*$ and satisfy $\epsilon_h(\xi_i) = 0$,

$$\langle \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1^{a_1} \dots X_5^{a_5})\rangle \neq 0 \iff (a_1, a_2, a_3, a_4, a_5) = (1, 0, 0, 0, 0)$$

and $\langle \xi_2 \otimes \xi_4 - \xi_4 \otimes \xi_2, \Delta^R(X_1)\rangle = 2h$. Hence, $\xi_2 \cdot_h \xi_4 - \xi_4 \cdot_h \xi_2 = 2h\xi_1$. The other relations are obtained similarly.

Let us now compute the coproduct Δ_h of $U_h(\mathfrak{g})^*$:

$$\langle \Delta_h(\xi_5), X_1^{a_1} X_2^{a_2} X_3^{a_3} X_4^{a_4} X_5^{a_5} \otimes X_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} X_5^{b_5}\rangle \neq 0 \iff$$

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = \begin{cases} (0, 0, 0, 0, 1, 0, 0, 0, 0, 0) & \text{or}\\ (0, 0, 0, 0, 0, 0, 0, 0, 0, 1) & \text{or}\\ (0, 0, 0, 1, 0, 1, 0, 0, 0, 0) & \text{or}\\ (0, 0, 1, 0, 0, 2, 0, 0, 0, 0) & \text{or}\\ (0, 1, 0, 0, 0, 3, 0, 0, 0, 0). \end{cases}$$

Moreover,

$$\langle \Delta_h(\xi_5), X_4 \otimes X_1\rangle = -1, \quad \langle \Delta_h(\xi_5), X_3 \otimes X_1^2\rangle = 1, \quad \langle \Delta_h(\xi_5), X_2 \otimes X_1^3\rangle = -1.$$

Hence,

$$\Delta_h(\xi_5) = \xi_5 \otimes 1 + 1 \otimes \xi_5 - \xi_4 \otimes \xi_1 + \tfrac{1}{2}\xi_3 \otimes \xi_1 \cdot_h \xi_1 - \tfrac{1}{6}\xi_2 \otimes \xi_1 \cdot_h \xi_1 \cdot_h \xi_1.$$

$\Delta_h(\xi_1)$, $\Delta_h(\xi_2)$, $\Delta_h(\xi_3)$ and $\Delta_h(\xi_4)$ are computed similarly.

We set $\check{\xi}_i = h^{-1}\xi_i$ and $e_i = \check{\xi}_i \bmod h\ (U(\mathfrak{g})^*)^\vee$. From what we have reviewed in the first paragraph of this section, the first part of this theorem is proved.

Then $\Psi : (U(\mathfrak{g})^*)^\vee \to U(\mathfrak{a})[\![h]\!]$, defined by

$$\Psi\left(\sum_{r\in\mathbb{N}} P_r(\check{\xi}_1, \dots, \check{\xi}_n)h^r\right) = \sum_{r\in\mathbb{N}} P_r(e_1, \dots, e_n)h^r,$$

is an isomorphism of topological $k[\![h]\!]$-modules. Let $\cdot_h$ be the transposition of the multiplication of F_h to $U(\mathfrak{a})[\![h]\!]$. If u and v are in $U(\mathfrak{a})$, one sets

$$u \cdot_h v = uv + \sum_{r=1}^{\infty} h^r \mu_r(u, v).$$

One has $\mu_1(e_3, e_4) = 0$, $\mu_1(e_4, e_3) = e_1^2$ and $\mu_1(e_2, e_5) = 0$, $\mu_1(e_5, e_2) = e_1^2$. Let us show that μ_1 is a coboundary in the Hochschild cohomology. The Hochschild cohomology $HH^*(U(\mathfrak{a}), U(\mathfrak{a}))$ is computed by the complex

$$\big(\mathrm{Hom}(U(\mathfrak{a})^{\otimes *}, U(\mathfrak{a})), b\big),$$

where

$$b(f)(a_0, \dots, a_n) = a_0 f(a_1, \dots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \dots, a_{i-1}a_i, \dots a_n) + f(a_0, \dots, a_{n-1})a_n(-1)^n$$

if $f \in \mathrm{Hom}(U(\mathfrak{a})^{\otimes n+1}, U(\mathfrak{a}))$. Using the explicit isomorphism between the Hochschild cohomology $HH^*(U(\mathfrak{a}), U(\mathfrak{a}))$ and the Lie algebra cohomology of $\mathfrak{a}$ with coefficients in $U(\mathfrak{a})^{ad}$ (with the adjoint action) and $H^*(\mathfrak{a}, U(\mathfrak{a})^{ad})$ [Loday 1998, Lemma 3], one can show that $\mu_1 = b(\alpha)$. The map $\alpha \in \mathrm{Hom}(U(\mathfrak{a}), U(\mathfrak{a}))$ is determined by

$$\alpha_{|\mathfrak{a}} = -\tfrac{1}{2}e_1e_2 \otimes e_3^* - \tfrac{1}{2}e_1e_4 \otimes e_5^*$$

and

$$\mu_1(u, v) = u\alpha(v) - \alpha(uv) + u\alpha(v) \quad \text{for all } (u, v) \in U(\mathfrak{a}).$$

We set $\beta_h = \mathrm{id} - h\alpha$. Then $\beta_h^{-1} = \sum_{i=0}^{\infty} h^i \alpha^i$. If u and v are elements of $U(\mathfrak{a})$, we put $u \cdot_h' v = \beta_h^{-1}(\beta_h(u) \cdot_h \beta_h(v))$. If i and j are different from 3 and 5, then $e_i \cdot_h' e_j = e_i \cdot_h e_j$. Computations lead to the relations:

$$e_1 \cdot_h' e_5 = e_5 \cdot_h' e_1, \quad e_2 \cdot_h' e_3 = e_3 \cdot_h' e_2, \quad e_2 \cdot_h' e_5 = e_5 \cdot_h' e_2, \quad e_3 \cdot_h' e_4 = e_4 \cdot_h' e_3,$$
$$e_1 \cdot_h' e_3 = e_3 \cdot_h' e_1, \quad e_3 \cdot_h' e_5 - e_5 \cdot_h' e_3 = \tfrac{1}{6}h^2e_1^3, \quad e_4 \cdot_h' e_5 - e_5 \cdot_h' e_4 = \tfrac{1}{6} - h^2e_1^3.$$

The topological algebras $[U(\mathfrak{a})[\![h]\!], \cdot_h]$ and $\big[U(\mathfrak{a})[\![h]\!], \cdot_h'\big]$ are isomorphic, hence, their centers are isomorphic. Using the commutation relations, one can compute the center $Z\big[U(\mathfrak{a})[\![h]\!], \cdot_h'\big]$ of $\big[U(\mathfrak{a})[\![h]\!], \cdot_h'\big]$:

$$Z\big[U(\mathfrak{a})[\![h]\!], \cdot_h'\big] = \left\{ \sum_{n\geq 0} P_r(e_1)h^r \;\middle|\; P_r \in k[X_1] \right\}.$$

But, the center of the trivial deformation of $U(\mathfrak{a})$ is

$$Z\big[U(\mathfrak{a})[\![h]\!], \mu_0\big] = \left\{ \sum_{n\geq 0} P_r(e_1, e_3, e_5)h^r \;\middle|\; P_r \in k[X_1, X_3, X_5] \right\}.$$

Hence, the algebras $[U(\mathfrak{a})[\![h]\!], \cdot'_h]$ and $[U(\mathfrak{a})[\![h]\!], \mu_0]$ are not isomorphic. □

Proposition 6.2. *We consider the quantized enveloping algebra of Proposition 6.1. We write the relations defining the ideal I as follows.*

$$e_i \otimes e_j - e_j \otimes e_i - \sum_a C^a_{i,j} e_a - P_{i,j}.$$

As all the $P_{i,j}$'s are monomials in e_1's, the notation $P_{i,j}/e_1$ makes sense. The complex

$$0 \to U_h(\mathfrak{a}) \otimes \textstyle\bigwedge^5 \mathfrak{a} \xrightarrow{\partial^h_5} U_h(\mathfrak{a}) \otimes \textstyle\bigwedge^4 \mathfrak{a} \xrightarrow{\partial^h_4} \cdots \xrightarrow{\partial^h_2} U_h(\mathfrak{a}) \otimes \mathfrak{a} \xrightarrow{\partial^h_1} U_h(\mathfrak{a}) \xrightarrow{\partial^h_0} k[\![h]\!] \to 0,$$

where the morphisms of $U_h(\mathfrak{a})$ and ∂^i_h are described below, is a resolution of the trivial $U_h(\mathfrak{a})$-module $k[\![h]\!]$.We set

$$\partial_n(1 \otimes e_{p_1} \wedge \cdots \wedge e_{p_n}) = \sum_{i=1}^{n} (-1)^{i-1} e_{p_i} \otimes e_{p_1} \wedge \cdots \wedge \widehat{e_{p_i}} \wedge \cdots \wedge e_{p_n} + \sum_{k<l} (-1)^{k+l} \sum_a C^a_{p_k,p_l} 1 \otimes e_a \wedge e_{p_1} \wedge \cdots \wedge \widehat{e_{p_k}} \wedge \cdots \wedge \widehat{e_{p_l}} \wedge \cdots \wedge e_{p_n}.$$

Then,

$$\partial^h_0 = \epsilon_h,$$

$$\partial^h_1(1 \otimes e_i) = e_i,$$

$$\partial^h_2(1 \otimes e_i \wedge e_j) = \partial_2(1 \otimes e_i \wedge e_j) - \frac{P_{i,j}}{e_1} \otimes e_i,$$

$$\partial^h_3(1 \otimes e_i \wedge e_j \wedge e_k) = \partial_3(1 \otimes e_i \wedge e_j \wedge e_k) - \frac{P_{i,j}}{e_1} \otimes e_1 \wedge e_k + \frac{P_{i,k}}{e_1} \otimes e_1 \wedge e_j - \frac{P_{j,k}}{e_1} \otimes e_1 \wedge e_i,$$

$$\partial^h_4(1 \otimes e_1 \wedge e_i \wedge e_j \wedge e_k) = \partial_4(1 \otimes e_1 \wedge e_i \wedge e_j \wedge e_k),$$

$$\partial^h_4(1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_4(1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5) + \frac{P_{3,5}}{e_1} \otimes e_1 \wedge e_2 \wedge e_4 - \frac{P_{3,4}}{e_1} \otimes e_1 \wedge e_2 \wedge e_5 - \frac{P_{4,5}}{e_1} \otimes e_1 \wedge e_2 \wedge e_3 - \frac{P_{2,5}}{e_1} \otimes e_1 \wedge e_3 \wedge e_4,$$

$$\partial^h_5(1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5) = \partial_5(1 \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5).$$

The character defined by right multiplication on $\operatorname{Ext}^5_{U_h(\mathfrak{a})}(k[\![h]\!], U_h(\mathfrak{a}))$ *of* $U_h(\mathfrak{a})$ *is zero.*

Proof of Proposition 6.2. The resolution of $k[\![h]\!]$ is obtained as in the proof of Theorem 5.1. The rest of the proposition follows by easy computations. □

7. Applications

Poincaré duality. Let M be an A_h^{op}-module and N an A_h-module. The right exact functor $M \underset{A_h}{\otimes} -$ has a left derived functor. We set

$$\operatorname{Tor}^i_{A_h}(M, N) = L^i(M \underset{A_h}{\otimes} -)(N).$$

Theorem 7.1. *Let* A_h *be a deformation algebra of* A_0 *satisfying the hypothesis of Theorem 4.1. Assume moreover that the* A_h*-module* K *is of finite projective dimension. Let* M *be an* A_h*-module. The* K*-modules* $\operatorname{Ext}^i_{A_h}(K, M)$ *and* $\operatorname{Tor}^{A_h}_{d_{A_h}-i}(\Omega_{A_h}, M)$ *are isomorphic.*

Remark. Theorem 7.1 generalizes classical Poincaré duality [Knapp 1988].

Proof of Theorem 7.1. As the A_h-module K admits a finite-length resolution by finitely generated projective A_h-modules, $P^\bullet \to K$, the canonical arrow

$$R\operatorname{Hom}_{A_h}(K, A_h) \otimes^L_{A_h} M \to R\operatorname{Hom}_{A_h}(K, M)$$

is an isomorphism in $D(\operatorname{Mod} A_h)$. □

Duality property for induced representations of quantum groups. From now on, we assume that A_h is a topological Hopf algebra.

In this section, we keep the notation of Theorem 4.5. Let V be a left A_h-module, then, by transposition, $V^* = \operatorname{Hom}_K(V, K)$ is naturally endowed with a right A_h-module structure. Using the antipode, we can also see V^* as a left module structure. Thus,

$$u \cdot f = f \cdot S(u) \quad \text{for all } u \in A_h \text{ and } f \in V^*.$$

We endow $\Omega_{A_h} \otimes V^*$ with the right A_h-module structure given by

$$(\omega \otimes f) \cdot u = \lim_{n \to +\infty} \sum_j \theta_{A_h}(u'_{j,n})\, \omega \otimes f \cdot S_h^2(u''_{j,n})$$

and $\Delta(u) = \lim_{n\to+\infty} \sum_j u'_{j,n} \otimes u''_{j,n}$, for all $u \in A_h$, all $f \in V^*$, and all $\omega \in \Omega_{A_h}$.

Let A_h be a topological Hopf deformation of A_0, and let B_h be a topological Hopf deformation of B_0. We assume, moreover, that there exists a morphism of Hopf algebras from B_h to A_h and that A_h is a flat B_h^{op}-module (by Proposition 3.6

this is verified if the induced B_0-module structure on A_0 is flat). If V is an A_h-module, we can define the induced representation from V as follows:

$$\mathrm{Ind}_{B_h}^{A_h}(V) = A_h \otimes_{B_h} V,$$

on which A_h acts by left multiplication.

Proposition 7.2. *Let A_h be a topological Hopf deformation of A_0 and let B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h, such that A_h is a flat B_h^{op}-module. In addition, we assume that B_h satisfies the hypothesis of Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then, $D_{B_h}(\mathrm{Ind}_{A_h}^{B_h}(V))$ is isomorphic to $(\Omega_{B_h} \otimes V^*)_{\otimes_{B_h}} A_h[-d_{B_h}]$ in $D(\mathrm{Mod}\, B_h^{op})$.*

Corollary 7.3. *Let A_h be a topological Hopf deformation of A_0 and let B_h be a topological deformation of B_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h, such that A_h is a flat B_h^{op}-module. We also assume that B_h satisfies the condition of Theorem 4.1. Let V be a B_h-module which is a free finite-dimensional K-module. Then,*

(a) $\mathrm{Ext}^i_{A_h}(A_h \otimes_{B_h} V, A_h)$ *is reduced to* 0 *if i is different from d_{B_h}.*

(b) *The right A_h-module* $\mathrm{Ext}^{d_{B_h}}_{A_h}(A_h \otimes_{B_h} V, A_h)$ *is isomorphic to* $(\Omega_{B_h} \otimes V^*) \otimes_{B_h} A_h$.

Remark. Proposition 7.2 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and A and B are the corresponding enveloping algebras. In this case, one has $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, It was proved by Brown and Levasseur [1985, page 410] and Kempf [1991] in the case where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, and $\mathrm{Ind}_{U(\mathfrak{h})}^{U(\mathfrak{g})}(V)$ is a Verma-module. In addition, Proposition 7.4 is proved in full generality for Lie superalgebras in [Chemla 1994].

Here are some examples of situations where we can apply Proposition 7.2.

Example. Let k be a field of characteristic 0. We set $K = k[\![h]\!]$. Etingof and Kazhdan have constructed a functor Q from the category $LB(k)$ of Lie bialgebras over k to the category $HA(K)$ of topological Hopf algebras over K. If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, its image by Q will be denoted by $U_h(\mathfrak{g})$.

Let $\mathfrak{g}$ be a Lie bialgebra and let $\mathfrak{h}$ be a Lie sub-bialgebra of $\mathfrak{g}$. The functoriality of the quantization implies the existence of an embedding of Hopf algebras from $U_h(\mathfrak{h})$ to $U_h(\mathfrak{g})$ which satisfies all our hypothesis.

Example. If $\mathfrak{g}$ is a Lie bialgebra, we will denote by $\mathcal{F}(\mathfrak{g})$ the formal group attached to it and by $\mathcal{F}_h(\mathfrak{g})$ its Etingof Kazhdan quantization. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras, and assume that there exists a surjective morphism of Lie bialgebras from $\mathfrak{g}$ to $\mathfrak{h}$. Then, $\mathscr{F}_h(\mathfrak{g})$ is a flat $\mathscr{F}_h(\mathfrak{h})$-module, and $A_h = \mathscr{F}_h(\mathfrak{g})$ and $B_h = \mathscr{F}_h(\mathfrak{h})$ satisfies the hypothesis of the theorem.

Example. If G is an affine algebraic Poisson group, we will denote by $\mathscr{F}(G)$ the algebra of regular functions on G and by $\mathscr{F}_h(G)$ its Etingof Kazhdan quantization. Let G and H be affine algebraic Poisson groups. Assume that there is a Poisson group map $G \to H$ such that $\mathscr{F}(G)$ is a flat $\mathscr{F}(H)^{op}$-module. By functoriality of Etingof Kazhdan quantization, $A_h = \mathscr{F}_h(G)$, and $B_h = \mathscr{F}_h(H)$ satisfies the hypothesis of the theorem.

The proof of Proposition 7.2 is analogous to that of [Chemla 2004, Proposition 3.2.4].

We now extend to Hopf algebras another duality property for induced representations of Lie algebras [Chemla 1994].

Proposition 7.4. *Let A_h be a Hopf deformation of A_0, B_h be a Hopf deformation of B_0 and C_h be a Hopf deformation of C_0. We assume that there exists a morphism of Hopf algebras from B_h to A_h and a morphism of Hopf algebras from C_h to A_h such that A_h is a flat B_h^{op}-module and a flat C_h^{op}-module. We also assume that B_h and C_h satisfy the hypothesis of Theorem 4.1. Let V (respectively W) be an B_h-module (respectively C_h-module) which is a free finite dimensional K-module. Then, for all integers n, one has an isomorphism*

$$\operatorname{Ext}^{n+d_{B_h}}_{A_h}\Big(A_h \underset{B_h}{\otimes} V, A_h \underset{C_h}{\otimes} W\Big) \simeq \operatorname{Ext}^{n+d_{C_h}}_{A_h^{op}}\Big((\Omega_{C_h} \otimes W^*) \underset{C_h}{\otimes} A_h, (\Omega_{B_h} \otimes V^*) \underset{C_h}{\otimes} A_h\Big).$$

Remark. Proposition 7.4 is already known in the case where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$, and A, B and C are the corresponding enveloping algebras. In this case one has $d_{B_h} = \dim \mathfrak{h}$ and $d_{C_h} = \dim \mathfrak{k}$. More precisely, generalizing a result of G. Zuckerman [Boe and Collingwood 1985], A. Gyoja [2000] proved a part of this theorem (namely the case where $\mathfrak{h} = \mathfrak{g}$ and $n = \dim \mathfrak{h} = \dim \mathfrak{k}$) under the assumptions that $\mathfrak{g}$ is split semisimple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. D. H. Collingwood and B. Shelton [1990] also proved a duality of this type (still under the semisimple hypothesis) but in a slightly different context.

M. Duflo [1987] proved Proposition 7.4 for a $\mathfrak{g}$ general Lie algebra, $\mathfrak{h} = \mathfrak{k}$, $V = W^*$ being one-dimensional representations.

Proposition 7.4 is proved in full generality in the context of Lie superalgebras in [Chemla 1994]. The proof in the present case is very similar to that of [Chemla 2004, Corollary 3.2.5].

Hochschild cohomology. In this subsection, A_h is a topological Hopf algebra. We set $A_h^e = A_h \otimes_{k[\![h]\!]} A_h^{op}$ and $\widehat{A_h^e} = A_h \widehat{\otimes_{k[\![h]\!]}} A_h^{op}$. If M is an $\widehat{A_h^e}$-module, we set

$$HH^i_{A_h}(M) = \operatorname{Ext}^i_{\widehat{A_h^e}}(A_h, M) \quad \text{and} \quad HH_i^{A_h}(M) = \operatorname{Tor}_i^{\widehat{A_h^e}}(A_h, M).$$

The next result was obtained in [Dolgushev and Etingof 2005] for a deformation of the algebra of regular functions on a smooth algebraic affine variety. Its proof in our setting is analogous to that of [Chemla 2004, Theorem 3.3.2].

Proposition 7.5. *Assume that A_0 satisfies the conditions of Theorem 4.1. Assume moreover that $A_0 \otimes A_0^{op}$ is noetherian. Consider $A_h \widehat{\otimes_{k[[h]]}} A_h$ with the $\widehat{A_h^e}$-module structure given by $\alpha \cdot (x \otimes y) \cdot \beta = \alpha x \otimes y\beta$. for $\alpha, \beta, x, y \in A_h$.*

(a) *$HH^i_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is zero if $i \neq d_{A_h}$.*

(b) *The $\widehat{A_h^e}$-module $U = HH^{d_{A_h}}_{A_h}(A_h \widehat{\otimes_{k[[h]]}} A_h)$ is isomorphic to $\Omega_{A_h} \otimes A_h$ with the $\widehat{A_h^e}$-module structure given by*

$$\alpha \cdot (\omega \otimes x) \cdot \beta = \omega \theta_{A_h}(\beta_i') \otimes S(\beta_i'') x S^{-1}(\alpha)$$

for $\alpha, \beta, x \in A_h$, where $\beta = \sum_i \beta_i' \otimes \beta_i''$.

Proof. Using the antipode S_h of A_h, we have in $D(\text{Mod}\,\widehat{A_h^e})$ the isomorphism

$$R\operatorname{Hom}_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes} A_h) \simeq R\operatorname{Hom}_{A_h \widehat{\otimes} A_h}\big((A_h)^{\#}, (A_h \widehat{\otimes} A_h)^{\#}\big),$$

where the structures on $(A_h)^{\#}$ and $(A_h \widehat{\otimes} A_h)^{\#}$ are given by $(\alpha \otimes \beta) \cdot u = \alpha u S_h(\beta)$, $(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes v S_h(\beta)$, and $(u \otimes v) \cdot \alpha \otimes \beta = u\alpha \otimes S_h(\beta) v$, for all $\alpha, \beta, u, v \in A_h$. Using the version of Lemma 4.6 for right modules [Chemla 2004, Lemma 1.1], one sees that $(A_h)^{\#}$ is isomorphic to $(A_h \widehat{\otimes} A_h) \otimes_{A_h} K$ as an $A_h \widehat{\otimes} A_h$-module. We get

$$\begin{aligned} R\operatorname{Hom}_{\widehat{A_h^e}}(A_h, A_h \widehat{\otimes} A_h) &\simeq R\operatorname{Hom}_{A_h \widehat{\otimes} A_h}(A_h \widehat{\otimes} A_h \otimes_{A_h} K, (A_h \widehat{\otimes} A_h)^{\#}) \\ &\simeq R\operatorname{Hom}_{A_h}(K, (A_h \widehat{\otimes} A_h)^{\#}) \\ &\simeq R\operatorname{Hom}_{A_h}(K, A_h) \otimes_{A_h} (A_h \widehat{\otimes} A_h)^{\#} \\ &\simeq \Omega_h \otimes_{A_h} (A_h \widehat{\otimes} A_h)^{\#}. \end{aligned}$$

Furthermore, the isomorphism $\mathrm{id} \otimes S_h^{-1}$ transforms $(A_h \widehat{\otimes} A_h)^{\#}$ into the natural $(A_h \widehat{\otimes} A_h) \otimes (A_h \widehat{\otimes} A_h)^{op}$-module $(A_h \widehat{\otimes} A_h)^{\text{nat}}$, given by

$$(\alpha \otimes \beta) \cdot (u \otimes v) = \alpha u \otimes \beta v, \quad (u \otimes v) \cdot \alpha \otimes \beta = u\alpha \otimes v\beta$$

for all $(\alpha, \beta, u, v) \in A_h$.

Using Lemma 4.6, one sees that $\Omega_h \otimes_{A_h} (A_h \widehat{\otimes} A_h)^{\text{nat}}$ is isomorphic to $\Omega_h \otimes A_h$ endowed with the $(A_h \widehat{\otimes} A_h)^{op}$-module structure given by

$$(u \otimes v) \cdot \alpha \otimes \beta = \sum_i u \theta_{A_h}(\alpha_i') \otimes S(\alpha_i'') v\beta \quad \text{for all } \alpha, \beta \in A_h.$$

This finishes the proof of the proposition. □

We are in the case where $\mathrm{Ext}^i_{\widehat{A_h^e}}(A_h, \widehat{A_h^e})$ is 0 except when $i = d_{A_h}$, so we get a duality between Hochschild homology and Hochschild cohomology [van den Bergh 1998].

Corollary 7.6. *Let A_0 be a k-algebra satisfying the hypothesis of Theorem 4.1. Assume moreover that $A_0^e = A_0 \otimes A_0^{op}$ is noetherian and that the $\widehat{A_h^e}$-module A_h is of finite projective dimension. Let M be an $\widehat{A_h^e}$-module. One has*

$$HH^i(M) \simeq HH_{d_{A_h}-i}(U \otimes_{A_h} M), \quad \text{where } U = \mathrm{Ext}^{d_{A_h}}_{\widehat{A_h^e}}(A_h, \widehat{A_h^e}).$$

Proof. The proof is similar to that of [van den Bergh 1998]. Assume first that M is a finite-type $\widehat{A_h^e}$-module. Let $P^\bullet \to A_h \to 0$ be a finite-length and finite-type projective resolution of the $\widehat{A_h^e}$-module A_h, and let $Q^\bullet \to M \to 0$ be a finite-type projective resolution of the $\widehat{A_h^e}$-module M. As Q^i and $U \otimes_{A_h} Q^i$ are complete, one has the following sequence of isomorphisms:

$$\begin{aligned} HH^i_{\widehat{A_h^e}}(M) &\simeq H^i\big(\mathrm{Hom}_{\widehat{A_h^e}}(P^\bullet, M)\big) \simeq H^i\big(\mathrm{Hom}_{\widehat{A_h^e}}(P^\bullet, \widehat{A_h^e}) \otimes_{\widehat{A_h^e}} M\big) \\ &\simeq H^i(U[-d] \otimes^L_{\widehat{A_h^e}} M) \simeq H^{i-d_{A_h}}\big(U \otimes_{\widehat{A_h^e}} Q^\bullet\big) \\ &\simeq H^{i-d_{A_h}}\big((A_h \otimes_{A_h} U) \otimes_{\widehat{A_h^e}} Q^\bullet\big) \\ &\simeq H^{i-d_{A_h}}\big(A_h \otimes_{\widehat{A_h^e}} (U \otimes_{A_h} Q^\bullet)\big) \simeq HH_{d_{A_h}-i}(U \otimes_{A_h} M). \end{aligned}$$

In the general case, when M is no longer a finite-type $\widehat{A_h^e}$-module. We have $M = \varinjlim M'$, where M' runs over all finitely generated $\widehat{A_h^e}$-submodules of M. This allows us to finish the proof. □

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