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Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the projection, let D be an integral domain contained in T/I, and let $R = \varphi^{-1}(D)$. We characterize when R is an almost Prüfer v-multiplication domain, an almost valuation domain, and an almost Prüfer domain, in the context of pullbacks.

1. Introduction

Let I be an ideal of an integral domain T, let $\varphi : T \to T/I$ be the natural projection, let D be an integral domain contained in T/I, and let $k = qf(D)$ be the quotient field of D. Let $R = \varphi^{-1}(D)$ be the integral domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

It is well-known that $D = R/I$ and that I is a prime ideal of R. Notice that I is a common ideal of R and T, and hence T is an overring of R. We assume that R is properly contained in T, and we refer to this as a pullback diagram of type ($\triangle$). For the diagram ($\triangle$), if $qf(D) \subseteq T/I$, then we refer to this as a diagram of type ($\triangle'$). For the diagram ($\triangle$), if I is a prime ideal of T and $qf(D) = qf(T/I)$, then we refer to this as a diagram of type ($\triangle^*$). Here $qf(T/I)$ denotes the quotient field of T/I. For the diagram ($\triangle$), if $I = M$ is a maximal ideal of T, we refer to this as a diagram of type ($\triangle_M$). For the diagram ($\triangle_M$), if $qf(D) = T/M$, then we refer to this as a diagram of type ($\triangle_M^*$).

Pullbacks are an important tool in constructing interesting examples and counterexamples. They have become so important that in recent years there have been many papers devoted to ring- and ideal-theoretic properties in pullback domains.

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For more details on pullbacks, see [Mimouni 2004; Houston and Taylor 2007; Fontana and Gabelli 1996; Gilmer 1972; Gabelli and Houston 1997].

Zafrullah [1985] began a general theory of almost factoriality and introduced the notion of an almost GCD-domain. Zafrullah defined R to be an almost GCD-domain (AGCD-domain for short) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Anderson and Zafrullah [AZ 1991] introduced several classes of integral domains related to almost GCD-domains, including almost Bézout domains (AB-domains), almost Prüfer domains (AP-domains), and almost valuation domains (AV-domains). As in [AZ 1991], an integral domain R is an AB-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal; while R is an AP-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible. Following [AZ 1991], an integral domain R is said to be an AV-domain if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n | b^n$ or $b^n | a^n$. Similarly, in [Li 2012] we defined an integral domain R to be an almost Prüfer v-multiplication domain (APVMD) if for each $a, b \in R \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^n D \cap b^n D$ is t-invertible, or equivalently, (a^n, b^n) is t-invertible. Recall that an integral domain R is said to be a Prüfer v-multiplication domain (PVMD) if each $a, b \in R \setminus \{0\}$, (a, b) is t-invertible. The class of APVMDs includes a lot of important rings, such as AV-domains, AB-domains, AGCD-domains, AP-domains, PVMDs, and so on.

Anderson and Zafrullah [1991, Theorem 4.9] proved that D is an AB-domain (respectively, AP-domain) if and only if $R = D + Xk[X]$ is an AB-domain (respectively, AP-domain). However, we notice that the $(D + Xk[X])$-construction is a special case of the pullback of type $(\triangle_M)$. Mimouni [2004, Theorem 2.2] generalized these results and proved that for the diagram $(\triangle_M)$, R is an AP-domain if and only if T and D are AP-domains and the extension $k \subseteq T/M$ is a root extension. He also gave a similar characterization for AV-domains. Mimouni [2004, Corollary 2.6] continued to show that for the diagram $(\triangle_M)$, assuming that $D = k$ is a field, then R is an AB-domain if and only if T is an AB-domain and the extension $k \subseteq T/M$ is a root extension. In [Li 2012, Theorem 3.10], we proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD.

From this we notice that the characterization of AV-domains and AP-domains is known only in the context of the special pullback of type $(\triangle_M)$, and that the study of APVMDs is only in the $(D + Xk[X])$-construction, a special case of type $(\triangle_M)$. So the main purpose of this paper is to characterize APVMDs in pullbacks in greater generality and to generalize the characterization of AV-domains and AP-domains for the pullback of type $(\triangle_M)$ to that for the pullback of type $(\triangle')$.

In Section 2, we mainly prove that in the pullback of type $(\triangle_M)$, R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension

$qf(D) \subseteq T/M$ is a root extension. Using this fact, we give Example 2.2 to show that an APVMD is not necessarily a PVMD. We also show that for the diagram $(\triangle_M^*)$, R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain. Using this result, we prove that D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.

In Section 3, we mainly indicate that for the diagram $(\triangle')$, if T is an AV-domain, then R is an APVMD if and only if D is an APVMD and the extension $qf(D) \subseteq T/I$ is a root extension. We prove that for the diagram $(\triangle')$, R is an AV-domain if and only if T and D are AV-domains and the extension $k = qf(D) \subseteq T/I$ is a root extension. We also show that for the diagram $(\triangle')$, assuming that T is an AV-domain, then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.

Following [Zafrullah 1988, p. 95], assume that D is the ring of entire functions and S is the multiplicative set generated by the principal primes of D; then D is integrally closed, and hence $R = D + XD_S[X]$ is integrally closed, but $R = D + XD_S[X]$ is not a PVMD. Because an integrally closed APVMD is a PVMD by [Li 2012, Theorem 2.4], R is not an APVMD. Consider the following pullback:

$$\begin{array}{ccc} R = D + XD_S[X] & \longrightarrow & D \\ \downarrow & & \downarrow \\ T = D_S[X] & \longrightarrow & D_S \cong T/I \end{array}$$

Here I denotes $XD_S[X]$. The example indicates that $qf(D) = qf(T/I)$, D and T are APVMDs, I is principal in T, and $T = D_S[X]$ is a PVMD. It follows that T_I is an AV-domain by [Li 2012, Theorem 2.3]. However, R is not an APVMD. The pullback above belongs to the pullback of type $(\triangle^*)$. Therefore, for the diagram $(\triangle^*)$, without some other assumption on T, D or T/I, there is no hope of proving that R is an APVMD even when T and D are APVMDs and T_I is an AV-domain. So in Section 4, we prove that in a pullback of type $(\triangle^*)$, if $T = (I_v : I_v)$, then R is an APVMD if and only if T is an APVMD, T_I is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of D, either (1) $D_{\bar{P}}$ and $T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains, or (2) there exists a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = T$.

For details on star operations, see [Gilmer 1972, Sections 32 and 34].

2. Pullbacks of type (Δ_M)

We begin with the characterization of APVMDs in a pullback of type $(\triangle_M)$.

Theorem 2.1. *For the diagram $(\triangle_M)$, R is an APVMD if and only if D and T are APVMDs, T_M is an AV-domain, and the extension $qf(D) \subseteq T/M$ is a root extension.*

Proof. (⇒) Assume that R is an APVMD. Let $x, y \in D \setminus \{0\}$; then $\varphi(a) = x$ and $\varphi(b) = y$ for some $a, b \in R \setminus M$. Because R is an APVMD, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is t-invertible in R. By [Wang 2006, Theorem 10.3.11], $(\varphi(a^n), \varphi(b^n))$ is t-invertible in D. Because $(x^n, y^n) = (\varphi(a)^n, \varphi(b)^n) = (\varphi(a^n), \varphi(b^n))$, it follows that (x^n, y^n) is t-invertible in D. Thus D is an APVMD. Let $c, d \in T \setminus \{0\}$. Because T and R have the same quotient field, there is an element $r \in R \setminus \{0\}$ with $rc, rd \in R$. Then $((rc)^n, (rd)^n)R$ is a t-invertible ideal of R for some positive integer n. According to [Wang 2006, Theorem 10.3.11], $((rc)^n, (rd)^n)T$ is t-invertible in T. It is well-known that $((rc)^n, (rd)^n)T = r^n(c^n, d^n)T$, so $(c^n, d^n)T$ is t-invertible in T. Therefore T is an APVMD. As we know, M is a v-ideal of R. Then R_M is an AV-domain by [Li 2012, Theorem 2.3]. By [Wang 2006, Theorem 10.2.2], we have the pullback

$$\begin{array}{ccc} R_M & \longrightarrow & D_{R\setminus M} \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

By [Mimouni 2004, Theorem 2.2], T_M and $D_{R\setminus M}$ are AV-domains and the extension $qf(D) = qf(D_{R\setminus M}) \subseteq T/M$ is a root extension.

(⇐) Let P be a maximal t-ideal of R.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorem 10.2.4(3)], there is a prime ideal Q of T with $P = Q \cap R$. Clearly, $M \nsubseteq Q$. In fact $P \nsubseteq M$. Because M is a v-ideal of R, M is a t-ideal of R. As the maximality, $P \nsubseteq M$. So $Q \nsubseteq M$. Hence Q is incomparable to M. According to [Fontana et al. 1998, Lemma 3.3], Q is a maximal t-ideal of T. Since T is an APVMD, T_Q is an AV-domain. By [Wang 2006, Theorem 10.2.1(6)], $R_P = T_Q$. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Because P is a t-ideal of R, $P = P_t$. Then $\varphi^{-1}(p) = (\varphi^{-1}(p))_t = \varphi^{-1}(p_t)$ by [Wang 2006, Theorem 10.3.5(3)]. So $p = p_t$. Thus p is a t-ideal of D. Since D is an APVMD, D_p is an AV-domain. In this case, consider the following pullback:

$$\begin{array}{ccc} R_P & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

Since T_M and D_p are AV-domains and the extension $qf(D) \subseteq T/M$ is a root extension, R_P is an AV-domain by [Mimouni 2004, Theorem 2.2]. Therefore R is an APVMD. □

Gabelli and Houston [1997, Theorem 4.13] showed that for the diagram $(\triangle_M)$, R is a PVMD if and only if T and D are PVMDs, $k = T/M$, and T_M is a valuation domain. Using this result and Theorem 2.1, we can easily get the following result.

Example 2.2. Let $R = K + XL[X]$, where K and L are fields, $K \subseteq L$, and for some prime p, $L^p \subseteq K$. Consider the pullback

$$\begin{array}{ccc} K + XL[X] & \longrightarrow & K \\ \downarrow & & \downarrow \\ L[X] & \longrightarrow & L \end{array}$$

Then R is an APVMD but not a PVMD. Thus an APVMD need not be a PVMD.

Corollary 2.3. *For the diagram $(\triangle_M^*)$, R is an APVMD if and only if D and T are APVMDs and T_M is an AV-domain.*

Proof. It easily follows from Theorem 2.1 and [Mimouni 2004, Lemma 2.3]. □

Corollary 2.4. *For the diagram $(\triangle_M^*)$, R is an AP-domain if and only if D and T are AP-domains.*

Proof. $(\Rightarrow)$ It follows from [Mimouni 2004, Theorem 2.2].

$(\Leftarrow)$ Let P be a maximal ideal of R.

Case 1. Suppose that $M \nsubseteq P$. By [Wang 2006, Theorems 10.2.4(3) and 10.2.1(6)], there is a prime ideal Q of T with $P = Q \cap R$ and $R_P = T_Q$. Since T is an AP-domain, T_Q is an AV-domain by [AZ 1991, Theorem 5.8]. Hence R_P is an AV-domain.

Case 2. Suppose that $M \subseteq P$. There exists a prime ideal p of D such that $P = \varphi^{-1}(p)$. Since D is an AP-domain, D_p is an AV-domain. In this case, consider the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_p \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & T/M \end{array}$$

Since T_M and D_p are AV-domains and $qf(D) = qf(D_p) = T/M$, R_P is an AV-domain by [Mimouni 2004, Lemma 2.3]. Therefore R is an AP-domain. □

Proposition 2.5. *For the diagram $(\triangle_M)$, suppose that (T, M) is a quasilocal domain and $D = k$ is a proper field of T/M. Then R is an APVMD if and only if R is an AV-domain.*

Proof. $(\Leftarrow)$ It easily follows from their definitions.

$(\Rightarrow)$ Assume that D is a field. Since $D = R/M$, M is a maximal ideal of R. Because T is quasilocal, R is quasilocal by [Wang 2006, Corollary 10.2.1]. Also

$M = (R : T)$ is a v-ideal of R. Hence M is the unique maximal t-ideal of R. Therefore $R = R_M$ is an AV-domain. □

In [Li 2012, Theorem 3.10], we considered the polynomial ring case and proved that D is an APVMD if and only if $R = D + Xk[X]$ is an APVMD. Similarly, we consider the power series ring case and get the following result.

Corollary 2.6. *Let D be an integral domain with quotient field k. Then D is an APVMD if and only if $R = D + Xk[\![X]\!]$ is an APVMD.*

Proof. Consider the pullback

$$\begin{array}{ccc} R = D + Xk[\![X]\!] & \longrightarrow & D \\ \downarrow & & \downarrow \\ T = k[\![X]\!] & \longrightarrow & k = k[\![X]\!]/Xk[\![X]\!] \end{array}$$

$T = k[\![X]\!]$ is a UFD, so T is an APVMD. The rest follows from Corollary 2.3. □

3. Pullbacks of type (Δ')

Mimouni [2004] considered the pullbacks of type ($\triangle_M$) in AP-domains and AV-domains. He proved that for the diagram ($\triangle_M$), R is an AV-domain (respectively AP-domain) if and only if T and D are AV-domains (respectively AP-domains) and the extension $k \subseteq T/M$ is a root extension. We generalize these results for the special pullback of type ($\triangle_M$) to those for the pullback of type ($\triangle'$).

Lemma 3.1. *For the diagram ($\triangle'$), if R is an AP-domain (resp. AGCD-domain), then the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. Assume that R is an AP-domain (resp. AGCD-domain). By way of contradiction, suppose that the extension $k \subseteq T/I$ is not a root extension. So there is $\lambda \in T/I$ such that λ^n is not in k for each positive integer n. Set $\lambda = \varphi(a)$ for some $a \in T \setminus I$. Let b be a nonzero fixed element of I. Since R is an AP-domain (resp. AGCD-domain), $((ab)^n, b^n)$ is invertible (resp. $((ab)^n, b^n)_v$ is principal) for some positive integer n. Let J denote $((ab)^n, b^n)$. Then $JJ^{-1} = R$ (resp. $J_v = cR$ for some $c \in R$). By [Wang 2006, Example 8.1.10(1)], $J^{-1} = (ab)^{-n}R \cap b^{-n}R$. Let $f \in J^{-1}$; then $f = (ab)^{-n} f_1 = b^{-n} f_2$ for some $f_1, f_2 \in R$. Thus $a^{-n} f_1 = f_2$ and so $f_1 = a^n f_2$. If f_2 is not in I, then $\varphi(f_2) \in D \setminus \{0\}$. Hence $\varphi(f_1) = \varphi(a^n f_2) = \varphi(a)^n \varphi(f_2) = \lambda^n \varphi(f_2)$. So $\lambda^n \in qf(D) = k$, a contradiction. Therefore $f_2 \in I$. So $J^{-1} \subseteq b^{-n}I$. We claim $b^{-n}I \subseteq J^{-1}$. Let $z \in I$ and $x \in J$ and write $x = \alpha(ab)^n + \beta b^n$ for some $\alpha, \beta \in R$. Then $(b^{-n}z)x = (b^{-n}z)(\alpha(ab)^n + \beta b^n) = z\alpha a^n + z\beta \subseteq I \subseteq R$, so $b^{-n}z \in J^{-1}$. Then $b^{-n}I \subseteq J^{-1}$. Therefore $b^{-n}I = J^{-1}$. So $J_v = b^n I^{-1}$. Since $JJ^{-1} = R$ (resp. $J_v = cR$), we have $1 = g_1h_1 + \cdots + g_mh_m$ for $g_1, \ldots, g_m \in J$, $h_1, \ldots, h_m \in J^{-1}$ (resp. $b^n I^{-1} = cR$). For each $i \in \{1, 2, \ldots, m\}$, write $g_i =$

$\alpha_i(ab)^n + \beta_i b^n$ and $h_i = b^{-n} f_i$, where $\alpha_i, \beta_i \in R$, $f_i \in I$. Then we have $1 = g_1h_1+\cdots+g_mh_m = (\alpha_1(ab)^n+\beta_1 b^n)(b^{-n} f_1)+\cdots+(\alpha_m(ab)^n+\beta_m b^n)(b^{-n} f_m) = (\alpha_1 a^n + \beta_1) f_1 + \cdots + (\alpha_m a^n + \beta_m) f_m \in I$, which is absurd. (Respectively, for each $y \in I^{-1}$, $TyI \subseteq yI \subseteq R$, so $Ty \in I^{-1}$, hence $T \subseteq (I^{-1} : I^{-1})$. Then $R \subset T \subseteq (I^{-1} : I^{-1}) = (b^n I^{-1} : b^n I^{-1}) = (J^{-1} : J^{-1}) = (cR : cR) = R$, which is absurd.) Therefore the extension $k \subseteq T/I$ is a root extension. □

Lemma 3.2. *For the diagram* $(\triangle')$, *assume that* $D = k$ *is a field. Then* R *is an AV-domain if and only if* T *is an AV-domain and the extension* $k \subseteq T/I$ *is a root extension.*

Proof. (⇒) It follows from Lemma 3.1 and the fact that T is an overring of R.

(⇐) Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\}$. Since the extension $k \subseteq T/I$ is a root extension, there is a positive integer m such that $\varphi(x^{nm}) = \varphi(x)^{nm} \in k$. Hence $x^{nm} \in \varphi^{-1}(k) = R$. It follows that R is an AV-domain. □

Theorem 3.3. *For the diagram* $(\triangle')$, R *is an AV-domain if and only if* T *and* D *are AV-domains and the extension* $k = qf(D) \subseteq T/I$ *is a root extension.*

Proof. (⇒) By [AZ 1991, Lemma 4.5], T is an AV-domain as an overring of R; and by [AZ 1991, Theorem 4.10], $D = R/I$ is an AV-domain. Also by Lemma 3.1, the extension $k = qf(D) \subseteq T/I$ is a root extension.

(⇐) We use the fact that the diagram $(\triangle')$ splits into two parts as follows:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Consider the second part of this diagram:

$$\begin{array}{ccc} R_0 & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Since T is an AV-domain and the extension $k \subseteq T/I$ is a root extension, by Lemma 3.2 R_0 is an AV-domain. The first part of the diagram —

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & k \end{array}$$

— is a pullback diagram of type $(\triangle_M^*)$. Since D and R_0 are AV-domains, R is an AV-domain by [Mimouni 2004, Lemma 2.3]. □

Lemma 3.4. *For the diagram* $(\triangle)$, *let* $Q(A) = \{x \in T \mid xI \subseteq A\}$ *for an ideal* A *of* R. *Then if* P *is a prime ideal of* R *and* $I \nsubseteq P$, *then* $Q(P)$ *is a prime ideal of* T, $P = Q(P) \cap R$ *and* $R_P = T_{Q(P)}$.

Proof. Let $I \nsubseteq P$, let $x, y \in T$, and let $xy \in Q(P)$. Then $xyI^2 \subseteq xyI \subseteq P$. Since $xI, yI \subseteq I \subseteq R$ and P is a prime ideal of R, we have $xI \subseteq P$ or $yI \subseteq P$. So $x \in Q(P)$ or $y \in Q(P)$. Thus $Q(P)$ is a prime ideal of T. We claim $P = Q(P) \cap R$. Because $PI \subseteq P$, we have $P \subseteq Q(P) \cap R$. Let $x \in Q(P) \cap R$; then $xI \subseteq P$. Since $I \nsubseteq P$, we have $x \in P$. Hence $Q(P) \cap R \subseteq P$. Thus $P = Q(P) \cap R$. Next we show that $R_P = T_{Q(P)}$. It easily follows that $R_P \subseteq T_{Q(P)}$. For the reverse inclusion, let $x \in T_{Q(P)}$. Then $x = z_1/z_2$ for some $z_1 \in T$, $z_2 \in T \setminus Q(P)$. Since $I \nsubseteq P$, there exists $u \in I \setminus P$. Of course $u \in I \setminus Q(P)$. Then $uz_1 \in I \subseteq R$, $uz_2 \in I \setminus Q(P) \subseteq R \setminus Q(P)$. Thus $uz_2 \in R \setminus P$. So $x = uz_1/uz_2 \in R_P$. Thus $T_{Q(P)} \subseteq R_P$, so $R_P = T_{Q(P)}$. □

Theorem 3.5. *For the diagram* $(\triangle')$, *assume that* T *is an AV-domain. Then* R *is an APVMD if and only if* D *is an APVMD and the extension* $k = qf(D) \subseteq T/I$ *is a root extension.*

Proof. As in Theorem 3.3, we consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

($\Leftarrow$) Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Because D is an APVMD, by Corollary 2.3 R is an APVMD.

($\Rightarrow$) Assume that R is an APVMD; by Corollary 2.3 D and R_0 are APVMDs and $(R_0)_I$ is an AV-domain. Set $S = R \setminus I$. Then $R_S = R_I$ and $(R_0)_I = (R_0)_S$. By

[Houston and Taylor 2007, Lemma 1.2], consider the pullback

$$\begin{array}{ccc} (R_0)_S & \longrightarrow & k = k_{\varphi(S)} \\ \downarrow & & \downarrow \\ T_S & \longrightarrow & (T/I)_{\varphi(S)} \end{array}$$

As $(R_0)_S = (R_0)_I$ is an AV-domain, the extension $k \subseteq (T/I)_{\varphi(S)}$ is a root extension by Lemma 3.2. So the extension $k \subseteq T/I$ is a root extension. □

Theorem 3.6. *For the diagram* $(\triangle')$, *assume that T is an AV-domain. Then R is an AP-domain if and only if D is an AP-domain and the extension $k = qf(D) \subseteq T/I$ is a root extension.*

Proof. $(\Leftarrow)$ As in Theorem 3.3, we consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_0 = \varphi^{-1}(k) & \longrightarrow & k = R_0/I \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Since T is an AV-domain, R_0 is an AV-domain by Lemma 3.2. Then R is an AP-domain by Corollary 2.4.

$(\Rightarrow)$ Assume that R is an AP-domain; then $D = R/I$ is an AP-domain by [AZ 1991, Theorem 4.10]. Also by Lemma 3.1, the extension $k \subseteq T/I$ is a root extension. □

4. Pullbacks of type (Δ^*)

Lemma 4.1. *For a diagram* $(\triangle^*)$, *R is an AV-domain if and only if T and D are AV-domains.*

Proof. The proof is similar to that of Lemma 3.2.

$(\Rightarrow)$ If R is an AV-domain, so are its homomorphic image of D and its overring T.

$(\Leftarrow)$ Let $x \in qf(R)$; then $x \in qf(T)$. Since T is an AV-domain, there is a positive integer $n = n(x)$ such that $x^n \in T$ or $x^{-n} \in T$. Assume that, for example, $x^n \in T$. If $x^n \in I$, then $x^n \in R$. If $x^n \in T \setminus I$, then $\varphi(x)^n = \varphi(x^n) \in T/I \setminus \{0\} \subseteq qf(T/I) = qf(D)$. Since D is an AV-domain, there is a positive integer m such that $\varphi(x)^{nm} \in D$. Hence $x^{nm} \in \varphi^{-1}(D) = R$. It follows that R is an AV-domain. □

Proposition 4.2. *Let R be an integral domain and I a nonzero ideal of R. If R is an APVMD, then $(I_v : I_v)$ is an APVMD.*

Proof. Set $T = (I_v : I_v)$. Assume that $x, y \in T = (I_v : I_v)$. Choose a fixed element $a \in I_v$. Then $ax, ay \in I_v \subseteq R$. Since R is an APVMD, there is a positive integer $n = n(ax, ay)$ such that $((ax)^n, (ay)^n)$ is t-invertible in R. Let J denote $((ax)^n, (ay)^n)$. So $(JJ^{-1})_t = R$. There is a finitely generated ideal $H \subseteq JJ^{-1} \subseteq R$ such that $H_v = R$. By [Houston and Taylor 2007, Lemma 2.3], $(I_v : I_v)$ is t-linked over R. Then $(HT)_v = T$. So $(JJ^{-1}T)_t = T$. Thus $(a^n(x^n, y^n)J^{-1}T)_t = \big(((ax)^n, (ay)^n)J^{-1}T\big)_t = T$. So (x^n, y^n) is t-invertible in T. Therefore $T = (I_v : I_v)$ is an APVMD. □

Proposition 4.3. *For a diagram* $(\triangle^*)$, *if R is an APVMD, then I is a prime t-ideal of both R and T.*

Proof. We claim R_I is an AV-domain, and thus I is a t-ideal of R. Let $x, y \in R \setminus \{0\}$. If $(x^n, y^n)(x^n, y^n)^{-1} \subseteq I$ for each positive integer n, then $((x^n, y^n)(x^n, y^n)^{-1})^{-1} \supseteq I^{-1} \supseteq T \supsetneq R$, which contradicts that R is an APVMD. Hence there exists a positive integer n such that $(x^n, y^n)(x^n, y^n)^{-1} \nsubseteq I$. Thus $((x^n, y^n)(x^n, y^n)^{-1})R_I = R_I$. So $(x^n, y^n)R_I$ is invertible in R_I. Since R_I is quasilocal, $(x^n, y^n)R_I$ is principal. Then R_I is an AV-domain. So IR_I is a maximal t-ideal of R_I. By [Kang 1989, Lemma 3.17], $I = IR_I \cap R$ is a t-ideal of R. Since $qf(D) = qf(T/I)$, we have $R_I = T_I$ by [Houston and Taylor 2007, Lemma 1.2]. So T_I is an AV-domain. Then IT_I is a maximal t-ideal of T. Therefore I is a prime t-ideal of T. □

Houston and Taylor [2007, Theorem 2.8] characterized the PVMD-property in a pullback of type $(\triangle^*)$. Similarly, we are ready to study the APVMD-property in a pullback of type $(\triangle^*)$. For convenience, let E denote T/I.

Theorem 4.4. *For a diagram* $(\triangle^*)$, *assume that $T = (I_v : I_v)$. Then R is an APVMD if and only if T is an APVMD and T_I is an AV-domain, and for each nonzero prime ideal $\bar{P}$ of D, either*

(1) *$D_{\bar{P}}$ and $T_{\varphi^{-1}(D\setminus\bar{P})}$ are AV-domains, or*

(2) *there is a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = T$.*

Proof. (⇒) Assume that R is an APVMD. By Proposition 4.2, $T = (I_v : I_v)$ is an APVMD. Also, T_I is an AV-domain by Proposition 4.3. Let $\bar{P}$ be a prime ideal of D, and let $P = \varphi^{-1}(\bar{P})$.

Case 1. If P is a t-ideal of R, then R_P is an AV-domain. By [Houston and Taylor 2007, Lemma 1.2], we have the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_{\varphi(R\setminus P)} = D_{\bar{P}} \\ \big\downarrow & & \big\downarrow \\ T_{R\setminus P} = T_{\varphi^{-1}(D\setminus\bar{P})} & \longrightarrow & E_{\varphi(S)} = E_{D\setminus\bar{P}} \end{array}$$

By Lemma 4.1, $D_{\bar{P}}$ and $T_{R\setminus P} = T_{\varphi^{-1}(D\setminus \bar{P})}$ are AV-domains.

Case 2. Suppose that P is not a t-ideal of R. Since R is an APVMD, it is a UMT-domain by [Li 2012, Theorem 3.8]. By [Fontana et al. 1998, Corollary 1.6], $P_t = R$. Hence there is a finitely generated ideal $J \subseteq P$ such that $J^{-1} = R$. Since T is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], we have $(JT)^{-1} = T$. So $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Now let $A = \varphi(J)$ and $e \in A^{-1} \cap E$. Then $\varphi(t) = e$ for some $t \in T$ and $eA \subseteq D$. Hence $\varphi^{-1}(eA) \subseteq \varphi^{-1}(D) = R$. Also, $\varphi^{-1}(eA) = \varphi^{-1}(e)\varphi^{-1}(A) = \varphi^{-1}(\varphi(t))\varphi^{-1}(\varphi(J)) \supseteq tJ$. So $tJ \subseteq R$. Then $t \in J^{-1} = R$. Thus $e = \varphi(t) \in D$. Therefore $A^{-1} \cap E = D$.

($\Leftarrow$) Let P be a maximal t-ideal of R. It suffices to show that R_P is an AV-domain.

Case 1. Assume that $I \nsubseteq P$. By Lemma 3.4, there is a prime ideal Q of T such that $P = Q \cap R$ and $R_P = T_Q$. By Proposition 4.3, we know that I is a prime t-ideal of R. Then $(PT)_t \neq T$ by [Houston and Taylor 2007, Lemma 2.6]. Hence $PT \subseteq Q_1$ for some prime t-ideal Q_1 of T. Since $T = (I_v : I_v)$ is t-linked over R by [Houston and Taylor 2007, Lemma 2.3], it follows that $(Q_1 \cap R)_t \neq R$. However, $P \subseteq Q_1 \cap R$ and P is a maximal t-ideal of R. It follows that $Q = Q_1$. Then Q is t-ideal of T. Therefore $R_P = T_Q$ is an AV-domain.

Case 2. Assume that $I \subseteq P$. Let $\bar{P}$ denote $\varphi(P)$. By way of contradiction, suppose that condition (2) of the hypothesis holds: there is a finitely generated ideal A of D such that $A \subseteq \bar{P}$, $A^{-1} \cap E = D$, and $(\varphi^{-1}(\bar{P})T)_t = (PT)_t = T$. Then $A = \varphi(J_1)$ and $(J_2T)^{-1} = T$ for some finitely generated ideals J_1, J_2 of R. Also $J_1 + J_2 \subseteq P$. Set $J = J_1 + J_2$. Then $J^{-1} \subseteq J_2^{-1}$. Let $x \in J_2^{-1}$; then $xJ_2 \subseteq R$, and hence $xJ_2T \subseteq T$. So $x \in (J_2T)^{-1} = T$. So $J^{-1} \subseteq J_2^{-1} \subseteq T$. Since $J \subseteq P$ and P is a prime t-ideal of R, then $J^{-1} \neq R$. Otherwise, if $J^{-1} = R$, then $R = J_v \subseteq P_t = P$, a contradiction. So $R \subsetneq J^{-1}$. Therefore, there is an element $t \in J^{-1} \setminus R$ with $tJ \subseteq R$. So $\varphi(t)A \subseteq \varphi(t)\varphi(J_1) \subseteq \varphi(t)\varphi(J) = \varphi(tJ) \subseteq D$. Then $\varphi(t) \in A^{-1} \cap E = D$. So $t \in R$, a contradiction. Hence condition (1) must hold. Localize the diagram at P and consider the pullback

$$\begin{array}{ccc} R_P & \longrightarrow & D_{\varphi(R\setminus P)} = D_{\bar{P}} \\ \downarrow & & \downarrow \\ T_{R\setminus P} = T_{\varphi^{-1}(D\setminus \bar{P})} & \longrightarrow & E_{\varphi(S)} = E_{D\setminus \bar{P}} \end{array}$$

By Lemma 4.1, R_P is an AV-domain. Therefore, R is an APVMD. □

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