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**THE KOSZUL PROPERTY AS A TOPOLOGICAL INVARIANT
AND MEASURE OF SINGULARITIES**

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Cassidy, Phan and Shelton have associated to any regular cell complex X a quadratic K -algebra $R(X)$. They gave a combinatorial solution to the question of when this algebra is Koszul. The algebra $R(X)$ is a combinatorial invariant but not a topological invariant. We show that, nevertheless, the property that $R(X)$ be Koszul is a topological invariant.

In the process, we establish some conditions on the types of local singularities that can occur in cell complexes X such that $R(X)$ is Koszul, and more generally in cell complexes that are pure and connected by codimension-one faces.

1. Introduction

Let X be a finite regular cell complex of dimension d , and let K be a field. Following Cassidy, Phan and Shelton (see [Cassidy et al. 2010], henceforth abbreviated [CPS]) we will associate to X , under certain global assumptions, a quadratic K -algebra $R(X)$ (defined below). The focus of [CPS] is to determine the combinatorial properties required for this algebra to be Koszul. The primary focus of this paper is to show that the Koszul property is actually a topological invariant, even though the algebra is not. In the process we see that our global assumptions also imply some restrictions on singularities of appropriate spaces X .

After a definition of our two technical assumptions we can state our main theorem. Our complexes will be finite throughout. We will generally not restate this hypothesis.

Definition 1. Let X be a regular cell complex of dimension d .

- (1) X is pure if X is the closure of its open d -cells.
- (2) A pure, finite regular cell complex X , is *connected through codimension-one faces* if the space $X - X^{(d-2)}$ is path connected (where $X^{(d-2)}$ is the $(d-2)$ -skeleton of X).

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Theorem 2. *Let X be a pure regular cell complex of dimension d , connected through codimension-one faces. Then $R(X)$ is Koszul (for the field \mathbf{K}) if and only if both of the following conditions hold.*

- (1) $\tilde{H}_i(X; \mathbf{K}) = 0$ for $i < d$.
- (2) $H_i(X, X - \{p\}; \mathbf{K}) = 0$ for each $p \in X$ and each $i < d$.

Because our hypotheses on the cell complex structure and on homology are obviously homeomorphism invariant, [Theorem 2](#) shows that the Koszul property for $R(X)$ is a homeomorphism invariant. Also, $H_i(X, X - \{p\}; \mathbf{K})$ depends only on the open cell containing p , so (2) involves checking only finitely many conditions. We point out, however, that one does not have any nice homotopy invariance:

Example 3. There are homotopy equivalent, pure regular cell complexes X and Y of dimension three such that $R(X)$ is not Koszul but $R(Y)$ is Koszul. For instance, let Y be the union of two 3-cells attached by some 2-dimensional face (so $R(Y)$ is Koszul by [Theorem 2](#)) and let X be the complex in [Example 22](#), taken from [\[CPS, Example 5.9\]](#). X is homotopy equivalent to Y since one gets a space homeomorphic to Y by collapsing the contractible subcomplex a of X to a point. But [\[CPS\]](#) shows that $R(X)$ is not Koszul (as one can also see by [Theorem 2](#)).

Although our argument does not make direct use of the definition of $R(X)$, we review that definition here in the interest of self-containedness.

Let P be any finite ranked poset with minimal element $\bar{0}$. For each $x \in P$ let $s_1(x) = \{y \in P \mid y < x, \text{rank}(x) - \text{rank}(y) = 1\}$ (the elements immediately below x in the ranked poset). We define $R(P)$ to be the quadratic \mathbf{K} -algebra on generators $r_x, x \in P - \{\bar{0}\}$ with relations

$$r_x r_y = 0 \text{ for all } y \notin s_1(x)$$

and

$$r_x \sum_{z \in s_1(x)} r_z = 0 \text{ for all } x.$$

The set of all closed cells of a regular cell complex, together with the empty set, form a finite ranked poset under set inclusion. Following [\[CPS\]](#), this is denoted $\bar{P}(X)$. (In this poset the rank of a cell is one more than its dimension, so that the rank of the empty set is 0).

If we assume that X is pure, we may adjoin one additional (maximal) element to the poset $\bar{P}(X)$. The resulting poset, which is denoted $\hat{P}(X)$ is still a ranked poset. If we also assume that X is connected through codimension-one faces, then $\hat{P}(X)$ has the combinatorial property known as *uniform* (cf. [\[Gelfand et al. 2005\]](#)). Then we define $R(X)$ to be $R(\hat{P}(X))$. While $R(\bar{P}(X))$ is always Koszul under the hypotheses that X is pure and connected through codimension-one faces (see

[CPS] and [Retakh et al. 2010]), the Koszul property for $R(X)$ is substantially more subtle. In [CPS, Theorem 5.3] a precise statement was given in terms of the combinatorial cell structure of X describing when $R(X)$ is Koszul (refer to Theorem 5 below).

We note that there are results of a similar nature about different sorts of combinatorially derived algebras in the literature. One example of results in a similar spirit but of a different nature can be found in [Reiner and Stamate 2010].

2. CPS cohomology and local homology

We fix X , a finite regular CW complex of dimension d . We begin by recalling the definitions of the groups $H_X(n, k)$ from [CPS, Section 4], which we will write as $H_k^n(X)$.

Assign orientations to each cell of X . If β is an n cell and α is an $n + 1$ cell, let $[\alpha : \beta]$ be the incidence number of β in α . Because X is regular, this is either 0, 1 or -1 . These incidence numbers are usually defined in the context of cellular homology so that, if $C_*(X)$ is the cellular chain complex of X and α is an $n + 1$ cell,

$$d(\alpha) = \sum_{n-1 \text{ cells } \beta} [\alpha : \beta]\beta.$$

Because X is finite and we have a chosen basis for the cellular chains, (given by the cells of X) we have an isomorphism between the cellular chains and the cellular cochains of X . We consider the cochains in dimension n to be generated by the basis dual to the cellular basis for $C_n(X)$, but we will use the same notation. That is, an n -cell α when considered as a generator of $C^n(X)$ has value 1 on α considered as a generator of $C_n(X)$, and value 0 on other cells generating $C_n(X)$. With this identification, the coboundary map $\delta : C^n(X) \rightarrow C^{n+1}(X)$ is given by

$$\delta(\alpha) = \sum_{n+1 \text{ cells } \beta} [\beta : \alpha]\beta.$$

We define $C_k^n(X)$ to be the submodule of $C^n(X) \otimes C_k(X)$ generated by $\alpha \otimes \beta$, such that $\beta \subseteq \partial\alpha$ (that is, the cell associated to β is a subset of the boundary of the cell associated to α). Then, d induces a differential $C_k^n(X) \rightarrow C_{k-1}^n(X)$ and δ induces a differential $C_k^n(X) \rightarrow C_k^{n+1}(X)$.

Definition 4 [CPS, Definition 4.1]. For each k and n , let

$$L_k^n(X) = \text{coker}(C_{k+1}^n(X) \xrightarrow{d} C_k^n(X)).$$

Then, $L_k^*(X)$ is a cochain complex with differential induced by δ . The CPS cohomology groups of X are defined by

$$H_k^n(X) = H^n(L_k^*(X)).$$

These cohomology groups are defined with coefficients in \mathbb{Z} . We write $H_k^n(X; R)$ to denote the same groups when calculated with coefficients in a commutative coefficient ring R .

Theorem 5 [CPS, Theorem 5.3]. *Let X be a pure regular cell complex of dimension d , connected by codimension-one faces. Then, the \mathbf{K} -algebra $R(X)$ is Koszul if and only if $H_k^n(X, \mathbf{K}) = 0$ for $0 \leq k < n < d$.*

For our purposes, it is convenient to present a reformulation of Theorem 5 in terms of relative cohomology groups involving the stars of the cells of X .

Definition 6. The *star* of a cell σ in a regular cell complex X is

$$\text{st}(\sigma) = \{y \in X : y \text{ is in some open cell whose closure contains } \sigma\}.$$

We note that $\text{st}(\sigma)$ is an open subset of X . We also use $\text{st}^l(\sigma)$ to denote the union of the open cell σ with all open cells in $\text{st}(\sigma)$ of dimension $\leq l$.

Theorem 7. *Let X be a pure regular cell complex of dimension d , connected by codimension-one faces. Then, the \mathbf{K} -algebra $R(X)$ is Koszul if and only if*

- (1) $\tilde{H}^n(X, \mathbf{K}) = 0$ for $n < d$,
- (2) For every k -cell σ and $k + 1 < n < d$, $H^n(X, X - \text{st}(\sigma); \mathbf{K}) = 0$.

Remark 1. Theorem 7 is a reformulation of [CPS, Corollary 5.8], where the condition $k + 1 < n$ was inadvertently omitted.

As we will see in the next section, the cohomology groups $H^n(X, X - \text{st}(\sigma))$ can be replaced by the *local homology* groups $H_n(X, X - \{x\})$ for any $x \in \sigma$ (see Lemma 10). This suggests the following definition.

Definition 8. We define the set S_n (relative to the ring of coefficients R) by $x \in S_n$ if $H_i(X, X - \{x\}; R) = 0$ for $i \leq n$ and $H_{n+1}(X, X - \{x\}; R) \neq 0$.

Now we can state and prove a proposition equivalent to Theorem 2. We will leave certain technical aspects of the proof to the subsequent two sections, as well as a more extensive discussion of the structure and significance of the sets S_n .

Proposition 9. *Let X be a pure regular cell complex of dimension d which is connected through codimension-one faces. Then, $R(X)$ is Koszul if and only if*

- (1) $\tilde{H}^n(X, \mathbf{K}) = 0$ for $n < d$
- (2) The sets S_k (relative to \mathbf{K}) are empty for $0 \leq k \leq d - 2$.

Proof. In this proof all homology and cohomology groups should be computed relative to the field \mathbf{K} . We suppress this from the notation.

We must see that condition (2) of Theorem 7 and condition (2) of Proposition 9 are equivalent under the hypotheses on X and condition (1). Suppose the sets S_k are empty for $0 \leq i \leq d - 2$. Then by Lemma 10, $H_i(X, X - \text{st}(\sigma)) = 0$ for every

cell σ and every $i < d$. The same follows by the universal coefficient theorem for $H^i(X, X - \text{st}(\sigma))$.

Conversely, assume that for some $i \leq d - 2$, the set S_i is not empty. Let m be minimal such that $S_m \neq \emptyset$. By [Lemma 14](#) and [Proposition 17](#), S_m is a union of cells and must contain a cell α of dimension $k < m$. By [Lemma 10](#), $H^{m+1}(X, X - \text{st}(\alpha))$ does not vanish, contradicting (2) of [Theorem 7](#). \square

3. Preliminary homotopy results

Let X be a regular cell complex of dimension d . If $x \in X$, we write $\sigma(x)$ for the unique open cell of X containing x . The following is a standard lemma of piecewise linear topology.

Lemma 10. *Given a cell σ , $\text{st}(\sigma)$ is contractible (and in fact has a strong deformation retract to σ). Also, given any point $x \in \sigma$, there is a strong deformation retract*

$$X - \{x\} \rightarrow X - \text{st}(\sigma(x)).$$

Proof. To see that $\text{st}(\sigma)$ has a strong deformation retract to σ , we want a homotopy

$$H : \text{st}(\sigma) \times I \rightarrow \text{st}(\sigma).$$

Of course $H|_{\sigma \times I}$ will just be the projection to σ .

Now suppose H has been defined on the subset of $\text{st}(\sigma)$ consisting of σ together with other open cells up through cells of dimension l . Since X was a regular cell complex, for each open $l + 1$ cell E of $\text{st}(\sigma)$, H is defined on a contractible subset of the boundary, $E' \subseteq \partial E$. So H is defined on $W = E \times \{0\} \cup E' \times I$. The pair $(E \times I, W)$ has the homotopy extension property (see [[Hatcher 2002](#), page 23]), so we use that to define H on $E \times I$.

To define our retract

$$X - \{x\} \rightarrow X - \text{st}(\sigma(x)),$$

we begin by noting there is a strong deformation retract $\overline{\sigma(x)} - \{x\}$ to $\overline{\sigma(x)}$. Now assume the homotopy is defined on $\text{st}^l(\sigma(x)) - \{x\}$ (and of course is the identity on $X - \text{st}(\sigma(x))$). Note that $\text{st}^0(\sigma(x)) = \sigma(x)$.

Let E be the closure of an $l + 1$ cell of $\text{st}(\sigma(x))$. Since the pair

$$((E - \{x\}) \times I, (E - \{x\}) \times \{0\} \cup (\partial E - \{x\}) \times I)$$

has the homotopy extension property, extend H across $E - \{x\}$. Continue until the homotopy is defined on all of $X - \{x\}$. \square

Corollary 11. *Given a cell σ there is a strong deformation retract*

$$X - \sigma \rightarrow X - \text{st}(\sigma),$$

such that if $x \in E$ for some cell E , then the image of $x \times I$ is in \bar{E} , and meets no cells of $\text{st}(\sigma)$ other than E .

Proof. Apply the strong deformation retract of [Lemma 10](#) to the space $X - \sigma$. \square

As an application of [Corollary 11](#) we have:

Corollary 12. *Let D be an open n -cell of X and σ a 0-cell in ∂D . Then,*

$$X - (\sigma \cup D) \simeq X - \sigma.$$

Proof. Apply the homotopy from [Corollary 11](#) to the space $X - (\sigma \cup D)$. This gives a retract of $X - (\sigma \cup D)$ to $X - (\text{st}(\sigma))$, and since that space is also a retract of $X - \sigma$, we get $X - (\sigma \cup D) \simeq X - \sigma$. \square

Proposition 13. *Let X be the realization of a simplicial complex Δ , and let A be a closed i -simplex in Δ' (the first barycentric subdivision of Δ). Let v be the vertex that A shares with an i -simplex of Δ . Then,*

$$X - \{v\} \simeq X - A.$$

Proof. In the complex given by Δ we can construct the deformation retract of $X - \{v\}$ to $X - \text{st}(v)$ (where $\text{st}(v)$ is defined using the simplicial complex Δ)

$$H : (X - \{v\}) \times I \rightarrow X - \{v\}$$

explicitly by using barycentric coordinates in each simplex of Δ .

Specifically, if σ is a simplex of Δ not containing v , then $H(p, t) = p$ for $p \in \sigma$.

If σ does contain v , let the vertices of σ be $v = v_0, v_1, \dots, v_k$. Then a typical point of $\sigma - \{v\}$ is given by $sv_0 + (1-s) \sum_{i=1}^k a_i v_i$, where $\sum_{i=1}^k a_i = 1$. Then,

$$H\left(sv_0 + (1-s) \sum_{i=1}^k a_i v_i, t\right) = (1-t)sv_0 + (1-s+ts) \sum_{i=1}^k a_i v_i.$$

Applying this homotopy to $X - A$ gives a deformation retract to $X - \text{st}(v)$. So $X - \{v\} \simeq X - A$. \square

4. Singularities detected by local homology

The singular sets S_n are composed of cells of dimension less than or equal to n . We continue to assume that X is a finite regular cell complex of dimension d . Throughout this section we assume further that X is pure. Recall that we refer to $H_*(X, X - x)$ as the local homology at x . Since X is locally contractible, we can choose a contractible neighborhood of x , say U . Then, by excision, we have

$$H_*(X, X - \{x\}) \cong H_*(U, U - \{x\}) \cong \tilde{H}_{*-1}(U - \{x\}).$$

From this we see that any x in the interior of a d -cell of X has local homology $H_*(X, X - x) \cong \tilde{H}(S^d)$ and $x \in S_{d-1}$. Similarly, if x is on the boundary of exactly one d -cell then $H_*(X, X - x) = 0$ and x is none of the sets S_k . So if we think of X as a singular manifold with boundary, the point with neighborhoods homeomorphic to \mathbb{R}^d or the corresponding half-space is not in S_k when $k < d - 1$. The sets S_i , $0 \leq i \leq d - 2$, form a stratification of those singularities of X that are detected by local homology.

Of course it is also possible for X to be topologically singular and still have local homology zero in dimensions below d at every point. A simple but illustrative example (for $d = 1$) is the space consisting of three copies of the unit interval identified at one end point:

$$([0, 1] \times \{a, b, c\}) / \{(0, a) \sim (0, b) \sim (0, c)\}.$$

The identification point is a singular point and has no local homology below dimension one. This singularity is still detected by local homology of course, but not until dimension one.

As is well known, there are also spaces with singularities so that all the local homology groups are those of a manifold. A standard source of examples is the suspension of any homology sphere which isn't actually a sphere.

We begin by showing that the sets S_n put restrictions on the cell structure of X . Recall first that S_n does not depend on the cellular structure of X ([Definition 8](#)). Nevertheless, we have the following.

Lemma 14. *The set S_n is a union of open cells (in any cell structure on X).*

Proof. If x is in some open cell D , then $X - D \simeq X - \{x\}$ by [Lemma 10](#) together with [Corollary 11](#). So applying the same argument to $x' \in D$ and using the appropriate long exact sequences, we get

$$H_*(X, X - \{x\}) \cong H_*(X, X - D) \cong H_*(X, X - \{x'\}). \quad \square$$

Lemma 15. *If $x \in S_n$ for $n < d - 1$, then x must be in the interior of a cell of dimension n or lower.*

Note that this fact depends on X being pure. For example, If we take X to be the union of a two cell and a one cell at a vertex, then points in the interior of the one cell will be in S_0 . Geometrically, $x \in S_n$ says that if we take a contractible neighborhood of x and remove x from that neighborhood then the resulting set is no longer n -connected.

Proof. Recall $st^k(\sigma)$ is the union of σ and the open cells of dimension k and lower that are contained in $st(\sigma)$. This is the same as $st(\sigma)$ within the space $X^{(k)}$ if σ is a cell of dimension less than or equal to k .

Suppose x is in the interior of a cell of dimension $k < d$. We have a commutative square of spaces

$$\begin{array}{ccc} X^{(k+1)} - \{x\} & \longrightarrow & X^{(k+1)} - \text{st}^{(k+1)}(\sigma(x)) \\ \downarrow & & \downarrow \\ X - \{x\} & \longrightarrow & X - \text{st}(\sigma(x)). \end{array}$$

The horizontal maps are homotopy equivalences by [Lemma 10](#). The spaces on the right are subcomplexes of X , and the right hand vertical map is inclusion of the $k+1$ -skeleton. So, by cellular approximation, all maps induce isomorphisms in H_i for $i < k+1$.

From the long exact sequence of a pair, it follows that

$$H_i(X^{(k+1)}, X^{(k+1)} - \{x\}) \rightarrow H_i(X, X - \{x\})$$

is an isomorphism for $i \leq k$. Now let $U = \text{st}^{(k+1)}(\sigma(x))$, which is an open neighborhood of x in $X^{(k+1)}$. U consists of the open k -cell containing x and any open $k+1$ cells that have that k -cell as a face. So, U looks like a finite collection of $k+1$ -cells identified along part of their boundary, and x is in that part of the common boundary. It follows that $U - \{x\}$ is homotopy equivalent to a wedge of k -spheres (one fewer than the number of $k+1$ -cells attached to $\sigma(x)$).

So,

$$H_i(X^{(k+1)}, X^{(k+1)} - \{x\}) = H_i(U, U - \{x\})$$

is 0 for $i < k+1$ (and is free abelian on one fewer generator than the number of $k+1$ -cells attached to $\sigma(x)$ for $i = k+1$).

It follows that x is not in S_n for $n < k$. □

See the appendix for examples where S_n contains the interiors of cells of dimension strictly smaller than n .

The implications of connectivity by codimension-one faces. We have already assumed the global topological condition: X is pure. Our final goal is to understand the effect of the extra global topological condition: connected by codimension-one faces. Under that condition we can prove a remarkable strengthening of [Lemma 15](#) (see [Proposition 17](#) and its Corollary). We require one technical lemma.

Lemma 16. *Let X be a pure regular cell complex of dimension d . Let $n < d$. Suppose $S_0 = \cdots = S_{n-1} = \emptyset$, $\tilde{H}_k(X) = 0$ for $k < d$, and D is an open n -cell of S_n with $S_n \cap \partial D = \emptyset$. Let $Y = X - D$.*

Let $A \subseteq \partial D$ be a subspace homeomorphic to D^i (the closed i disk) and also a subcomplex of ∂D under some cell structure on ∂D which subdivides the given cell structure.

Then, $\tilde{H}_j(Y - A) = 0$ for $j < n+1-i$.

Proof. The proof is by a double induction with the outer induction on i and the inner induction on the number of i -cells in A , which we'll denote by r .

Let i be 0. Note that $r = 1$ by our hypotheses that $A \cong D^0$. Then by [Corollary 12](#), $Y - A = X - (A \cup D) \simeq X - A$. Since A is a single point in ∂D , and by hypothesis that point isn't in $S_0 \cup \cdots \cup S_n$, we have $\tilde{H}_j(X - A) = 0$ for $j < n + 1$.

Now suppose the lemma is established for $i - 1 \geq 0$. Consider first the following special case. Subdivide the cell complex structure on ∂D so that it is a simplicial complex. Then take the first barycentric subdivision of that simplicial complex. Let A be the closure of an i -cell in that complex, so $A \cong D^i$.

Let v be the vertex that A shares with the i -simplex (before subdivision) that A is part of. By [Proposition 13](#), $Y - A \simeq Y - \{v\}$, which is in turn homotopy equivalent to $X - \{v\}$ by the previous case. So $\tilde{H}_j(Y - A) = 0$ for $j < n + 1$.

Now let A be as in the hypotheses, with the additional assumption that it is a subcomplex of a barycentric subdivision of a simplicial subdivision of ∂D , as in the special case above. Suppose A has $r + 1$ cells, and that the lemma is true in the case of r cells. Write $A = A' \cup A''$ where A' is a single cell, A'' has r -cells, and $A' \cap A''$ is homeomorphic to D^{i-1} .

We look at the Mayer–Vietoris sequence for

$$Y - (A' \cap A'') = (Y - A') \cup (Y - A''),$$

which gives

$$\begin{aligned} H_{j+1}(Y - A') \oplus H_{j+1}(Y - A'') &\rightarrow H_{j+1}(Y - (A' \cap A'')) \rightarrow H_j(Y - A) \\ &\rightarrow H_j(Y - A') \oplus H_j(Y - A''). \end{aligned}$$

By our two inductive hypotheses (on $i - 1$ and r), we have $H_{j+1}(Y - (A' \cap A'')) = 0$ for $j + 1 < n + 1 - (i - 1)$ (or $j < n + 1 - i$), and $H_j(Y - A'') = H_j(Y - A') = 0$ for $j < n + 1 - i$.

It follows that $H_j(Y - A) = 0$ for $j < n + 1 - i$. By induction on r , this holds for any A which is an appropriate subcomplex of our subdivision (of the cell structure on ∂D).

Finally, if $A \subseteq \partial D$ is any appropriate subcomplex of a subdivision of ∂D so that $A \cong D^i$, then A is also an appropriate subcomplex of a finer subdivision of ∂D which is itself a barycentric subdivision of a simplicial complex. So our special case covers this subcomplex A of ∂D . \square

Proposition 17. *Let X be a complex as above. In addition, assume that $\tilde{H}_i(X) = 0$ for $i < d$, and that X is connected through codimension-one faces. If there is an $n < d - 1$ so that $S_n \neq \emptyset$, then there is some point in some such S_n which is in an open cell of dimension smaller than n .*

Proof. Let n be minimal so that $S_n \neq \emptyset$. If there is no such n , or if $n \geq d - 1$, we're done. So assume $n < d - 1$. By Lemmas 14 and 15 S_n must contain an open cell D of dimension n at most. If D has dimension less than n , then we are done. So assume D has dimension n . Let $Y = X - D$. From the hypothesis $\tilde{H}_k(X) = 0$ for $k < d$, we get $H_n(Y) = H_{n+1}(X, Y) \neq 0$.

We wish to prove that $S_n \cap \partial D \neq \emptyset$. Choose a sequence of subsets A^i, B^i , $i = 0, \dots, n - 1$ subcomplexes of ∂D (or of some subdivision) so that

- $A^{n-1} \cup B^{n-1} = \partial D \cong S^{n-1}$,
- $A^i \cup B^i \cong S^i$,
- $A^i \cap B^i = A^{i-1} \cup B^{i-1}$.

Notice that A^0 and B^0 are distinct singleton sets.

Assume $S_n \cap \partial D = \emptyset$. Consider

$$(18) \quad Y = (Y - A^0) \cup (Y - B^0).$$

The space $Y - A^0 \simeq X - A^0$ by Corollary 12, so since the point of A^0 is not in $S_0 \cup \dots \cup S_n$, we get $\tilde{H}_j(Y - A^0) = 0$ for $j \leq n$, and of course the same result for $\tilde{H}_j(Y - B^0)$.

Then, in the Mayer–Vietoris sequence for (18), we get

$$H_n(Y) \cong H_{n-1}(Y - (A^0 \cup B^0)).$$

We do a similar analysis for

$$(19) \quad Y - (A^0 \cup B^0) = (Y - A^1) \cup (Y - B^1).$$

We have $\tilde{H}_j(Y - A^1) = 0$ for $j < n + 1 - 1 = n$ by Lemma 16.

Then, in the Mayer–Vietoris sequence for (19), we get

$$H_{n-1}(Y - (A^0 \cup B^0)) \cong H_{n-2}(Y - (A^1 \cup B^1)).$$

Similarly, we have

$$(20) \quad Y - (A^{k-1} \cup B^{k-1}) = (Y - A^k) \cup (Y - B^k).$$

Lemma 16 tells us that $\tilde{H}_j(Y - A^k) = 0$ for $j < n + 1 - k$ (and a similar result for B^k).

So, by the Mayer–Vietoris sequence for (20), we get

$$\tilde{H}_{n-k}(Y - (A^{k-1} \cup B^{k-1})) \cong \tilde{H}_{n-k-1}(Y - (A_k \cup B_k)).$$

Assembling this information, we get

$$0 \neq H_n(Y) = \tilde{H}_0(Y - (A^{n-1} \cup B^{n-1})) = \tilde{H}_0(X - \bar{D}).$$

The hypothesis that X is connected through codimension-one faces tells us that $\tilde{H}_0(X - \bar{D}) = 0$ unless (possibly) D has codimension 1. But D was assumed to have dimension $n < d - 1$, so we have a contradiction to our assumption that $S_n \cap \partial D = \emptyset$. \square

Corollary 21. *Suppose X is a pure regular cell complex of dimension d , connected through codimension-one faces, and with $\tilde{H}_i(X) = 0$ for $i < d$.*

If for each $0 \leq i < d - 1$, S_i contains no cells of dimension less than i , then for $0 \leq i < d - 1$, each S_i is empty.

Appendix: Examples of singularities

As is clear from the above, $x \in S_n$ does not determine the dimension of the open cell containing x . For example, S_{d-1} contains all open d cells and all interior open $d - 1$ cells. Similarly, the 3-dimensional complex X of [Example 22](#) below has points $x \in S_1$ so that x is in an open 1-cell, and also at least one $x \in S_1$ which is in an open 0-cell.

Below we give examples of contractible cell complexes connected through codimension-one faces, where (regardless of the chosen cell structure) the singular set S_r , for some r , is composed of cells of varying dimensions.

Example 22. Let T_1 be a 3-simplex with vertices $\{v_0, \dots, v_3\}$ and T_2 be a 3-simplex with vertices $\{w_0, \dots, w_3\}$. Define X by

$$(T_1 \sqcup T_2) / \sim,$$

where the relation \sim is given by identifying the 2-simplex spanned by $\{v_0, v_1, v_2\}$ linearly (preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2\}$, and by identifying the 1-simplex spanned by $\{v_0, v_3\}$ with that spanned by $\{w_0, w_3\}$ (again preserving the order of simplices).

If we just made the first identification we would have something homeomorphic to a 3-disk. With both identifications, we have something homotopy equivalent to a 3-disk since it is a homotopy equivalence to identify the contractible subcomplex consisting of the closed 1-cell that is the image of the 1-simplex spanned by $\{v_0, v_3\}$ to a point.

In this space, S_1 is the open 1-cell that is the image of the open 1-simplex spanned by $\{v_0, v_3\}$ (or equivalently by $\{w_0, w_3\}$) together with the image of v_0 . The point that is the image of v_0 will be a 0-cell in any cell structure on X , so that S_1 (in this example) will always have points belonging to 0-cells.

Example 23. We can mimic [Example 22](#) in higher dimensions. For example to create a space of dimension 4 such that S_1 must necessarily contain both points of 1-cells and 0-cells, we can define X as follows. Let T_1 be a 4-simplex with vertices

$\{v_0, \dots, v_4\}$, and let T_2 be a 4-simplex with vertices $\{w_0, \dots, w_4\}$.

$$X = (T_1 \sqcup T_2) / \sim,$$

where the relation \sim is given by linearly identifying the 3-simplex spanned by $\{v_0, v_1, v_2, v_3\}$ (while preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2, w_3\}$, and by identifying the 1-simplex spanned by $\{v_0, v_4\}$ with that spanned by $\{w_0, w_4\}$ (again preserving the order of simplices).

Then S_1 is analogous to the previous example; it contains the image of the 1-simplex spanned by $\{v_0, v_4\}$ together with the image of v_0 . $S_2 = \emptyset$. As in the previous example, the point that is the image of v_0 will be a 0-cell in any cell structure on X , and the rest of the points of S_1 will be in 1-cells or 0-cells.

Example 24. We can also create a complicated S_2 .

Let T_1 be a 4-simplex with vertices $\{v_0, \dots, v_4\}$, and let T_2 be a 4-simplex with vertices $\{w_0, \dots, w_4\}$.

$$X = (T_1 \sqcup T_2) / \sim,$$

where the relation \sim is given by linearly identifying the 3-simplex spanned by $\{v_0, v_1, v_2, v_3\}$ (while preserving the order of the vertices) with that spanned by $\{w_0, w_1, w_2, w_3\}$, and by identifying the 2-simplex spanned by $\{v_0, v_1, v_4\}$ with that spanned by $\{w_0, w_1, w_4\}$ (preserving the order of simplices).

Then $S_1 = \emptyset$, but S_2 consisted of the open 2-cell that is the image of the simplex $\{v_0, v_1, v_4\}$ together with the open 1-cell that is the image of the simplex $\{v_0, v_1\}$. This subset of X will be a union of a nonzero number of open 2-cells and a nonzero number of open 1-cells in any cell complex on X .

Example 25. It is also possible to create a complex X of dimension 4, where S_2 will be a union of a nonzero number of 2-cells, a nonzero number of 1-cells and a nonzero number of 0-cells for any cell structure on X .

We begin by creating the subcomplex most of which will become S_2 . We will glue three 2-simplices together along a common edge. So let A be the simplicial complexes with vertices a, b, c, d, e , 2-simplices $\{a, d, e\}$, $\{b, d, e\}$, $\{c, d, e\}$. This determines the 1-simplices, and, thus, the 1-simplex common to the three 2-simplices is $\{d, e\}$. This is illustrated by the three shaded 2-simplices in [Figure 1](#).

Next we attach three 4-simplices to A by attaching adjacent 2-faces (sharing a 1-face) of each 4-simplex to pairs of 2-simplices in A . Let T_1 be the 4-simplex with vertices $\{u_0, \dots, u_3\}$, let T_2 have vertices $\{v_0, \dots, v_3\}$ and let T_3 have vertices $\{w_0, \dots, w_3\}$.

We identify the 2-face spanned by u_0, u_1, u_2 with the simplex e, d, a , and the 2-face spanned by u_0, u_1, u_3 with the simplex e, d, c (preserving the given order in both cases).

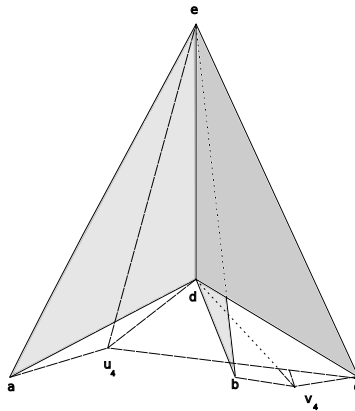


Figure 1. A with $\{u_0, \dots, u_3\}$ and $\{v_0, \dots, v_3\}$ attached.

Similarly for v_0, v_1, v_2 we use e, d, c and for v_0, v_1, v_3 we use e, d, b . Finally, for w_0, w_1, w_2 we use e, d, b and for w_0, w_1, w_3 we use e, d, a . We sketch part of this complex in Figure 1, but note that we have no realistic way to sketch the 4-simplices involved, so we are only showing a two dimensional sketch of a three dimensional picture of the space.

Of course this is contractible, and S_2 consists of the three open 2-simplices from A together with the open 1-simplex e, d . But this space is not connected through codimension-one faces. To fix this, we add a last 4-simplex with vertices $\{x_0, \dots, x_4\}$. We identify the face with vertices x_0, \dots, x_3 with u_1, u_2, u_3, u_4 , the face x_0, x_2, x_3, x_4 with v_1, v_2, v_3, v_4 and the face x_0, x_1, x_3, x_4 with w_1, w_3, w_2, w_4 .

We'll call the resulting space X . X is now dimension 4, contractible and connected through codimension-one faces. $S_0 = S_1 = \emptyset$ and S_2 is the union of the open 2-cells of A together with the open 1-cells $\{a, d\}$, $\{b, d\}$, $\{c, d\}$ and the 0-cell $\{d\}$. In any cell structure on X , $\{d\}$ will be a zero cell, and all but finitely many points of the open 1-cells we just listed will be in open 1-cells. Of course almost all points in the interiors of the 2-cells listed will be in open 2-cells.

Acknowledgements

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