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**A COMPLETELY POSITIVE MAP
ASSOCIATED WITH A POSITIVE MAP**

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We show that each positive map from $B(K)$ to $B(H)$ is a scalar multiple of a map of the form $\text{Tr} - \psi$ with ψ completely positive. This is used to give necessary and sufficient conditions for maps to be \mathcal{C} -positive for a large class of mapping cones; in particular, we apply the results to k -positive maps.

Introduction

In [Skowronek and Størmer 2010], we studied several norms on positive maps from $B(K)$ into $B(H)$, where K and H are finite-dimensional Hilbert spaces. These norms were very useful in the study of maps of the form $\text{Tr} - \lambda\psi$, where Tr is the usual trace on $B(K)$, $\lambda > 0$, and ψ is a completely positive map of $B(K)$ into $B(H)$. Herein we shall see that every positive map is a positive scalar multiple of a map of the above form with $\lambda = 1$; hence the results in that reference are applicable to all positive maps. In particular, they yield a simple criterion for some maps to be k -positive but not $(k + 1)$ -positive. As an illustration, we give a new proof that the Choi map of $B(\mathbb{C}^3)$ into itself is atomic, that is, not the sum of a 2-positive and a 2-copositive map.

\mathcal{C} -positive maps

Let K and H be finite-dimensional Hilbert spaces. We denote by $B(B(K), B(H))$ (resp. $B(B(K), B(H))^+$) the linear (resp. positive linear) maps of $B(K)$ into $B(H)$. In the case $K = H$, we write $P(H) = B(B(H), B(H))^+$. Following [Størmer 1986], we say that a closed cone $\mathcal{C} \subset P(H)$ is a *mapping cone* if $\alpha \circ \phi \circ \beta \in \mathcal{C}$ for all $\phi \in \mathcal{C}$ and $\alpha, \beta \in CP$ — the completely positive maps in $P(H)$. A map ϕ in $B(B(K), B(H))$ defines a linear functional $\tilde{\phi}$ on $B(K) \otimes B(H)$, identified with $B(K \otimes H)$ in the sequel, by $\tilde{\phi}(a \otimes b) = \text{Tr}(\phi(a)b^t)$, where Tr is the usual trace on $B(H)$ and t denotes the transpose. Let $P(B(K), \mathcal{C})$ denote the closed cone

$$P(B(K), \mathcal{C}) = \{a \in B(K \otimes H) : \iota \otimes \alpha(a) \geq 0 \text{ for all } \alpha \in \mathcal{C}\},$$

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where ι denotes the identity map on $B(K)$. Then a map $\phi \in B((B(K), B(H)))$ is said to be \mathcal{C} -positive if $\tilde{\phi}$ is positive on $P(B(K), \mathcal{C})$. We denote by $\mathcal{P}_{\mathcal{C}}$ the cone of \mathcal{C} -positive maps.

If (e_{ij}) is a complete set of matrix units for $B(K)$, then the *Choi matrix* for a map ϕ is

$$C_{\phi} = \sum e_{ij} \otimes \phi(e_{ij}) \in B(K \otimes H).$$

By [Størmer 2008; 2009], the transpose C_{ϕ}^t of C_{ϕ} is the density operator for $\tilde{\phi}$, and by [Choi 1975], ϕ is completely positive if and only if $C_{\phi} \geq 0$ if and only if $\tilde{\phi} \geq 0$ as a linear functional on $B(K \otimes H)$. When $\mathcal{C} = CP$, we have $P(B(K), CP) = B(K \otimes H)^+$, so ϕ is CP -positive if and only if ϕ is completely positive.

If $\mathcal{C}_1 \subset \mathcal{C}_2$ are two mapping cones on $B(H)$, then $P(B(K), \mathcal{C}_1) \supset P(B(K), \mathcal{C}_2)$, because if $\iota \otimes \alpha(a) \geq 0$ for all $\alpha \in \mathcal{C}_2$, then the same inequality holds for all $\alpha \in \mathcal{C}_1$. Thus $\tilde{\phi} \geq 0$ on $P(B(K), \mathcal{C}_1)$ implies $\tilde{\phi} \geq 0$ on $P(B(K), \mathcal{C}_2)$, so $\mathcal{P}_{\mathcal{C}_1} \subset \mathcal{P}_{\mathcal{C}_2}$.

Let \mathcal{C} be a mapping cone on $B(H)$. Let $\mathcal{P}_{\mathcal{C}}^o$ be the *dual cone* of $\mathcal{P}_{\mathcal{C}}$ defined as

$$\mathcal{P}_{\mathcal{C}}^o = \{\phi \in B(B(K), B(H)) : \text{Tr}(C_{\phi} C_{\psi}) \geq 0 \text{ for all } \psi \in \mathcal{P}_{\mathcal{C}}\}.$$

Thus, if $\mathcal{C}_1 \subset \mathcal{C}_2$ then $\mathcal{P}_{\mathcal{C}_1}^o \supset \mathcal{P}_{\mathcal{C}_2}^o$. In the particular case when $\mathcal{C} \supset CP$, we thus get $\mathcal{P}_{\mathcal{C}}^o \subset \mathcal{P}_{CP}^o = CP(K, H)$ — the completely positive maps of $B(K)$ into $B(H)$.

As in [Skowronek and Størmer 2010], \mathcal{C} defines a norm on $B(B(K), B(H))$ by

$$\|\phi\|_{\mathcal{C}} = \sup\{|\text{Tr}(C_{\phi} C_{\psi})| : \psi \in \mathcal{P}_{\mathcal{C}}^o, \text{Tr}(C_{\psi}) = 1\}.$$

In the special case when $\mathcal{C} \supset CP$, it follows that

$$\|\phi\|_{\mathcal{C}} = \sup |\rho(C_{\phi})|,$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_{ψ} with $\psi \in \mathcal{P}_{\mathcal{C}}^o$. Let $\phi \in B(B(K), B(H))$ be a self-adjoint map, that is, $\phi(a)$ is self-adjoint for a self-adjoint. Then C_{ϕ} is a self-adjoint operator, and so is a difference $C_{\phi}^+ - C_{\phi}^-$ of two positive operators with orthogonal supports. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_{\phi}$. Then $c = \|C_{\phi}^+\|$. Hence, if $c \neq 0$, there exists a map $\phi_{cp} \in B(B(K), B(H))$ such that the Choi matrix for ϕ_{cp} equals $1 - c^{-1}C_{\phi}$, which is a positive operator. Thus, if we let Tr denote the map $x \mapsto \text{Tr}(x)1$, then ϕ_{cp} is completely positive, and $c^{-1}\phi = \text{Tr} - \phi_{cp}$, since $C_{\text{Tr}} = 1$, as is easily shown. Combining this discussion with [Skowronek and Størmer 2010, Proposition 2], we get the following theorem.

Theorem 1. *Let ϕ be a self-adjoint map of $B(K)$ into $B(H)$. Then if $-\phi$ is not completely positive, we have:*

(i) *There exists a completely positive map $\phi_{cp} \in B(B(K), B(H))$ such that*

$$\|C_{\phi}^+\|^{-1}\phi = \text{Tr} - \phi_{cp}.$$

(ii) If \mathcal{C} is a mapping cone on $B(H)$ containing CP , then ϕ is \mathcal{C} -positive if and only if

$$1 \geq \|\phi_{cp}\|_{\mathcal{C}} = \sup \rho(C_{\phi_{cp}}),$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_{ψ} with $\psi \in \mathcal{P}_{\mathcal{C}}^o$.

We did not need to take the absolute value of $\rho(C_{\phi_{cp}})$ because $C_{\phi_{cp}} \geq 0$ and $\psi \in \mathcal{P}_{\mathcal{C}}^o \subset CP$.

We next spell out the theorem for some well-known mapping cones. Recall that a map ϕ is *decomposable* if $\phi = \phi_1 + \phi_2$ with ϕ_1 completely positive and ϕ_2 copositive, that is, $\phi_2 = t \circ \psi$ with ψ completely positive. Also recall that a state ρ on $B(K \otimes H)$ is a *PPT-state* if $\rho \circ (t \otimes t)$ is also a state.

Corollary 2. *Let $\phi \in B(B(K), B(H))$ be a self-adjoint map. Then we have:*

- (i) ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all separable states ρ on $B(K \otimes H)$.
- (ii) ϕ is decomposable if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all PPT-states ρ on $B(K \otimes H)$.
- (iii) ϕ is completely positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all states ρ on $B(K \otimes H)$.

Proof. (i) That ϕ is positive is the same as saying that ϕ is $P(H)$ -positive. Since the dual cone of $P(H)$ is the cone of separable states, (i) follows.

(ii) A state ρ is PPT if and only if its density operator is of the form C_{ψ} with ψ a map that is both positive and copositive [Størmer 2008, Proposition 4]. But the dual of those maps is the cone of decomposable maps [Skowronek et al. 2009]. Thus (ii) follows from the theorem.

(iii) This follows because the dual cone of the completely positive maps is the cone of completely positive maps, and the density operator for a state is positive; hence the corresponding map ψ is completely positive. \square

k -positive maps

A map $\phi \in B(B(K), B(H))$ is said to be *k -positive* if

$$\phi \otimes \iota \in B(B(K \otimes L), B(H \otimes L))^+$$

whenever L is a k -dimensional Hilbert space. The k -positive maps in $P(H)$ form a mapping cone P_k containing CP . Denote by $P_k(K, H)$ the cone of k -positive maps in $B(B(K), B(H))$. Then we have (see also [Itôh 1987]):

Lemma 3. $\mathcal{P}_{P_k} = P_k(K, H)$.

Proof. We have $P_k^o = SP_k$, the k -superpositive maps in $P(H)$, which is the mapping cone generated by maps of the form AdV defined by $AdV(a) = VaV^*$, where

$V \in B(H)$, $\text{rank } V \leq k$ [Skowronek et al. 2009]. By [Størmer 2009], the dual cone of $\mathcal{P}_{P_k^o}$ is given by

$$\mathcal{P}_{P_k^o}^o = \{ \phi \in B(B(K), B(H)) : \text{Ad } V \circ \phi \in CP(K, H) \text{ for all } V \in B(H), \text{rank } V \leq k \}.$$

By [Skowronek 2010, Theorem 3] or [Skowronek and Størmer 2010, Theorem 2], it follows that $\mathcal{P}_{P_k^o}^o = P_k(K, H)$. By [Størmer 1986, Theorem 3.6], \mathcal{P}_{P_k} is generated by maps of the form $\alpha \circ \beta$ with $\alpha \in P_k$, $\beta \in CP(K, H)$. Let $\text{Ad } V \circ \gamma$, $\text{Ad } V \in SP_k$, $\gamma \in CP(K, H)$ be a generator for $\mathcal{P}_{P_k^o}$. Then

$$\text{Tr}(C_{\alpha \circ \beta} C_{\text{Ad } V \circ \gamma}) = \text{Tr}(C_{\text{Ad } V^* \circ \alpha \circ \beta} C_\gamma) \geq 0,$$

since $\text{Ad } V^* \circ \alpha$ is completely positive because $\alpha \in P_k$ and $\text{rank } V \leq k$. Since the above inequality holds for the generators of the two cones, it follows that $\mathcal{P}_{P_k} = \mathcal{P}_{P_k^o}^o = P_k(K, H)$, completing the proof of the lemma. \square

It follows from the above description of $\mathcal{P}_{P_k^o}^o$ that the states with density operators C_ψ , $\psi \in \mathcal{P}_{P_k^o}^o$, are the same as the vector states generated by vectors in the Schmidt class $S(k)$, that is, the vectors $y = \sum_{i=1}^k x_i \otimes y_i$, $x_i \in K$, $y_i \in H$, where the x_i and y_i are not necessarily all $\neq 0$.

Theorem 4. *Let $\phi \in B(B(K), B(H))^+$. Then we have:*

- (i) ϕ is k -positive if and only if $\sup_{x \in S(k), \|x\|=1} (C_{\phi_{cp}} x, x) \leq 1$.
- (ii) Suppose $k < \min(\dim K, \dim H)$, and that there exists a unit vector $y = \sum_{i=1}^k x_i \otimes y_i \in S(k)$ such that $y \perp C_\phi y \notin X \otimes Y$, where $X = \text{span}(x_i)$, $Y = \text{span}(y_i)$. Then ϕ is not $(k+1)$ -positive.

In order to prove the theorem we first prove a lemma.

Lemma 5. *Let A be a self-adjoint operator in $B(K \otimes H)$. Suppose $y = \sum_{i=1}^k x_i \otimes y_i$ satisfies $(Ay, y) = 1$, and $Ay \notin X \otimes Y$ with X, Y as in Theorem 4. Then there exist a unit product vector $x \perp X \otimes Y$ and $s \in (0, 1)$ such that*

$$(A(sx + (1-s^2)^{1/2}y), sx + (1-s^2)^{1/2}y) > 1.$$

Proof. Because $Ay \notin X \otimes Y$, there exists a product vector $x \perp X \otimes Y$ such that $\text{Re}(x, Ay) > 0$. Let $s \in (-1, 1)$ and $t = t(s) = (1-s^2)^{1/2}$, and let f denote the function

$$f(s) = (A(sx + ty), s + ty) = s^2(Ax, x) + t^2(Ay, y) + 2st \text{Re}(Ax, y).$$

Because $(Ay, y) = 1$, we get

$$f'(0) = 2 \text{Re}(Ax, y) > 0.$$

Therefore, for $s > 0$ and near 0 we have $(A(sx + ty), s + ty) > f(0) = 1$, proving the lemma. \square

Proof of Theorem 4. (i) is a direct consequence of [Theorem 1](#), since, as noted in the proof of [Lemma 3](#), the vector states ω_x with $x \in S(k)$ generate the set of states with density operators C_ψ with $\psi \in \mathcal{P}_{P_k}^o$.

(ii) By [Theorem 1](#), we have $C_{\phi_{cp}} = 1 - \|C_\phi^+\|^{-1}C_\phi$, so that $(C_{\phi_{cp}}y, y) = 1$, using the assumption that $C_\phi y \perp y$. Furthermore, $C_{\phi_{cp}}y = y - \|C_\phi^+\|^{-1}C_\phi y$. Since $C_\phi y \notin X \otimes Y$, we have $C_{\phi_{cp}}y \notin X \otimes Y$. Thus by [Lemma 5](#), there exist a unit product vector $x \perp X \otimes Y$ and $s, t = (1 - s^2)^{1/2} > 0$ such that $(C_{\phi_{cp}}(sx + ty), sx + ty) > 1$. Since $sx + ty$ is a unit vector in $S(k + 1)$, ϕ is not $(k + 1)$ -positive by part (i), completing the proof of the theorem. \square

Example. We illustrate the above results by an application to the Choi map $\phi \in B(B(C^3), B(C^3))$ defined by

$$\phi((x_{ij})) = \begin{bmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{11} + x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{22} + x_{33} \end{bmatrix}.$$

We have $C_{t \circ \phi} = (t \otimes t)C_\phi$. So if $y = x \otimes x$ with $x = 3^{-1/2}(1, 1, 1) \in C^3$, then $(C_\phi y, y) = (C_{t \circ \phi} y, y) = 0$, and $C_\phi y \neq 0 \neq C_{t \circ \phi} y$. Hence, by [Theorem 4](#), neither ϕ nor $t \circ \phi$ is 2-positive, that is, ϕ is neither 2-positive nor 2-copositive. Since ϕ is an extremal positive map of $B(C^3)$ into itself [[Choi and Lam 1977](#)], ϕ cannot be the sum of a 2-positive and a 2-copositive map, and hence ϕ is atomic, a result first proved in [[Tanahashi and Tomiyama 1988](#)], and then extended to more general maps by others (see [[Ha 1998](#)] for references).

The Choi map ϕ also yields an example of a PPT-state on $B(C^3) \otimes B(C^3)$, which is not separable. Indeed, in [[Størmer 1982](#)] we gave an example of a positive matrix in A in $B(C^3) \otimes B(C^3)$ such that its partial transpose $t \otimes \iota(A)$ is also positive, and that $\phi \otimes \iota(A)$ is not positive. Then A cannot be of the form $\sum A_i \otimes B_i$ with A_i and B_i positive, and hence the state $\rho(x) = \text{Tr}(A)^{-1} \text{Tr}(Ax)$ is PPT but not separable. An example of a PPT state on $B(C^3) \otimes B(C^3)$ that is not separable was later exhibited in [[Horodecki 1997](#)].

References

- [Choi 1975] M. D. Choi, “Completely positive linear maps on complex matrices”, *Linear Algebra and Appl.* **10** (1975), 285–290. [MR 51 #12901](#) [Zbl 0327.15018](#)
- [Choi and Lam 1977] M. D. Choi and T. Y. Lam, “Extremal positive semidefinite forms”, *Math. Ann.* **231**:1 (1977), 1–18. [MR 58 #16512](#) [Zbl 0347.15009](#)
- [Ha 1998] K.-C. Ha, “Atomic positive linear maps in matrix algebras”, *Publ. Res. Inst. Math. Sci.* **34**:6 (1998), 591–599. [MR 2000b:46098](#) [Zbl 0963.46042](#)
- [Horodecki 1997] P. Horodecki, “Separability criterion and inseparable mixed states with positive partial transposition”, *Phys. Lett. A* **232**:5 (1997), 333–339. [MR 98g:81018](#) [Zbl 1053.81504](#)

- [Itoh 1987] T. Itoh, “ K^n -positive maps in C^* -algebras”, *Proc. Amer. Math. Soc.* **101**:1 (1987), 76–80. [MR 89f:46115](#) [Zbl 0645.46043](#)
- [Skowronek 2010] L. Skowronek, “Theory of generalized mapping cones in the finite dimensional case”, preprint, 2010. [arXiv 1008.3237](#)
- [Skowronek and Størmer 2010] L. Skowronek and E. Størmer, “Choi matrices, norms and entanglement associated with positive maps of matrix algebras”, preprint, 2010. [arXiv 1008.3126](#)
- [Skowronek et al. 2009] Ł. Skowronek, E. Størmer, and K. Życzkowski, “Cones of positive maps and their duality relations”, *J. Math. Phys.* **50**:6 (2009), Art. # 062106. [MR 2010k:46051](#) [Zbl 1216.46052](#)
- [Størmer 1982] E. Størmer, “Decomposable positive maps on C^* -algebras”, *Proc. Amer. Math. Soc.* **86**:3 (1982), 402–404. [MR 84a:46123](#) [Zbl 0526.46054](#)
- [Størmer 1986] E. Størmer, “Extension of positive maps into $B(\mathcal{H})$ ”, *J. Funct. Anal.* **66**:2 (1986), 235–254. [MR 87f:46105](#) [Zbl 0637.46061](#)
- [Størmer 2008] E. Størmer, “Separable states and positive maps”, *J. Funct. Anal.* **254**:8 (2008), 2303–2312. [MR 2009c:46083](#) [Zbl 1143.46033](#)
- [Størmer 2009] E. Størmer, “Duality of cones of positive maps”, *Münster J. Math.* **2** (2009), 299–309. [MR 2010j:46113](#) [Zbl 1191.46048](#)
- [Tanahashi and Tomiyama 1988] K. Tanahashi and J. Tomiyama, “Indecomposable positive maps in matrix algebras”, *Canad. Math. Bull.* **31**:3 (1988), 308–317. [MR 90a:46156](#) [Zbl 0679.46044](#)

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Remarks on a Künneth formula for foliated de Rham cohomology	257
MÉLANIE BERTELSON	
K -groups of the quantum homogeneous space ${}_q(n)/_q(n-2)$	275
PARTHA SARATHI CHAKRABORTY and S. SUNDAR	
A class of irreducible integrable modules for the extended baby TKK algebra	293
XUEWU CHANG and SHAOBIN TAN	
Duality properties for quantum groups	313
SOPHIE CHEMLA	
Representations of the category of modules over pointed Hopf algebras over \mathbb{S}_3 and \mathbb{S}_4	343
AGUSTÍN GARCÍA IGLESIAS and MARTÍN MOMBELLI	
(p, p) -Galois representations attached to automorphic forms on n	379
EKNATH GHATE and NARASIMHA KUMAR	
On intrinsically knotted or completely 3-linked graphs	407
RYO HANAKI, RYO NIKKUNI, KOUKI TANIYAMA and AKIKO YAMAZAKI	
Connection relations and expansions	427
MOURAD E. H. ISMAIL and MIZAN RAHMAN	
Characterizing almost Prüfer v -multiplication domains in pullbacks	447
QING LI	
Whitney umbrellas and swallowtails	459
TAKASHI NISHIMURA	
The Koszul property as a topological invariant and measure of singularities	473
HAL SADOFSKY and BRAD SHELTON	
A completely positive map associated with a positive map	487
ERLING STØRMER	
Classification of embedded projective manifolds swept out by rational homogeneous varieties of codimension one	493
KIWAMU WATANABE	
Note on the relations in the tautological ring of \mathcal{M}_g	499
SHENGMAO ZHU	