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We show that each positive map from $B(K)$ to $B(H)$ is a scalar multiple of a map of the form $\mathrm{Tr} - \psi$ with ψ completely positive. This is used to give necessary and sufficient conditions for maps to be $\mathscr{C}$-positive for a large class of mapping cones; in particular, we apply the results to k-positive maps.

Introduction

In [Skowronek and Størmer 2010], we studied several norms on positive maps from $B(K)$ into $B(H)$, where K and H are finite-dimensional Hilbert spaces. These norms were very useful in the study of maps of the form $\mathrm{Tr} - \lambda\psi$, where Tr is the usual trace on $B(K)$, $\lambda > 0$, and ψ is a completely positive map of $B(K)$ into $B(H)$. Herein we shall see that every positive map is a positive scalar multiple of a map of the above form with $\lambda = 1$; hence the results in that reference are applicable to all positive maps. In particular, they yield a simple criterion for some maps to be k-positive but not $(k+1)$-positive. As an illustration, we give a new proof that the Choi map of $B(\mathbb{C}^3)$ into itself is atomic, that is, not the sum of a 2-positive and a 2-copositive map.

$\mathscr{C}$-positive maps

Let K and H be finite-dimensional Hilbert spaces. We denote by $B(B(K), B(H))$ (resp. $B(B(K), B(H))^+$) the linear (resp. positive linear) maps of $B(K)$ into $B(H)$. In the case $K = H$, we write $P(H) = B(B(H), B(H))^+$. Following [Størmer 1986], we say that a closed cone $\mathscr{C} \subset P(H)$ is a *mapping cone* if $\alpha \circ \phi \circ \beta \in \mathscr{C}$ for all $\phi \in \mathscr{C}$ and $\alpha, \beta \in CP$ — the completely positive maps in $P(H)$. A map ϕ in $B(B(K), B(H))$ defines a linear functional $\tilde{\phi}$ on $B(K) \otimes B(H)$, identified with $B(K \otimes H)$ in the sequel, by $\tilde{\phi}(a \otimes b) = \mathrm{Tr}(\phi(a) b^t)$, where Tr is the usual trace on $B(H)$ and t denotes the transpose. Let $P(B(K), \mathscr{C})$ denote the closed cone

$$P(B(K), \mathscr{C}) = \{a \in B(K \otimes H) : \iota \otimes \alpha(a) \geq 0 \ \text{ for all } \ \alpha \in \mathscr{C}\},$$

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where ι denotes the identity map on $B(K)$. Then a map $\phi \in B((B(K), B(H))$ is said to be *$\mathscr{C}$-positive* if $\tilde{\phi}$ is positive on $P(B(K), \mathscr{C})$. We denote by $\mathscr{P}_{\mathscr{C}}$ the cone of $\mathscr{C}$-positive maps.

If (e_{ij}) is a complete set of matrix units for $B(K)$, then the *Choi matrix* for a map ϕ is

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) \in B(K \otimes H).$$

By [Størmer 2008; 2009], the transpose C_ϕ^t of C_ϕ is the density operator for $\tilde{\phi}$, and by [Choi 1975], ϕ is completely positive if and only if $C_\phi \geq 0$ if and only if $\tilde{\phi} \geq 0$ as a linear functional on $B(K \otimes H)$. When $\mathscr{C} = CP$, we have $P(B(K), CP) = B(K \otimes H)^+$, so ϕ is CP-positive if and only if ϕ is completely positive.

If $\mathscr{C}_1 \subset \mathscr{C}_2$ are two mapping cones on $B(H)$, then $P(B(K), \mathscr{C}_1) \supset P(B(K), \mathscr{C}_2)$, because if $\iota \otimes \alpha(a) \geq 0$ for all $\alpha \in \mathscr{C}_2$, then the same inequality holds for all $\alpha \in \mathscr{C}_1$. Thus $\tilde{\phi} \geq 0$ on $P(B(K), \mathscr{C}_1)$ implies $\tilde{\phi} \geq 0$ on $P(B(K), \mathscr{C}_2)$, so $\mathscr{P}_{\mathscr{C}_1} \subset \mathscr{P}_{\mathscr{C}_2}$.

Let $\mathscr{C}$ be a mapping cone on $B(H)$. Let $\mathscr{P}_{\mathscr{C}}^o$ be the *dual cone* of $\mathscr{P}_{\mathscr{C}}$ defined as

$$\mathscr{P}_{\mathscr{C}}^o = \{\phi \in B(B(K), B(H)) : \operatorname{Tr}(C_\phi C_\psi) \geq 0 \ \text{ for all } \ \psi \in \mathscr{P}_{\mathscr{C}}\}.$$

Thus, if $\mathscr{C}_1 \subset \mathscr{C}_2$ then $\mathscr{P}_{\mathscr{C}_1}^o \supset \mathscr{P}_{\mathscr{C}_2}^o$. In the particular case when $\mathscr{C} \supset CP$, we thus get $\mathscr{P}_{\mathscr{C}}^o \subset \mathscr{P}_{CP}^o = CP(K, H)$ — the completely positive maps of $B(K)$ into $B(H)$.

As in [Skowronek and Størmer 2010], $\mathscr{C}$ defines a norm on $B(B(K), B(H))$ by

$$\|\phi\|_{\mathscr{C}} = \sup\{|\operatorname{Tr}(C_\phi C_\psi)| : \psi \in \mathscr{P}_{\mathscr{C}}^o, \operatorname{Tr}(C_\psi) = 1\}.$$

In the special case when $\mathscr{C} \supset CP$, it follows that

$$\|\phi\|_{\mathscr{C}} = \sup |\rho(C_\phi)|,$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathscr{P}_C^o$. Let $\phi \in B(B(K), B(H))$ be a self-adjoint map, that is, $\phi(a)$ is self-adjoint for a self-adjoint. Then C_ϕ is a self-adjoint operator, and so is a difference $C_\phi^+ - C_\phi^-$ of two positive operators with orthogonal supports. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_\phi$. Then $c = \|C_\phi^+\|$. Hence, if $c \neq 0$, there exists a map $\phi_{cp} \in B(B(K), B(H))$ such that the Choi matrix for ϕ_{cp} equals $1 - c^{-1} C_\phi$, which is a positive operator. Thus, if we let Tr denote the map $x \mapsto \operatorname{Tr}(x)1$, then ϕ_{cp} is completely positive, and $c^{-1}\phi = \operatorname{Tr} - \phi_{cp}$, since $C_{\operatorname{Tr}} = 1$, as is easily shown. Combining this discussion with [Skowronek and Størmer 2010, Proposition 2], we get the following theorem.

Theorem 1. *Let ϕ be a self-adjoint map of $B(K)$ into $B(H)$. Then if $-\phi$ is not completely positive, we have*:

(i) *There exists a completely positive map $\phi_{cp} \in B(B(K), B(H))$ such that*

$$\|C_\phi^+\|^{-1} \phi = \operatorname{Tr} - \phi_{cp}.$$

(ii) *If $\mathscr{C}$ is a mapping cone on $B(H)$ containing CP, then ϕ is $\mathscr{C}$-positive if and only if*

$$1 \geq \|\phi_{cp}\|_{\mathscr{C}} = \sup \rho(C_{\phi_{cp}}),$$

where the sup *is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathscr{P}^o_{\mathscr{C}}$.*

We did not need to take the absolute value of $\rho(C_{\phi_{cp}})$ because $C_{\phi_{cp}} \geq 0$ and $\psi \in \mathscr{P}^o_{\mathscr{C}} \subset CP$.

We next spell out the theorem for some well-known mapping cones. Recall that a map ϕ is *decomposable* if $\phi = \phi_1 + \phi_2$ with ϕ_1 completely positive and ϕ_2 copositive, that is, $\phi_2 = t \circ \psi$ with ψ completely positive. Also recall that a state ρ on $B(K \otimes H)$ is a *PPT -state* if $\rho \circ (\iota \otimes t)$ is also a state.

Corollary 2. *Let $\phi \in B(B(K), B(H))$ be a self-adjoint map. Then we have*:

(i) *ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all separable states ρ on $B(K \otimes H)$.*

(ii) *ϕ is decomposable if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all PPT-states ρ on $B(K \otimes H)$.*

(iii) *ϕ is completely positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all states ρ on $B(K \otimes H)$.*

Proof. (i) That ϕ is positive is the same as saying that ϕ is $P(H)$-positive. Since the dual cone of $P(H)$ is the cone of separable states, (i) follows.

(ii) A state ρ is PPT if and only if its density operator is of the form C_ψ with ψ a map that is both positive and copositive [Størmer 2008, Proposition 4]. But the dual of those maps is the cone of decomposable maps [Skowronek et al. 2009]. Thus (ii) follows from the theorem.

(iii) This follows because the dual cone of the completely positive maps is the cone of completely positive maps, and the density operator for a state is positive; hence the corresponding map ψ is completely positive. □

k-positive maps

A map $\phi \in B(B(K), B(H))$ is said to be *k-positive* if

$$\phi \otimes \iota \in B(B(K \otimes L), B(H \otimes L))^+$$

whenever L is a k-dimensional Hilbert space. The k-positive maps in $P(H)$ form a mapping cone P_k containing CP. Denote by $P_k(K, H)$ the cone of k-positive maps in $B(B(K), B(H))$. Then we have (see also [Itoh 1987]):

Lemma 3. $\mathscr{P}_{P_k} = P_k(K, H)$.

Proof. We have $P_k^o = SP_k$, the k-superpositive maps in $P(H)$, which is the mapping cone generated by maps of the form AdV defined by $AdV(a) = VaV^*$, where

$V \in B(H)$, rank $V \leq k$ [Skowronek et al. 2009]. By [Størmer 2009], the dual cone of $\mathcal{P}_{P_k^o}$ is given by

$$\mathcal{P}^o_{P_k^o} = \{\phi \in B(B(K), B(H)) : Ad V \circ \phi \in CP(K, H) \text{ for all } V \in B(H), \text{rank } V \leq k\}.$$

By [Skowronek 2010, Theorem 3] or [Skowronek and Størmer 2010, Theorem 2], it follows that $\mathcal{P}^o_{P_k^o} = P_k(K, H)$. By [Størmer 1986, Theorem 3.6], $\mathcal{P}_{P_k}$ is generated by maps of the form $\alpha \circ \beta$ with $\alpha \in P_k$, $\beta \in CP(K, H)$. Let $Ad V \circ \gamma$, $Ad V \in SP_k$, $\gamma \in CP(K, H)$ be a generator for $\mathcal{P}_{P_k^o}$. Then

$$\text{Tr}(C_{\alpha \circ \beta} C_{Ad V \circ \gamma}) = \text{Tr}(C_{Ad V^* \circ \alpha \circ \beta} C_\gamma) \geq 0,$$

since $Ad V^* \circ \alpha$ is completely positive because $\alpha \in P_k$ and rank $V \leq k$. Since the above inequality holds for the generators of the two cones, it follows that $\mathcal{P}_{P_k} = \mathcal{P}^o_{P_k^o} = P_k(K, H)$, completing the proof of the lemma. □

It follows from the above description of $\mathcal{P}^o_{P_k}$ that the states with density operators C_ψ, $\psi \in \mathcal{P}^o_{P_k}$, are the same as the vector states generated by vectors in the Schmidt class $S(k)$, that is, the vectors $y = \sum_{i=1}^k x_i \otimes y_i$, $x_i \in K$, $y_i \in H$, where the x_i and y_i are not necessarily all $\neq 0$.

Theorem 4. *Let $\phi \in B(B(K), B(H))^+$. Then we have*:

(i) *ϕ is k-positive if and only if* $\sup_{x \in S(k), \|x\|=1} (C_{\phi_{cp}} x, x) \leq 1$.

(ii) *Suppose $k < \min(\dim K, \dim H)$, and that there exists a unit vector $y = \sum_{i=1}^k x_i \otimes y_i \in S(k)$ such that $y \perp C_\phi y \notin X \otimes Y$, where $X = \text{span}(x_i)$, $Y = \text{span}(y_i)$. Then ϕ is not $(k+1)$-positive.*

In order to prove the theorem we first prove a lemma.

Lemma 5. *Let A be a self-adjoint operator in $B(K \otimes H)$. Suppose $y = \sum_{i=1}^k x_i \otimes y_i$ satisfies $(Ay, y) = 1$, and $Ay \notin X \otimes Y$ with X, Y as in Theorem 4. Then there exist a unit product vector $x \perp X \otimes Y$ and $s \in (0, 1)$ such that*

$$\big(A(sx + (1-s^2)^{1/2} y), sx + (1-s^2)^{1/2} y\big) > 1.$$

Proof. Because $Ay \notin X \otimes Y$, there exists a product vector $x \perp X \otimes Y$ such that $\text{Re}(x, Ay) > 0$. Let $s \in (-1, 1)$ and $t = t(s) = (1-s^2)^{1/2}$, and let f denote the function

$$f(s) = (A(sx + ty),\ s + ty) = s^2(Ax, x) + t^2(Ay, y) + 2st\ \text{Re}(Ax, y).$$

Because $(Ay, y) = 1$, we get

$$f'(0) = 2\,\text{Re}(Ax, y) > 0.$$

Therefore, for $s > 0$ and near 0 we have $(A(sx + ty),\ s + ty) > f(0) = 1$, proving the lemma. □

Proof of Theorem 4. (i) is a direct consequence of Theorem 1, since, as noted in the proof of Lemma 3, the vector states ω_x with $x \in S(k)$ generate the set of states with density operators C_ψ with $\psi \in \mathcal{P}^o_{P_k}$.

(ii) By Theorem 1, we have $C_{\phi_{cp}} = 1 - \|C_\phi^+\|^{-1} C_\phi$, so that $(C_{\phi_{cp}} y, y) = 1$, using the assumption that $C_\phi y \perp y$. Furthermore, $C_{\phi_{cp}} y = y - \|C_\phi^+\|^{-1} C_\phi y$. Since $C_\phi y \notin X \otimes Y$, we have $C_{\phi_{cp}} y \notin X \otimes Y$. Thus by Lemma 5, there exist a unit product vector $x \perp X \otimes Y$ and $s, t = (1-s^2)^{1/2} > 0$ such that $(C_{\phi_{cp}}(sx+ty), sx+ty) > 1$. Since $sx + ty$ is a unit vector in $S(k+1)$, ϕ is not $(k+1)$-positive by part (i), completing the proof of the theorem. $\square$

Example. We illustrate the above results by an application to the Choi map $\phi \in B(B(C^3), B(C^3))$ defined by

$$\phi((x_{ij})) = \begin{bmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{11}+x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{22}+x_{33} \end{bmatrix}.$$

We have $C_{t\circ\phi} = (\iota \otimes t) C_\phi$. So if $y = x \otimes x$ with $x = 3^{-1/2}(1, 1, 1) \in C^3$, then $(C_\phi y, y) = (C_{t\circ\phi} y, y) = 0$, and $C_\phi y \neq 0 \neq C_{t\circ\phi} y$. Hence, by Theorem 4, neither ϕ nor $t \circ \phi$ is 2-positive, that is, ϕ is neither 2-positive nor 2-copositive. Since ϕ is an extremal positive map of $B(C^3)$ into itself [Choi and Lam 1977], ϕ cannot be the sum of a 2-positive and a 2-copositive map, and hence ϕ is atomic, a result first proved in [Tanahashi and Tomiyama 1988], and then extended to more general maps by others (see [Ha 1998] for references).

The Choi map ϕ also yields an example of a PPT-state on $B(C^3) \otimes B(C^3)$, which is not separable. Indeed, in [Størmer 1982] we gave an example of a positive matrix in A in $B(C^3) \otimes B(C^3)$ such that its partial transpose $t \otimes \iota(A)$ is also positive, and that $\phi \otimes \iota(A)$ is not positive. Then A cannot be of the form $\sum A_i \otimes B_i$ with A_i and B_i positive, and hence the state $\rho(x) = \mathrm{Tr}(A)^{-1}\,\mathrm{Tr}(Ax)$ is PPT but not separable. An example of a PPT state on $B(C^3) \otimes B(C^3)$ that is not separable was later exhibited in [Horodecki 1997].

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