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Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety and let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual. Let $L \subset \mathbb{P}^N$ be a linear space such that $\langle L, T_{X,x}\rangle \neq \mathbb{P}^N$ for all $x \in X_{\text{smooth}}$ and such that the lines in X meeting L do not cover X. If $x \in X$ is general, we prove that the multiplicity of X^* at a general point of $\langle L, T_{X,x}\rangle^{\perp}$ is strictly greater than the multiplicity of X^* at a general point of $L^{\perp}$. This is a strong refinement of Bertini's theorem.

1. Introduction

1.1. ***Multiplicities of the projective dual.*** Let $X \subset \mathbb{P}^N$ be an irreducible projective variety over the field of complex numbers. Let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual, let $L \subset \mathbb{P}^N$ be a linear space and H be a general hyperplane containing L. Bertini's classical theorem asserts that the tangency locus of H with X is included in $X \cap L$. Very little is known about the hyperplanes whose tangency locus with X lies outside $L \cap X$. It is tempting to think that the multiplicity in X^* of such a hyperplane is strictly larger than the multiplicity of a general hyperplane containing L. The following example shows that this is not true for every L.

Example 1.1.1. Let $X \subset \mathbb{P}^4$ be a smooth hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The variety X is a ruled surface of degree 3. Its dual is a hypersurface of degree 3 in $\mathbb{P}^{4^*}$ which does not contain any points of multiplicity higher than 2. Let L be the exceptional section of X. If $H \subset \mathbb{P}^4$ is a general hyperplane which contains L, then $H \cap X = L \cup D_1 \cup D_2$, where D_1 and D_2 are two distinct lines on X such that $D_1.D_2 = 0$ and $L.D_i = 1$ for $i = 1, 2$. As a consequence, a general point of $L^{\perp}$ is of multiplicity 2 in X^*. Now, let $D \subset X$ be a line such that $D.L = 1$ and let $x \in D$ such that $x \notin L$. The hyperplane containing L and $T_{X,x}$ is a point of multiplicity exactly 2 in X^*, that is, the multiplicity of a general point of $L^{\perp}$.

This example shows that, even for general $x \in X$, the multiplicity in X^* of a hyperplane containing L and tangent to X at x may well be equal to the multiplicity of a general hyperplane containing L. Thus, without extra hypotheses on L, it

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seems hopeless to say something about the multiplicity in X^* of special points of $L^\perp$. For this purpose, we introduce a definition:

Definition 1.1.2. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and let $L \subset \mathbb{P}^N$ be a linear space. Consider the conormal diagram

$$\begin{array}{ccc} & I(X/\mathbb{P}^N) := \overline{\{(H,x) \in \mathbb{P}^{N*} \times X_{\text{smooth}} : T_{X,x} \subset H\}} & \\ {}^{q}\swarrow & & \searrow^{p} \\ X^* \subset \mathbb{P}^{N*} & & X \subset \mathbb{P}^N \end{array}$$

Let $F_1, \ldots, F_m$ be all the irreducible components of $q^{-1}(L^\perp)$ such that the restrictions

$$q|_{F_i} : F_i \to L^\perp$$

are surjective. The *contact locus* of L with X, which we denote by $\text{Tan}(L, X)$, is the union of the $p(F_i)$, for $1 \le i \le m$.

In the case where L is a hyperplane, the contact locus $\text{Tan}(L, X)$ is called the *tangency locus* of L with X. A *tangent hyperplane* to X is a hyperplane $H \subset \mathbb{P}^N$ such that $\text{Tan}(H, X) \neq \varnothing$.

The contact locus $\text{Tan}(L, X)$ can be thought as the variety covered by the tangency loci of general hyperplanes containing L. In case $L^\perp \not\subset X^*$, this locus is empty. We always have the inclusion

$$\overline{\{x \in X_{\text{smooth}} : T_{X,x} \subset L\}} \subset \text{Tan}(L, X),$$

but if $\dim(L) < N-1$ or if X is not smooth, the former locus can be strictly smaller than the latter. Note also that Bertini's theorem says that $\text{Tan}(L, X) \subset L \cap X$. Finally, the contact locus is well behaved. If for a general hyperplane H' containing L, we have $\dim \text{Tan}(H', X) > 0$, then

$$\text{Tan}(H \cap L, H \cap X) = H \cap \text{Tan}(L, X),$$

for any general hyperplane $H \subset \mathbb{P}^N$.

Example 1.1.3. If $X \subset \mathbb{P}^N$ is such that X^* is a hypersurface and $L = T_{X,x}$, where $x \in X$ is a general point, then $\text{Tan}(L, X) = x$.

- If $X = G(1,7) \subset \mathbb{P}^{27}$ and $L = \langle T_{X,y_1}, T_{X,y_2} \rangle$, where $y_1, y_2 \in \mathbb{G}(1,7)$ are two general points, then $\text{Tan}(L, X) = \{x \in X : T_{X,x} \subset L\}$ is a 4-dimensional quadric, the entry locus of a general point $z \in \langle y_1, y_2 \rangle$.
- If $X = G(1,4) \subset \mathbb{P}^9$ and $L = T_{X,y}$, for any $y \in X$, then $\dim \text{Tan}(L, X) > 0$, whereas $\{x \in X : T_{X,x} \subset L\} = \{y\}$.

Definition 1.1.4. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety, and let $L \subset \mathbb{P}^N$ be a linear subspace. The *shadow* of L on X, which we denote by $\mathrm{Sh}_X(L)$, is the closed variety covered by the linear spaces $M \subset X$ such that $\dim(M) = \mathrm{def}(X)+1$ and $\dim(M \cap \mathrm{Tan}(L, X)) = \mathrm{def}(X)$.

Here $\mathrm{def}(X) = \mathrm{codim}(X^*) - 1$. The shadow is also well behaved. Namely, assume that $\mathrm{def}(X) > 0$; then

$$\mathrm{Sh}_L(X) = X \iff \mathrm{Sh}_{H\cap L}(H \cap X) = H \cap X,$$

for any general hyperplane $H \subset \mathbb{P}^N$. Note also that if $x \in X$ is a general point and $L = T_{X,x}$, then $\mathrm{Sh}_L(X) \neq X$, unless X is a linear space. Indeed, if X^* is a hypersurface, this is obvious since $\mathrm{Tan}(T_{X,x}, X) = x$ for general $x \in X$. If X^* is not a hypersurface, take enough general hyperplane sections of X passing through x, so that the corresponding dual is a hypersurface.

Main Theorem 1.1.5. *Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^N$ be a linear space such that $\mathrm{Sh}_X(L) \neq X$. Then, for all $x \in X_{\mathrm{smooth}}$ such that $x \notin \mathrm{Sh}_X(L)$ and such that $\langle L, T_{X,x}\rangle \neq \mathbb{P}^N$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x}\rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing L.*

If X is the ruled cubic surface considered in Example 1.1.1 and L is the directrix of X, one notices easily that $\mathrm{Sh}_X(L) = X$. This shows that the hypothesis $\mathrm{Sh}_X(L) \neq X$ can not be withdrawn. Here is an obvious consequence of Main Theorem 1.1.5:

Corollary 1.1.6. *Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^N$ be a linear space such that $\langle L, T_{X,x}\rangle \neq \mathbb{P}^N$ for general $x \in X$, and such that the lines in X meeting L do not cover X. Then, for general $x \in X$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x}\rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing L.*

1.2. *Variety of multisecant spaces and duals.*

Definition 1.2.1. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. Let

$$S_X^{k\ 0} = \bigl\{(x_0, \dots, x_k, u) \in X \times \cdots \times X \times \mathbb{P}^N : \dim\langle x_0, \dots, x_k\rangle = k,\ u \in \langle x_0, \dots x_k\rangle\bigr\},$$

and let S_X^k be its Zariski closure in $X \times \cdots \times X \times \mathbb{P}^N$. Denote by ϕ the projection onto $\mathbb{P}^N$. The variety $S^k(X) = \phi(S_X^k)$ is the *k-th secant variety* to X.

Theorem 1.2.2 (Terracini's lemma [Zak 1993]). *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety, and let $(x_0, \dots, x_k) \in X \times \cdots \times X$, be general points. If u is general in $\langle x_0, \dots, x_k\rangle$, we have the equality*

$$\langle T_{X,x_0}, \dots, T_{X,x_k}\rangle = T_{S^k(X),u}.$$

Definition 1.2.3. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety, and let k be an integer such that $S^k(X) \neq \mathbb{P}^N$. We say that X is *dual k-defective* if $\mathrm{def}(S^k(X)) > t(S^k(X))$, where $t(S^k(X))$ is the dimension of the general fiber of the Gauss map of $S^k(X)$.

Note that when X is smooth, then dual 0-defectiveness is the classical dual defectiveness. I don't know if there exist smooth varieties which are dual k-defective for some $k \geq 1$, but which are not dual 0-defective. I believe it would be interesting to find some examples of such varieties.

Note also that the notion of dual k-defectiveness seems to be related to that of R_k regularity explored in [Chiantini and Ciliberto 2010].

Here is a consequence of the Main Theorem 1.1.5 and Terracini's lemma:

Proposition 1.2.4. *Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume moreover that for all k such that $S^k(X) \neq \mathbb{P}^N$ the variety X is not dual $k-1$-defective. Then, for any such k, we have*

$$S^k(X)^* \subset X^*_{k+1},$$

where X^*_{k+1} is the set of points which have multiplicity at least $k+1$ in X^*.

Proof. The case $k = 0$ is the definition of $S^0(X)^* = X^*$. Let $k \geq 1$ be an integer such that $S^k(X) \neq \mathbb{P}^N$, let $z \in S^{k-1}(X)$ be a general point and H be a general hyperplane containing $T_{S^{k-1}(X),z}$. Let's prove that

$$\mathrm{Tan}(H, X) = \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$$

Let $x_0, \ldots, x_{k-1}$ be k general points in $\mathrm{Tan}(H, X)$. Let z' be a general point in $\langle x_0, \ldots, x_{k-1}\rangle$, by Terracini's lemma we have

$$T_{S^{k-1}(X),z'} = \langle T_{X,x_0}, \ldots, T_{X,x_{k-1}}\rangle.$$

So $z' \in \mathrm{Tan}(H, S^{k-1}(X))$. But $\mathrm{def}(S^{k-1}(X)) = t(S^{k-1}(X))$ by hypothesis, and this implies

$$z' \in \overline{\{y \in S^{k-1}(X)_{\mathrm{smooth}} : T_{S^{k-1}(X),y} = T_{S^{k-1}(X),z}\}},$$

so that $x_0, \ldots, x_{k-1} \in \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}$.

We now prove that $\mathrm{Sh}_X(T_{S^{k-1}(X),z}) \neq X$. The argument above shows that

$$\mathrm{Tan}(T_{S^{k-1}(X),z}, X) = \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$$

Assume $\mathrm{Sh}_X(T_{S^{k-1}(X),z}) = X$. For all $x'' \in X$, there is $x' \in \{x \in X : T_{X,x} \subset T_{S^k(X),z}\}$ such that the line $\langle x'', x'\rangle$ lies in X. But since X is smooth, this line $\langle x'', x'\rangle$ lies in $T_{X,x'}$. So we have $X \subset T_{S^{k-1}(X),z}$, which contradicts the nondegeneracy.

As a consequence of Main Theorem 1.1.5, we get that for a general $x \in X$, the multiplicity in X^* of a general hyperplane containing $\langle T_{S^{k-1}(X),z}, T_{X,x}\rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing $T_{S^{k-1}(X),z}$. We apply Terracini's lemma to find that $S^k(X)^* \subset X^*_{k+1}$. This concludes the proof. $\square$

A stronger result than Proposition 1.2.4 has been stated for the first time in [Zak 2004], but no proof was given there.

In the second part of this paper we present a proof of Main Theorem 1.1.5, while in the third part we discuss some consequences and open questions.

2. Proof of the Main Theorem

When $Z \subset \mathbb{P}^N$, we denote by $\mathscr{C}_z(Z) \subset \mathbb{P}^N$ the embedded tangent cone to Z at z and if $H \subset \mathbb{P}^N$ is a hyperplane, then $[h]$ is the corresponding point in $(\mathbb{P}^N)^*$.

The proof of Main Theorem 1.1.5 is obvious if $L^\perp \not\subset X^*$. Thus, we only deal with the case where $L^\perp \subset X^*$. Moreover, we can restrict to the case where X^* is a hypersurface. Indeed, assume that X^* has codimension $p \geq 2$. Let $z \in L^\perp$ and $z_x \in \langle L, T_{X,x}\rangle^\perp$ be general points, let $M \subset \mathbb{P}^N$ be a general $\mathbb{P}^{N+1-p}$ passing through x, let $X' = M \cap X$ and $L' = M \cap L$. We have $\mathrm{Sh}_{X'}(L') \neq X'$ and $\langle T_{X',x}, L'\rangle \neq \mathbb{P}^{N+1-p}$. Moreover, we have

$$(X')^* = \pi_{M^\perp}(X^*),$$

where $\pi_{M^\perp}$ is the projection from $M^\perp$ in $\mathbb{P}^{N^*}$. Since M is general, the map $\pi_{M^\perp}$ is locally an isomorphism around z_x. Hence

$$\mathrm{mult}_z\, X^* = \mathrm{mult}_{z_x}\, X^* \iff \mathrm{mult}_{\pi_{M^\perp}(z)}(X')^* = \mathrm{mult}_{\pi_{M^\perp}(z_x)}(X')^*.$$

Finally, note that $\pi_{M^\perp}(z)$ is a general point of $(L')^\perp$ and that $\pi_{M^\perp}(z_x)$ is a general point of $\langle L', T_{X',x}\rangle^\perp$. As a consequence, it is sufficient to prove the theorem for X', whose dual is a hypersurface.

Let's start with a plan of the proof. We assume that X^* has constant multiplicity along a smooth curve $S \subset L^\perp$ passing through $\langle L, T_{X,x}\rangle^\perp$ and through a general point of $L^\perp$ and we find a contradiction. More precisely:

- We prove that the equimultiplicity of X^* along S implies that the family of the tangent cones to X^* at the points of S is flat.
- Then, we show that the flatness of the family of the tangent cones to X^* at the points of S leads to the flatness of the family of the conormal spaces of these tangent cones. As a consequence, we have $|\mathscr{C}_s(X^*)|^* \subset L$ for all $s \in S$.
- Finally, we relate the tangent cone to X^* at z to the set of tangent hyperplanes to X^* at z (when z is a smooth point of X^*; this is the reflexivity theorem [Kleiman 1986]). Using the fact that $\mathrm{Sh}_L(X) \neq X$, we deduce that $|\mathscr{C}_s(X^*)|^* \not\subset L$ for $s \in \langle L, T_{X,x}\rangle^\perp$ and thus a contradiction.

2.1. ***Normal flatness and Lagrangian specialization principle.*** Let $S \subset Z \subset \mathbb{P}^N$ be two varieties. We recall some properties of the tangent cones $\mathscr{C}_s(Z)$, $s \in S$ when Z is equimultiple along S.

Definition 2.1.1. Let $S \subset Z$ be two varieties. We say that Z is *equimultiple* along S if the multiplicity of the local ring $\mathbb{O}_{Z,s}$ is constant for $s \in S$.

Proposition 2.1.2 [Hironaka 1964, Corollary 2, p. 197]. *Let $Z \subset \mathbb{P}^N$ be a hypersurface and S a connected smooth subvariety (not necessarily closed) of Z such that Z is equimultiple along S.*

Then, for all $s \in S$, there exists an open neighborhood U of s in S containing s and a closed subscheme $\mathscr{G}(Z) \subset \mathbb{P}^N \times U$ such that the natural projection $p : \mathscr{G}(Z) \to U$ is a flat and surjective morphism whose fiber $\mathscr{G}(Z)_{s'}$ over any $s' \in U$ is $\mathscr{C}_{s'}(Z)$.

We assume that our theorem is not true, that is for general $x \in X$, the multiplicity of X^* at a general point of $\langle L, T_{X,x}\rangle^{\perp}$ is equal to the multiplicity at a general point of $L^{\perp}$.

Let $[h]$ be a general point of $\langle L, T_{X,x}\rangle^{\perp}$ and let $S \subset L^{\perp}$ be a smooth (not necessarily closed) connected curve passing through $[h]$ and through a general point of $L^{\perp}$. We apply the proposition to X^* and S. Then there exists a scheme $\mathscr{G}(X^*) \subset \mathbb{P}^{N^*} \times S$ such that the natural projection $p : \mathscr{G}(X^*) \to S$ is a flat and surjective morphism whose fiber over $s \in S$ is the tangent cone to X^* at s. Let $\Gamma(X^*) = |\mathscr{G}(X^*)|$. The induced morphism $\Gamma(X^*) \to S$ is flat and for general $s \in S$ the fiber $\Gamma(X^*)_s$ is exactly $|\mathscr{C}_s(X^*)|$.

Now we study the family of the duals of the reduced tangent cones of X^* at points of S. Applying the Lagrangian specialization principle [Lê and Teissier 1988; Kleiman 1984] to $\Gamma(X^*)$ and S, we find:

Theorem 2.1.3. *Let $S \subset X^*$ be a smooth curve such that X^* is equimultiple along S. There esists a variety $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)$ with the following properties.*

(i) *For general $s \in S$, the following equality holds in $\mathbb{P}^N \times \Gamma(X^*)_s$:*

$$I(|\mathscr{C}_s(X^*)|/\mathbb{P}^{N^*}) = I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s.$$

(ii) *The morphism $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S) \to S$ is flat and surjective.*

(iii) *For all $s \in S$, the conormal space $I(|\mathscr{C}_s(X^*)|/\mathbb{P}^{N^*})$ is a union of irreducible components of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$.*

As a consequence, the image in $\mathbb{P}^N$ of the fiber $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s$, for general $s \in S$, is $|\mathscr{C}_s(X^*)|^*$. Moreover, for any $s \in S$, the image of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$ contains $|\mathscr{C}_s(X^*)|^*$.

2.2. *Polar varieties and duals of tangent cones.* We discuss an extension of the reflexivity theorem proved in [Lê and Teissier 1988]. The main results of this section will be applied to X^* when it is a hypersurface, so we restrict our study to that case.

Definition 2.2.1. Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a linear space. The *polar variety* of Z associated to D, which we denote by $P(Z, D)$, is the closure of the set $\{z \in Z_{\text{smooth}} : D \subset T_{Z,z}\}$.

If $D = \varnothing$ (that is, D has dimension -1), then we put $P(Z, D) = Z$.

Remark 2.2.2. If Z is normal, if $u = [u_0, \dots, u_N]$ in an homogeneous system of coordinates on $\mathbb{P}^N$ and f is an equation of Z in this system then $P(Z, u)$ is given by the equations $f = 0$ and $u_0\, \partial f/\partial x_0 + \cdots + u_N\, \partial f/\partial x_N = 0$.

If Z is not normal, then all irreducible components of Z_{sing} which are of dimension $N-2$ are irreducible components of the scheme defined by $f = 0$ and $u_0\, \partial f/\partial x_0 + \cdots + u_N\, \partial f/\partial x_N = 0$, but they are not irreducible components of $P(X, u)$.

Proposition 2.2.3. *Let $Z \subset \mathbb{P}^N$ be a reduced, irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a general linear space of dimension k. Then $P(Z, D)$ is empty or of codimension $k+1$ in Z.*

We state a result of Lê and Teissier which relates the duals of the tangent cones at z of some polar varieties of Z with the tangency locus of $z^\perp$ with Z^*. See [Lê and Teissier 1988, Proposition 2.2.1]. For any $z \in Z$, recall that $\operatorname{Tan}(z^\perp, Z^*)$ is the tangency locus of z along Z^* (see conormal diagram on page 2).

Theorem 2.2.4. *Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $z \in Z$ be a point.*

(i) *The dual of $|\mathscr{C}_z(Z)|$ is a union of reduced spaces underlying (possibly embedded) components of* $\operatorname{Tan}(z^\perp, Z^*)$.

(ii) *Any irreducible component of $|\operatorname{Tan}(z^\perp, Z^*)|$ is dual to an irreducible component of $|\mathscr{C}_z(P(Z, D))|$ for general $D \in \mathbb{G}(k, N)$ and for some integer $k \in \{-1, \dots, N-2\}$.*

Remark 2.2.5. Part (ii) of the theorem has to be explained. Assume that there is an irreducible component (say T) of $|\operatorname{Tan}(z^\perp, Z^*)|$ which is not dual to an irreducible component of $|\mathscr{C}_z(Z)|$. Then, there is $k \in \{0, \dots, N-2\}$ such that for general $D \in \mathbb{G}(k, N)$, we have $z \in P(Z, D)$. Moreover, as D varies in a dense open subset of $\mathbb{G}(k, D)$, the cones $\mathscr{C}_z(P(D, Z))$ have a fixed irreducible component in common whose reduced locus is T^*.

Note also that if $z \in Z_{\text{smooth}}$ then for $k \geq 0$ and for D general in $\mathbb{G}(k, N)$, we have $z \notin P(Z, D)$. As a consequence of the (ii) of the above theorem, we find

$\mathrm{Tan}(z^{\perp}, Z^*) = T^{\perp}_{Z,z}$ for $z \in Z_{\mathrm{smooth}}$. This is the way the (obvious corollary of the) reflexivity theorem is often stated.

When $\mathrm{Tan}(z^{\perp}, Z^*)$ is irreducible, one may expect $|\mathscr{C}_z(Z)|^* = |\mathrm{Tan}(z^{\perp}, Z^*)|$. But this is not true:

Example 2.2.6. Let $X \subset \mathbb{P}^4$ be the smooth ruled surface of degree 3 considered in example 1.1.1 and let X^* its dual. The hypersurface X^* has also degree 3 and its singular locus is a $\mathbb{P}^2$, the dual of the exceptional section of X (which we denote by L). Let $C \subset L^{\perp} = X^*_{\mathrm{sing}}$ be the conic corresponding to the hyperplanes which are tangent to X along a ruling of X and let $z \in C$.

The tangent cone $\mathscr{C}_z(X^*)$ is a doubled $\mathbb{P}^3$ so that $|\mathscr{C}_z(X^*)|^* \neq \mathrm{Tan}(z^{\perp}, X)$. We also note that the scheme-theoretic tangency locus of $z^{\perp}$ along X is a line with an embedded point. The embedded point is dual to $|\mathscr{C}_z(X^*)|$ and the line is dual to $|\mathscr{C}_z(P(X^*, u))|$, for general $u \in \mathbb{P}^{4*}$.

Notations 2.2.7. Let $f : Y \to T$ be a quasiprojective morphism between quasi-projective schemes, let $T' \subset T$ be a smooth variety and let $s \in T'$ be any point. Let $Y_1, \dots, Y_m$ be the irreducible components of $f^{-1}(T')$ such that the restrictions

$$f|_{Y_i} : Y_i \to T',$$

are surjective. Define the scheme

$$\mathrm{limflat}_{\{t \to s, t \in T'\}} f^{-1}(t) := f|^{-1}_{Y_1 \cup \dots \cup Y_m}(s).$$

If $\dim(T') = 1$ and the Y_i are all reduced, this is the classical flat limit taken along a smooth curve. If $f|_{f^{-1}(T')} : f^{-1}(T') \to T'$ is flat, then

$$\mathrm{limflat}_{\{t \to s, t \in T'\}} f^{-1}(t) = f|^{-1}_{f^{-1}(T')}(s).$$

Proof of Main Theorem 1.1.5. We recall the setting for the convenience of the reader. The projective variety $X \subset \mathbb{P}^N$ is irreducible and nondegenerate. The linear space $L \subset \mathbb{P}^N$ is such that $\mathrm{Sh}_X(L) \neq X$ and $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$ for all $x \in X_{\mathrm{smooth}}$. We want to prove that for all $x \in X_{\mathrm{smooth}}$ such that $x \notin \mathrm{Sh}_X(L)$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x} \rangle$ is strictly greater than that of a general hyperplane containing L.

The result is obvious if $L^{\perp} \not\subset X^*$ and we have already seen that we can restrict to the case where X^* is a hypersurface. So we only consider the case where $L^{\perp} \subset X^*$ and X^* is a hypersurface and we assume that our result is not true. Let $x \in X_{\mathrm{smooth}}$ with $x \notin \mathrm{Sh}_X(L)$ and let $[h]$ be a general point in $\langle L, T_{X,x} \rangle^{\perp}$. By the results of the previous section, there exists a smooth (not necessarily closed) curve $S \subset L^{\perp}$ with $[h] \in S$ and a flat morphism

$$I_S(\Gamma(X^*)/\mathbb{P}^{N*} \times S) \to S,$$

whose fiber $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s$ is the conormal space of $|\mathscr{C}_s(X^*)|$, for general $s \in S$. Further, the conormal space of $|\mathscr{C}_s(X^*)|$ is included in $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$ for all $s \in S$.

Theorem 2.2.4(i) implies that

$$|\mathscr{C}_s(X^*)|^* \subset p(|q^{-1}(s)|),$$

for all $s \in S$, where p and q are as in the conormal diagram of page 2. The flatness of $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S) \to S$ gives the inclusion

$$|\mathscr{C}_{[h]}(X^*)|^* \subset p(\operatorname{limflat}_{\{s\to[h], s\in S\}} |q^{-1}(s)|).$$

By Definition 1.1.2, the right-hand side is contained in $\operatorname{Tan}(L, X) \subset L$.

Let $\mathscr{F}$ be an irreducible component of $\operatorname{Tan}(H, X)$ passing through x. By Theorem 2.2.4, there is an integer $k \in \{-1, \dots, N-2\}$ such that $|\mathscr{F}|$ is dual to an irreducible component of $|\mathscr{C}_{[h]}(P(X^*, D))|$, for general $D \in \mathbb{G}(k, N)$. Since $|\mathscr{C}_{[h]}(X^*)|^* \subset L$, we have $k \geq 0$.

Let $x_0 \in \mathscr{F}$ be a general point. Duality implies $T_{|\mathscr{C}_{[h]}(P(X^*,D))|,z} \subset x_0^\perp$ for some general z in the irreducible component of $\mathscr{C}_{[h]}(P(X^*, D))$ whose reduced locus is $|\mathscr{F}|^*$. Note that $\mathscr{C}_{[h]}(P(X^*, D)) \subset \mathscr{C}_{[h]}(X^*)$. Let $T_{|\mathscr{C}_{[h]}(X^*)|,z}$ be a limit of tangent spaces to $|\mathscr{C}_{[h]}(X^*)|$ at z. The point z is general in $|\mathscr{C}_{[h]}(P(X^*, D))|$, so $T_{|\mathscr{C}_{[h]}(P(X^*,D))|,z} \subset T_{|\mathscr{C}_{[h]}(X^*)|,z}$.

As a consequence of this, we have $T_{|\mathscr{C}_{[h]}(P(X^*,D))|,z} \subset x_0^\perp \cap T_{\mathscr{C}_{[h]}(X^*),z}$. That is,

$$\langle x_0, T^\perp_{|\mathscr{C}_{[h]}(X^*)|,z}\rangle \subset \mathscr{F} \subset X.$$

But $|\mathscr{C}_{[h]}(X^*)|^* \subset \operatorname{Tan}(L, X)$, so $T^\perp_{|\mathscr{C}_{[h]}(X^*)|,z} \in \operatorname{Tan}(L, X)$, and the inclusion above says that $x_0 \in \operatorname{Sh}_X(L)$. This is a contradiction. □

3. Corollaries and open questions

We present here some corollaries of the Main Theorem and related open questions.

3.1. ***Zak's conjecture on varieties with minimal codegree.*** Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. We recall, following Zak, that the order of X is $\operatorname{ord} X = \min\{k, S^{k-1}(X) = \mathbb{P}^N\}$ and the k-th secant-defect is $\delta_k = \dim X + \dim S^{k-1}(X) + 1 - \dim S^k(X)$, for all $k \leq \operatorname{ord} X - 1$.

Zak [1993] proved an important result related to secant defects.

Theorem 3.1.1 (Zak's superadditivity theorem). *Let $X \subset \mathbb{P}^N$ an irreducible, nondegenerate projective variety such that $\delta_1 > 0$. For all $k \leq \operatorname{ord} X - 1$, we have the inequality*

$$\delta_k \geq \delta_{k-1} + \delta_1.$$

The varieties on the boundary are called Scorza varieties. More precisely:

Definition 3.1.2. An irreducible, smooth, nondegenerate projective variety $X \subset \mathbb{P}^N$ is a Scorza variety if the following conditions hold:

(i) $\delta_1 > 0$ and $N > 2n + 1 - \delta_1$,

(ii) $\delta_k = \delta_{k-1} + \delta_1$ for all $k \leq \operatorname{ord} X - 1$,

(iii) $\operatorname{ord} X - 1 = [\dim X/\delta_1]$, where [ ] denotes the integral part.

Theorem 3.1.3 (Classification of Scorza varieties [Zak 1993]). *Any Scorza variety X is of one of the following types*:

(i) $X = v_2(\mathbb{P}^n) \subset \mathbb{P}^{n(n+3)/2}$ (2^{nd} *Veronese*) *and* $\deg X^* = n + 1$;

(ii) $X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n(n+2)}$ *and* $\deg X^* = n + 1$;

(iii) $X = \mathbb{G}(1, 2n + 1) \subset \mathbb{P}(\Lambda^2\mathbb{C}^{2n+2})$ *and* $deg(X^*) = n + 1$;

(iv) $X \subset \mathbb{P}^{26}$ *is the* 16*-dimensional variety corresponding to the orbit of highest weight vector in the lowest nontrivial representation of the group of type* E_6 *and* $\deg X^* = 3$.

In [Zak 2004] an important consequence of the assertion $S^k(X)^* \subset X^*_{k+1}$ (where X^*_k is the set of points of multiplicity at least k in X^*) was discovered. We state that result in the setting where we are able to prove it.

Proposition 3.1.4. *Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume that X is not k dual defective for $k <$* $\operatorname{ord} X - 1$, *then*

$$\deg X^* \geq \operatorname{ord} X.$$

Proof. With the assumptions above, Proposition 1.2.4 implies that there is a point of multiplicity $\operatorname{ord} X - 1$ in X^*. Since X is nondegenerate, its dual is not a cone and so $\deg X^* \geq \operatorname{ord} X$. □

If X is a Scorza variety then $\deg X^* = \operatorname{ord} X$. The converse statement in conjectured in [Zak 2004]. We formulate the conjecture in the setting where we can prove the inequality: $\deg X^* \geq \operatorname{ord} X$.

Conjecture 3.1.5 [Zak 2004]. *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate, projective variety. Assume that X is not k dual defective for all $k <$* $\operatorname{ord} X$ *and that* $\deg X^* = \operatorname{ord} X + 1$, *then X is a hyperquadric or a Scorza variety.*

It is proved in [Zak 1993], without any hypothesis on the dual defectiveness of X, that smooth varieties with $\deg(X^*) = 3$ and $\operatorname{ord} X = 3$ are Severi varieties. In particular, they are Scorza varieties. Note, however, that the smoothness assumption seems to be necessary in his proof. I believe it would be very interesting to have a classification of all varieties whose duals have degree 3.

3.2. *Varieties with unexpected equisingular linear spaces.* We come back to our usual setting. Let $L \subset \mathbb{P}^N$ be a linear space such that for all $x \in X_{\text{smooth}}$, we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We have seen in example 1.1.1 that a hyperplane containing the join $\langle L, T_{X,x} \rangle$ may have the same multiplicity in X^* as the general hyperplane containing L, even if x is a general point of X. The following definition is convenient to describe this situation.

Definition 3.2.1. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset \mathbb{P}^N$ be a linear space such that for all $x \in X_{\text{smooth}}$, we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We say that $L^{\perp}$ is an *unexpected equisingular linear space* in X^* if for all $x \in X_{\text{smooth}}$, the general hyperplane containing $\langle L, T_{X,x} \rangle$ has the same multiplicity in X^* as the general hyperplane containing L.

The variety in Example 1.1.1 is rather special since it is a scroll surface (see [Zak 2004] for interesting discussions about this variety). It is not a coincidence that the directrix of this variety is an unexpected equisingular linear space in its dual. Indeed, we have:

Theorem 3.2.2. *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with* $\dim(L) = \dim(X) - 1$. *Assume that $L^{\perp}$ is an unexpected equisingular linear space in X^* such that* $\operatorname{mult}_{L^{\perp}} X^* = 2$. *Then X is the cubic scroll surface in $\mathbb{P}^4$.*

Here $\operatorname{mult}_{L^{\perp}} X^*$ denotes the multiplicity in X^* of a general point of $L^{\perp}$. Before diving into the proof of Theorem 3.2.2, we describe the tangency locus of any point $[h] \subset X^*$, such that $\operatorname{mult}_{[h]} X^* = 2$.

Proposition 3.2.3. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that* $\operatorname{mult}_{[h]} X^* = 2$. *The scheme theoretic tangency locus of H with X is either*

(i) *an irreducible hyperquadric and in this case* $|\mathscr{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$,

(ii) *the union of two (not necessarily distinct) linear spaces, or*

(iii) *a linear space with at least one embedded component.*

We postpone the proof of this result to the Appendix.

Proof of Theorem 3.2.2. Let H be a general hyperplane containing L. We have $H \cap X = L \cup D_H$, where D_H is a divisor such that

$$D_H \cap L = \operatorname{Tan}(H, X).$$

Let $x \in X$ be a general point and let H_x be a general hyperplane containing $\langle L, T_{X,x} \rangle$. Then $\operatorname{Tan}(H_x, X)$ contains x and

$$\xi := p\big(\operatorname{limflat}_{\{[h] \to [h_x], [h] \in L^{\perp}\}} q^{-1}([h])\big).$$

By hypothesis, we have

$$\operatorname{mult}_{[h_x]} X^* = \operatorname{mult}_{[h]} X^* = 2,$$

for all $[h] \in L^{\perp}$. Proposition 3.2.3 hence implies that the irreducible component of $\operatorname{Tan}(H_x, X)$ containing x, which we denote by R_{H_x}, also contains ξ. Moreover, $\xi \subset L$, so $\dim R_{H_x} > \dim \xi$, for general $[h] \in L^{\perp}$. As a consequence, $\dim R_{H_x} = n-1$.

On the other hand, since

$$\operatorname{mult}_{[h_x]} X^* = \operatorname{mult}_{[h]} X^* = 2,$$

for all $[h] \in L^{\perp}$, we have $|\mathscr{C}_{[h_x]}(X^*)|^* \neq |R_{H_x}|$. We apply again Proposition 3.2.3 and we find that $|R_{H_x}|$ is necessarily a linear space of dimension $n-1$. Thus,

$$\dim\langle L, T_{X,x}\rangle = n+1.$$

Note that Bertini's theorem implies that

$$R_{H_x} \subset \langle L, T_{X,x}\rangle \cap X,$$

for general H_x containing $\langle L, T_{X,x}\rangle$. As a consequence R_{H_x} is an irreducible component of $\langle L, T_{X,x}\rangle \cap X$, for general H_x. Thus R_{H_x} does not depend on H_x, for general H_x containing $\langle L, T_{X,x}\rangle$. We deduce that $\langle L, T_{X,x}\rangle$ is tangent to X along a linear space of dimension $n-1$. By the theorem on tangencies, we have $n-1 \leq 1$, that is $n = 2$ (obviously, X is not a curve). So $X \subset \mathbb{P}^N$ is a nondegenerate surface containing a distinguished line L, such that for general $x \in X$, there is a $\mathbb{P}^3$ tangent to X along a line passing through x and meeting L. This means that X is the projection of a scroll of type $S_{1,d-1}$. By hypothesis, we have $\operatorname{mult}_{L^{\perp}} X^* = 2$, hence of [Ciliberto et al. 2008, Proposition 1.6] implies that $X = S_{1,2} \subset \mathbb{P}^4$. □

Appendix: Tangency loci of points of multiplicity 2 in the dual

The goal of this appendix is to prove the following proposition.

Proposition 3.2.3. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that* $\operatorname{mult}_{[h]} X^* = 2$. *The scheme theoretic tangency locus of H with X is either*

(i) *an irreducible hyperquadric and in this case* $|\mathscr{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$,

(ii) *the union of two (not necessarily distinct) linear spaces, or*

(iii) *a linear space with at least one embedded component.*

Example A.1. All three cases are encountered in nature:

(i) If $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, then for all $[h] \in v_2(\mathbb{P}^{2^*}) \subset X^*$, we have $\operatorname{mult}_{[h]} X^* = 2$ and $\operatorname{Tan}(H, X)$ is a smooth conic.

(ii) If X is a complete intersection of large multidegree and large codimension, then there are points $[h_1], [h_2] \in X^*$ such that $\mathrm{mult}_{[h_i]} X^* = 2$ and $\mathrm{Tan}(H_1, X)$ is exactly two distinct points, whereas $\mathrm{Tan}(H_2, X)$ is a single double point.

(iii) If X is the cubic scroll of Example 1.1.1, then there is a conic $C \subset X^*$, such that for all $[h] \in C$, we have $\mathrm{mult}_{[h]} X^* = 2$ and $\mathrm{Tan}(H, X)$ is a line with an embedded point.

A doubled linear space will be considered as the union of two (not distinct) linear spaces. By Theorem 2.2.4, we know that the irreducible components of $\mathrm{Tan}(H, X)$ are dual to irreducible components of the reduced spaces underlying some $\mathscr{C}_{[h]}(P(X^*, D_k))$ for general $D_k \in \mathbb{G}(k, N)$. When $\mathrm{mult}_{[h]} X^* = 2$, the cones $\mathscr{C}_{[h]}(P(X^*, D_k))$ are rather easy to describe. Let's start with some notation.

Notations A.2. Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface. Let $D \in \mathbb{G}(k, N)$ and let f_Z be an equation for Z in some coordinate system of $\mathbb{P}^N$. We denote by $P(f_Z, D)$ the subscheme of $\mathbb{P}^N$ whose ideal is generated by the equations

$$u_0 \frac{\partial f_Z}{\partial t_0} + \cdots + u_N \frac{\partial f_Z}{\partial t_N},$$

for $u = [u_0, \ldots, u_N]$ varying in D.

Let $D \in \mathbb{G}(k, N)$ be a general k-plane. Note that if $\dim(Z_{\mathrm{sing}}) < \dim P(Z, D)$ (that is $\dim Z_{\mathrm{sing}} \leq N-k-3$), then $P(Z, D) = P(f_Z, D) \cap Z$. In the other case, the irreducible components of maximal dimension of Z_{sing} are irreducible components of $P(f_Z, D) \cap Z$.

Lemma A.3. *Let $Z \subset \mathbb{P}^N$ be an irreducible and reduced hypersurface. Let $z \in Z$ and let $k \in \{-1, \ldots, N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have*

1. *$z \notin P(Z, D)$, or*
2. *$\mathrm{mult}_z P(Z, D) = \mathrm{mult}_z(Z). \mathrm{mult}_z P(f_Z, D)$, if $\dim(Z_{\mathrm{sing}}^{(z)}) < \dim P(Z, D)$, where $Z_{\mathrm{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through z, or*
3. *$\mathrm{mult}_z P(Z, D) < \mathrm{mult}_z(Z). \mathrm{mult}_z P(f_Z, D)$, if $\dim(Z_{\mathrm{sing}}^{(z)}) \geq \dim P(Z, D)$, where $Z_{\mathrm{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through z.*

Proof. If $z \in P(Z, D)$ for general $D \in \mathbb{G}(k, N)$, we will prove the lemma only in the case $P(f_Z, D)$ is smooth at z, for two reasons. The general case is obtained by the same methods, this is only more technical, and we will use the result only in the case $P(f_Z, D)$ is smooth at z.

Moreover if $z \in P(Z, D)$ for general D, we will only concentrate on the case $\dim(Z_{\mathrm{sing}}^{(s)}) < \dim P(Z, D)$. In this case, we have locally around z the equality

$P(Z, D) = P(f_Z, D) \cap Z$ for general $D \in \mathbb{G}(k, N)$. The situation where an irreducible component Z_{sing} containing z is an irreducible component of $P(f_Z, D) \cap Z$ — this is case (3) of the lemma — is dealt with exactly in the same way.

Now, we work locally around z, so that $P(f_Z, D) \cap Z = P(Z, D) \subset \mathbb{A}^N$, for general $D \in \mathbb{G}(k, N)$. Let $(Z_i)_{i \in I}$ be a stratification of Z such that Z_i is smooth and Z is normally flat along Z_i, for all $i \in I$. Such a stratification exists, due to the open nature of normal flatness (see [Hironaka 1964, Chapter II]). Consider the Gauss map $G : Z \to \mathbb{P}^{N*}$. It restricts to a map $G_i : Z_i \to \mathbb{P}^{N*}$. We have

$$P(f_Z, D) \cap Z = P(Z, D) = G^{-1}(D^{\perp}),$$

so that $P(f_Z, D) \cap Z_i = G_i^{-1}(D^{\perp})$, for all i.

Now, we apply Kleiman's transversality theorem to find that for all i and for general $D \in \mathbb{G}(k, N)$, the inverse images $G_i^{-1}(D^{\perp})$ are either empty or smooth of the expected dimension.

Let i such that z is in Z_i. If $z \notin G_i^{-1}(D^{\perp})$ for general $D \in \mathbb{G}(k, N)$, then $z \notin P(Z, D)$ and we are in the case 1 of the lemma. Otherwise, z is a smooth point of $G_i^{-1}(D^{\perp})$, so $T_{P(f_Z, D), z}$ and $T_{Z_i, z}$ are transverse.

Assume that $\mathrm{mult}_z P(Z, D) > \mathrm{mult}_z Z . \mathrm{mult}_z P(f_Z, D)$. Since $P(f_Z, D)$ is smooth at z, this implies that $T_{P(f_Z, D), z}$ and $\mathscr{C}_z(Z)$ are not transverse. In particular, the linear spaces $T_{P(f_Z, D), z}$ and $\mathrm{Vert}(\mathscr{C}_z(Z))$ are not transverse (here $\mathrm{Vert}(\mathscr{C}_z(Z))$ is the vertex of the cone $\mathscr{C}_z(Z)$). But Z is normally flat along Z_i, so we have $T_{Z_i, z} \subset \mathrm{Vert}(\mathscr{C}_z(Z))$ (see [Hironaka 1964, Theorem 2, p. 195]). This is a contradiction. □

Corollary A.4. *Let $Z \subset \mathbb{P}^N$ be a reduced, irreducible hypersurface. Let $z \in Z$ such that* $\mathrm{mult}_z Z = 2$ *and let $k \in \{-1, \ldots, N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have*

$$\mathrm{mult}_z P(Z, D) \leq 2.$$

Proof. The result is obvious for $k = -1$, since in this case $P(Z, D) = Z$. Assume that $k \geq 0$ and let $D \in \mathbb{G}(k, N)$ be a general k-plane. Let $u \in D$ be a general point ans let π_u be the projection from u. Then, the projections

$$\pi_u|_{P(Z,u)} : P(Z, u) \to \pi_u(P(Z, u))$$

and

$$\pi_u|_{P(Z,D)} : P(Z, D) \to \pi_u(P(Z, D))$$

are locally isomorphisms around z. Moreover, we have the following equality (see [Teissier 1982]):

$$\pi_u(P(Z, D)) = P(\pi_u(P(Z, u)), \pi_u(D)).$$

As a consequence, it is sufficient to prove the result for $k = 0$. But in this case, this is an obvious application of the lemma above. Indeed, for general $u \in \mathbb{P}^N$,

$$\operatorname{mult}_z P(f_Z, u) = \operatorname{mult}_z Z - 1 = 1. \qquad \square$$

We also need the following result.

Proposition A.5. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that* $\operatorname{Tan}(H, X)$ *has m components* (*some of which may be embedded components*), *then there exists $k \in \{-1, \dots, N-2\}$, such that for general $D \in G(k, N)$, we have*

$$\operatorname{mult}_{[h]} P(X^*, D) \geq m.$$

Proof. We only prove the result when $\operatorname{Tan}(H, X)$ is reduced and pure dimensional. The general case is done using the same ideas; it's just more technical.

Assume that

$$\operatorname{Tan}(H, X) = Y_1 \cup \cdots \cup Y_m,$$

where the Y_i have the same codimension, say c. Let $D \subset \mathbb{P}^{N^*}$ be a general $\mathbb{P}^{N-1-c}$. Then

$$\pi_D(P(X^*, D)) = (D^\perp \cap X)^*,$$

where π_D is the projection from D. Moreover, we have $[h] \in P(X^*, D)$ and

$$\operatorname{Tan}(D^\perp \cap H, D^\perp \cap X) = D^\perp \cap \operatorname{Tan}(H, X).$$

As a consequence, $\operatorname{Tan}(D^\perp \cap H, D^\perp \cap X)$ is a 0-dimensional scheme of degree at least m. In this case, it is clear that

$$\operatorname{mult}_{\pi_D([h])} \pi_D(P(X^*, D)) \geq m.$$

On the other hand, since D is general, the morphism

$$\pi_D : P(X^*, D) \to \pi_D(P(X^*, D))$$

is locally an isomorphism around $[h]$, so that

$$\operatorname{mult}_{[h]} P(X^*, D) \geq m. \qquad \square$$

Proof of Proposition 3.2.3. Let $T_1 \cup \cdots \cup T_m$ be the decomposition of $\operatorname{Tan}(H, X)$ into irreducible components. If $m \geq 3$, then Proposition A.5 implies that $\operatorname{mult}_{[h]}(X^*) \geq 3$, this is impossible, so that $m \leq 2$.

Assume that $m = 2$. The proof of Proposition A.5 shows that these two irreducible components are scheme-theoretically linear spaces.

Assume that $m = 1$ and let $k \in \{-1, \dots, N-2\}$ such that T_1 is dual to some irreducible components of the reduced space underlying $\mathscr{C}_{[h]} P(X^*, D)$, for general $D \in \mathbb{G}(k, N)$. By Corollary A.4, the cone $\mathscr{C}_{[h]} P(X^*, D)$ is either a hyperquadric or a linear space. Assume that it is an irreducible hyperquadric. If $k \geq 0$, we know by Theorem 2.2.4 that $|\mathscr{C}_{[h]}(X^*)|^*$ is the reduced space underlying some embedded component of $\mathrm{Tan}(H, X)$. Taking $q = \dim \mathrm{Tan}(H, X)$ general hyperplane sections of $\mathrm{Tan}(H, X)$ passing through $|\mathscr{C}_{[h]}(X^*)|^*$, we see as in the proof of Proposition A.5 that for general $D' \in \mathbb{G}(q-1, N)$, we have

$$\mathrm{mult}_{[h]}\, P(X^*, D') \geq 3.$$

This is impossible by Corollary A.4. Thus, if $\mathscr{C}_{[h]}P(X^*, D)$ is an irreducible hyperquadric, then $k = -1$, and we are in the case 1 of the proposition.

Finally, if $\mathscr{C}_{[h]}P(X^*, D)$ is a the union of two linear spaces or a unique linear space, then we are in case 2 or 3 of the proposition. This concludes the proof of Proposition 3.2.3. □

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