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EIGENVALUE ESTIMATES FOR HYPERSURFACES IN $\mathbb{H}^m \times \mathbb{R}$ AND APPLICATIONS

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In the first part of this paper, we give a lower bound for the spectrum of the Laplacian on minimal hypersurfaces immersed into $\mathbb{H}^m \times \mathbb{R}$. As an application, in dimension 2, we prove that a complete minimal surface with finite total extrinsic curvature has finite index. In the second part, we consider the operator $L = \Delta_g + a + bK_g$ on a complete noncompact surface (M^2, g). Assuming that L is nonnegative for some constants $a > 0$ and $b > 1/4$, we show that the infimum of the spectrum of M^2 is bounded from above by $a/(4b-1)$. We apply this result to stable minimal surfaces immersed into homogeneous 3-manifolds.

1. Introduction

In this paper, we mainly consider orientable, minimal hypersurfaces immersed into $\mathbb{H}^m \times \mathbb{R}$. Let ν denote a unit normal field along M and let $v = \hat{g}(\nu, \partial_t)$ be the component of ν with respect to the unit vector field ∂_t tangent to the $\mathbb{R}$-direction in the ambient space.

In Section 3, we give a lower bound of the spectrum of Δ_g which relies on the inequality $-\Delta_g b \geq (m-2) + v^2$ satisfied by a "horizontal" Busemann function b (see Theorem 3.1). As an application, in Section 4, we get a finiteness result for the index of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite total *extrinsic* curvature (answering a question raised in [Bérard and Sá Earp 2008]):

Theorem 1.1. *Let $(M^2, g) \looparrowright (\mathbb{H}^2 \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal surface with second fundamental form A. If $\int_M |A|^2 \, dv_g$ is finite, then the immersion has finite index.*

We also obtain a lower bound for the spectrum of the Laplacian on a complete minimal surface contained in a slab (Proposition 4.3).

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The stability operator of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ has the form $\Delta_g + a + bK_g$ with $a \geq 0$ and $b > \frac{1}{4}$. In Section 5A, we consider such operators on an arbitrary complete Riemannian surface, and we show that their positivity implies an upper bound on the infimum of the spectrum of Δ_g.

Theorem 1.2. *Let (M, g) be a complete noncompact Riemannian surface. Let $a \geq 0$ and $b > \frac{1}{4}$. Denote by Δ_g the (nonnegative) Laplacian and by K_g the Gaussian curvature of (M, g). Denote by $\lambda_\sigma(\Delta_g)$ the infimum of the spectrum of Δ_g and by $\lambda_e(\Delta_g)$ the infimum of the essential spectrum of Δ_g.*

(i) *If the operator $\Delta_g + a + bK_g$ is nonnegative on $C_0^\infty(M)$, then*

$$\lambda_\sigma(\Delta_g) \leq \frac{a}{4b-1}.$$

(ii) *If the operator $\Delta_g + a + bK_g$ has finite index on $C_0^\infty(M)$ and if M has infinite volume, then*

$$\lambda_e(\Delta_g) \leq \frac{a}{4b-1}.$$

We also prove that the positivity of such operators implies an upper bound on the volume growth of the surface (Proposition 5.3). In Section 5B, applying this result to stable minimal surfaces in $\mathbb{H}^3$ or in $\mathbb{H}^2 \times \mathbb{R}$, we generalize and extend results of [Candel 2007]. While Candel used Pogorelov's method [1981], we use the method of Colding and Minicozzi [2002] (see also [Castillon 2006]).

In Section 6, we give some applications of our general lower bounds on the spectrum to higher dimensional hypersurfaces. In Section 2, we provide some preliminary technical lemmas.

2. Preliminary computations

In this section, we make some preliminary computations for later reference. For the sake of simplicity, we work in the following model for the hyperbolic space $\mathbb{H}^{m+1}$:

$$\begin{cases} \mathbb{H}^{m+1} = \mathbb{R}^m \times \mathbb{R}, \\ h = e^{2s}(dx_1^2 + \cdots + dx_m^2) + ds^2 \text{ at the point } (x, s) \in \mathbb{H}^{m+1}. \end{cases} \tag{1}$$

These coordinates are called *horocyclic* because the slices $\mathbb{R}^m \times \{s\}$ are horospheres and the coordinate function s is a Busemann function. They are quite natural when some Busemann function plays a special role, as will be the case in the sequel. Let

$$\gamma_0 : [0, \infty) \to \mathbb{H}^{m+1}, \quad u \mapsto \gamma_0(u) = (0, \ldots, 0, u). \tag{2}$$

be a geodesic ray. The Busemann function (see [Ballmann et al. 1985, page 23]) associated with γ_0 is the function

$$B : \mathbb{H}^{m+1} \to \mathbb{R}, \quad (x, s) \mapsto B(x, s) = s. \tag{3}$$

In the sequel, we denote by

$$\begin{cases} D^h & \text{the Levi-Civita connection,} \\ \Delta_h & \text{the geometric (that is, nonnegative) Laplacian,} \end{cases} \tag{4}$$

for the hyperbolic metric h on $\mathbb{H}^{m+1}$.

Lemma 2.1. *With the notation above, we have the formulas*

$$\Delta_h B = -m, \tag{5}$$

$$\operatorname{Hess}_h B = e^{2s}(dx_1^2 + \cdots + dx_m^2) \tag{6}$$

at the point $(x, s) \in \mathbb{H}^{m+1}$. In particular, if we decompose the vector $u \in T_{(x,s)}\mathbb{H}^{m+1}$ h-orthogonally as $u = (u_x, u_s)$, we have

$$\operatorname{Hess}_h B(u, u) = h(u_x, u_x). \tag{7}$$

The proof is straightforward.

Recall the following general lemmas.

Lemma 2.2. *Let $(M^m, g) \looparrowright (\widehat{M}^{m+1}, \hat{g})$ be an orientable, isometric immersion with unit normal field ν and corresponding normalized mean curvature H. Let $\hat{F} : \widehat{M} \to \mathbb{R}$ be a smooth function and let $F := \hat{F}|_M$ be its restriction to M. Then, on M,*

$$\Delta_g F = \Delta_{\hat{g}} \hat{F}|_M + \operatorname{Hess}_{\hat{g}} \hat{F}(\nu, \nu) - mH d\hat{F}(\nu).$$

Proof. See for example [Choe and Gulliver 1992, Lemma 2]. □

Lemma 2.3. *Assume that the manifold (M, g) carries a function f which satisfies*

$$|df|_g \le 1 \quad \text{and} \quad -\Delta_g f \ge c \text{ for some constant } c > 0.$$

Then, any smooth, relatively compact domain $\Omega \subset M$ satisfies the isoperimetric inequalities

$$\operatorname{Vol}_{m-1}(\partial\Omega) \ge c \operatorname{Vol}_m(\Omega) \quad \text{and} \quad \lambda_1(\Omega) \ge \frac{c^2}{4},$$

where $\lambda_1(\Omega)$ is the least eigenvalue of Δ_g in Ω, with Dirichlet boundary condition.

Proof. Use integration by parts and Cauchy–Schwarz. □

3. Hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$

We consider orientable, isometric immersions $(M^m, g) \looparrowright (\widehat{M}^{m+1}, \hat{g})$ with unit normal ν, where $\widehat{M} = \mathbb{H}^m \times \mathbb{R}$ with the product metric $\hat{g} = h + dt^2$. We take the model (1) for the hyperbolic space (here with dimension m), so that $\widehat{M}$ is the product $\mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}$ with the Riemannian metric $\hat{g}$ given by

$$\hat{g} = e^{2s}(dx_1^2 + \cdots + dx_{m-1}^2) + ds^2 + dt^2.$$

We define the function $\hat{b}$ on $\widehat{M}$ by

$$\hat{b}(x_1, \dots, x_{m-1}, s, t) = s. \tag{8}$$

This function is in fact a Busemann function of $\widehat{M}$ (seen as a Cartan–Hadamard manifold) associated with a "horizontal" geodesic (justifying the name "horizontal" Busemann function used in the introduction).

Denote by $b := \hat{b}|_M$ the restriction of $\hat{b}$ to M. Decompose the unit vector ν according to the product structure $\mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}$, orthogonally with respect to $\hat{g}$, as

$$\nu = \nu_x + w\partial_s + v\partial_t. \tag{9}$$

Applying Lemma 2.2, we obtain the equation

$$\Delta_g b = \Delta_{\hat{g}} \hat{b}|_M + \operatorname{Hess}_{\hat{g}} \hat{b}(\nu, \nu) - mH\hat{g}(\nu, \partial_s). \tag{10}$$

Using (7) and (9), this inequality can be rewritten as

$$-\Delta_g b = (m-1) - |\nu_x|^2 + mHw. \tag{11}$$

From $|\nu_x|^2 + v^2 + w^2 = 1$, it follows that

$$-\Delta_g b \geq (m-2) + v^2 + w^2 - mH|w|. \tag{12}$$

Theorem 3.1. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal hypersurface with normal vector ν. Let v be the vertical component of the normal vector, that is, $v = \hat{g}(\nu, \partial_t)$.*

(i) *The function b satisfies the inequality*

$$-\Delta_g b \geq (m-2) + v^2, \tag{13}$$

and the infimum $\lambda_\sigma(\Delta_g)$ of the spectrum of the Laplacian Δ_g of M satisfies

$$\lambda_\sigma(\Delta_g) \geq \left(\frac{m-2+\inf_M v^2}{2}\right)^2 \geq \left(\frac{m-2}{2}\right)^2. \tag{14}$$

In particular, if $m \geq 3$, (M, g) is nonparabolic.

(ii) *The spectrum of the operator $\Delta_g + (m-1)(1-v^2)$ is bounded from below by $((m-1)/2)^2$.*

Proof. Assertion (i) follows immediately from (12) with $H = 0$. For the last statement, apply (14) and [Grigoryan 1999, Proposition 10.1]. To prove (ii), we start from the inequality (13), $-\Delta_g b \geq (m-2) + v^2$. Multiplying this inequality by f^2, where $f \in C_0^\infty(M)$, and integrating by parts using the fact that $|db|_g \leq 1$ gives (all integrals are taken with respect to the Riemannian measure dv_g)

$$(m-2)\int_M f^2 + \int_M v^2 f^2 \leq \int_M |df^2| \leq 2\int_M |f||df|.$$

Rewrite this inequality as

$$(m-1)\int_M f^2 \leq 2\int_M |f||df| + \int_M (1-v^2)f^2.$$

Using the Cauchy–Schwarz inequality $2|f|\,.\,|df| \leq \dfrac{1}{a}|df|^2 + af^2$ for $a > 0$, we obtain

$$a(m-1-a)\int_M f^2 \leq \int_M \big(|df|^2 + a(1-v^2)f^2\big) \leq \int_M \big(|df|^2 + (m-1)(1-v^2)f^2\big),$$

provided that $0 \leq a \leq m-1$. We can now maximize the constant on the left-hand side by choosing $a = (m-1)/2$. $\square$

When the mean curvature H is nonzero, we also obtain the following result from inequality (12):

Proposition 3.2. *Let $(M^m, g) \looparrowright (\mathbb{H}^m \times \mathbb{R}, \hat{g})$ be a complete, orientable hypersurface with normal vector ν and constant mean curvature H for $0 \leq H \leq (m-1)/m$. Recall that $v = \hat{g}(\nu, \partial_t)$. Then,*

$$-\Delta_g b \geq (m-2)(1-\sqrt{1-v^2}) + (m-2)\Big(1 - \frac{mH}{m-2}\Big)\sqrt{1-v^2}. \tag{15}$$

Remarks. (i) Inequalities (13), (14) and Theorem 3.1(ii) are sharp. Indeed, taking the horizontal slice $M = \mathbb{H}^m \times \{0\}$ gives the case $v = 1$, and taking $M = \mathbb{P} \times \mathbb{R}$, where $\mathbb{P}$ is some totally geodesic $(m-1)$-space in $\mathbb{H}^m$, gives the case $v = 0$.

(ii) If we assume that $v^2 \leq \alpha^2 < 1$, the spectrum of $\Delta_g + (m-1)(1-v^2)$ is bounded from below by $(m-1)(1-\alpha^2)$.

(iii) Inequality (15) generalizes an earlier result of [Castillon 1997] for submanifolds immersed into Hadamard manifolds. For other estimates, see [Bessa and Costa 2009]. We point out that it is more convenient in our context to use the "horizontal" Busemann function rather than the hyperbolic distance function as in [Castillon 1997].

(iv) Inequality (15) still holds if M^m is only assumed to have mean curvature bounded from above by H.

4. Applications to minimal hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$

4A. ***Index of minimal surfaces immersed in $\mathbb{H}^m \times \mathbb{R}$.*** The Jacobi (or stability) operator of a minimal hypersurface $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$ is given by

$$J_M = \Delta + (m-1)(1-v^2) - |A|^2, \tag{16}$$

where v is the vertical component of the unit normal ν and A the second fundamental form of the immersion (see [Bérard and Sá Earp 2008]). As a corollary of Theorem 3.1, we obtain:

Theorem 4.1. *Let $(M^2, g) \looparrowright (\mathbb{H}^2 \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal surface with second fundamental form A. If $\int_M |A|^2\, dv_g$ is finite, then the immersion has finite index.*

Proof. When $\int_M |A|^2$ is finite, the second fundamental form tends to zero uniformly at infinity (see [Bérard and Sá Earp 2008, Theorem 4.1]). Theorem 3.1(ii) with $m = 2$ implies that the essential spectrum of the Jacobi operator J_M is bounded from below by $\frac{1}{4}$. Since the operator J_M is also bounded from below, it has only finitely many negative eigenvalues (see [Bérard et al. 1997, Proposition 1]). ☐

Remark. This theorem answers a question raised in [Bérard and Sá Earp 2008], where the finiteness of the index of J_M is proved in dimension $m \geq 3$ under the assumption that $\int_M |A|^m$ is finite, and in dimension 2 under the assumption that both $\int_M v^2$ and $\int_M |A|^2$ are finite. In dimension $m \geq 3$, the index of J_M is bounded from above by a constant times $\int_M |A|^m$ (see [Bérard and Sá Earp 2008]). In the next section, we investigate bounds on the index in dimension 2.

4B. *Bounds on the index of minimal surfaces immersed in $\mathbb{H}^m \times \mathbb{R}$.*

Proposition 4.2. *Let $(M^2, g) \looparrowright (\mathbb{H}^2 \times \mathbb{R}, \hat{g})$ be a complete, orientable, minimal surface with second fundamental form A. If $\int_M |A|^2\, dv_g$ is finite, then for any $r > 1$, there exists a constant C_r such that the index of the immersion is bounded from above by $C_r \int_M |A|^{2r}\, dv_g$.*

Remarks. (i) The assumption that $\int_M |A|^2\, dv_g$ is finite implies that A tends to zero uniformly at infinity. Thus the integrals $\int_M |A|^{2r}\, dv_g$ are all finite.

(ii) Our proof provides a constant C_r that tends to infinity when r tends to 1. We do not know whether there is a bound for the index in terms of $\int_M |A|^2\, dv_g$, as is the case for minimal surfaces in $\mathbb{R}^3$ [Tysk 1987].

Proof. As in Section 4A, we write the Jacobi operator as $J = \Delta_g + 1 - v^2 - |A|^2$. The closure $\tilde{Q}$ of the quadratic form

$$Q[f] = \int_M \big(|df|^2 + (1 - v^2) f^2\big)\, dv_g$$

with domain $C_0^1(M)$ satisfies the Beurling–Deny condition (if f is in the domain of $\tilde{Q}$, then so is $|f|$ and $\tilde{Q}[|f|] = \tilde{Q}[f]$; see [Davies 1989, Theorem 1.3.2]) and, by Theorem 3.1, the Cheeger inequality

$$\int_M f^2\, dv_g \leq 4Q[f] \quad \text{for all } f \in C_0^1(M). \tag{17}$$

On the other hand, the surface M satisfies the Sobolev inequality

$$\int_M f^2\,dv_g \le S\Big(\int_M |df|_g^2\,dv_g\Big)^2 \quad \text{for all } f \in C_0^1(M), \tag{18}$$

for some constant $S > 0$. Indeed, this follows from the Sobolev inequality for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, using the fact that the ambient space has nonpositive curvature and infinite injectivity radius (see [Hoffman and Spruck 1974]).

From the above Cheeger and Sobolev inequalities, we can establish that for any $q \ge 1$, there exists a constant D_q such that for any $f \in C_0^1(M)$,

$$\Big(\int_M |f|^{2q}\,dv_g\Big)^{1/q} \le D_q\,Q[f]. \tag{19}$$

When q is an integer, the inequality follows from an induction argument and we can conclude by interpolation.

We can then apply [Levin and Solomyak 1997, Theorem 1.2] to conclude that the index is less than $e^p D_q^p \int_M |A|^{2p}\,dv_g$, where $p = q/(q-1)$. □

4C. ***Hypersurfaces in a slab.*** In this section, we use the computations of Section 3 to give a lower bound on the spectrum of the Laplacian on a complete minimal surface immersed in a slab $\mathbb{H}^2 \times [-a, a]$, where $a > 0$.

First, consider functions on $\mathbb{H}^m \times \mathbb{R}$ depending only on the height t, namely $\hat\beta(x, s, t) = f(t)$. In this case, $d\hat\beta = f'(t)\,dt$, and

$$\mathrm{Hess}_{\hat g}\,\hat\beta(X, Y) = f''(t)\hat g(X, \partial_t)\hat g(Y, \partial_t).$$

In particular,

$$\Delta_{\hat g}\hat\beta = -f''(t) \quad \text{and} \quad \mathrm{Hess}_{\hat g}\,\hat\beta(\nu, \nu) = v^2 f''(t).$$

Define $\beta = \hat\beta|_M$. Using Lemma 2.2, we have

$$-\Delta_g\beta = (1 - v^2) f''(t) + mHvf'(t). \tag{20}$$

To estimate the first eigenvalue of a *minimal* hypersurface $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, use the identity (20) with some particular choice of f. For instance, let

$$\hat\beta(x, s, t) = \tfrac{1}{2}t^2.$$

In this case,

$$-\Delta_g\beta = (1 - v^2).$$

Assume now that $M^m \looparrowright \mathbb{H}^m \times [-a, a]$, for some $a > 0$. Then,

$$-\Delta_g\beta = (1 - v^2) \quad \text{and} \quad |d\beta| \le a.$$

Defining $Z = b + \beta$, where b is the restriction of the Busemann function $\hat{b}$ to M^m, we can use the last inequality in (12) to obtain

$$-\Delta Z \geq m-1 \quad \text{and} \quad |dZ| \leq \sqrt{1+a^2}. \tag{21}$$

Using the above notation and Lemma 2.3, we have the following estimate:

Proposition 4.3. *Given $a > 0$, let $(M^m, g) \looparrowright (\mathbb{H}^m \times [-a, a], \hat{g})$ be a complete, immersed, orientable, minimal hypersurface. Then, the infimum of the spectrum of Δ_g on M is positive. More precisely,*

$$\lambda_\sigma(\Delta_g) \geq \frac{(m-1)^2}{4(1+a^2)}. \tag{22}$$

5. Bounds derived from a stability assumption

Let (M, g) be a complete Riemannian surface with (nonnegative) Laplace operator Δ_g and Gaussian curvature K_g. Let a, b be real numbers with $a \geq 0$ and $b > 1/4$. Let L be the operator $L = \Delta_g + a + bK_g$.

Let $\mathrm{Ind}(L, \Omega)$ denote the number of negative eigenvalues of the operator L in Ω with Dirichlet boundary conditions on $\partial\Omega$. The index $\mathrm{Ind}(L)$ of the operator L is defined to be the supremum

$$\mathrm{Ind}(L) = \sup\{\, \mathrm{Ind}(L, \Omega) \mid \Omega \Subset M \,\}$$

taken over the relatively compact subdomains Ω in M.

In Section 5A, we state two intrinsic consequences of the assumption that the operator L has finite index. In Sections 5B and 5C, we consider applications to minimal and CMC surfaces.

5A. ***Intrinsic results.*** We prove the next theorem using the method of [Colding and Minicozzi 2002], and more precisely [Castillon 2006, Lemma 1.8].

Theorem 5.1. *Let (M, g) be a complete noncompact Riemannian surface. Let $a \geq 0$ and $b > \frac{1}{4}$. Denote by Δ_g the (nonnegative) Laplacian and by K_g the Gaussian curvature of (M, g). Denote by $\lambda_\sigma(\Delta_g)$ the infimum of the spectrum of Δ_g and by $\lambda_e(\Delta_g)$ the infimum of the essential spectrum of Δ_g.*

(i) *If the operator $\Delta_g + a + bK_g$ is nonnegative on $C_0^\infty(M)$, then*

$$\lambda_\sigma(\Delta_g) \leq \frac{a}{4b-1}.$$

(ii) *If the operator $\Delta_g + a + bK_g$ has finite index on $C_0^\infty(M)$ and if M has infinite volume, then*

$$\lambda_e(\Delta_g) \leq \frac{a}{4b-1}.$$

Proof. (i) We can assume the surface M has infinite volume (otherwise $\lambda_\sigma(\Delta_g) = 0$ because the function 1 is in $L^2(M, v_g)$, and the estimate is trivial). Fix a point x_0 in M and let $r(x)$ denote the Riemannian distance to the point x_0. Given $S > R > 0$, let $B(R)$ denote the open geodesic ball in M with center x_0 and radius R. Let $C(R, S)$ denote the open annulus $B(S) \setminus \bar{B}(R)$. Let $V(R)$ denote the volume of $B(R)$ and $L(R)$ the length of its boundary $\partial B(R)$. Let $G(R)$ denote the integral curvature of $B(R)$, that is, $G(R) = \int_{B(R)} K_g(x)\, dv_g(x)$, where dv_g is the Riemannian measure. The main idea in [Castillon 2006] is to use the work of [Shiohama and Tanaka 1989; 1993] on the length of geodesic circles, which shows that the function $L(r)$ is differentiable almost everywhere and is related to the Euler characteristic and to the integral curvature of geodesic balls by the formula

$$L'(r) \leq 2\pi\chi(B(r)) - G(r) \leq 2\pi - G(r)$$

(see [Castillon 2006, Theorem 1.7]; the second inequality comes from the fact that the Euler characteristic of balls is less than or equal to 1).

Now choose ξ as in the following lemma:

Lemma 5.2 [Castillon 2006, Lemma 1.8]. *For $0 < R < S$, let $\xi : [R, S] \to \mathbb{R}$ be such that $\xi \geq 0$, $\xi' \leq 0$, $\xi'' \geq 0$ and $\xi(S) = 0$. Then*

$$\int_{C(R,S)} K_g \xi^2(r)\, dv_g \\ \leq -\xi^2(R)G(R) + 2\pi\xi^2(R) - 2\xi(R)\xi'(R)L(R) - \int_{C(R,S)} (\xi^2)''(r)\, dv_g. \quad \square$$

Choose a function $f : B(S) \to \mathbb{R}$ such that $f(r) \equiv \xi(R)$ on $B(R)$, $f(r) = \xi(r)$ on $C(R, S)$, and write the positivity assumption

$$0 \leq \int_M |df|_g^2\, dv_g + a \int_M f^2\, dv_g + b \int_M K_g f^2\, dv_g.$$

On the ball $B(R)$, we have $\int_{B(R)} K_g f^2\, dv_g = \xi^2(R)G(R)$ and $\int_{B(R)} |df|^2\, dv_g = 0$. Using Lemma 5.2, we obtain

$$0 \leq \int_{C(R,S)} (\xi')^2(r)\, dv_g + a \int_M f^2\, dv_g + b\xi^2(R)G(R) - b\xi^2(R)G(R) \\ + 2\pi b\xi^2(R) - 2b\xi(R)\xi'(R)L(R) - b \int_{C(R,S)} (\xi^2)''(r)\, dv_g,$$

and hence,

$$0 \leq (1 - 2b) \int_{C(R,S)} (\xi')^2(r)\, dv_g + a \int_M f^2\, dv_g + 2\pi b\xi^2(R) \\ - 2b\xi(R)\xi'(R)L(R) - 2b \int_{C(R,S)} \xi(r)\xi''(r)\, dv_g. \tag{23}$$

Choose $\xi(r) = (S-r)^k$ in $[R, S]$ for $k \geq 1$ big enough (we will eventually let k tend to infinity). Then $\xi(r)\xi''(r) = (1-\frac{1}{k})(\xi'(r))^2$. It follows that

$$0 \leq \Big(1-4b+\frac{2b}{k}\Big)\int_M |df|^2\,dv_g + a\int_M f^2\,dv_g + 2b\big(\pi(S-R)^{2k} + kL(R)(S-R)^{2k-1}\big).$$

Using the fact that $\int_M f^2\,dv_g \geq (S-R)^{2k}V(R)$, we obtain

$$\begin{aligned}\lambda_\sigma(\Delta_g) &\leq \frac{\int_M |df|^2\,dv_g}{\int_M f^2\,dv_g} \\ &\leq \frac{a}{4b-1-2b/k} + \frac{2b}{(4b-1-2b/k)V(R)}\Big(\pi + \frac{kL(R)}{S-R}\Big).\end{aligned} \tag{24}$$

First letting S tend to infinity, then letting R tend to infinity, using the fact that M has infinite volume, and finally letting k tend to infinity gives

$$\lambda_\sigma(\Delta_g) \leq \frac{a}{4b-1}.$$

(ii) The finiteness of the index of the operator $\Delta_g + a + bK_g$ implies that it is nonnegative outside a compact set (see [Fischer-Colbrie 1985, Proposition 1], for instance). Choose R_0 big enough for $\Delta_g + a + bK_g$ to be nonnegative in $M \setminus B(R_0)$. Next, for $S > R > R_1 + 1 > R_0 + 1$, choose ξ as in Lemma 5.2 and a test function

$$f(r) = \begin{cases} 0 & \text{in } B(R_1), \\ \xi(R)(r-R_1) & \text{in } C(R_1, R_1+1), \\ \xi(R) & \text{in } C(R_1+1, R), \\ \xi(r) & \text{in } C(R, S). \end{cases} \tag{25}$$

Following the same scheme as for (i), and under the assumption that the volume of M is infinite, we can prove that the bottom of the spectrum of Δ_g in $M \setminus B(R_1)$ with Dirichlet boundary conditions on $\partial B(R_1)$ satisfies the inequality

$$\lambda_\sigma(\Delta_g, M \setminus B(R_1)) \leq \frac{a}{4b-1}.$$

To conclude the proof of Theorem 5.1, we use the fact that

$$\lambda_e(\Delta_g) = \lim_{R\to\infty} \lambda_\sigma(\Delta_g, M \setminus B(R)). \qquad \square$$

Proposition 5.3. *Let (M, g) be a complete Riemannian surface with (nonnegative) Laplace operator Δ_g and Gaussian curvature K_g. Let $V(r)$ denote the volume of the geodesic ball of radius r in M centered at some point. Let a, b be positive real numbers with $b > \frac{1}{4}$. Let $\alpha_0 = \sqrt{a/(4b-1)}$. If the operator $L := \Delta_g + a + bK_g$ has finite index, then*

$$\int_0^\infty e^{-2\alpha r}V(r)\,dr < \infty \quad \textit{for all } \alpha > \alpha_0,$$

and hence, *the lower volume growth of M satisfies*

$$\liminf_{r\to\infty} \frac{\ln V(r)}{r} \leq 2\alpha_0.$$

Proof. It follows from our assumptions that the operator L is positive outside some compact set (see [Fischer-Colbrie 1985, Proposition 1]). In particular, it is positive on $M \setminus B(R_0)$ for some radius R_0. Choose $R > R_0 + 1$ and define the function

$$\xi(r) = \begin{cases} 0 & \text{for } r \leq R_0, \\ (1-(R_0+1)/R)^{\alpha R}(r-R_0) & \text{for } R_0 \leq r \leq R_0+1, \\ (1-r/R)^{\alpha R} & \text{for } R_0+1 \leq r \leq R, \end{cases} \tag{26}$$

where the parameter α will be chosen later on. The positivity of the operator L on $M \setminus B(R_0)$ implies that

$$0 \leq \int_M \big((\xi'(r))^2 + a\xi^2(r) + bK_g\xi^2(r)\big)\, dv_g.$$

Write this integral as the sum of $\int_{C(R_0,R_0+1)}$ and $\int_{C(R_0+1,R)}$. The first integral can be written as

$$\int_{C(R_0,R_0+1)} = \Big(1 - \frac{R_0+1}{R}\Big)^{\alpha R} C(B(R_0)),$$

where $C(B(R_0))$ is a constant which only depends on the geometry of M on the ball $B(R_0)$. Using Lemma 5.2 and the fact that $\chi(B(r)) \leq 1$ for all r, the second integral can be estimated as follows:

$$\begin{aligned}\int_{C(R_0+1,R)} \leq \int_{C(R_0+1,R)} &\big((\xi')^2 + a\xi^2 - b(\xi^2)''\big)\, dv_g \\ &+ 2\pi b - \xi^2(R_0+1)G(R_0+1) + 2\alpha L(R_0+1).\end{aligned}$$

Using (26), the definition for the function ξ, the integral in the first line of the above inequality can be written as

$$-\Big((4b-1)\alpha^2 - \frac{2b\alpha}{R} - a\Big)\int_{R_0+1}^{R} \Big(1-\frac{r}{R}\Big)^{2\alpha R-2} L(r)\, dr.$$

Taking α big enough so that the constant is positive, and using $L(r) = V'(r)$, we obtain the inequality

$$\frac{2\alpha R - 2}{R}\Big((4b-1)\alpha^2 - \frac{2b\alpha}{R} - a\Big)\int_{R_0+1}^{R}\Big(1-\frac{r}{R}\Big)^{2\alpha R-3} V(r)\, dr \leq D(B(R_0),\alpha),$$

where $D(B(R_0),\alpha)$ is a constant which only depends on the geometry of M in the ball $B(R_0)$ and α. Letting R tend to infinity finally shows

$$2\alpha\big((4b-1)\alpha^2 - a\big)\int_{R_0+1}^{\infty} e^{-2\alpha r} V(r)\, dr < \infty,$$

provided that $\alpha > \alpha_0$, which proves the first assertion in the proposition. The second assertion follows easily. $\square$

Remark. In Proposition 5.3, we assumed that $a > 0$. In the case $a = 0$, the volume growth is at most quadratic (see [Castillon 2006, Proposition 2.2]).

5B. ***Applications to stable minimal surfaces in $\mathbb{H}^3$ or $\mathbb{H}^2 \times \mathbb{R}$.*** Let M be a complete, orientable, minimal immersion into either 3-dimensional hyperbolic space $\mathbb{H}^3$ or into $\mathbb{H}^2 \times \mathbb{R}$. Let J_M denote the Jacobi operator of the immersion.

In the case of a minimal immersion $M \looparrowright \mathbb{H}^3(-1)$, the operator J_M takes the form $J_M = \Delta_M + 2 - |A|^2$, where A is the second fundamental form. Using the Gauss equation, we have that $K_M = -1 - \frac{1}{2}|A|^2$, so that we can rewrite the Jacobi operator of $M \looparrowright \mathbb{H}^3(-1)$ as

$$J_M = \Delta_M + 4 + 2K_M. \tag{27}$$

In the case of a minimal immersion $M \looparrowright \mathbb{H}^2(-1) \times \mathbb{R}$, the Jacobi operator is given by $J_M = \Delta_M + 1 - v^2 - |A|^2$, where v is the vertical component of the unit normal vector to the surface. Using the Gauss equation, we have that $K_M = -v^2 - \frac{1}{2}|A|^2$, so that we can rewrite the Jacobi operator of $M \looparrowright \mathbb{H}^2(-1) \times \mathbb{R}$ as

$$J_M = \Delta_M + 2 + 2K_M - (1 - v^2) \le \tilde{J}_M := \Delta_M + 2 + 2K_M. \tag{28}$$

In this case, the positivity of the operator J_M implies the positivity of the operator $\tilde{J}_M$.

Applying Theorem 5.1 to the operator J_M in the form (27) when M is a minimal surface in $\mathbb{H}^3$, respectively, to the operator $\tilde{J}_M$ in the form (28) when M is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, we obtain the following corollary.

Corollary 5.4. *Let $(M, g) \looparrowright (\widehat{M}, \hat{g})$ be a complete, orientable, minimal immersion. Assume that the immersion is stable.*

(i) *If $\widehat{M} = \mathbb{H}^3$, then $\lambda_\sigma(\Delta_g) \le \frac{4}{7}$.*
(ii) *If $\widehat{M} = \mathbb{H}^2 \times \mathbb{R}$, then $\lambda_\sigma(\Delta_g) \le \frac{2}{7}$.*

If the immersion is only assumed to have finite index, then the same inequalities hold with $\lambda_\sigma(\Delta_g)$ replaced by $\lambda_e(\Delta_g)$, the infimum of the essential spectrum.

Remarks. (i) The first assertion improves an earlier result from [Candel 2007], to the effect that $\lambda_\sigma(M) \le \frac{4}{3}$, provided that M is a complete, *simply connected*, stable minimal surface in $\mathbb{H}^3$.

(ii) In both cases, the bottom of the spectrum of a totally geodesic $\mathbb{H}^2$ is $\frac{1}{4}$.

Applying Proposition 5.3, we have the following corollary.

Corollary 5.5. *Let* $(M, g) \looparrowright (\widehat{M}, \hat{g})$ *be a complete, orientable, minimal immersion. Let* μ *denote the lower volume growth rate of* M, *given by*

$$\mu = \liminf_{r\to\infty} \frac{\ln V(r)}{r},$$

where $V(r)$ *is the volume of the geodesic ball* $B(x_0, r)$ *about some point. Assume that the immersion has finite index.*

(i) *If* $\widehat{M} = \mathbb{H}^3$, *then* $\mu \le 2\sqrt{\frac{4}{7}}$.

(ii) *If* $\widehat{M} = \mathbb{H}^2 \times \mathbb{R}$, *then* $\mu \le 2\sqrt{\frac{2}{7}}$.

Remarks. (i) Assertion (i) in Corollary 5.5 improves an earlier upper bound on μ given in [Candel 2007] under the assumption that M is simply connected.

(ii) Recall from [Brooks 1981; Kumura 2007] that the volume growth is related to the infimum of the essential spectrum by the formula

$$\lambda_e(\Delta_g) \le \left(\frac{1}{2} \liminf_{r\to\infty} \frac{\ln V(r)}{r}\right)^2.$$

5C. ***Further applications.*** The argument above also works for surfaces with constant mean curvature $|H| \le 1$ in hyperbolic space. In that case,

$$K_M = -(1 - H^2) - \tfrac{1}{2}|A|^2 \quad \text{and} \quad J_M = \Delta_M + 4(1 - H^2) + 2K_M.$$

Proposition 5.6. *Let* $(M, g) \looparrowright \mathbb{H}^3$ *be a complete, orientable, stable CMC immersion with* $|H| \le 1$. *Then*

$$\lambda_\sigma(\Delta_g) \le \frac{4(1 - H^2)}{7}.$$

The space $\mathbb{H}^2 \times \mathbb{R}$ is a simply connected 3-dimensional homogeneous manifold, whose isometry group has dimension 4. Such manifolds have been well studied (see for instance [Daniel 2007] and references therein) and can be parametrized by two real parameters, say κ and τ with $\kappa \ne 4\tau^2$. Denote these manifolds by $\mathbb{E}^3(\kappa, \tau)$. When $\tau = 0$, the manifold $\mathbb{E}^3(\kappa, 0)$ is the product space $\mathbb{E}^2(\kappa) \times \mathbb{R}$, where $\mathbb{E}^2(\kappa)$ is the space form of constant curvature κ. In particular, $\mathbb{H}^2 \times \mathbb{R} = \mathbb{E}^3(-1, 0)$.

If $(M, g) \looparrowright \mathbb{E}^3(\kappa, \tau)$ is an immersed CMC H surface, then its Jacobi operator is given by (see [Daniel 2007, Proposition 5.11])

$$J_M := \Delta_g + 2K - 4H^2 - \kappa - (\kappa - 4\tau^2)v^2.$$

In the next proposition we give an upper bound for the bottom of the spectrum in this general framework.

Proposition 5.7. *Let $(M, g) \looparrowright \mathbb{E}^3(\kappa, \tau)$ be a complete, orientable, stable CMC H immersion such that $\kappa < 4\tau^2$. Assume furthermore that $2H^2 \leq (2\tau^2 - \kappa)$. Then*

$$\lambda_\sigma(\Delta_g) \leq \frac{4\tau^2 - 2\kappa - 4H^2}{7}.$$

Proof. Under the hypotheses we have the inequalities

$$0 \leq \Delta_g + 2K - 4H^2 - \kappa - (\kappa - 4\tau^2)v^2 \leq \Delta_g + 2K - 4H^2 - 2(\kappa - 2\tau^2),$$

and we may apply Theorem 5.1 again. □

6. Applications in higher dimensions

In this section, we give some further applications of the inequalities proved in Section 3. In the following proposition, we give a structure theorem for minimal hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$.

Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$ be a complete, orientable minimal hypersurface with second fundamental form A, where $\|A\|_m < \infty$ and $m \geq 3$. By [Bérard and Sá Earp 2008], the Ricci curvature of M^m satisfies the inequality (30) below and furthermore $|A|$ tends to zero uniformly at infinity. It follows that M satisfies the assumption of [Li and Wang 2001, Theorem 3.1] provided that $m \geq 7$ and hence that M has only finitely many infinite volume ends. On the other-hand, all ends of M must have infinite volume [Cheng et al. 2008, Proposition 2.1]. Thus M has only finitely many ends provided that $m \geq 7$. The next result gives a sufficient condition for M to have only one end.

Proposition 6.1. *Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, $m \geq 3$, be a complete, orientable minimal hypersurface with unit normal field ν and second fundamental form A. Let v denote the component of ν along ∂_t. Assume that*

(i) $m \geq 7$ *and* $0 \leq \alpha \leq 1$, *or*

(ii) $m = 6$ *and* $0.083 \leq \alpha \leq 1$, *or*

(iii) $m = 5$ *and* $0.578 \leq \alpha \leq 1$.

There exists a constant $c(m, \alpha) > 0$ such that, if M satisfies $\|A\|_m \leq c(m, \alpha)$ and $v^2 \geq \alpha^2$, then M carries no nontrivial L^2-harmonic 1-form and hence has at most one end.

Sketch of proof. *Step 1:* According to [Hoffman and Spruck 1974], the manifold M^m satisfies the Sobolev inequality

$$\|\varphi\|^2_{2m/(m-2)} \leq S(2, m)\|d\varphi\|^2_2 \quad \text{for all } \varphi \in C^1_0(M). \tag{29}$$

Step 2: Let $u \in T_1 M$ be a unit tangent vector to M. By the Gauss equation, we have

$$\mathrm{Ric}(u, u) = \widehat{\mathrm{Ric}}(u, u) - \hat{R}(u, \nu, u, \nu) - |A(u)|^2,$$

where Ric denotes the Ricci curvature of M, $\widehat{\mathrm{Ric}}$ the Ricci curvature of $\widehat{M} = \mathbb{H}^m \times \mathbb{R}$ and $\hat{R}$ the curvature tensor of $\widehat{M}$, and where A denotes the Weingarten operator of the immersion. Using the curvature computations in [Bérard and Sá Earp 2008] and the fact that A has trace zero, we obtain the inequality

$$\mathrm{Ric}(u,u) \geq -(m-1) - \frac{m-1}{m}|A|^2. \tag{30}$$

Step 3: Let ω be an L^2 harmonic 1-form on M. Using the Weitzenböck formula for 1-forms, the improved Kato inequality

$$\frac{1}{m-1}|d|\omega||^2 \leq |D\omega|^2 - |d|\omega||^2, \tag{31}$$

and inequality (30), we find that ω satisfies the inequality

$$\frac{1}{m-1}|d|\omega||^2 + |\omega|\Delta|\omega| \leq (m-1)|\omega|^2 + \frac{m-1}{m}|A|^2|\omega|^2 \tag{32}$$

in the weak sense. The following formal calculation can easily be made rigorous by using cut-off functions. Integrate (32) over M using integration by parts and using the notation $f := |\omega|$, we obtain

$$\frac{m}{m-1}\int_M |df|^2 \leq (m-1)\int_M f^2 + \frac{m-1}{m}\int_M |A|^2 f^2.$$

Plug the assumption $|v| \geq \alpha$ and the inequality (14) into the preceding inequality. Use Hölder's inequality to estimate the integral $\int_M |A|^2 f^2$ and the Sobolev inequality (29). It follows that

$$\left(\frac{m}{m-1} - \frac{4(m-1)}{(m-2+\alpha)^2}\right)\|f\|^2_{2m/(m-2)} \leq S(2,m)\frac{m-1}{m}\|A\|^2_m\|f\|^2_{2m/(m-2)}$$

and we can conclude the proof with the constant

$$c(m,\alpha) = \frac{m}{(m-1)S(2,m)}\,\frac{m(m-2+\alpha)^2 - 4(m-1)^2}{(m-1)(m-2+\alpha)^2}. \qquad \square$$

Proposition 6.2. *Let $M^m \looparrowright \mathbb{H}^m \times \mathbb{R}$, $m \geq 3$, be a complete, orientable minimal hypersurface with unit normal field ν and second fundamental form A. Let v denote the component of ν along ∂_t. The immersion M is stable if*

(i) $\|A\|_\infty \leq \frac{1}{4}(m-1)^2$, *or*

(ii) $\|A\|_\infty \leq \frac{1}{4}(m-2+\alpha)^2$ *and* $v^2 \geq \alpha$, *or*

(iii) $|A|^2 + (m-1)v^2 \leq \frac{1}{4}m^2$ *on* M.

Proof. Recall that the Jacobi operator J_M of the immersion M is given by $J_M = \Delta_g + (m-1)(1-v^2) - |A|^2$. The proposition follows from Theorem 3.1. $\square$

Remark 1. Condition (ii) is not so interesting. Indeed, if M is connected, we may assume that $v > 0$ and it follows that M is stable because v is a Jacobi field.

Remark 2. We can write the operator J_M as

$$J_M = \Delta_g - \left(\frac{m-2}{2}\right)^2 + \left(\left(\frac{m}{2}\right)^2 - |A|^2\right).$$

In view of the results à la Lieb or Li and Yau, one can show that if the integral

$$\int_M \left(\left(\frac{m}{2}\right)^2 - |A|^2\right)_-^{m/2}$$

is small enough, then M is stable.

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