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REGULARITY OF WEAKLY HARMONIC MAPS FROM A FINSLER SURFACE INTO AN n-SPHERE

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We study harmonic maps on Finsler surfaces. Using Berwald frames on Finsler surfaces, we prove conformal invariance for the energy of Finsler harmonic maps. As an application, we show that weakly harmonic maps from a Finsler surface to a sphere $\mathbb{S}^n$ are in fact smooth by establishing a new Jacobi structure, generalizing the regularity result previously known for the case of a harmonic map from a Riemannian surface.

1. Introduction

Harmonic maps between Riemannian manifolds are very important both in classical and modern differential geometry. They are defined as the critical points of the energy functionals. One of the fundamental problems in the theory of harmonic maps is to study regularity [Evans 1991; Hélein 1990; 1991b; 1991a]. Harmonic maps on Riemann surfaces are the natural extension of minimal surfaces in Riemannian manifolds. More importantly, the energy of a harmonic map on a two-dimensional domain is a conformal invariant, i.e. the set of harmonic maps from a Riemannian surface M depends only on the conformal structure on M.

Finsler manifolds are just Riemannian manifolds with metrics without the quadratic restriction [Chern 1996]. Many Finslerian geometers have made effort in the investigation of the geometric and analytic properties (such as, existence, stability, etc) of harmonic maps on Finsler manifolds [He and Shen 2005; Mo 2001; Mo and Yang 2005; Shen and Wei 2008; Shen and Zhang 2004].

In this paper, we are going to study the theory of harmonic maps on Finsler surfaces. Let (M, F) be a Finsler manifold and (N, h) a Riemannian manifold. We will say that $\phi : (M, F) \to (N, h)$ is *Finsler harmonic* if ϕ is a critical point of the energy functional with respect to any compactly supported variation of ϕ [Mo 2001]. First, we obtain conformal invariance for Finsler harmonic maps from surfaces by using the Berwald frames on surfaces (see Theorem 3.1).

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As an application, we obtain a regularity result of a weakly Finsler harmonic map on a Finsler surface M, generalizing a theorem of Hélein for the case of M being a Riemannian surface [Hélein 1990]. More precisely:

Theorem 1.1. *For a Finsler surface* $(M,\ F)$ *with* $\partial M = \varnothing$, *if* $\phi \in W^{1,2}(M, \mathbb{S}^n)$ *be a weakly Finsler harmonic map, then* $\phi \in C^\infty(M, \mathbb{S}^n)$.

Recall that $W^{1,2}(M,\ \mathbb{R}^l)$ is the usual Sobolev space of L^2 functions on M to $\mathbb{R}^l$ having first derivatives in the sense of distributions in L^2. A map $\phi \in W^{1,2}(M,\ N)$ is *weakly Finsler harmonic map* if it is a weak solution to the Euler–Lagrange equation (4-8), where the Sobolev space $W^{1,2}(M,\ N)$ of maps from M to $N \subset \mathbb{R}^l$ is defined by

$$W^{1,2}(M,\ N) = \{\varphi \in W^{1,2}(M,\ \mathbb{R}^l) : \varphi(x) \in N \text{ a.e. } x \in M\}$$

(see Section 4).

It is well-known that the Jacobi (determinant) structure plays an important role in studying the regularity of harmonic maps from Riemannian surfaces to Riemannian spheres [Lin and Wang 2008]. But, in general, we cannot get the Jacobi structure of Finsler harmonic maps directly. So we have to construct a new Jacobi structure by using the existence and regularity theory of elliptic equations (see Lemma 4.1). After obtaining this interesting structure, we can prove Theorem 1.1 successfully.

2. Preliminaries

Finsler harmonic maps. Let $\phi : (M,\ F) \to (N, h)$ be a map from an m-dimensional Finsler manifold (M, F) to an n-dimensional Riemannian manifold (N, h). In natural coordinates, the energy density of ϕ is given by

$$e(\phi) = \tfrac{1}{2} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} h_{\alpha\beta} \tag{2-1}$$

where $\phi = (\phi^1, \ldots, \phi^n)$, $g_{ij} = \left(\tfrac{1}{2}F^2\right)_{y^i y^j}$ and $(g^{ij}) = (g_{ij})^{-1}$. We introduce a dual adapted orthonormal frame e_i on the Riemannian vector bundle $(\pi^*TM,\ g)$ and coframe ω_i with $\omega_m := (\partial F/\partial y^i) dx^i$ where $\pi : SM \to M$ is the canonical projection map, SM the projective sphere bundle of M and g is the fundamental tensor of F. Put $\omega_i = v_{ij} dx^j$. Then $\det(v_{ij}) = \sqrt{\det(g_{kl})}$ [Mo 2001, p. 1333] and $g = \sum_{i=1}^m \omega_i^2$.

Taking the exterior derivative of ω_m yields the Chern connection on π^*TM described by an $m \times m$ matrix of 1-forms (ω_{ij}) on SM. The coframe ω_i and connection forms $\omega_{m\alpha}$ determine the volume form with respect to the Riemannian metric $G := \sum_{i=1}^m \omega_i^2 + \sum_{\lambda=1}^{m-1} \omega_{m\lambda}^2$ on SM as follows:

$$\Pi := \omega_1 \wedge \cdots \wedge \omega_m \wedge \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1}.$$

A direct calculation yields

$$\begin{aligned}\Pi &= v_{1i_1}dx^{i_1}\wedge\cdots\wedge v_{mi_m}dx^{i_1}\wedge\omega_{m1}\wedge\cdots\wedge\omega_{m,m-1}\\ &= \det(v_{ij})dx^1\wedge\cdots\wedge dx^m\wedge\chi = \sqrt{\det(g_{kl})}\,dx\wedge\chi,\end{aligned}\tag{2-2}$$

where $dx = dx^1\wedge\cdots\wedge dx^m$ and

$$\chi \equiv \omega_{m1}\wedge\cdots\wedge\omega_{m,m-1} \quad \mathrm{mod}\ dx^j.\tag{2-3}$$

If Ω is a compact domain in M, define the energy of $\phi : (\Omega,\ F)\to(N,\ h)$ by

$$E(\phi,\ \Omega) = \frac{1}{c}\int_{S\Omega} e(\phi)\Pi,\tag{2-4}$$

where c is the volume of the standard $(m-1)$-dimensional sphere and $S\Omega$ is the projective sphere bundle of Ω. If M is compact, we write $E(\phi) = E(\phi,\ M)$.

Plugging (2-1) into (2-4) and using (2-2) we have

$$E(\phi) = \frac{1}{2c}\int_{SM} g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}\sqrt{\det(g_{kl})}\,dx\wedge\chi.\tag{2-5}$$

Conformal transformations of Finsler metrics. Let F be a Finsler metric on a manifold M and g its fundamental tensor. For two arbitrary nonzero vectors $v,\ y$ in T_xM, the angle $\theta(y,\ v)$ between y and v is defined by [Antonelli et al. 1993; Bácsó and Cheng 2007]

$$\cos\theta(y,\ v) := \frac{g_{(x,\,y)}(y,\ v)}{\sqrt{g_{(x,\,y)}(y,\ y)}\sqrt{g_{(x,\,y)}(v,\ v)}}.$$

Let F and $\bar F$ be two Finsler metrics on a manifold M. If the angle $\theta(y,\ v)$ with respect to F is equal to the angle $\bar\theta(y,\ v)$ with respect to $\bar F$ for any nonzero vectors $v,\ y\in T_xM$ and any $x\in M$, then F is called *conformal* to $\bar F$ and the transformation $F\to\bar F$ of the metric is called a *conformal transformation*.

Now assume that F is conformal to $\bar F$. From [Antonelli et al. 1993], we have

$$\bar F(x,\ y) = e^{c(x)}F(x,\ y)\tag{2-6}$$

for some scalar function $c(x)$. Furthermore, we have (see [Matsumoto 2003])

$$\bar g_{ij}(x,\ y) = e^{2c(x)}g_{ij}(x,\ y),\tag{2-7}$$

$$\bar g^{ij}(x,\ y) = e^{-2c(x)}g^{ij}(x,\ y).\tag{2-8}$$

3. Conformal invariance

In this section we are going to show an important property for Finsler harmonic maps on two-dimensional domains: the conformal invariance required in the proof of the regularity result.

Theorem 3.1. *The energy functional E on the set of Finsler harmonic maps from a surface M to a Riemannian manifold N depends only on the conformal structures on M.*

Proof. This follows from the conformal invariance of the energy functional E in dimension two. We assume that $\varphi : F \to \bar{F}$ is given by (2-6) and denote the corresponding objects with respect to $\bar{F}$ by adding a bar $^{-}$.

By using (2-7), we obtain

$$\sqrt{\det(\bar{g}_{kl})} = \sqrt{\det(e^{2c(x)} g_{kl})} = \sqrt{e^{4c(x)}\det(g_{kl})} = e^{2c(x)}\sqrt{\det(g_{kl})}. \tag{3-1}$$

The first structure equation for $(M,\ F)$ can be written as

$$d\omega_i = \sum \omega_j \wedge \omega_{ji}, \quad \omega_{ij} + \omega_{ji} = -2\sum A_{ijk}\omega_{2k}, \tag{3-2}$$

where $A_{ijk} = A(e_i,\ e_j,\ e_k)$ and A is the Cartan tensor of F. From [Mo 2006, (3.79)], we have

$$\omega_{12} \equiv -{u_1}^i F_{y^i y^j} dy^j \mod dx^i. \tag{3-3}$$

By using (2-3), (3-2) and (3-3) we obtain

$$\chi \equiv \omega_{21} = -\omega_{12} - 2\sum A_{21i}\omega_{2i} = -\omega_{12} \equiv {u_1}^i F_{y^i y^j} dy^j, \tag{3-4}$$

where the equivalences are mod dx^i and the ${u_j}^i$ are defined by

$$e_j = {u_j}^i \frac{\partial}{\partial x^i}.$$

It is not difficult to conclude that the section e_1 is in fact globally defined [Mo 2006]. The frame $\{e_1,\ e_2\}$ is call the *Berwald frame* [Bao and Chern 1996]. Explicitly,

$$e_1 = \frac{F_{y^2}}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^1} + \frac{-F_{y^1}}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^2}.$$

It follows that

$${u_1}^1 = \frac{F_{y^2}}{\sqrt{\det(g_{ij})}}, \qquad {u_1}^2 = \frac{-F_{y^1}}{\sqrt{\det(g_{ij})}}.$$

Under the conformal transformation (2-6), from (3-1) we have

$$\bar{u}_1{}^1 = \frac{\bar{F}_{y^2}}{\sqrt{\det(\bar{g}_{ij})}} = \frac{e^{c(x)} F_{y^2}}{e^{2c(x)}\sqrt{\det(g_{ij})}} = e^{-c(x)}{u_1}^1,$$

$$\bar{u}_1{}^2 = -\frac{\bar{F}_{y^1}}{\sqrt{\det(\bar{g}_{ij})}} = e^{-c(x)}{u_1}^2,$$

$$\bar{F}_{y^i y^j} = (e^{c(x)}F)_{y^i y^j} = e^{c(x)} F_{y^i y^j}.$$

Hence

$$\bar{u}_1{}^i \bar{F}_{y^i y^j} = u_1{}^i F_{y^i y^j}.$$

From this together with (3-4) we get

$$\bar{\chi} \equiv \chi \qquad \text{mod} \quad dx^i. \tag{3-5}$$

Thus, for any $\phi \in W^{1,2}(M,\ N)$,

$$\begin{aligned}\bar{E}(\phi) &= \frac{1}{2c}\int_{SM} \bar{g}^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}\sqrt{\det(\bar{g}_{kl})}\,dx\wedge\bar{\chi}\\ &= \frac{1}{2c}\int_{SM} e^{-2c(x)}g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}e^{2c(x)}\sqrt{\det(g_{kl})}\,dx\wedge\chi\\ &= \frac{1}{2c}\int_{SM} g^{ij}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}h_{\alpha\beta}\sqrt{\det(g_{kl})}\,dx\wedge\chi = E(\phi).\end{aligned}$$ □

Let $(M,\ F)$ be a Finsler surface and set

$$\tilde{g}^{ij}(x) := \frac{\int_{S_xM} g^{ij}(x,\ y)\sqrt{\det(g_{kl}(x,\ y))}\chi}{\int_{S_xM}\sqrt{\det(g_{kl}(x,\ y))}\chi},$$
$$\tilde{g} := \tilde{g}_{ij}dx^i\otimes dx^j, \quad \text{where } (\tilde{g}_{ij}) = (\tilde{g}^{ij})^{-1}.$$

Then (see [Mo and Yang 2005, Lemma 5.2]) $\tilde{g}$ is a Riemannian metric on M, called the (Riemannian) metric *induced* by F.

Proposition 3.2. *Let F and $\bar{F}$ be two Finsler metrics on a surface. If $\bar{F}(x,\ y) = e^{c(x)}F(x,\ y)$, then*

$$\tilde{\bar{g}}(x) = e^{2c(x)}\tilde{g},$$

where $\tilde{\bar{g}}$ and $\tilde{g}$ are the induced metrics by $\bar{F}$ and F respectively.

Proof. Using (2-7), (3-1) and (3-5) we get

$$\begin{aligned}\tilde{\bar{g}}^{ij}(x) &= \frac{\int_{S_xM}\bar{g}^{ij}(x,\ y)\sqrt{\det(\bar{g}_{kl}(x,\ y))}\bar{\chi}}{\int_{S_xM}\sqrt{\det(\bar{g}_{kl}(x,\ y))}\chi}\\ &= \frac{\int_{S_xM}e^{-2c(x)}g^{ij}(x,\ y)e^{2c(x)}\sqrt{\det(g_{kl}(x,\ y))}\chi}{\int_{S_xM}e^{2c(x)}\sqrt{\det(g_{kl}(x,\ y))}\chi} = e^{-2c(x)}\tilde{g}^{ij}(x).\end{aligned}$$

It follows that

$$\tilde{\bar{g}}(x) := \tilde{\bar{g}}_{ij}(x)dx^i\otimes dx^j = e^{2c(x)}\tilde{g}_{ij}(x)dx^i\otimes dx^j = e^{2c(x)}\tilde{g}(x).$$ □

Note that locally all Riemannian conformal structures are equivalent [Hélein 2002, Theorem 1.1.3].

Corollary 3.3. *Let (M,F) be a Finsler surface. For each point x_0, there is a local coordinate system $(U,\ x^i)$ in M, and a conformal transformation $F \to \bar{F}$, such that the Riemannian metric induced by $\bar{F}$ is a canonical Euclidean metric on U.*

4. Proof of Theorem 1.1

Before giving the proof proper, we state several lemmas. Throughout this section, D will denote a small disc in $\mathbb{R}^2$, but its radius may shrink as we go along if necessary.

Lemma 4.1. *Let $D \subset \mathbb{R}^2$ be a disk. If $\phi \in W^{1,2}(D, \mathbb{S}^n)$ is a weakly Finsler harmonic map, then for any $1 \leq \alpha, \beta \leq n$, the field*

$$V^{\alpha\beta} \equiv \phi^\beta \nabla\phi^\alpha - \phi^\alpha \nabla\phi^\beta - \nabla\varphi^{\alpha\beta} \in L^2(D,\ \mathbb{R}^2),$$

is divergence-free, where $\varphi^{\alpha\beta}$ is a solution of

$$\Delta\varphi^{\alpha\beta} = \nabla\psi \cdot (\phi^\beta \nabla\phi^\alpha - \phi^\alpha \nabla\phi^\beta) \tag{4-1}$$

in which

$$\psi(x) = -\log\sqrt{\det(\tilde{g}_{kl})} - \log\int_{S_xM}\sqrt{\det(g_{kl}(x,\ y))}\chi.$$

Moreover, ϕ satisfies

$$-\Delta(\phi-\eta)^\alpha = \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot V^{\alpha\beta} \tag{4-2}$$

where $\eta = (\eta^1,\ \dots,\ \eta^{n+1})$ is a solution of

$$-\Delta\eta^\alpha = \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot \nabla\varphi^{\alpha\beta} - \nabla\psi \cdot \nabla\phi^\alpha. \tag{4-3}$$

Proof. For a round spherical target manifold, the Euler–Lagrange equation of ϕ is given by $\Delta\phi = |\nabla\phi|^2\phi + \nabla\psi \cdot \nabla\phi$ [Mo and Yang 2005], equivalently,

$$\Delta\phi^\alpha = |\nabla\phi|^2\phi^\alpha + \nabla\psi \cdot \nabla\phi^\alpha \tag{4-4}$$

for $\alpha = 1,\ \dots,\ n+1$. Set $\Omega^{\alpha\beta} \equiv \phi^\beta\nabla\phi^\alpha - \phi^\alpha\nabla\phi^\beta$. Then

$$\begin{aligned}
\operatorname{div}(\Omega^{\alpha\beta}) &= \operatorname{div}(\phi^\beta\nabla\phi^\alpha) - \operatorname{div}(\phi^\alpha\nabla\phi^\beta)\\
&= \phi^\beta\Delta\phi^\alpha + \nabla\phi^\beta\cdot\nabla\phi^\alpha - (\phi^\alpha\Delta\phi^\beta + \nabla\phi^\alpha\cdot\nabla\phi^\beta)\\
&= \phi^\beta\Delta\phi^\alpha - \phi^\alpha\Delta\phi^\beta\\
&= \phi^\beta(|\nabla\phi|^2\phi^\alpha + \nabla\psi\cdot\nabla\phi^\alpha) - \phi^\alpha(|\nabla\phi|^2\phi^\beta + \nabla\psi\cdot\nabla\phi^\beta)\\
&= \nabla\psi\cdot(\phi^\beta\nabla\phi^\alpha - \phi^\alpha\nabla\phi^\beta) = \nabla\psi\cdot\Omega^{\alpha\beta}.
\end{aligned}$$

Together with (4-1) we obtain

$$\begin{aligned}\operatorname{div}(V^{\alpha\beta}) &= \operatorname{div}(\Omega^{\alpha\beta} - \nabla\varphi^{\alpha\beta}) = \operatorname{div}(\Omega^{\alpha\beta}) - \operatorname{div}(\nabla\varphi^{\alpha\beta}) \\ &= \nabla\psi \cdot \Omega^{\alpha\beta} - \Delta\varphi^{\alpha\beta} = \nabla\psi \cdot (\Omega^{\alpha\beta} - \phi^\beta\nabla\phi^\alpha + \phi^\alpha\nabla\phi^\beta) = 0.\end{aligned}$$

This implies that $V^{\alpha\beta}$ is divergence-free. From (4-1) and the regularity theory of elliptic equations, we obtain $\varphi^{\alpha\beta} \in W^{2,2}(D)$. To see (4-2), observe that

$$\sum_{\beta=1}^{n+1} \phi^\beta \nabla\phi^\beta = 0, \tag{4-5}$$

since $|\phi|^2 = 1$. We rewrite (4-4) as

$$-\Delta\phi^\alpha = -K - \nabla\psi \cdot \nabla\phi^\alpha, \tag{4-6}$$

where

$$\begin{aligned}K = |\nabla\phi|^2\phi^\alpha &= \Big(\sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot \nabla\phi^\beta\Big)\phi^\alpha = \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot (\phi^\alpha\nabla\phi^\beta) \\ &= \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot (\phi^\alpha\nabla\phi^\beta - \phi^\beta\nabla\phi^\alpha) = -\sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot (V^{\alpha\beta} + \nabla\varphi^{\alpha\beta}),\end{aligned} \tag{4-7}$$

where we used (4-5). Plugging the last expression into (4-6) and using (4-3) yields

$$-\Delta\phi^\alpha = \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot (V^{\alpha\beta} + \nabla\varphi^{\alpha\beta}) - \nabla\psi \cdot \nabla\phi^\alpha = \sum_{\beta=1}^{n+1} \nabla\phi^\beta \cdot V^{\alpha\beta} - \Delta\eta^\alpha.$$

Thus $\phi - \eta$ satisfies (4-2). □

Remark. From (4-3) and the regularity theory of elliptic equations, we have $\eta^\alpha \in C^0(D)$.

The right-hand side of (4-2) has a special structure: it is the product of a curl-free vector field and a divergence-free vector field. It follows from Lemma 4.3 below that the product belongs to the Hardy space $\mathscr{H}^1(\mathbb{R}^m)$ [Lin and Wang 2008, Definition 3.2.3]. (The next three lemmas can be found in [Lin and Wang 2008].)

Lemma 4.2 [Stein 1993]. *For $f \in \mathscr{H}^1(\mathbb{R}^m)$, let $u \in L^1(\mathbb{R}^m)$ be a solution of*

$$-\Delta u = f \quad \text{in } \mathbb{R}^m,$$

where Δu is the Laplacian of u. Then all second-order derivatives $\nabla^2 u$ of u satisfy $\nabla^2 u \in L^1(\mathbb{R}^m)$ and

$$\|\nabla^2 u\|_{L^1(\mathbb{R}^m)} \le C\|f\|_{\mathscr{H}^1(\mathbb{R}^m)}.$$

Lemma 4.3 [Coifman et al. 1989]. *For* $1 < p < +\infty$ *and* $q = p/(p-1)$, *suppose* $h \in W^{1,p}(\mathbb{R}^m)$ *and* $G \in L^q(\mathbb{R}^m, \mathbb{R}^m)$ *is a divergence-free vector field. Then the function* $f := \nabla h \cdot G \in \mathcal{H}^1(\mathbb{R}^m)$. *Moreover, there exists a positive constant* C_m *depending only on m such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^m)} \le C_m \|h\|_{W^{1,p}(\mathbb{R}^m)} \|G\|_{L^q(\mathbb{R}^m)}.$$

Lemma 4.4 [Hélein 2002, Theorem 3.3.4]. *If* $f \in W^{1,2}(\mathbb{R}^2)$ *has compact support and the weak derivative* ∇f *of* f *belongs to the Lorentz space* $L^{(2,1)}(\mathbb{R}^2)$, *then* $f \in C^0(\mathbb{R}^2)$.

Proof of Theorem 1.1. Since regularity is a local property, it suffices to show that it is valid in the neighborhood of each point in M. But, since every sufficiently small geodesic ball in $(M, \tilde{g})$ is conformal equivalent to the unit ball D of $\mathbb{R}^2$ equipped with the canonical metric via a conformal transformation of corresponding Finsler metric (Corollary 3.3), and Finsler harmonic maps are preserved under Finsler conformal transformation (Theorem 3.1), we assume that $M = D$ is a disc in $\mathbb{R}^2$.

The Euler–Lagrange equation for Finsler harmonic map ϕ is given by [Mo and Yang 2005, (5.8)]

$$\Delta\phi = A(\phi)(\nabla\phi, \nabla\phi) + \nabla\psi \cdot \nabla\phi, \tag{4-8}$$

where $A(\phi)$ is the second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Note that $\mathbb{S}^n$ is a totally umbilical hypersurface in $\mathbb{R}^{n+1}$ with unit normal field ϕ. Hence

$$A(\phi)(\nabla\phi, \nabla\phi) = |\nabla\phi|^2\phi.$$

Plugging this into (4-8) yields (4-4).

For $1 \le \alpha, \ \beta \le n+1$, let $V^{\alpha\beta} \in L^2(D, \ \mathbb{R}^2)$ be divergence-free vector fields given by Lemma 4.2. Let $\hat{\phi} \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^n)$ be an extension of ϕ such that $\|\nabla\hat{\phi}\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\phi\|_{L^2(D)}$ and $\hat{V}^{\alpha\beta} \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ be an extension of $V^{\alpha\beta}$ such that

$$\operatorname{div}\,(\hat{V}^{\alpha\beta}) = 0, \quad \|\hat{V}^{\alpha\beta}\|_{L^2(\mathbb{R}^2)} \le C\|V^{\alpha\beta}\|_{L^2(D)}. \tag{4-9}$$

The existence of such $\hat{V}^{\alpha\beta}$ can be obtained as follows.

We write $V^{\alpha\beta}$ as

$$V^{\alpha\beta} = (V_1^{\alpha\beta}, \ V_2^{\alpha\beta}). \tag{4-10}$$

Then

$$0 = \operatorname{div}(V^{\alpha\beta}) = \frac{\partial V_1^{\alpha\beta}}{\partial x^1} + \frac{\partial V_2^{\alpha\beta}}{\partial x^2}.$$

It follows that

$$\frac{\partial V_1^{\alpha\beta}}{\partial x^1} = \frac{\partial(-V_2^{\alpha\beta})}{\partial x^2};$$

therefore, there exists $\rho \in W^{1,2}(D)$ such that

$$\frac{\partial \rho}{\partial x^1} = -V_2^{\alpha\beta}, \quad \frac{\partial \rho}{\partial x^2} = V_1^{\alpha\beta}.$$

Substituting these into (4-10) yields

$$V^{\alpha\beta} = \left(\frac{\partial \rho}{\partial x^2}, -\frac{\partial \rho}{\partial x^1}\right)$$

in D. Let $\hat{\rho} \in W^{1,2}(\mathbb{R}^2)$ be an extension of ρ such that

$$\|\nabla \hat{\rho}\|_{L^2(\mathbb{R}^2)} \le C\|\nabla \rho\|_{L^2(D)}.$$

Then

$$\hat{V}^{\alpha\beta} :\equiv \left(\frac{\partial \hat{\rho}}{\partial x^2}, -\frac{\partial \hat{\rho}}{\partial x^1}\right)$$

satisfies (4-9).

For $1 \le \alpha \le n+1$, let $\omega^\alpha \in L^1(\mathbb{R}^2)$ be a solution of

$$-\Delta\omega^\alpha = \sum_{\beta=1}^{n+1} \nabla\hat{\phi}^\beta \cdot \hat{V}^{\alpha\beta}$$

in $\mathbb{R}^2$. By Lemmas 4.2 and 4.3, we know that $\nabla^2\omega^\alpha \in L^1(\mathbb{R}^2)$ and

$$\begin{aligned}\|\nabla^2\omega^\alpha\|_{L^1(\mathbb{R}^2)} &\le C\sum_{\beta=1}^{n+1}\|\nabla\hat{\phi}^\beta \cdot \hat{V}^{\alpha\beta}\|_{\mathscr{H}^1(\mathbb{R}^2)}\\ &\le C\sum_{\beta=1}^{n+1}\|\nabla\hat{\phi}^\beta\|_{L^2(\mathbb{R}^2)}\|\hat{V}^{\alpha\beta}\|_{L^2(\mathbb{R}^2)}\\ &\le C\sum_{\beta=1}^{n+1}\|\nabla\phi^\beta\|_{L^2(D)}\|V^{\alpha\beta}\|_{L^2(D)} \le C\|\nabla\phi\|^2_{L^2(D)}.\end{aligned}$$

It follows that $\nabla\omega^\alpha \in W^{1,1}(\mathbb{R}^2)$. Note that for $m \ge 2$, $W^{1,1}(\mathbb{R}^m)$ is continuously embedded in $L^{(\frac{m}{m-1},1)}(\mathbb{R}^m)$ [Hélein 2002]. This implies that $\nabla\omega^\alpha \in L^{(2,1)}(\mathbb{R}^2)$. Hence by Lemma 4.4 we have $\omega^\alpha \in C^0_{loc}(\mathbb{R}^2)$.

Let $\nu^\alpha = \phi^\alpha - \eta^\alpha - \omega^\alpha : D \to \mathbb{R}$. By using (4-2) we have

$$\begin{aligned}-\Delta\nu^\alpha &= -\Delta(\phi^\alpha - \eta^\alpha - \omega^\alpha) = -\Delta(\phi^\alpha - \eta^\alpha) + \Delta\omega^\alpha\\ &= \sum_{\beta=1}^{n+1}\nabla\phi^\beta \cdot V^{\alpha\beta} - \sum_{\beta=1}^{n+1}\nabla\hat{\phi}^\beta \cdot \hat{V}^{\alpha\beta}\\ &= \sum_{\beta=1}^{n+1}\nabla\phi^\beta \cdot V^{\alpha\beta} - \sum_{\beta=1}^{n+1}\nabla\phi^\beta \cdot V^{\alpha\beta} = 0 \quad \text{in } D,\end{aligned}$$

so $\nu^\alpha \in C^0(D)$. This implies that $\phi \in C^0(D, \mathbb{S}^n)$ and $\phi \in C^\infty(D, \mathbb{S}^n)$ by higher-order regularity theory for harmonic maps [Jost 2008, Section 8.6]. □

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