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SPRINGER VARIETIES

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Springer varieties appear in both geometric representation theory and knot theory. Motivated by knot theory and categorification, Khovanov provides a topological construction of (m, m) Springer varieties. Here we extend his construction to all two-row Springer varieties. Using the combinatorial and diagrammatic properties of this construction we provide a particularly useful homology basis and construct the Springer representation using this basis. We also provide a skein-theoretic formulation of the representation in this case.

1. Introduction

Springer varieties (or Springer fibers) are certain subvarieties of the variety of full flags in $\mathbb{C}^n$. Given a partition λ of the number n the Springer variety $\mathcal{S}_\lambda$ is the collection of full flags in $\mathbb{C}^n$ fixed by a nilpotent linear operator with Jordan blocks given by λ. Springer varieties were first introduced by Springer who constructed irreducible representations of the symmetric group on their top nonzero cohomology classes [Springer 1976]. This remarkable construction has motivated the study of these varieties by geometric representation theorists; see, for example, [Fung 2003; Hotta 1981; Kazhdan and Lusztig 1980].

This representation of the symmetric group, which has come to be known as the Springer representation, appears in the literature in a variety of forms. This includes Springer's original representation [1976] on the cohomology of Springer varieties, the tensor product of Springer's representation with the sign representation [Lusztig 1981; Slodowy 1980; Borho and MacPherson 1981], and a representation on homology which is isomorphic to Springer's cohomology representation [Kazhdan and Lusztig 1980]. Many of these equivalences are proven in [Hotta 1981].

Springer varieties have also begun to appear with increasing frequency in the literature on knot homologies and categorification. For $n = 2m$ Khovanov constructs a functor-valued invariant of tangles using an arc algebra H^m with center isomorphic to the cohomology of the (m, m) Springer variety [Khovanov 2002]. In proving this isomorphism he provides a topological construction of the (m, m) Springer variety as a subspace of a product of spheres [Khovanov 2004]. Springer varieties also appear in Cautis–Kamnitzer knot homology via derived categories of coherent sheaves and in the Seidel–Smith link invariant from the symplectic geometry of nilpotent slices [Seidel and Smith 2006].

Stroppel gave a Lie-theoretic generalization [2005] of Khovanov's functor-valued invariant of tangles, and described diagrammatically [2009] the categories related to this construction and the top-degree cohomology of (m, m)-Springer varieties, using them to build the Springer representation. In this paper we consider homology rather than cohomology and all degrees rather than just the top degree.

The structure of Springer varieties is not well understood. In particular for general classes of Springer varieties the topology of individual components and the interaction of those components in not known. Finding topological models for Springer varieties has the potential to aid geometric representation theory as well as deepen our understanding of the connections between Springer varieties and knot theory.

In previous work we use Khovanov's topological construction to prove an isomorphism between the homology of the (m, m) Springer variety and the Bar-Natan skein module of the solid torus with boundary web $2m$ copies of the longitude [Russell 2009]. We also use this construction to give a completely explicit and combinatorial construction of Springer's representation on this class of Springer varieties [Russell and Tymoczko 2008].

Now consider n not necessarily even. The first main result of this paper is an extension of Khovanov's construction of a topological model for (m, m) Springer varieties to all Springer varieties $\mathcal{S}_\lambda$ where λ is a two element partition of the number n. This construction can be found in Section 2. The proof that our construction is homeomorphic to the Springer variety follows the structure and approach of [Russell and Tymoczko 2008, Appendix].

We draw the reader's attention to the fact that the object we construct and the Springer variety are *homeomorphic as topological spaces* and not isomorphic as algebraic varieties. For our purposes, this is sufficient since we are interested in studying the homology of the Springer variety. Throughout the paper we consider singular homology with complex coefficients, and we refer to the homeomorphic image of the irreducible components of the variety as irreducible components.

In Section 3 we analyze the intersection of irreducible components of two-row Springer varieties and build an exact sequence on homology. In order to construct

this sequence we generalize certain lemmas stated but not proven in [Khovanov 2004]. We prove these results in the Appendix.

In Section 4 we introduce diagrammatic generators for the homology of each component of a two-row Springer variety. We give three simple relations between the generators from different components within the same variety; these relations generalize those given in [Russell 2009]. We then use the exact sequence constructed in Section 3 to prove Theorem 4.3 which shows that these diagrammatic generators and relations describe the homology of the associated two-row Springer variety. We use these generators and relations to reduce to a basis.

Section 5 gives a completely diagrammatic construction of a symmetric group action on the homology of the $(n-k, k)$ Springer variety. As in [Russell and Tymoczko 2008], we prove it is isomorphic to the Springer representation by comparing it with the de Concini–Procesi construction of Springer representations [De Concini and Procesi 1981], making use of Garsia and Procesi's combinatorial analysis [1992] of de Concini and Procesi's work.

In response to a question of Stephan Wehrli we conclude Section 5 with the following theorem, which provides a skein-theoretic formulation of the Springer representation in the two-row case.

Theorem. *Given a diagrammatic homology generator $M \in H_*(\mathscr{S}_{n-k,k})$ and $\sigma \in S_n$ glue a flattened braid corresponding to σ to the bottom of M forming M'. Then the Springer action $\sigma \cdot M$ is equal to $s(M')$, where s is defined as follows.*

$$s\left(\rlap{\raise2pt\hbox{$\times$}}\bigcirc\right) = s\left(\asymp\right) + s\left(\,)\,(\,\right) \qquad\qquad s\left(\diagdown\!\bullet\right) = 0$$

$$s\left(M' \sqcup \bigcirc\right) = s(-2M') \qquad\qquad s\left(M' \sqcup \bullet\!\!\bigcirc\right) = 0$$

2. Extending Khovanov's topological construction

For $n = 2m$ Khovanov constructs a topological space with cohomology isomorphic to that of the (m, m) Springer variety and conjectures that the two are actually homeomorphic [Khovanov 2004, Conjecture 1]. The main ideas behind a proof of this fact can be found in [Cautis and Kamnitzer 2008], and detailed proofs are given in [Russell and Tymoczko 2008, Appendix] and independently in [Wehrli 2009]. In this section we generalize Khovanov's construction to all two-row Springer varieties. The proof that our construction is homeomorphic to the Springer variety follows that of [Russell and Tymoczko 2008, Appendix].

Let $n \geq 1$ be some positive integer. A complete flag in $\mathbb{C}^n$, denoted by $V_\bullet$, is a collection of nested subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n$$

such that the complex dimension of V_i is i. The collection of all such objects is the variety we denote by $\mathscr{F}_n$. Partial flags have the same nesting property but are not required to have subsets in every intermediate dimension.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ be a partition of the number n. Let $\Gamma : \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent linear operator with Jordan blocks of sizes specified by the partition λ. Then we have the following definition.

Definition 2.1. The Springer variety associated to the partition λ is

$$\mathscr{S}_\lambda = \{V_\bullet \in \mathscr{F}_n : \Gamma V_i \subseteq V_i \text{ for all } i\}.$$

We focus on Springer varieties associated to two-element partitions which we call two-row Springer varieties. Note that every two-element partition of n can be written as $(n-k, k)$ for some positive integer $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. We call $\mathscr{S}_{n-k,k}$ the $(n-k, k)$ Springer variety.

A Young diagram for the partition λ of the number n is a top and left justified collection of boxes such that the i-th row has λ_i boxes. A standard filling of this diagram is a filling of the boxes with the numbers 1 through n such that each occurs exactly once and the rows and columns are strictly decreasing from left to right and top to bottom respectively.

The irreducible components of Springer varieties are indexed by standard Young tableaux [Spaltenstein 1976; Vargas 1979]. In the two-row case, the tableaux of shape $(n-k, k)$ are in one-to-one correspondence with certain noncrossing matchings which we will call noncrossing matchings of type $(n-k, k)$. This was noticed in [Stroppel and Webster 2010].

Definition 2.2. Consider n vertices evenly spaced along a horizontal line. A noncrossing matching of type $(n-k, k)$ is a nonintersecting arrangement of k arcs and $n-2k$ rays incident on the n vertices lying above the horizontal line. We assume that the rays are "infinitely high", so that arcs cannot cross rays. Here is an example of a noncrossing matching of type $(6, 5)$:

You can get a unique noncrossing matching of type $(n-k, k)$ from a standard Young tableau via the following procedure, illustrated in Figure 1. Start with the smallest number in the bottom row, draw an arc with right endpoint incident on that

1	2	5	6	7	9
3	4	8	10	11	

$\leftrightarrow$

Figure 1. Going from a Young tableau to a noncrossing matching.

vertex and left endpoint incident on the nearest unoccupied vertex. Repeat this for all remaining numbers in the bottom row, moving from left to right in the bottom row. Fill any remaining unoccupied positions with rays. One can easily define an inverse operation and see this is indeed a bijection. We generalize this procedure in the proof Lemma 4.7, so the reader can go there for more details.

Matchings will be denoted by lowercase letters. For clarity we will make these bold throughout. Let $B^{n-k,k}$ be the set of noncrossing matchings of type $(n-k, k)$. Given $\boldsymbol{a} \in B^{n-k,k}$ write $(i, j) \in \boldsymbol{a}$ if $\boldsymbol{a}$ has an arc connecting vertices i and j. Write $(i) \in \boldsymbol{a}$ if $\boldsymbol{a}$ has a ray incident on the vertex i.

Let $N > 2n$ be a large fixed integer. Let $X : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ be a nilpotent linear operator with two Jordan blocks of size N. Let $\{e_1, \ldots, e_N, f_1, \ldots, f_N\}$ be an orthonormal basis for $\mathbb{C}^{2N}$ with the property that $Xe_i = e_{i-1}$ and $Xf_i = f_{i-1}$. Here we define $e_{-1} = f_{-1} = 0$.

Consider the following variety Y_n of partial flags in $\mathbb{C}^{2N}$.

$$Y_n = \left\{V_\bullet = V_1 \subset \cdots \subset V_n : \dim_{\mathbb{C}}(V_i) = i \text{ and } XV_i \subseteq V_{i-1}\right\}.$$

From now on flags will be written $(V_1, \ldots, V_n)$ to save space. For $0 \le k \le \lfloor \frac{n}{2} \rfloor$, let

$$V_{n-k,k} = \langle e_1, \ldots, e_{n-k}, f_1, \ldots, f_k \rangle$$

and define $Y_{n-k,k} = \{V_\bullet \in Y_n : V_n = V_{n-k,k}\}$. Since $Y_{n-k,k}$ is the set of all complete flags on $V_{n-k,k}$ fixed by X, it is diffeomorphic to $\mathcal{S}_{n-k,k}$.

For $1 \le i \le n$ define a subvariety of Y_n by

$$Z_n^i = \{V_\bullet \in Y_n : V_{i+1} = X^{-1}V_{i-1}\}.$$

Define the map

$$q : Z_n^i \to Y_{n-2}, \quad V_\bullet = (V_1, \ldots V_n) \mapsto V_\bullet' = (V_1, \ldots, V_{i-1}, XV_{i+2}, \ldots, XV_n).$$

This map is a $\mathbb{P}^1$ bundle [Cautis and Kamnitzer 2008, page 5]. To see this, suppose that we have $V_\bullet' = (V_1', \ldots, V_{n-2}') \in Y_{n-2}$ and are considering possible choices of $V_\bullet \in Z_n^i$ with $q(V_\bullet) = V_\bullet'$. Then we must have $XV_{i+2} = V_i'$ and $V_{i-1} = V_{i-1}'$. Since $V_{i-1}' \subset V_i'$ and $V_\bullet \in Z_n^i$ we have a $\mathbb{P}^1$ worth of choices for V_i.

Let $p = (0, 0, 1)$ be the north pole of the standard unit two-sphere S^2 embedded in $\mathbb{R}^3$. Let $-p = (0, 0, -1)$ be the south pole in S^2 where $-p$ is understood as the antipodal map.

Definition 2.3. Given $\boldsymbol{a} \in B^{n-k,k}$, define subspaces of $(S^2)^n$ by

$$S_{\boldsymbol{a},n-k,k} = \{(x_1, \ldots, x_n) \in (S^2)^n : x_i = x_j \text{ if } (i, j) \in \boldsymbol{a} \text{ and } x_i = (-1)^i p \text{ if } (i) \in \boldsymbol{a}\},$$

$$S_{\boldsymbol{a},n-k,k}' = \{(x_1, \ldots, x_n) \in (S^2)^n : x_i = -x_j \text{ if } (i, j) \in \boldsymbol{a} \text{ and } x_i = p \text{ if } (i) \in \boldsymbol{a}\}.$$

Taking unions over all $\boldsymbol{a} \in B^{n-k,k}$ define

$$X_{n-k,k} = \bigcup_{\boldsymbol{a}} S_{\boldsymbol{a},n-k,k} \subset (S^2)^n \qquad \text{and} \qquad X'_{n-k,k} = \bigcup_{\boldsymbol{a}} S'_{\boldsymbol{a},n-k,k} \subset (S^2)^n.$$

The space $X_{n-k,k}$ is a generalization of Khovanov's construction for $n = 2m$ of the (m, m) Springer variety [Khovanov 2004, page 4]. Indeed in the (m, m) case matchings have no rays, so each component is built by identifying coordinates pairwise as prescribed by the arcs of the associated matching.

Using the basis $\{e_1, \ldots, e_N, f_1, \ldots, f_N\}$ for $\mathbb{C}^{2N}$, write points in $\mathbb{P}^{2N-1}$ in the form $\langle \sum a_i e_i + \sum b_i f_i \rangle$, for $a_i, b_i \in \mathbb{C}$. For $\mathbb{P}^1$ write $e_1 = e$ and $f_1 = f$. Define the map $C : \mathbb{P}^{2N-1} \to \mathbb{P}^1$ by

$$\left\langle \sum a_i e_i + \sum b_i f_i \right\rangle \mapsto \left\langle \left(\sum a_i \right) e + \left(\sum b_i \right) f \right\rangle.$$

For each $V_\bullet \in Y_n$ define lines $L_1, \ldots, L_n$ by $V_i = V_{i-1} \oplus L_i$ and $L_i \perp V_{i-1}$.

Proposition 2.4 [Cautis and Kamnitzer 2008, Theorem 2.1]. *The map*

$$\ell : Y_n \to (\mathbb{P}^1)^n$$

defined by

$$V_\bullet \mapsto (C(L_1), C(L_2), \ldots, C(L_n))$$

is a diffeomorphism. Furthermore the image of Z_n^i under the diffeomorphism ℓ is exactly the elements in $(\mathbb{P}^1)^n$ satisfying $C(L_i) = -C(L_{i+1})$.

Let $s : S^2 - \{p\} \to \mathbb{C}$ be the stereographic projection, and let $\varphi : \mathbb{P}^1 - \{\langle e \rangle\} \to \mathbb{C}$ be defined by $\varphi(\langle xe + yf \rangle) = x/y$. Let $\Phi : \mathbb{P}^1 \to S^2$ be the diffeomorphism defined by

$$\Phi(\langle xe + yf \rangle) = \begin{cases} s^{-1} \circ \varphi(\langle xe + yf \rangle) & \text{if } \langle xe + yf \rangle \neq \langle e \rangle, \\ (0, 0, 1) & \text{if } \langle xe + yf \rangle = \langle e \rangle. \end{cases}$$

Define the diffeomorphism $\widetilde{\Phi} : (\mathbb{P}^1)^n \to (S^2)^n$ by

$$\widetilde{\Phi}(x_1, \ldots, x_n) = (\Phi(x_1), \ldots, \Phi(x_n)).$$

Given $\boldsymbol{a} \in B^{n-k,k}$ define $C_{\boldsymbol{a},n-k,k}$ to be the preimage $\ell^{-1} \circ \widetilde{\Phi}^{-1}(S'_{\boldsymbol{a},n-k,k})$.

For $k > 0$ let $\boldsymbol{a} \in B^{n-k,k}$ such that $(i, i+1) \in \boldsymbol{a}$, and let $\boldsymbol{a}' \in B^{n-k-1,k-1}$ be the noncrossing matching obtained from $\boldsymbol{a}$ by erasing the arc $(i, i+1)$. If $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n-2\}$ is the map

$$f(j) = \begin{cases} j & \text{if } j \leq i-1, \\ j-2 & \text{if } j \geq i+2, \end{cases}$$

then $\boldsymbol{a}'$ is comprised of the set of arcs

$$\{(f(j), f(j')) : (j, j') \in \boldsymbol{a} \text{ and } (j, j') \neq (i, i+1)\}$$

and the set of rays $\{(f(j)) : (j) \in \boldsymbol{a}\}$. Define the projection map $q' : S'_{\boldsymbol{a},n-k,k} \to S'_{\boldsymbol{a}',n-k-1,k-1}$ by $q'(x_1, \ldots, x_n) = (x_1, \ldots, \widehat{x_i}, \widehat{x_{i+1}}, \ldots, x_n)$ where $\widehat{x_j}$ omits the j-th coordinate.

Lemma 2.5. *For noncrossing matchings $\boldsymbol{a}$ and $\boldsymbol{a}'$ as above there is a commutative diagram*

$$
\begin{array}{ccc}
C_{\boldsymbol{a},n-k,k} & \xrightarrow{\widetilde{\Phi} \circ \ell} & S'_{\boldsymbol{a},n-k,k} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\
Y_{n-2} & \xrightarrow{\widetilde{\Phi} \circ \ell} & S'_{\boldsymbol{a}',n-k-1,k-1}
\end{array}
$$

and the image $q(C_{\boldsymbol{a},n-k,k})$ is $C_{\boldsymbol{a}',n-k-1,k-1}$.

Proof. We provide a short proof here. For more details see [Russell and Tymoczko 2008, Lemma 5.2], which contains an identical argument.

Since the coordinates x_i and x_{i+1} in $S'_{\boldsymbol{a},n-k,k}$ are antipodes, Proposition 2.4 allows us to conclude $C_{\boldsymbol{a},n-k,k} \subseteq Z_n^i$. Let $V_\bullet \in Z_n^i$. By the definition of Z_n^i we know that $X^{-1}V_{i-1} = V_{i+1}$ and thus $\ker X \subseteq V_{i+1}$. It follows that L_j is spanned by $\sum_{i \geq 2} a_i e_i + b_i f_i$ and $XL_j \perp XV_{j-1}$ for all $j \geq i+2$. This shows that if $V_\bullet \in C_{\boldsymbol{a},n-k,k}$ then $(\widetilde{\Phi} \circ \ell)(q(V_\bullet)) = q'((\widetilde{\Phi} \circ \ell)(V_\bullet))$. By commutativity of the diagram, we conclude that $q(C_{\boldsymbol{a},n-k,k})$ is $C_{\boldsymbol{a}',n-k-1,k-1}$. $\qquad\square$

Lemma 2.6. *The union $\bigcup_{\boldsymbol{a} \in B^{n-k,k}} C_{\boldsymbol{a},n-k,k}$ is equal to $Y_{n-k,k}$. The $C_{\boldsymbol{a},n-k,k}$ are the irreducible components of $Y_{n-k,k}$.*

Proof. Consider the partition $(n, 0)$. The unique noncrossing matching $\boldsymbol{a} \in B^{n,0}$ has n rays and no arcs. Then $X'_{n,0} = S'_{\boldsymbol{a},n,0} = \{(p, \ldots, p) \in (S^2)^n\}$ and

$$
\widetilde{\Phi}^{-1}(S'_{\boldsymbol{a},n,0}) = \{(\langle e \rangle, \ldots, \langle e \rangle) \in (\mathbb{P}^1)^n\}.
$$

We want to find $V_\bullet \in Y_n$ such that $\ell(V_\bullet) = (\langle e \rangle, \ldots, \langle e \rangle)$.

Since $C(L_1) = \langle e \rangle$ we must have $L_1 = V_1 = \langle e_1 \rangle$. Inductively assume that $V_{i-1} = \langle e_1, \ldots, e_{i-1} \rangle$. We must have $L_i \perp V_{i-1}$ and $XL_i \subset V_{i-1}$ so $L_i = \langle x_i e_i + y_1 f_1 \rangle$. Since $C(L_i) = \langle e \rangle$ this forces $y_1 = 0$. This shows

$$
\begin{aligned}
(\ell^{-1} \circ \widetilde{\Phi}^{-1})(S'_{\boldsymbol{a},n,0}) = C_{\boldsymbol{a},n,0} &= \{(\langle e_1 \rangle, \langle e_1, e_2 \rangle, \ldots, \langle e_1, \ldots, e_n \rangle)\} \\
&= \{(V_{1,0}, \ldots, V_{n,0})\} \\
&= \{V_\bullet \in Y_n : XV_i \subset V_i \text{ and } V_n = V_{n,0}\} = Y_{n,0}.
\end{aligned}
$$

Thus we have proven $C_{\boldsymbol{a},n,0}$ is diffeomorphic to $Y_{n,0}$ for all n. There is a unique noncrossing matching $\boldsymbol{a} \in B^{1,1}$ namely the one with arc $(1, 2)$. From [Russell and Tymoczko 2008, Lemma 5.3] $C_{\boldsymbol{a},1,1}$ is diffeomorphic to $Y_{1,1}$.

We proceed by induction assuming the statement is true for $(n-k-1, k-1)$ in order to prove it for $(n-k, k)$. Assume that the claim holds for $(n-k-1, k-1)$

where $k \geq 1$. Let $\boldsymbol{a} \in B^{n-k,k}$ be a matching of type $(n-k, k)$. Because $\boldsymbol{a}$ is noncrossing, it necessarily has an arc of the form $(i, i+1)$. Let $\boldsymbol{a}' \in B^{n-k-1,k-1}$ be the matching of type $(n-k-1, k-1)$ obtained by erasing the arc $(i, i+1)$ in $\boldsymbol{a}$. Then we have the $\mathbb{P}^1$ bundle $C_{\boldsymbol{a},n,k} \xrightarrow{q} C_{\boldsymbol{a}',n-k-1,k-1}$.

If $q(V_\bullet) = V'_\bullet$ then for each j there exists some m with $V_j \subseteq X^{-1} V'_m$. Since $V'_\bullet = (V'_1, \ldots, V'_{n-2}) = (V_1, \ldots, V_{i-1}, XV_{i+2}, \ldots, XV_n)$ we may choose $m = j$ for $j \leq i$, $m = i$ for $i < j < i+3$, and $m = i-2$ for $j > i+2$.

Since $V'_\bullet \in C_{\boldsymbol{a}',n-k-1,k-1}$ each V'_m is contained in $V_{n-k-1,k-1}$, so each V_j is contained in $X^{-1} V_{n-k-1,k-1} = V_{n-k,k}$. Thus for all noncrossing matchings $\boldsymbol{a} \in B^{n-k,k}$, each $C_{\boldsymbol{a},n-k,k}$ is contained in $Y_{n-k,k}$. Since $\widetilde{\Phi}^{-1} \circ \ell$ is a diffeomorphism, all the $C_{\boldsymbol{a},n-k,k}$ are compact irreducible subvarieties of $Y_{n-k,k}$ with the same dimension as $X'_{n-k,k}$. Thus each $C_{\boldsymbol{a},n-k,k}$ is an irreducible component of the Springer variety $Y_{n-k,k}$. Since $S'_{\boldsymbol{a},n-k,k} \neq S'_{\boldsymbol{b},n-k,k}$ for $\boldsymbol{a} \neq \boldsymbol{b}$ and $\widetilde{\Phi} \circ \ell$ is a diffeomorphism, it follows that $C_{\boldsymbol{a},n-k,k} \neq C_{\boldsymbol{b},n-k,k}$ for $\boldsymbol{a} \neq \boldsymbol{b}$.

Recall that the irreducible components of $\mathcal{S}_{n-k,k}$ are in bijection with standard Young tableaux of shape $(n-k, k)$. [Stroppel and Webster 2010, Proposition 1.3] shows the set of noncrossing matchings of type $(n-k, k)$ are in bijection with standard Young tableaux of shape $(n-k, k)$. These results together show that the irreducible components of $\mathcal{S}_{n-k,k}$ are indexed by noncrossing matchings of type $(n-k, k)$. Thus $\bigcup_{\boldsymbol{a} \in B^{n-k,k}} C_{\boldsymbol{a},n-k,k} \subseteq Y_{n-k,k}$ is isomorphic to the $(n, n-k)$ Springer variety, and we conclude $\bigcup_{\boldsymbol{a} \in B^{n-k,k}} C_{\boldsymbol{a},n-k,k} = Y_{n-k,k}$. $\square$

Theorem 2.7. *The $(n, n-k)$ Springer variety $\mathcal{S}_{n-k,k}$ is homeomorphic to $X_{n-k,k}$.*

Proof. Lemmas 2.5 and 2.6 imply that $\mathcal{S}_{n-k,k}$ is homeomorphic to $X'_{n-k,k}$. Now we use an antipodal map to show that $X'_{n-k,k}$ and $X_{n-k,k}$ are homeomorphic.

Define $\gamma : (S^2)^n \to (S^2)^n$ as $\gamma((x_1, \ldots, x_n)) = (-x_1, x_2, \ldots, (-1)^n x_n)$. This map is its own inverse and thus is a diffeomorphism. Since $\boldsymbol{a} \in B^{n-k,k}$ is noncrossing each arc in $\boldsymbol{a}$ has some number of arcs and no rays between its endpoints. This means that there are an even number of vertices between the endpoints of each arc and every arc $(i, j) \in \boldsymbol{a}$ has one even and one odd endpoint. Therefore γ is the identity on exactly one of the coordinates x_i, x_j and the antipodal map on the other. Furthermore given a ray $(i) \in \boldsymbol{a}$ the map γ is the identity on x_i if i is even and the antipodal map if i is odd. Hence $\gamma(X_{n-k,k}) = X'_{n-k,k}$. $\square$

Again we alert the reader to the fact that this is a *homeomorphism of topological spaces* and not an isomorphism of algebraic varieties. One immediate consequence of our topological construction of the space $X_{n-k,k}$ is that given any two $\boldsymbol{a}, \boldsymbol{b} \in B^{n-k,k}$ the spaces $S_{\boldsymbol{a}}$ and $S_{\boldsymbol{b}}$ are homeomorphic to the product $(S^2)^k$. Contrast this with [Stroppel and Webster 2010, Example 8] where the authors show that the two irreducible components of the (2,2) Springer variety are both iterated $\mathbb{P}^1$-bundles but are not isomorphic as algebraic varieties.

We finish this section by using our new construction to get a *topological* picture of the (3, 1) Springer variety. The topological model $X_{3,1}$ for the (3, 1) Springer variety is built from the homeomorphic images of three irreducible components indexed by the three elements of $B^{3,1}$ shown here:

$$a = \quad \overset{\frown}{} \;|\;| \qquad b = \;|\; \overset{\frown}{} \;|\; \qquad c = \;|\;|\; \overset{\frown}{}$$

Each of these is homeomorphic to S^2:

$$S_a = \{(x, x, -p, p) : x \in S^2\},$$
$$S_b = \{(-p, x, x, p) : x \in S^2\},$$
$$S_c = \{(-p, p, x, x) : x \in S^2\}.$$

These spaces intersect in two different points:

$$S_a \cap S_b = \{(-p, -p, -p, p) \in (S^2)^4\},$$
$$S_b \cap S_c = \{(-p, p, p, p) \in (S^2)^4\},$$
$$S_a \cap S_c = \varnothing.$$

Putting this information together we see that $X_{3,1}$ is the wedge of three spheres as shown in Figure 2. Note that $H_0(X_{3,1}) = \mathbb{C}$ and that $H_2(X_{3,1}) = \mathbb{C}^{|B^{3,1}|} = \mathbb{C}^3$.
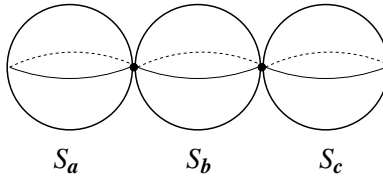
Figure 2. $X^{3,1}$ is a wedge of three copies of S^2.

3. Intersections of components

From now on we regard $X_{n-k,k}$ as the $(n-k, k)$ Springer variety. We reiterate that these two objects are the same in the topological sense and not the algebrogeometric sense. For fixed n and k we refer to $S_{a,n-k,k}$ simply as S_a. We call these S_a which are the homeomorphic images of the irreducible components of the Springer variety the irreducible components (or components) of $X_{n-k,k}$.

This section follows [Khovanov 2004, Section 3] proving a sequence of lemmas about intersection between components and uses this information to set up a Mayer–Vietoris type exact sequence on homology. Proposition 3.9 and the (m, m) case of Lemma 3.14 are stated without proof in that reference. Because we need these results for our arguments, we include proofs in the Appendix. Fung [2003]

and Stroppel and Webster [2010] have studied intersections of the components of the Springer variety in the two-row case. Much of what we do in this section rediscovers their results via our topological model $X_{n-k,k}$.

Matchings and their associated components.

Definition 3.1. Given $a, b \in B^{n-k,k}$ let $aw(b)$ be the result of reflecting b horizontally and gluing this reflection to a, as in this example:
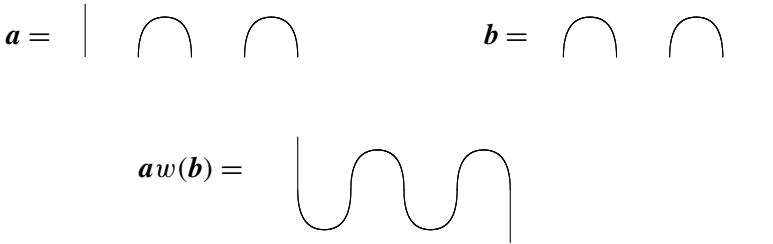

The 1-manifold $aw(b)$ will consist of circles, lines with both endpoints pointing up, lines with both endpoints pointing down, and lines with one endpoint pointing in each direction. We define $|aw(b)|$ to be the number of connected components in $aw(b)$.

Definition 3.2 (order on noncrossing matchings). Given $a, b \in B^{n-k,k}$, write $a \to b$ if one of the following is true:

- There is a quadruple $i < j < k < l$ such that $(i, j), (k, l) \in a$, $(i, l), (j, k) \in b$, and a and b are identical on all vertices not equal to i, j, k, l.
- There is a triple $i < j < k$ where $(i), (j, k) \in a$, $(i, j), (k) \in b$, and a and b are identical on all vertices not equal to i, j, k.

Define a partial order $a \prec b$ if there exist a chain of arrows $a \to \cdots \to b$. Extend this to a total order $<$. Note that the extension to a total order is not unique. Figure 3 shows an example.
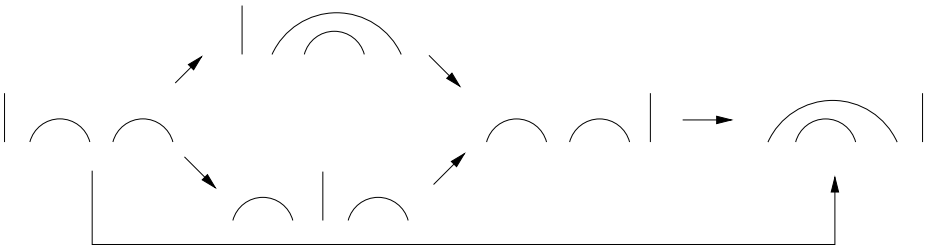
Figure 3. The $\to$ relation on elements of $B^{3,2}$.

Definition 3.3. Given $a, b \in B^{n-k,k}$ let the distance from a to b, denoted $d(a, b)$, be the minimal length m of a sequence $(a = a_0, a_1, \ldots, a_{m-1}, a_m = b)$ such that $a_i \to a_{i+1}$ or $a_i \leftarrow a_{i+1}$ for all i. We call a sequence that realizes the distance between a and b a minimal sequence.

The following observation of Fung shows that the line segments in the one manifold $aw(b)$ tell us when two components have empty intersection. A refinement of this result given in [Stroppel and Webster 2010, Lemma 19] further characterizes the intersection of components by looking at orientations of the 1-manifold $aw(b)$.

Proposition 3.4 [Fung 2003, Theorem 7.3]. *Given $a, b \in B^{n-k,k}$ if $aw(b)$ has a line with both endpoints pointing in the same direction then $S_a \cap S_b = \varnothing$.*

We give a special name to the subset of $B^{n-k,k}$ consisting of matchings associated to components that have nonempty intersection with a specific component.

Definition 3.5. Given $a \in B^{n-k,k}$ define the subset $B_a^{n-k,k}$ of $B^{n-k,k}$ as

$$B_a^{n-k,k} = \left\{ \begin{array}{r} b \in B^{n-k,k} \colon \text{each line in } aw(b) \text{ has endpoints} \\ \text{pointing in opposite directions} \end{array} \right\}.$$

Applying Proposition 3.4 we see that given $a \in B^{n-k,k}$

$$\left(\bigcup_{\substack{b < a \\ b \in B^{n-k,k}}} S_b \right) \cap S_a = \left(\bigcup_{\substack{b < a \\ b \in B_a^{n-k,k}}} S_b \right) \cap S_a.$$

For $b \in B_a^{n-k,k}$ the intersection $S_a \cap S_b$ is homeomorphic to $(S^2)^{\# \text{ of circles in } aw(b)}$.

Lemma 3.6. *Let $a \in B^{n-k,k}$ and $b \in B_a^{n-k,k}$. Then $aw(b)$ is a union of circles and line segments. If we enumerate the rays in the matchings a and b from left to right as $i_i, \ldots, i_{n-2k}$ and $j_i, \ldots, j_{2n-k}$ respectively then in $aw(b)$ each line segment will contain rays i_t and j_t for some $1 \le t \le n - 2k$.*

Proof. Each line segment in $aw(b)$ consists of two rays connected by some number of arcs. By assumption one ray is in a and one ray is in b. Say that rays i_{t_1} and j_{t_2} are part of the same segment in $aw(b)$. Then the rays $i_i, \ldots, i_{t_1-1}, j_1, \ldots, j_{t_2-1}$ lie on one side of this line segment. Since $aw(b)$ is noncrossing these rays must be connected pairwise in a noncrossing manner. Since $b \in B_a^{n-k,k}$ each i-ray must be connected to a j-ray. Therefore $t_1 = t_2$. $\square$

Lemma 3.7. *Let $a \in B^{n-k,k}$ and $b \in B_a^{n-k,k}$, and consider a sequence*

$$(a = a_0, a_1, \ldots, a_{m-1}, a_m = b).$$

If $|a_i w(b)| = |a_{i+1} w(b)| - 1$ for all i, the sequence is minimal.

Proof. For each $0 \leq i \leq m-1$ either $a_i \rightarrow a_{i+1}$ or $a_i \leftarrow a_{i+1}$. Thus either an arc and a ray are changing position, two unnested arcs are replaced by two nested arcs, or two nested arcs are replaced by unnested arcs. Consider the associated sequence

$$\big(aw(b) = a_0w(b), \ a_1w(b), \ \ldots, \ a_{m-1}w(b), \ a_mw(b) = bw(b)\big).$$

Analyzing each possibility, one can check that $|a_iw(b)| = |a_{i+1}w(b)| \pm 1$. Moreover, given $a_iw(b)$ and $a_{i+1}w(b)$, if the arc-ray or arc-arc pair that change from a_i to a_{i+1} are part of the same connected component in $a_iw(b)$ then $a_{i+1}w(b)$ will have one more component because we are splitting one component into two. On the other hand, if they are part of different connected components in $a_iw(b)$ then $a_{i+1}w(b)$ will have one less component because we are joining two components.

The one manifold $bw(b)$ has the maximum number of components possible, so $|aw(b)| \leq |bw(b)|$. Since each arrow move changes the number of components by 1, no sequence from a to b can be shorter than $|bw(b)| - |aw(b)|$. The sequence given in the statement of the lemma has $|aw(b)| = |bw(b)| - m$. Since no shorter sequence can exist the given sequence is minimal. $\qquad\square$

Given $a, b \in B^{n-k,k}$ label the k connected components of $aw(b)$ with $c_1, \ldots, c_k$. Associated to each c_i let

$$C_i = \{1 \leq j \leq n : c_i \text{ passes through vertex } j \text{ in } aw(b)\}.$$

Let the collection of all of these sets be denoted by $\mathscr{C}_{a,b} = \{C_i\}$. A useful way to understand the distance between matchings is to study the sequence of collections $(\mathscr{C}_{a,b}, \ldots, \mathscr{C}_{b,b})$ associated to a sequence $(a = a_0, a_1, \ldots a_{m-1}, a_m = b)$. Restating Lemma 3.7 in the language of the sets $\mathscr{C}_{a_i,b}$ we have Lemma 3.8.

Lemma 3.8. *Let $a \in B^{n-k,k}$ and $b \in B_a^{n-k,k}$, and consider a sequence $(a = a_0, \ldots, a_m = b)$. If for each $0 \leq i \leq m-1$ the vertices of the arc-ray or arc-arc pair in a_i being changed when transitioning from a_i to a_{i+1} are part of a single set in $\mathscr{C}_{a_i,b}$ then the sequence is minimal.*

The following formula, which we prove in the Appendix, is given in [Khovanov 2004, page 9].

Proposition 3.9. *For $n = 2m$ and $a, b \in B^{m,m}$ the distance $d(a, b)$ is equal to $m - |aw(b)|$.*

We will extend this formula to the distance between matchings of any type. In order to prove this we appeal to the notion of completing matchings of type $(n-k, k)$ to matchings of type $(n-k, n-k)$. This idea is also used in [Khovanov 2004, Section 5.4]. Completions are a natural way to see the set $B^{n-k,k}$ sitting inside of $B^{n-k,n-k}$.

Definition 3.10 (completion). Given $a \in B^{n-k,k}$ enumerate the rays in a from left to right by $(i_1), (i_2), \dots, (i_{n-2k})$. Define the function

$$\varphi : B^{n-k,k} \rightarrow B^{n-k,n-k}$$

where $\varphi(a)$ is the matching with arcs

- $(i+n-2k, j+n-2k)$ if (i, j) is an arc in a,
- $(n-2k-t+1, i_t+n-2k)$ if (i_t) is a ray in a.

In other words, we complete each ray in a to an arc by adding additional vertices to the left of the matching and connecting unanchored endpoints of the rays to these vertices in the only way that keeps the matching noncrossing. We call $\varphi(a)$ the *completion of a*. Figure 4 has an example.

Let $\widetilde{B}^{n-k,n-k}$ be the subset of $B^{n-k,n-k}$ consisting of matchings with no arcs of the form (i, j) for $0 \le i < j \le n-2k$. There is a natural inverse operation to completion on this subset of $B^{n-k,n-k}$.

Definition 3.11 (restriction). Consider $a \in \widetilde{B}^{n-k,n-k}$ Define the function

$$\psi : \widetilde{B}^{n-k,n-k} \rightarrow B^{n-k,k}$$

where $\psi(a)$ is the matching with

- arcs $(i-n+2k, j-n+2k)$ if (i, j) is an arc in a with $n-2k < i$,
- rays $(j-n+2k)$ if (i, j) is an arc in a with $1 \le i \le n-2k$.

In other words, we remove the first $n-2k$ vertices of a so that each arc with left endpoint incident on one of those vertices becomes a ray. We call $\psi(a)$ the *restriction of a*. Figure 4 has an example.
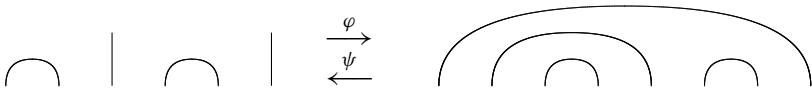
Figure 4. A matching $a \in B^{4,2}$ and its completion $\varphi(a) \in B^{4,4}$.

An immediate consequence of the definitions of restriction and completion is that φ gives a one-to-one correspondence between the sets $B^{n-k,k}$ and $\widetilde{B}^{n-k,n-k}$. Note also that when $n = 2m$ and $k = m$ the sets $B^{n-k,k}$, $B^{n-k,n-k}$, and $\widetilde{B}^{n-k,n-k}$ are the same. Furthermore the maps φ and ψ are both the identity in this case.

Proposition 3.12. *Given $a \in B^{n-k,k}$ and $b \in B_a^{n-k,k}$, every minimal sequence $(\varphi(a), \dots, \varphi(b))$ gives a minimal sequence $(a, \dots, b)$.*

Proof. Consider a minimal sequence $(\varphi(\boldsymbol{a}) = \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m-1}, \boldsymbol{x}_m = \varphi(\boldsymbol{b}))$. Since this sequence is minimal Lemma 3.8 guarantees that for all $0 \le i < m$ the collection $\mathscr{C}_{\boldsymbol{x}_i, \varphi(\boldsymbol{b})}$ is a refinement of $\mathscr{C}_{\boldsymbol{x}_{i-1}, \varphi(\boldsymbol{b})}$ in the sense that every set in $\mathscr{C}_{\boldsymbol{x}_{i-1}, \varphi(\boldsymbol{b})}$ is a union of sets in $\mathscr{C}_{\boldsymbol{x}_i, \varphi(\boldsymbol{b})}$. In particular, $\mathscr{C}_{\boldsymbol{x}_i, \varphi(\boldsymbol{b})}$ contains exactly one more set than $\mathscr{C}_{\boldsymbol{x}_{i-1}, \varphi(\boldsymbol{b})}$.

By Lemma 3.6 we know that the i-th ray in $\boldsymbol{a}$ and the i-th ray in $\boldsymbol{b}$ are part of the same line segment in $\boldsymbol{a}w(\boldsymbol{b})$. Under φ this line segment is completed to a circle passing through the newly added vertex $n - 2k - i + 1$. Thus each set in $\mathscr{C}_{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})}$ contains at most one of the vertices $1, \ldots, n - 2k$. Indeed the sets coming from circles in $\varphi(\boldsymbol{a})w(\varphi(\boldsymbol{b}))$ that were line segments in $\boldsymbol{a}w(\boldsymbol{b})$ each contain one of these vertices.

Each set in $\mathscr{C}_{\boldsymbol{x}_i, \varphi(\boldsymbol{b})}$ is a refinement of the collection $\mathscr{C}_{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})}$, so the sets in $\mathscr{C}_{\boldsymbol{x}_i, \varphi(\boldsymbol{b})}$ also each contain at most one of the vertices $1, \ldots, n - 2k$. This means that for all $0 \le i \le m$ the arcs in the matching $\boldsymbol{x}_i$ have at most one vertex incident on the vertices $1, \ldots, n - 2k$. Therefore $\boldsymbol{x}_i \in \widetilde{B}^{n-k, n-k}$. We can further conclude that whenever $\boldsymbol{x}_i \to \boldsymbol{x}_{i+1}$ then $\psi(\boldsymbol{x}_i) \to \psi(\boldsymbol{x}_{i+1})$. Similarly whenever $\boldsymbol{x}_i \leftarrow \boldsymbol{x}_{i+1}$ then $\psi(\boldsymbol{x}_i) \leftarrow \psi(\boldsymbol{x}_{i+1})$.

We conclude that the sequence $(\varphi(\boldsymbol{a}) = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_m = \varphi(\boldsymbol{b}))$ has a corresponding sequence $(\boldsymbol{a}, \psi(\boldsymbol{x}_1), \ldots, \psi(\boldsymbol{x}_{m-1}), \boldsymbol{b})$. Finally, using the formula in Proposition 3.9 we see that

$$d(\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})) = n - k - |\varphi(\boldsymbol{a})w(\varphi(\boldsymbol{b}))| = n - k - |\boldsymbol{a}w(\boldsymbol{b})|.$$

Thus we conclude that this sequence is minimal. $\qquad\square$

Corollary 3.13. *Let $\boldsymbol{a} \in B^{n-k,k}$ and $\boldsymbol{b} \in B_{\boldsymbol{a}}^{n-k,k}$. Then $d(\boldsymbol{a}, \boldsymbol{b}) = n - k - |\boldsymbol{a}w(\boldsymbol{b})|$.*

Next we want to show that a very specific minimal sequence between $\boldsymbol{a}$ and $\boldsymbol{b}$ always exists.

Lemma 3.14. *Given $\boldsymbol{a} \in B^{n-k,k}$ and $\boldsymbol{b} \in B_{\boldsymbol{a}}^{n-k,k}$ there exists $\boldsymbol{c} \in B_{\boldsymbol{a}}^{n-k,k}$ with $d(\boldsymbol{a}, \boldsymbol{b}) = d(\boldsymbol{a}, \boldsymbol{c}) + d(\boldsymbol{c}, \boldsymbol{b})$ and $\boldsymbol{a} \succ \boldsymbol{c} \prec \boldsymbol{b}$.*

Proof. [Khovanov 2004, Lemma 1] asserts this statement for the (m, m) case. This proof, which is fairly involved, is given in the Appendix.

Using this fact we can guarantee the existence of a minimal sequence

$$\varphi(\boldsymbol{a}) \leftarrow \boldsymbol{a}_1 \leftarrow \cdots \leftarrow \boldsymbol{c}' \to \cdots \to \boldsymbol{a}_{m-1} \to \varphi(\boldsymbol{b}).$$

The proof of Proposition 3.12 implies that for all $1 \le i \le m$ we have $\psi(\boldsymbol{a}_i) \in B_{\boldsymbol{a}}^{n-k,k}$ and $\psi(\boldsymbol{a}_i) \in B_{\boldsymbol{b}}^{n-k,k}$. This gives us the minimal sequence

$$\boldsymbol{a} \leftarrow \psi(\boldsymbol{a}_1) \leftarrow \cdots \leftarrow \psi(\boldsymbol{c}') \to \cdots \to \psi(\boldsymbol{a}_{m-1}) \to \boldsymbol{b}.$$

Setting $\psi(\boldsymbol{c}') = \boldsymbol{c}$ the lemma is proven. $\qquad\square$

Lemma 3.15. *Let $a \in B^{n-k,k}$ and $b \in B_a^{n-k,k}$. If $d(a, c) = d(a, b) + d(b, c)$ then*

$$S_a \cap S_c = S_a \cap S_b \cap S_c.$$

Proof. Clearly $S_a \cap S_b \cap S_b \subseteq S_a \cap S_c$.

Since $d(a, c) = d(a, b) + d(b, c)$, there is a minimal sequence $(a, \ldots, b, \ldots, c)$. Consider the collections $\mathcal{C}_{a,c} = \mathcal{C}_{c,a} = \{C_1, \ldots, C_{i_{a,c}}\}$, $\mathcal{C}_{a,b} = \{C_1', \ldots, C_{i_{a,b}}'\}$, and $\mathcal{C}_{b,c} = \{C_1'', \ldots, C_{i_{b,c}}''\}$. The minimal sequence ensures that the collections $\mathcal{C}_{a,b}$ and $\mathcal{C}_{b,c}$ are each refinements of the collection $\mathcal{C}_{a,c}$.

In terms of sets in $\mathcal{C}_{a,c}$ we have

$$S_a \cap S_c = \{x = (x_1, \ldots, x_n) : x_i = x_j \text{ if there is } C_k \in \mathcal{C}_{a,c} \text{ with } i, j \in C_k\} \subset S_a.$$

Whenever $i, j \in C_k$ there are some $C_{k_1}' \in \mathcal{C}_{a,b}$ and $C_{k_2} \in \mathcal{C}_{b,c}$ with $C_k \subset C_{k_1}'$ amd $C_k \subset C_{k_2}''$. Then $x \in S_a \cap S_b$ and $x \in S_b \cap S_c$. Finally we conclude that $S_a \cap S_c \subseteq S_a \cap S_b \cap S_c$. $\qquad\square$

Lemma 3.16. *Let $S_{<a} \cap S_a = (\cup_{b<a} S_b) \cap S_a$. Then $S_{<a} \cap S_a = \cup_{b \to a}(S_b \cap S_a)$.*

Proof. By basic set theory we have $S_{<a} \cap S_a = \cup_{b<a}(S_a \cap S_b)$. Since $b < a$ whenever $b \to a$ we conclude that $\cup_{b \to a}(S_b \cap S_a) \subseteq S_{<a} \cap S_a$.

Take $x \in S_{<a} \cap S_a$. Then there exists $b < a$ such that $x \in S_a \cap S_b$. By Lemma 3.14 we have a minimal sequence $(a \leftarrow a_1 \cdots \leftarrow c \to \cdots a_{m-1} \to b)$. This sequence must begin with $\leftarrow$ since otherwise we would have a minimal sequence consisting entirely of $\to$ moves. The existence of such a sequence would mean that $a < b$, but we have assumed $b < a$.

From the sequence we have $d(a, b) = d(a, a_1) + d(a_1, b)$. From Lemma 3.15 we know

$$S_a \cap S_b = S_a \cap S_{a_1} \cap S_b \subseteq S_a \cap S_{a_1}.$$

We have shown that whenever $b < a$ there exists $a_1 \to a$ with $S_a \cap S_b \subseteq S_a \cap S_{a_1}$. $\quad\square$

A useful cell decomposition. We recall the definition of a CW-decomposition, which we will also sometimes call a cell decomposition or simply a decomposition. Then we introduce a CW-decomposition of $X_{n-k,k}$ that generalizes the one found in [Khovanov 2004, Lemma 4].

Definition 3.17. A CW-decomposition of a space X is a partition of X into open cells (possibly of varying dimension) each of which is either homeomorphic to an open ball (when the cell has dimension larger than zero) or a single point (when the cell has dimension zero).

Additionally it is required that for each n dimensional open cell C there is a homeomorphism f from the closed ball of dimension n to X such that

- f restricted to the interior of the closed ball is a homeomorphism onto C, and

- the image under f of the boundary of the closed ball is a finite union of cells of lower dimension.

As an example we can choose any point in the two-sphere $p \in S^2$. Then p and $S^2 - p$ constitute a CW-decomposition of S^2 into one two-dimensional cell and one zero-dimensional cell. Since each component $S_a \in X_{n-k,k}$ is homeomorphic to a product of spheres, the components S_a have an obvious CW-decomposition into even dimensional cells coming from their cartesian product structure. In Section 4 we will explore the combinatorics of this decomposition in depth.

This section gives another CW-decomposition for the components of $X_{n-k,k}$. This decomposition makes use of the order on matchings given in the previous section and restricts to a CW-decomposition of intersections of certain components (see Lemma 3.20), making it easier to construct an iterated Mayer–Vietoris sequence on homology.

It is an interesting question what the relationship might be between the aforementioned CW-decompositions for components of $X_{n-k,k}$ and the Schubert cell decomposition of Grassmannians. A related issue is what connection the order defined in this paper on components has to do with the Bruhat order on Schubert cells. We are also interested in exploring the relationship between our cells and the attracting cells found in the paper [Stroppel and Webster 2010] on two-row Springer fibers. We hope to explore these connections in upcoming work.

Definition 3.18 (decomposition of S_a). Given a matching $a \in B^{n-k,k}$ we construct a graph $\Gamma = (V, E)$ with a vertex $v \in V$ for each arc in a. Given $v \in V$ let (v_1, v_2) with $v_1 < v_2$ be the arc in a corresponding to v. There is an edge between $v, w \in V$ if and only if there is some $b \in B^{n-k,k}$ with $b \to a$ where b is identical to a off of vertices v_1, v_2, w_1, w_2. In b these vertices are connected in the only other possible noncrossing manner.

Let M be the set of all root vertices. Figure 5 has an example. Let $p = (0, 0, 1) \in S^2$ once again be the north pole of the standard embedded unit sphere. We construct a set $c(J) \subset S_a \subset (S^2)^n$ where $J \subseteq (E \cup M)$.

Let $c(J) \subset \{x = (x_1, \ldots, x_n) \in S_a\}$ such that

- $x_{v_1} = x_{w_1}$ if edge $(v, w) \in J$,
- $x_{v_1} \neq x_{w_1}$ if edge $(v, w) \in E - J$,
- $x_{v_1} = (-1)^{v_1} p$ if vertex $v \in J$,
- $x_{v_1} \neq (-1)^{v_1} p$ if vertex $v \in M - J$.

Figure 5. A matching and its graph Γ.

Note that we only specify values for x_{v_1} and x_{w_1} because $x_{v_1} = x_{v_2}$ and $x_{w_1} = x_{w_2}$.

Lemma 3.19. *The construction above is indeed a CW-decomposition of* S_a.

Proof. These cells are products of the cells found in [Khovanov 2004] with some points in fixed positions. The CW-decomposition structure is thus inherited from the CW-decomposition structure on the cells in that reference.

One can also see this structure directly. Using induction on the number of arcs it is straightforward to prove that cells $c(J)$ are trivial $\mathbb{R}^2$ bundles over $\mathbb{R}^{2k-|J|-2}$ and are thus homeomorphic to $\mathbb{R}^{2k-|J|}$. The cells are disjoint by construction. Also by construction, the boundary of a cell is the union of smaller cells. More precisely, if we let $\mathscr{J} = \{J' \subset (E \sqcup M) : J \subsetneq J'\}$ then $\partial(c(J)) = \bigsqcup_{J' \in \mathscr{J}} c(J')$. □

Lemma 3.20. *Given* $b \to a$ *the cell decomposition for* S_a *restricts to a cell decomposition for* $S_a \cap S_b$. *Therefore the decomposition for* S_a *also restricts to a decomposition for* $S_{<a} \cap S_a$.

Proof. Consider $b \to a$. Then either

- there exist $i < j < k < l$ with $(i, j), (k, l) \in b$ and $(i, l), (j, k) \in a$, or
- there exist $i < j < k$ with $(i), (j, k) \in b$ and $(i, j), (k) \in a$.

The matchings a and b are identical otherwise.

In the first case, no rays change position as we move from b to a. Thus we get that $S_a \cap S_b$ is a subcomplex of S_a by [Khovanov 2004, Lemma 4].

Consider the second case. In order for both a and b to be noncrossing, there can be no arcs (i', k') or rays (i') where $i < i' < j < k < k'$. Every arc with left endpoint i' with $i < i' < j$ must have right endpoint k' with $k' < j$, so all arcs incident on vertices between i and j are completely contained between i and j. Therefore there is an even number of vertices between i and j, and so i and j have opposite parity. Since there is an even number of vertices between j and k the vertices i and k have the same parity.

Let $\Gamma_a = (V_a, E_a)$ with distinguished vertex set M_a be the graph for a. The vertex $v \in V_a$ corresponding to the arc $(i, j) \in a$ is a root, so $v \in M_a$. Given $J \subset (V_a \cup M_a)$ whenever $v \in J$ all points $x = (x_1, \ldots, x_n) \in c(J)$ have $x_i = x_j = (-1)^i p$. For all J and for all $x \in c(J)$ we also have $x_k = (-1)^k p = (-1)^i p$ since (k) is a ray in a. Since $S_a \cap S_b = \{x \in S_a : x_i = x_j = x_k = (-1)^i p\}$ the cells $c(J)$ for which $v \in J$ form a cell complex for $S_a \cap S_b$.

Appealing to Lemma 3.16 we see that $S_{<a} \cap S_a$ is a subcomplex of the complex for S_a. □

Building an exact sequence. The following two statements are consequences of Lemma 3.16.

Lemma 3.21. *The inclusion* $(S_{<a} \cap S_a) \hookrightarrow S_a$ *induces an injective homomorphism on homology* $H_*(S_{<a} \cap S_a) \hookrightarrow H_*(S_a)$.

Lemma 3.22. *The homomorphism* $\bigoplus_{b \to a} H_*(S_b \cap S_a) \xrightarrow{f'} H_*(S_{<a} \cap S_a)$ *induced by inclusion is surjective.*

Now consider the Mayer–Vietoris sequence:

$$\cdots \to H_m(S_{<a} \cap S_a) \xrightarrow{f} H_m(S_{<a}) \oplus H_m(S_a) \xrightarrow{g} H_m(S_{\le a}) \to \cdots$$

Lemma 3.23. *The space* $S_{\le a}$ *has nonzero homology only in even degrees. The Mayer–Vietoris sequence above reduces to short exact sequences of the form*

$$0 \to H_{2m}(S_{<a} \cap S_a) \xrightarrow{f} H_{2m}(S_{<a}) \oplus H_{2m}(S_a) \xrightarrow{g} H_{2m}(S_{\le a}) \to 0.$$

Proof. We prove this inductively with respect to the order $<$. The base case is obvious since for the first a in the order $S_{\le a} = S_a$. Assume the lemma is true for all matchings up to and including some matching e, and let a be the next matching in the order. Then $S_{<a} = S_{\le e}$, S_a, and $S_{<a} \cap S_a$ all have homology in even degrees only. Therefore the Mayer–Vietoris sequence decomposes into exact sequences

$$0 \to H_{2m+1}(S_{\le a}) \xrightarrow{\partial} H_{2m}(S_{<a} \cap S_a) \xrightarrow{f} H_{2m}(S_{<a}) \oplus H_{2m}(S_a) \xrightarrow{g} H_{2m}(S_{\le a}) \to 0.$$

By Lemma 3.21 the map $H_{2m}(S_{<a} \cap S_a) \to H_{2m}(S_a)$ is injective so $H_{2m+1}(S_{\le a}) = 0$ as desired. $\square$

Theorem 3.24. *The sequence*

$$\bigoplus_{b \to c \le a} H_*(S_b \cap S_c) \xrightarrow{\psi^-} \bigoplus_{b \le a} H_*(S_b) \xrightarrow{\phi} H_*(S_{\le a}) \to 0$$

is exact. Here ψ^- *is defined to be* $\Sigma_{b \to c \le a}(\psi_{b,c} - \psi_{c,b})$ *where* $\psi_{b,c} : H_*(S_b \cap S_c) \to H_*(S_b)$ *is induced by the inclusion* $S_b \cap S_c \subset S_b$ *and* $\psi_{c,b}$ *is similarly defined. The map* ϕ *is induced by the inclusions* $S_b \subset S_{\le a}$.

Proof. This proof follows the structure of the proof of [Khovanov 2004, Proposition 4]. We prove this result by induction with respect to the total order on the set $B^{n-k,k}$. Let a_0 be the first element with respect to this order. Then there do not exist $b, c \le a_0$ and $S_{\le a_0} = S_{a_0}$. The sequence

$$0 \to H_*(S_{a_0}) \xrightarrow{\phi} H_*(S_{a_0}) \to 0$$

is exact since ϕ is an isomorphism, and the base case is proven.

Assume for some element $a_{i-1} \in B^{n-k,k}$ that the sequence

$$\bigoplus_{b \to c \le a_{i-1}} H_*(S_b \cap S_c) \xrightarrow{\psi^-} \bigoplus_{b \le a_{i-1}} H_*(S_b) \xrightarrow{\phi} H_*(S_{\le a_{i-1}}) \to 0$$

is exact. Consider the next element a_i. By Lemma 3.23 we know that

$$0 \to H_*(S_{<a_i} \cap S_{a_i}) \xrightarrow{f} H_*(S_{<a_i}) \oplus H_*(S_{a_i}) \xrightarrow{g} H_*(S_{\leq a_i}) \to 0$$

is an exact sequence. By Lemma 3.22 the map $f' : \bigoplus_{b \to c \leq a_i} H_*(S_b \cap S_c) \to H_*(S_{<a_i} \cap S_{a_i})$ is surjective so $Im(f) = Im(f \circ f')$. Thus we get the exact sequence

$$\bigoplus_{b \to c \leq a_i} H_*(S_b \cap S_c) \xrightarrow{f' \circ f} H_*(S_{<a_i}) \oplus H_*(S_{a_i}) \xrightarrow{g} H_*(S_{\leq a_i}) \to 0.$$

Now consider the sequence

$$\bigoplus_{b \to c \leq a_i} H_*(S_b \cap S_c) \xrightarrow{\psi^-} \bigoplus_{b \leq a_i} H_*(S_b) \xrightarrow{\phi} H_*(S_{\leq a_i}) \to 0.$$

Since the inclusions $S_b \cap S_c \subset S_{<a_i} \subset S_{\leq a_i}$ and $S_b \cap S_c \subset S_b \subset S_{\leq a_i}$ and $S_b \cap S_c \subset S_c \subset S_{\leq a_i}$ all induce the same maps on homology we see that $Im(\psi^-) = Ker(\phi)$. Take some $y \in H_*(S_{\leq a_i})$. By exactness there exists $(x, x') \in H_*(S_{<a_i}) \oplus H_*(S_{a_i})$ with $g(x, x') = y$. Again by exactness, there exists some $x'' \in \bigoplus_{b \leq a_{i-1}} H_*(S_b)$ with $\phi(x'') = x$. Then we have $\phi(x'', x') = g(x, x') = y$, and the desired sequence is exact. $\qquad\square$

4. Diagrammatic homology generators

In the last section we proved an exact sequence on the collection of complex vector spaces $H_*(X_{n-k,k})$. In this section we will ultimately use this sequence to give diagrammatic bases for these vector spaces. We begin by giving a diagrammatic collection of generators and relations for $H_*(X_{n-k,k})$.

Let $a, b \in B^{n-k,k}$ with $a \to b$. Recall that S_a is homeomorphic to $(S^2)^k$ and $S_a \cap S_b$ is homeomorphic to $(S^2)^{k-1}$. We will use the cartesian product CW-decomposition for S_a to give a convenient diagrammatic way to represent cells. This is a different decomposition than the one found in Section 3. While that decomposition was useful for constructing an exact sequence on homology, the cartesian product decomposition lends itself nicely to the diagrammatic calculations we would like to do in the remainder of this paper.

Let E be the set of arcs in a, and for each $e \in E$ let e_ℓ be the left endpoint of the arc e and e_r be the right endpoint. Note that for each $x = (x_1, \ldots, x_n) \in S_a$ we have $x_{e_\ell} = x_{e_r}$. Therefore to describe a subset of S_a we only need to specify the value of x_{e_ℓ}. Consider the cell decomposition of the two-sphere given by the point $p = (0, 0, 1) \in S^2$ and the two-cell $c = S^2 - p$.

Definition 4.1 (diagrammatic homology basis for S_a). Given $I \subset E$ we define the following cell $c(I)$ which is homeomorphic to $\mathbb{R}^{2|I|}$.

$$c(I) = \{x = (x_1, \ldots, x_n) \in S_a : x_{e_\ell} = p \text{ if } e \notin I \text{ and } x_{e_\ell} \neq p \text{ if } e \in I\}.$$

Varying over all possible subsets of E we get the cartesian product cell decomposition for S_a. Note that this decomposition is in general not the same as the one described in Definition 3.18. We can diagrammatically represent the cartesian product homology generators by considering certain markings of the matching a.

Given $I \subset E$ let M be the matching a decorated as follows:

- Mark each ray with a dot.
- For each $e \in I$ mark the corresponding arc in a with a dot.
- Leave all other arcs unmarked.

We call such a diagram a dotted noncrossing matching for a. These will be denoted throughout by capital letters. Figure 6 has an example.
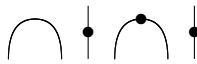
Figure 6. A dotted noncrossing matching M for a.

We use a similar diagrammatic convention to represent cartesian product cells for $(S^2)^n$ in [Russell and Tymoczko 2008, Definition 3.1]. These diagrams, which we call line diagrams, are collections of n dotted and undotted vertical lines. The common theme of dotted noncrossing matchings and line diagrams is that unmarked lines and arcs correspond to two-cells while dotted lines and arcs correspond to points. We want to consider the behavior of these generators under maps induced by inclusion.

Definition 4.2. Consider $a \to b \in B^{n-k,k}$. Then either a and b agree off of two arcs or an arc and a ray. The first two relations deal with the case when a and b differ on two arcs. The third relation deals with the case when they differ on an arc and a ray.

Type I relations: Let M_1 be the dotted noncrossing matching with dotted arc (i, j) and undotted arc (k, l). Let M_2 be the dotted noncrossing matching with dotted arc (k, l) and undotted arc (i, j). Define M_1' and M_2' analogously for b, so that they agree with the M_i off of i, j, k, l. Type I relations have the form

$$M_1 + M_2 - M_1' - M_2' = 0.$$
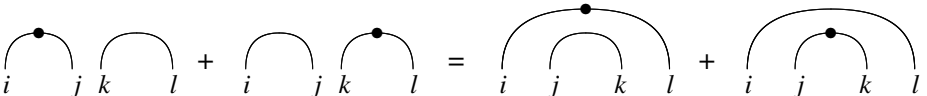
Figure 7. Type I relation.

Type II relations: Let M_3 be the dotted noncrossing matching with dotted arcs (i, j) and (k, l). Define M_3' analogously for b. Type II relations have the form

$$M_3 - M_3' = 0.$$
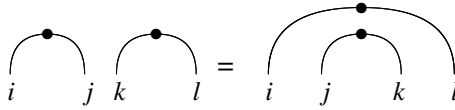
Figure 8. Type II relation.

Type III relations: Let M_4 be the dotted noncrossing matching with dotted ray (i) and dotted arc (j, k). Let M_4' be the dotted noncrossing matching identical to M_4 except with dotted arc (i, j) and dotted ray (k). Type III relations have the form

$$M_4 - M_4' = 0$$
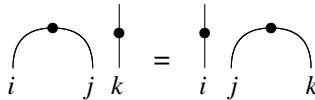
Figure 9. Type III relation.

Theorem 4.3. *Let $\mathcal{T}$ be the subspace of $\bigoplus_b H_*(S_b)$ generated by all complex linear combinations of Type I, Type II, and Type III relations. Then there is the following isomorphism of complex vector spaces.*

$$H_*(X_{n-k,k}) \cong \left(\bigoplus_b H_*(S_b) \right) \Big/ \mathcal{T}.$$

Proof. From Theorem 3.24 we see that $\bigoplus_b H_*(S_b)$ generates $H_*(X_{n-k,k})$ with relations coming from $\psi^-(H_*(S_a \cap S_b))$ where $a \to b$. The following calculations show that these relations always have the form of Type I, II, or III relations. We use the notation for cartesian product cells found in Definition 4.1.

Consider $a, b \in B^{2,2}$ where a has arcs $(1, 2)$, $(3, 4)$ and b has arcs $(1, 4)$, $(2, 3)$. Then $aw(b)$ consists of one circle, and $S_a \cap S_b$ is homeomorphic to S^2. Decompose $S_a \cap S_b$ into

$$c_1 = \{(p, p, p, p) \in S_a \cap S_b\} \cong \mathbb{R}^0 \quad \text{and} \quad c_2 = \{(x, x, x, x) \in S_a \cap S_b : x \neq p\} \cong \mathbb{R}^2.$$

Under the inclusion map $\iota_a : H_*(S_a \cap S_b) \hookrightarrow H_*(S_a)$ we have $\iota_a(c_1) = c(\varnothing)$ and $\iota_a(c_2) = c(\{(1, 2)\}) + c(\{(3, 4)\})$. Similarly under the inclusion $\iota_b : H_*(S_a \cap S_b) \hookrightarrow H_*(S_b)$ we have $\iota_b(c_1) = c(\varnothing)$ and $\iota_b(c_2) = c(\{(1, 4)\}) + c(\{(2, 3)\})$. This proves that $\psi^-(H_*(S_a \cap S_b))$ is generated by Type I and Type II relations whenever a and b differ on two arcs.

Consider $a, b \in B^{3,1}$ where a has ray (1) and arc (2, 3) while b has ray (3) and arc (1, 2). In this case $aw(b)$ consists of one line and no circles, so $S_a \cap S_b$ is just a point. Call this point c_1. Under the inclusion map $\iota_a : H_*(S_a \cap S_b) \hookrightarrow H_*(S_a)$ we have $\iota_a(c_1) = c(\varnothing)$. Under the inclusion $\iota_b : H_*(S_a \cap S_b) \hookrightarrow H_*(S_b)$ we get $\iota_b(c_1) = c(\varnothing)$. This proves that whenever a and b differ on an arc and a ray, the image $\psi^-(H_*(S_a \cap S_b))$ is generated by Type III relations.   □

Definition 4.4. A standard dotted noncrossing matching is a dotted noncrossing matching where no dotted arc is nested beneath any other arc and no ray is to the right of any dotted arc. Figure 10 has an example.

Lemma 4.5. *The set of standard noncrossing matchings contains a set of generators for the collection of complex vector spaces $H_*(X_{n-k,k})$.*

Proof. Type I and Type II relations allow us to reduce to a set of generators where no dotted arcs are nested beneath undotted or dotted arcs. Type III relations allow us to move rays to the left of all dotted arcs.   □

Young tableaux theory is often used to study irreducible representations of the symmetric group. Recall that a Young diagram corresponding to a partition of the number n is a collection of top and left aligned boxes with row lengths given by the partition. A standard Young tableau is a filling of a Young diagram with the numbers 1 through n so that numbers strictly increase from left to right and top to bottom. For more information on Young tableaux and the Specht module theory we will use in the next section we refer the reader to [Fulton 1997, Chapter 7].

Results from [De Concini and Procesi 1981, p. 213] and [Garsia and Procesi 1992, Equation 4.2], specialized to the two-row case, yield the following statement, which gives us the dimension of $H_*(X_{n-k,k})$ in each degree.

Proposition 4.6. *For each $m \leq k$ the Springer action on $H_{2m}(X_{n-k,k})$ is isomorphic to the irreducible representation of the symmetric group S_n corresponding to the partition $(n-m, m)$.*

Classical results about representations of the symmetric group together with Proposition 4.6 tell us that the dimension of $H_{2m}(X_{n-k,k})$ is therefore the number of standard fillings of a Young diagram of shape $(n-m, m)$. As we show in the following lemma, it turns out that this dimension is also equal to the number of standard noncrossing matchings of type $(n-k, k)$ with m undotted arcs.

Lemma 4.7. *Standard noncrossing matchings of type $(n-k, k)$ with m undotted arcs are in bijection with standard Young tableaux of shape $(n-m, m)$.*

Proof. We give an explicit bijection between standard dotted noncrossing matchings and standard Young tableaux.

Let M be a standard dotted noncrossing matching. Define $g(M)$ to be the tableau obtained by writing the right hand endpoint of each undotted arc of M on the

bottom row and all other numbers on the top. The tableau $g(M)$ is standard since for each number in the bottom row, the left endpoint of that arc is written in the top row. The tableau has shape $(n-m, m)$ where m is the number of undotted arcs in M.

Now let T be a standard tableau of shape $(n-m, m)$. We define $h(T)$ to be the standard dotted noncrossing matching with $n-2k$ rays and m undotted arcs constructed as follows. Start with the smallest number in the bottom row, draw an undotted arc with right endpoint incident on that number and left endpoint incident on the nearest unoccupied vertex. Repeat this for all remaining numbers in the bottom row. Then in the leftmost $n-2k$ empty positions place $n-2k$ rays. Fill the remaining empty positions with unnested dotted arcs. Figure 10 has an example of this map.
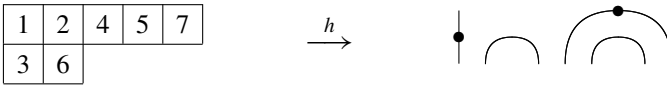
Figure 10. A standard tableau of shape $(5, 2)$ and its associated standard dotted noncrossing matching of type $(4, 3)$.

The proof that $g \circ h$ and $h \circ g$ produce the identity on the sets of standard tableaux and standard dotted noncrossing matchings respectively is a straightforward generalization of the one given in [Russell and Tymoczko 2008, Section 2.2]. □

Corollary 4.8. *The standard noncrossing matchings on n vertices with $n-2k$ rays and m undotted arcs form a basis for the complex vector space $H_{2m}(X_{n-k,k})$. In particular, when $m = k$ there are no dotted arcs and we recover the bijection from* [Stroppel and Webster 2010, Proposition 3].

Proof. By Proposition 4.6 the dimension of $H_{2m}(X_{n-k,k})$ is the same as the dimension of the irreducible representation of S_n corresponding to the partition $(n-m, m)$. Classical representation theory tells us that this dimension is equal to the number of standard Young tableaux of shape $(n-m, m)$ [Fulton 1997].

By Lemma 4.5 the set of standard dotted noncrossing matchings with m undotted arcs is a generating set for $H_{2m}(X_{n-k,k})$. Lemma 4.7 tells us that this generating set has the proper dimension and therefore is a basis. □

5. Springer action on $H_*(X_{n-k,k})$

Consider n and $0 \le k \le \lfloor n/2 \rfloor$. In [Russell and Tymoczko 2008] we define a graded action of the symmetric group S_{2n-2k} on $H_*(X_{n-k,n-k})$. We proved that this action which can be combinatorially and diagrammatically described is isomorphic to the Springer action. In other words we prove that the action of S_{2n-2k} that we define
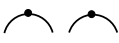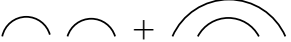
Vertex Labelings	M	$(i \quad i{+}1) \cdot M$
Case 1 $i, i{+}1$ both on dotted arcs		
Case 2 $(i, i{+}1)$ is an undotted arc		
Case 3 (j, i) and $(i{+}1, k)$ have one dot		
Case 4 (j, i) and $(i{+}1, k)$ have no dots		

Figure 11. The S_n-action on standard dotted noncrossing matchings. (Gray vertices indicate positions i and $i{+}1$.)

on $H_{2m}(X_{n-k,n-k})$ is the irreducible representation of S_{2n-2k} corresponding to the partition $(2n - 2k - m, m)$ for each $0 \leq m \leq n - k$. This action is summarized by the chart in Figure 11.

Using the same approach as [Russell and Tymoczko 2008, Section 4] we construct a symmetric group action on the homology of all two-row Springer varieties that is isomorphic to the Springer action. We once again describe this action diagrammatically. Finally we give a convenient skein-theoretic method for calculating the action.

Defining a symmetric group action. Define the map $\eta : X_{n-k,k} \to X_{n-k,n-k}$ to be

$$\eta(x_1, \ldots, x_n) = (\pm p, \mp p, \ldots, p, -p, x_1, \ldots, x_n).$$

Recall that the completion map φ from Section 3 builds a matching of type $(n - k, n - k)$ from one of type $(n-k, k)$ by anchoring rays to the left of the matching. Given some noncrossing matching $\boldsymbol{a} \in B^{n-k,k}$ it is an immediate consequence of the

definition of η that $\eta(S_a) \subset S_{\varphi(a)}$. In other words η maps the component associated to a into the component associated to its completion.

Recall the antipodal map $\gamma : X_{n-k,k} \to (S^2)^n$ defined in the proof of Theorem 2.7:

$$\gamma(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, (-1)^n x_n).$$

Finally consider the map $\iota_n : (S^2)^n \to (S^2)^{2n-2k}$ given by

$$\iota_n(x_1, \ldots, x_n) = ((-1)^{n-1}p, \ldots, (-1)^{n-1}p, (-1)^n x_1, \ldots, (-1)^n x_n).$$

Lemma 5.1. *The following diagram is commutative.*

$$
\begin{array}{ccc}
H_*(X_{n-k,k}) & \xrightarrow{\eta_*} & H_*(X_{n-k,n-k}) \\
\Big\downarrow{\gamma_*} & & \Big\downarrow{\gamma_*} \\
H_*((S^2)^n) & \xrightarrow{\iota_{n*}} & H_*((S^2)^{2n-2k})
\end{array}
$$

Proof. On one hand we have

$$
\begin{aligned}
\iota_n \circ \gamma(x_1, \ldots, x_n) &= \iota_n(-x_1, x_2, \ldots, (-1)^{n-1}x_{n-1}, (-1)^n x_n) \\
&= \big((-1)^{n-1}p, \ldots, (-1)^{n-1}p, \\
&\qquad (-1)^{n+1}x_1, (-1)^{n+2}x_2, \ldots, (-1)^{2n-1}x_{n-1}, (-1)^{2n}x_n\big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\gamma \circ \eta(x_1, \ldots, x_n) &= \gamma\big((-1)^{n-2k}p, (-1)^{n-2k-1}p, \ldots, (-1)^2 p, (-1)^1 p, x_1, \ldots, x_n\big) \\
&= \big((-1)^{n-2k+1}p, \ldots, (-1)^{n-2k+1}p, \\
&\qquad (-1)^{n-2k+1}x_1, \ldots, (-1)^{2n-2k-1}x_{n-1}, (-1)^{2n-2k}x_n\big).
\end{aligned}
$$

The parity of n and $n-2k$ must be the same. Therefore $(-1)^{n-2k+i} = (-1)^{n+i}$ and the diagram commutes. $\qquad\square$

Proposition 5.2. *The map $\gamma_* : H_*(X_{n-k,k}) \to H_*((S^2)^n)$ is injective, and the action of S_n on $H_*((S^2)^n)$ given by permutation of coordinates stabilizes the image of $H_*(X_{n-k,k})$ under γ_*.*

Proof. Viewing the maps η and ι_n on diagrammatic homology generators it is straightforward to see that η_* and ι_{n*} are injective. In [Russell and Tymoczko 2008, Corollary 3.13] we prove that $\gamma_* : H_*(X_{n-k,n-k}) \to H_*((S^2)^{2n-2k})$ is injective. Therefore by Lemma 5.1 we see that $\gamma_* : H_*(X_{n-k,k}) \to H_*((S^2)^n)$ is injective and the first half of the lemma is proven.

Consider the subgroup $G \le S_{2n-2k}$ fixing the numbers $1, \ldots, n-2k$. Then G is isomorphic to S_n via the map

$$(i \quad j) \mapsto (i+n-2k \quad j+n-2k).$$

The action of G on $H_*(X_{n-k,n-k})$ is given by pulling back the action of G on $H_*((S^2)^{2n-2k})$. Since the homology of a cartesian product of copies of S^2 is a tensor product of copies of $H_*(S^2)$, the symmetric group action that permutes tensor factors commutes with the inclusion map ι_{n*}. We can therefore prove the second part of the lemma by showing that the action of G on $H_*(X_{n-k,n-k})$ stabilizes the image of $H_*(X_{n-k,k})$ under η_*.

Again note that given $\boldsymbol{a} \in B^{n-k,k}$ we have $\eta(S_{\boldsymbol{a}}) \subset S_{\varphi(\boldsymbol{a})}$. On the level of homology η sends generators in $H_*(X_{n-k,k})$ to their associated completions in $H_*(X_{n-k,n-k})$. The new dotted arcs in the completed generators will have right endpoint at the vertex where the ray was formerly incident and left endpoint on one of the first $n-2k$ vertices. While this new dotted noncrossing matching is not necessarily standard, it is equivalent to a unique standard noncrossing matching via Type II relations.

Diagrammatically, the image $\eta_*(H_*(X_{n-k,k}))$ consists of exactly those generators in $H_*(X_{n-k,n-k})$ with no undotted arcs incident on the first $n-2k$ vertices. Indeed the preimage of such a generator under η_* is the standard dotted noncrossing matching with the first $n-2k$ vertices removed, identical undotted arcs, and the unique standard arrangement of rays and dotted arcs in the remaining positions.

In order to prove that the subgroup action of G stabilizes $\eta_*(H_*(X_{n-k,k}))$ we need to show that each simple transposition $(i \quad i+1)$, where $n-2k+1 \le i < 2n-2k$, maps elements of $\eta_*(H_*(X_{n-k,k}))$ to a linear combination of elements of $H_*(X_{n-k,n-k})$ with no undotted arcs incident on the first $n-2k$ vertices. Examining rows 1, 2, 4, and 7 on the chart in Figure 11 we see that it is not possible for such a simple transposition to map a generator without an undotted arc incident on the first $n-2k$ vertices to one with an undotted arc incident on the first $n-2k$ vertices. $\square$

Theorem 5.3. *There is a graded S_n-action on $H_*(X_{n-k,k})$ described by the chart in Figure 11 together with the chart in Figure 12.*

Proof. Using the commutative diagram in Lemma 5.1 together with the chart in Figure 11 we are able to calculate $\gamma_*^{-1}((i \quad i+1) \cdot \gamma_*(M))$ where $M \in H_*(X_{n-k,k})$. By Proposition 5.2 we have a well-defined S_n-action given by

$$(i \quad i+1) \cdot M := \gamma_*^{-1}\big((i \quad i+1) \cdot \gamma_*(M)\big). \qquad \square$$

Verifying the Springer action. Young tabloids are equivalence classes of Young tableaux where the equivalence relation is permutation of numbers within rows. The symmetric group S_n acts on tabloids; indeed given $\sigma \in S_n$ and a tabloid T we let $\sigma \cdot T$ be the tabloid with $\sigma(i)$ replacing i for all $i = 1, \dots, n$. Consider the vector space U_λ of complex linear combinations of tabloids of shape λ. For a

Vertex Labelings	M	$(i \quad i+1) \cdot M$	
Case 5 $i, i+1$ both on rays			
Case 6 $i, i+1$ on ray and dotted arc			
Case 7 $i, i+1$ on ray and undotted arc		$+$	
		$+$	

Figure 12. The S_n-action on matchings of type $(n-k, k)$ with rays. (Gray vertices indicate positions i and $i+1$.)

tabloid T write its corresponding element in U_λ as v_T. Then the action of S_n on tabloids induces an action on U_λ.

Consider $0 \le m \le \lfloor n/2 \rfloor$ and $\lambda = (n-m, m)$ a partition of the number n. For each tableau T let $\mathrm{Col}(T)$ be the subgroup of permutations in S_n that stabilize all columns of T. For each tableau T define the following vector in $U_{n-m,m}$.

$$e_T = \sum_{\sigma \in \mathrm{Col}(T)} \mathrm{sign}(\sigma) v_{\sigma \cdot T}$$

The subspace of $U_{n-m,m}$ generated by these vectors which we denote $V_{n-m,m}$ is known as the Specht module. A classical result says that the symmetric group action on $V_{n-m,m}$ given by permuting entries in tabloids is the irreducible representation of S_n corresponding to the partition $(n-m, m)$. Furthermore $V_{n-m,m}$ has a basis given by $\{e_T\}$ where T are standard tableaux.

Given a standard dotted noncrossing matching M of type $(n-k, k)$ with m undotted arcs let $\mathrm{Undot}(M)$ be the subgroup of S_n generated by all transpositions $(i \quad j)$ where (i, j) is an undotted arc in M. Let e_M be the vector in $U_{n-m,m}$ defined as

$$e_M = \sum_{\sigma \in \mathrm{Undot}(T)} \mathrm{sign}(\sigma) v_{\sigma \cdot h(M)}.$$

Definition 5.4. Let the subspace of $U_{n-m,m}$ generated by the vectors e_M the matching module. Denote the matching module by $W_{n-m,m}$.

Define a map

$$\zeta : H_{2m}(X_{n-k,k}) \to U_{n-m,m}$$

by $\zeta(M) = e_M$. Then the image of ζ is exactly the matching module.

Define the map

$$f : U_{n-m,m} \to U_{2n-2k-m,m}$$

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & 7 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array} \qquad T' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 & 8 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array}$$

Figure 13. The behavior of tabloids under the map f where $n = 7$ and $k = 3$.

as $f(v_T) = v_{T'}$, where T' is the tabloid with numbers 1 through $n - 2k$ in the top row and $i + n - 2k$ in the row where i appears in T. Note that f is injective. Figure 13 has an example.

Lemma 5.5. *Let M be a standard dotted noncrossing matching. Then $f(e_M) = e_{\varphi(M)}$ where $\varphi(M)$ is understood to be the completion of the associated matching along with dotting information.*

Proof. Recall that h is the bijection taking standard Young tableaux to standard dotted noncrossing matchings. If (i, j) is an undotted arc in M then $(i + n - 2k, j + n - 2k)$ is an undotted arc in $\varphi(M)$. This means that $h(M)' = h(\varphi(M))$. Furthermore if $\text{Undot}(M)$ is generated by transpositions $(i \quad j)$ then $\text{Undot}(\varphi(M))$ is generated by transpositions $(i + n - 2k \quad j + n - 2k)$.

The bijection $(i \quad j) \mapsto (i + n - 2k \quad j + n - 2k)$ between S_n and G restricts to a bijection between $\text{Undot}(M)$ and $\text{Undot}(\varphi(M))$. Moreover for $\sigma \in \text{Undot}(M)$ and its corresponding $\sigma' \in \text{Undot}\varphi(M)$ we have $(\sigma \cdot \theta(M))' = \sigma' \cdot \theta(\varphi(M))$. Thus we conclude that $f(e_M) = e_{\varphi(M)}$. $\square$

Corollary 5.6. *The following diagram is commutative*:

$$\begin{array}{ccc} H_*(X_{n-k,k}) & \xrightarrow{\eta_*} & H_*(X_{n-k,n-k}) \\ \Big\downarrow{\zeta} & & \Big\downarrow{\zeta} \\ U_{n-m,m} & \xrightarrow{\quad f \quad} & U_{2n-2k-m,m} \end{array}$$

Lemma 5.7. *The action of S_n on $U_{n-m,m}$ commutes with that of G on $U_{2n-2k-m,m}$.*

Proof. Let σ' be the element of G corresponding to $\sigma \in S_n$ via the bijection described above. For $v_T \in U_{n-m,m}$ we want to show that $f(\sigma \cdot v_T) = \sigma' \cdot f(v_T) = v_{\sigma' \cdot T'}$.

The tabloid $\sigma \cdot T$ has $\sigma(i)$ in the row that i occupied in T. The tabloid corresponding to $f(\sigma \cdot v_T)$ has $1, \ldots, n-2k$ in the top row and $\sigma(i) + n - 2k$ in the row that i occupied in T. On the other hand T' has $1, \ldots, n-2k$ in the top row and $i + n - 2k$ in the row that i occupied in T. The tabloid $\sigma' \cdot T'$ has $1, \ldots, n-2k$ on the top row and $\sigma'(i + n - 2k)$ in the row that i occupied in T. Since $\sigma'(i + n - 2k) = \sigma(i) + n - 2k$ we conclude that $f(\sigma \cdot v_T) = \sigma' \cdot f(v_T)$. $\square$

Corollary 5.8. *The map $\zeta : H_{2m}(X_{n-k,k}) \to W_{n-m,m}$ is an S_n-equivariant isomorphism of complex vector spaces.*

Proof. As we noted earlier η_* is injective, so $\eta_* : H_{2m}(X_{n-k,k}) \to \eta_*(H_{2m}(X_{n-k,k})$ is an isomorphism. In [Russell and Tymoczko 2008, Lemma 4.2] we prove that

$$\zeta : H_{2m}(X_{n-k,n-k}) \to W_{2n-2k-m,m} \subset U_{2n-2k-m,m}$$

is an S_{2n-2k}-equivariant isomorphism, so it follows that

$$\zeta : \eta_*(H_{2m}(X_{n-k,k})) \to f(W_{n-m,m})$$

is a G-equivariant isomorphism. Finally Lemma 5.5 implies that f is injective, so $f : W_{n-m,m} \to f(W_{n-m,m})$ is an isomorphism. Thus

$$\zeta = f^{-1} \circ \zeta \circ \eta_* : H_{2m}(X_{n-k,k}) \to W_{n-m,m}$$

is an isomorphism.

By construction the action of S_n on $H_{2m}(X_{n-k,k})$ commutes with the action of G on $\eta_*(H_{2m}(X_{n-k,k}))$. Since $\zeta : H_{2m}(X_{n-k,n-k}) \to U_{2n-2k-m,m}$ is S_{2n-2k}-equivariant it is also G-equivariant. This information together with Lemma 5.7 proves that $\zeta : H_{2m}(X_{n-k,k}) \to W_{n-m,m}$ is S_n-equivariant. □

Theorem 5.9. *The Specht module $V_{n-m,m}$ and the matching module $W_{n-m,m}$ are equal as S_n-representations. Thus the S_n action on $H_*(X_{n-k,k})$ is isomorphic to the Springer representation.*

Proof. Since $V_{n-m,m}$ and $W_{n-m,m}$ are subspaces of $U_{n-m,m}$ of the same dimension each equipped with an S_n-action, the intersection $V_{n-m,m} \cap W_{n-m,m}$ is also an S_n representation. The Specht module $V_{n-m,m}$ is irreducible, so $V_{n-m,m} \cap W_{n-m,m}$ is either 0 or $V_{n-m,m}$.

Let M be the standard dotted noncrossing matching with m undotted arcs $(1, 2)$, $(3, 4), \ldots, (2m-1, 2m)$. Then M represents a basis element in $H_{2m}(X_{n-k,k})$. Furthermore the columns of $h(M)$ are exactly the undotted arcs of M, so $\mathrm{Undot}(M) = \mathrm{Col}(h(M))$. This means that $e_M = e_{h(M)}$, so the matching module and Specht module always have a common vector.

Since $V_{n-m,m} \cap W_{n-m,m}$ is nonempty we conclude that the intersection is all of $V_{n-m,m}$. We therefore conclude that $V_{n-m,m} = W_{n-m,m}$. By the equivariance proved in Corollary 5.8 the S_n action we have defined on $H_*(X_{n-k,k})$ is isomorphic to the unique irreducible representation of S_n corresponding to the partition $(n-m, m)$. Therefore we conclude that the action we have defined on the homology of the Springer variety is isomorphic to the Springer action. □

A skein-theoretic formulation of the Springer action. The Springer action on $H_*(X_{n-k,k})$ is described diagrammatically in the charts in Figures 11 and 12. We can also describe the action skein-theoretically.

Theorem 5.10. *Given standard dotted noncrossing matching $M \in H_*(X_{n-k,k})$ and $\sigma \in S_n$ glue a flattened braid corresponding to σ to the bottom of M forming M'. Then the Springer action $\sigma \cdot M$ is equal to $s(M')$, where s is defined as follows.*

$$s\left(\vcenter{\hbox{⊗}}\right) = s\left(\vcenter{\hbox{⌣}}\right) + s\left(\vcenter{\hbox{)(}}\right) \qquad\qquad s\left(\vcenter{\hbox{↘}}\right) = 0$$

$$s\left(M' \sqcup \bigcirc\right) = s(-2M') \qquad\qquad s\left(M' \sqcup \bullet\!\!\bigcirc\right) = 0$$

As an example, consider the permutation $(1\;\;2\;\;3)$ acting on the following generator of $H_2(X_{2,1})$.

$$(1\,2\,3) \cdot \vcenter{\hbox{⌢⎪}} = s\left(\vcenter{\hbox{⊗⎪}}\right) = s\left(\vcenter{\hbox{⋂ ⎪}}\right) + s\left(\vcenter{\hbox{○ ⎪}}\right) + s\left(\vcenter{\hbox{⋂⌣}}\right) + s\left(\vcenter{\hbox{⌣}}\right)$$

$$= \vcenter{\hbox{⌢⎪}} - 2\vcenter{\hbox{⌢⎪}} + \vcenter{\hbox{⎪⌢}} + \vcenter{\hbox{⌢⎪}} = \vcenter{\hbox{⎪⌢}}$$

Appendix

In Section 3 we cited [Khovanov 2004] as the source for Proposition 3.9 and the base case of Lemma 3.14. Proofs of these results were left to the reader in that paper. For the sake of completeness we include the proofs here. Throughout the appendix we take $n = 2m$.

Proof of Proposition 3.9. Let $|aw(b)|$ be denoted by c. This proof proceeds by induction on c. We begin with the case that $c = 1$ and then argue by induction. The following algorithm finds a sequence of $m - 1$ moves on a that yields b. An arc (i, j) in b is said to be unpaired in $aw(b)$ if a does not have arc (i, j). Let $a = a_0$ and let $t = 0$.

(1) If b has no unpaired arcs in $a_t w(b)$, then $a_t = b$ and we are done. Otherwise begin with the narrowest leftmost arc (i, j) in b that is unpaired in $a_t w(b)$. Then a_t has arcs (i, k), (j, l) where i, j, k, l have no particular order.

(2) Perform $\rightarrow$ or $\leftarrow$ in a_t to produce a new matching a_{t+1} with arcs (i, j) and (k, l).

(3) Increment t by 1, and repeat the first two steps.

It is always possible to perform $\rightarrow$ or $\leftarrow$ as required. At each step the algorithm guarantees all arcs nested beneath (i, j) are paired in $a_t w(b)$ since arcs beneath (i, j) are narrower than (i, j). Thus each arc in a_t either has zero or two endpoints between vertices i and j. This means that the arcs (i, k) and (j, l) are adjacent in a_t, and $\rightarrow$ or $\leftarrow$ can be performed to get a new matching a_{t+1} with arcs (i, j) and (k, l).

The number of circles in $a_{t+1}w(b)$ is one more than the number of circles in $a_t w(b)$. To see this note that the arcs (i, k), (j, l) in a_t and the arc (i, j) in b are all part of the same circle in $a_t w(b)$. The move that produces a_{t+1} must change the number of circles. Since this move acts on two arcs that are part of a single circle, the only possibility is that the move increases the number of circles by one.

Since we increase the number of circles with each iteration of our algorithm, this process terminates after $m - 1$ iterations at which point we will have m circles - the maximum number of circles possible. This proves that $d(a, b) \leq m - 1$.

In an arbitrary sequence of moves, each time $a_t \rightarrow a_{t+1}$ or $a_t \leftarrow a_{t+1}$ is performed the number of circles in $a_t w(b)$ is either one greater or one less than $a_{t+1}w(b)$ depending on whether the move taking a_t to a_{t+1} joins two circles or splits a single circle. Since we start with one circle, it will require at least $m - 1$ moves to produce m circles. This means our algorithm generates a minimal sequence $(a = a_0, a_1, \ldots, a_{m-2}, a_{m-1} = b)$ and $d(a, b) = m - 1$.

Now consider $a, b \in B^{m,m}$ where $aw(b)$ has c circles, and enumerate the circles $\alpha_1, \ldots, \alpha_c$. Each circle α_r passes through the horizontal axis $2q_r$ times. By the argument above, we have

$$d(a, b) = \sum_{1 \leq r \leq c} (q_r - 1) = \left(\sum_{1 \leq r \leq c} q_r \right) - c = m - c. \qquad \square$$

Proof of the base case for Lemma 3.14. This is proven using induction on the distance between a and b. If there exists a minimal sequence of moves taking a to b that begins with $\leftarrow$ the inductive step is clear. The lemmas below are required in order to handle the case that no minimal sequences of moves taking a to b begin with $\leftarrow$. When this occurs, the following arguments prove that there always exists a minimal sequence for a and b of the form $(\rightarrow, \rightarrow, \ldots, \rightarrow, \rightarrow)$. In other words, we want to show that in this case we take $a = c$.

Lemma 5.11. *Let a be the matching with arcs*

$$(1, 2), (3, 4), \ldots, (n - 3, n - 2), (n - 1, n)$$

and b the matching with arcs

$$(2, 3), (4, 5), \ldots, (n - 4, n - 3), (n - 2, n - 1), (1, n).$$

Then there exists a minimal sequence $(a \rightarrow a_1 \ldots a_{m-1} \rightarrow b)$.

Proof. Consider the minimal sequence generated by the algorithm in the proof of Proposition 3.9. Each move takes the matching with arcs $(1, i)$, $(i+1, i+2)$ to the matching with arcs $(1, i+2)$, $(i, i+1)$. Since arcs $(1, i)$, $(i+1, i+2)$ are unnested these moves always have the form $\rightarrow$. $\qquad \square$

This next lemma addresses circles of this type within more complicated gluings of two matchings.

Lemma 5.12. *Let $i_1 < \cdots < i_p$. Let $\boldsymbol{a}$ be some matching with arcs $(i_1, i_2), \ldots,$ (i_{p-1}, i_p). Let $\boldsymbol{b}$ be some matching with arcs $(i_2, i_3), \ldots, (i_{p-2}, i_{p-1}), (i_1, i_p)$. There exists a minimal sequence for $\boldsymbol{a}$ and $\boldsymbol{b}$ where all moves on arcs incident on the vertices $i_1, \ldots, i_p$ have the form $\rightarrow$.*

Proof. We prove this using induction. Assume $i_1 < i_2 < i_3 < i_4$ and that $\boldsymbol{a}$ is a matching with arcs $(i_1, i_2), (i_3, i_4)$ while $\boldsymbol{b}$ is a matching with arcs $(i_2, i_3), (i_1, i_4)$. The algorithm given in the proof of Proposition 3.9 will eventually find arc (i_2, i_3) unpaired in $\boldsymbol{a}_t w(\boldsymbol{b})$. It will perform the move $\rightarrow$ taking the unnested pair of arcs $(i_1, i_2), (i_3, i_4)$ to the nested pair of arcs $(i_2, i_3), (i_1, i_4)$.

Assume that the result is true for some p. Consider $i_1 < \cdots < i_{p+2}$. Take $\boldsymbol{a}$ and $\boldsymbol{b}$ to be matchings as in the statement of the corollary. The algorithm from the proof of Proposition 3.9 will eventually find the narrowest arc in $\boldsymbol{b}$ incident on these vertices. Say this arc is (i_k, i_{k+1}). The move $\rightarrow$ will be performed on matching $\boldsymbol{a}_t$ producing new arcs $(i_{k-1}, i_{k+2}), (i_k, i_{k+1})$. The arc (i_k, i_{k+1}) is now paired in $\boldsymbol{a}_{t+1} w(\boldsymbol{b})$. After renumbering the remaining arcs are incident on p vertices as in the statement of the corollary. We apply the inductive hypothesis to the remaining arcs, and the result follows by induction. $\qquad\square$

Before the next lemma, we provide one technical definition that will make the statement of the lemma easier.

Definition 5.13. Let $i < j < k < l$ and suppose $(i, l), (j, k)$ are nested arcs in some matching $\boldsymbol{a}$. We say that an arc is between (i, l) and (j, k) if the arc in $\boldsymbol{a}$ with left endpoint between i and j and right endpoint between k and l.

Lemma 5.14. *Let $\boldsymbol{a}, \boldsymbol{b} \in B^{m,m}$. Let $i < j < k < l$ and suppose $(i, l), (j, k)$ are nested arcs in $\boldsymbol{a}$ that are part of the same circle in $\boldsymbol{a} w(\boldsymbol{b})$ such that there is no arc in $\boldsymbol{a}$ lying between (i, l) and (j, k) that is also part of that circle. Then the number of arcs between $(i, l), (j, k)$ is even.*

Proof. Given some circle c in $\boldsymbol{a} w(\boldsymbol{b})$ let $W(c, z)$ be the winding number of c about z. Consider a line segment between $z_0, z_1 \notin c$ which is transverse to c. Say that the number of intersections of that segment with the circle c is some number r. Then mod 2 intersection theory tells us that

$$W(c, z_0) \equiv W(c, z_1) + r \pmod 2.$$

Take some arc between (i, l) and (j, k). Say this arc is part of circle c in $\boldsymbol{a} w(\boldsymbol{b})$. Note that the circle c is not the same circle that (i, l) and (j, k) are part of. Vertices i and j are both either inside or outside the circle c. Consider the vertices i and j to be points in the plane. Then $W(i, c) \equiv W(j, c) \pmod 2$. Putting this together

with the previous statement, say the segment between i and j intersects c exactly r times. Then $W(i, c) \equiv W(j, c) + r \pmod 2$ and hence $r \equiv 0 \pmod 2$. We conclude that c has an even number of intersections with the line segment between i and j.

Each arc in c with both endpoints between i and j contributes an even number of intersections, so the number of arcs between (i, l) and (j, k) must also be even. Since this is true for each circle with some arc between (i, l) and (j, k), there is an even number of arcs between (i, l) and (j, k).   □

Lemma 5.15. *Within the collection of arcs between (i, l) and (j, k) there are two adjacent arcs that are part of the same circle.*

Proof. By the argument above, there is an even number of vertices between i and j. Given two circles c and c' with arcs that lie between (i, l) and (k, j) the argument in the previous proof says that there are an even number of arcs from c' between any pair of arcs in c.

Because there are finitely many vertices between i and j, there must be some pair of arcs from a single circle that have no arcs from any other circle between them. Otherwise we could find infinitely many arcs between (i, l) and (j, k).   □

Lemma 5.16. *Given $a, b \in B^{m,m}$ if every minimal sequence $(a, a_1, \ldots, a_{m-1}, b)$ begins with $a \to a_1$ then there exists a minimal sequence $(\to, \to, \cdots, \to, \to)$.*

Proof. Assume that all minimal sequences for a and b begin with $\to$. In particular this means that there does not exist a pair of nested arcs in a that are part of the same circle in $aw(b)$ and have no other arc from that same circle between them. (The existence of such a pair would violate our assumption that all minimal sequences begin with $\to$.)

Say there is some pair of nested arcs $(i, l), (j, k) \in a$ that are part of the same circle in $aw(b)$. By Lemma 5.15 there must be a pair of nested, adjacent arcs in a contributed by some other circle of $aw(b)$ lying between (i, l) and (j, k). In this case, we could perform $\leftarrow$ on this pair once again violating our assumption that all minimal sequences must begin with $\to$. We conclude that the no pair of nested arcs in a are part of the same circle in $aw(b)$.

Since we have just shown the arcs in each circle in $aw(b)$ are pairwise unnested, they have the form

$$(i_1, i_2), (i_3, i_4), \ldots, (i_{t-1}, i_t) \text{ where } i_1 < i_2 < \cdots < i_{t-1} < i_t.$$

The arcs in b that comprise the rest of this circle necessarily have the form

$$(i_2, i_3), \ldots, (i_{t-2}, i_{t-1}), (i_1, i_t).$$

Consider the minimal sequence of moves for a and b obtained by the algorithm in the proof of Proposition 3.9. By Lemma 5.12 each move has the form $\to$.

Therefore we have a minimal sequence consisting only of $\rightarrow$ moves, and the lemma is proven. $\square$

Having established Lemma 5.16 we can now easily finish the proof of the base case for Lemma 3.14.

We proceed using induction. If the distance between matchings is 1, the result is clear since there is only one move between matchings in any minimal sequence.

Now assume the lemma is true for all a, b with $d(a, b) \leq m - 1$. Assume that $d(a, b) = m$. Let $(a = a_0, a_1, \ldots, a_{m-1}, a_m = b)$ be a minimal sequence of moves from a to b such that the move $\leftarrow$ occurs as early as possible.

If $a_0 \leftarrow a_1$ we can apply the inductive assumption to get $a_1 \succ c \prec a_m$ with $d(a_1, a_m) = d(a_1, c) + d(c, a_m)$ and $(a_1 \leftarrow \cdots \leftarrow c \rightarrow \cdots \rightarrow a_m)$. This proves the lemma in this case since we have a minimal sequence of the form

$$(a \leftarrow a_1 \leftarrow \cdots \leftarrow c \rightarrow \cdots \rightarrow b).$$

If $a_0 \rightarrow a_1$ then there does not exist a minimal sequence for a and b with first move $\leftarrow$. By Lemma 5.16 there exists in this case a minimal sequence of the form $(a \rightarrow \cdots \rightarrow b)$. $\square$

Acknowledgements

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