Pacific Journal of Mathematics

B2-CONVEXITY IMPLIES STRONG AND WEAK LOWER SEMICONTINUITY OF PARTITIONS OF $\mathbb{R}^n$

DAVID G. CARABALLO

Volume 253 No. 2 October 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 253, No. 2, 2011

B2-CONVEXITY IMPLIES STRONG AND WEAK LOWER SEMICONTINUITY OF PARTITIONS OF $\mathbb{R}^n$

DAVID G. CARABALLO

We prove that B2-convexity is sufficient for lower semicontinuity of surface energy of partitions of $\mathbb{R}^n$, for any $n \geq 2$. We establish lower semicontinuity in the usual strong topology, assuming the regions converge in volume. We also establish lower semicontinuity in the more general situation in which we suppose integral currents associated with individual regions converge to some integral current in the weak topology of integral currents.

B2-convexity, formulated by F. Morgan in 1995, is a powerful condition since it is easy to work with and since many other conditions from the literature imply it. Our results therefore imply that each of those conditions is sufficient for strong and weak lower semicontinuity of surface energy.

We establish other results of independent interest, including a Lebesgue point theorem for partitions and a localization theorem, which shows that if lower semicontinuity holds locally then it holds globally.

1. Introduction

Many applications, both static and dynamic, involve surface area or surface energy of interfaces in space, $\mathbb{R}^n$. Energy minimization problems involving surface energy of partitions – or polycrystals – are central to materials science, physics, biology, computer science, image processing, and other fields. Some such applications include crystal growth, tumor growth, annealing of metals, image segmentation, noise reduction in images, as well as the study of cell structures, immiscible fluids, metal foams, and semiconductors; see, for example, [Almgren 1976; Almgren and Taylor 1996; Ambrosio and Braides 1995; Ambrosio et al. 2001; 2003; 2000; Aubert and Kornprobst 2002; Bellettini et al. 2002; 2006; Braides 1998; Brook et al. 2003; Gurtin 1993; 1986; Morgan 1997; 1998; Mumford and Shah 1989; Osher and Fedkiw 2001; Sethian 1999; Taylor 1978; 1993; 1999; 2003; White 1996]. Indeed, most materials are polycrystalline.

MSC2000: 49J45, 49Q20.

Keywords: lower semicontinuity, partitions, clusters, polycrystals, B2-convexity, surface energy, geometric measure theory.

Lower semicontinuity of energy of partitions, which under suitable compactness conditions ensures existence of energy minimizers, is therefore of fundamental importance. In this paper, we consider the surface energy functional

$$SE(P) = \sum_{1 \le u < v \le s} \int_{p \in \partial K_u \cap \partial K_v} \phi_{uv}(n_{K_u}(p))\, d\mathcal{H}^{n-1} p, \tag{1}$$

where $P = K_{1..s}$ is a polycrystal corresponding to the partition $\{K_1, K_2, \ldots, K_s\}$ of $\mathbb{R}^n$, $n_{K_u}(p)$ is a measure-theoretic exterior unit normal vector to K_u at p, and the functions ϕ_{uv} are surface energy density functions used to model surface energy dependence on orientation, or anisotropy, of the interface $\partial K_u \cap \partial K_v$ between regions u and v in $\mathbb{R}^n$. This surface energy functional was first rigorously considered by F. Almgren [1976] for the special case $\phi_{uv} = c_{uv}\phi$ for constants c_{uv} satisfying additional hypotheses and for a fixed norm ϕ satisfying additional regularity hypotheses. It has subsequently been considered more generally; see, for example, [Almgren et al. 1993; Ambrosio and Braides 1990a; 1990b; Ambrosio et al. 2000; Bellettini et al. 2006; Braides 1998; Caraballo 1997; 2008; 2009; Morgan 1997; 1998; Taylor 1993; 1999; White 1996].

Although Almgren's restrictions on the functions ϕ_{uv} were sufficient for lower semicontinuity of the surface energy functional (1) with respect to strong convergence (i.e., convergence in volume of each of the regions separately), his hypotheses were far from necessary [Caraballo 2008]. L. Ambrosio and A. Braides [1990a; 1990b] discovered the first necessary and sufficient conditions that the functions ϕ_{uv} must satisfy for strong lower semicontinuity, an integral condition they named BV-ellipticity. This condition, which ensures that certain perturbations of a planar interface cannot have less surface energy than the original planar interface, is analogous, for the setting of surface energy functionals defined on partitions, to C. B. Morrey's quasi-convexity [1952].

Unfortunately, BV-ellipticity is an integral condition and is often quite difficult to check in practice. As a result, many other conditions on the integrands ϕ_{uv} have been introduced and studied, such as (B)-convexity (introduced in [Ambrosio and Braides 1990b]; cf. [Ambrosio et al. 2000]), joint convexity (see [Ambrosio et al. 2000] but also [Ambrosio and Braides 1990b]), LSC1 and LSC3 (introduced in [Caraballo 1997]), B2-convexity (introduced in [Morgan 1997]), and A-convexity, A2-convexity, and directional control (introduced in [Caraballo 2008]). These other conditions are easier to work with than BV-ellipticity, and yet are less restrictive than Almgren's conditions.

F. Morgan's B2-convexity is particularly important, as it is quite general and relatively easy to work with. In [Caraballo 2008], we showed that Almgren's condition, (B)-convexity, LSC1, LSC3, A-convexity, A2-convexity, and directional control each imply B2-convexity. When those results are combined with the main results of this paper, Theorems 14 and 15, we conclude that each of those conditions on the integrands ϕ_{uv} is sufficient for both strong and weak lower semicontinuity of the surface energy functional (1).

Because it is implied by most other conditions on the integrands ϕ_{uv}, it seemed that B2-convexity might be necessary for lower semicontinuity. As observed in [Morgan 1997], the example given in [Ambrosio and Braides 1990b] to show that (B)-convexity is not necessary for lower semicontinuity does not apply to B2-convexity. When $\phi_{uv} = c_{uv}\phi$, for a norm ϕ and for positive constants c_{uv} satisfying the triangle inequality $c_{uv} \leq c_{uw} + c_{wv}$ for each triple (u, v, w), B2-convexity is necessary for lower semicontinuity [Morgan 1997]. (This important special case arises, for example, when one considers immiscible fluids; see also [Almgren 1976] and [White 1996].) The necessity of B2-convexity for lower semicontinuity of (1) in general had remained an open question ever since the condition was formulated in 1995, in an early version of [Morgan 1997]. This question was settled in [Caraballo 2010]: B2-convexity is not necessary for lower semicontinuity.

In this paper, we prove that B2-convexity is sufficient for lower semicontinuity of the surface energy functional (1). This is a very useful result since B2-convexity has turned out to be a convenient to check condition, implied by most other conditions considered in the literature; as noted above, our results here imply the other conditions are all sufficient for both strong and weak lower semicontinuity. There are currently no published proofs of the sufficiency of B2-convexity for lower semicontinuity, the only other proof being the author's original proof in the unpublished thesis [Caraballo 1997]. Our proof here is simpler, much clearer, and significantly shorter than our original proof, on which this proof was based.

We establish lower semicontinuity for both strong and weak topologies. In Theorem 14, we have a sequence $\{P_i\}$ of polycrystals $P_i = K_{1..s,i}$ (each corresponding to the partition $\{K_{1,i}, K_{2,i}, \ldots, K_{s,i}\}$ of $\mathbb{R}^n$) converging to a fixed polycrystal $P = K_{1..s}$ in the sense that $[K_{u,i}] \to [K_u]$ in the strong topology of integral currents for each $u = 1, \ldots, s-1$. This is equivalent to the volume convergence used in other lower semicontinuity results in the literature, i.e., $P_i = K_{1..s,i}$ converges to $P = K_{1..s}$ in volume provided

$$\mathscr{L}^n\big((K_{u,i} \setminus K_u) \cup (K_u \setminus K_{u,i})\big) \to 0 \quad \text{for each } u.$$

In Theorem 15, we relax the convergence assumptions somewhat and require (for each $u = 1, \ldots, s-1$) that $[K_{u,i}]$ converges in the weak topology of integral currents (i.e., pointwise for each compactly supported C^∞ differential form) to an integral n-current T_u as $i \to \infty$. Assuming the crystals $[K_{u,i}]$ stay in a bounded region and that their masses are uniformly bounded, we deduce strong convergence and lower semicontinuity. The weak lower semicontinuity is a little surprising, since in general a collection of integral n-currents $T_1, \ldots, T_{s-1}$ does not specify a polycrystal.

We also establish other results of independent interest, such as a Lebesgue point theorem (Theorem 10) which shows that individual interfaces have surface energy which may be locally approximated by that of a disk, and a powerful localization estimate (Theorem 13), which shows that if lower semicontinuity holds locally then it holds globally.

We work in the context of the sets of finite perimeter and integral and rectifiable currents of geometric measure theory. The setting is general enough to allow for realistically complex boundary and topological structures, such as those present in an annealing metal, in a metal foam, in a soap bubble cluster, in a tumor, or in image segmentation problems, and yet is sufficiently structured that suitable notions of convergence and compactness exist.

2. Crystals and polycrystals

2.1. ***Basic notation and sets of finite perimeter.*** We will measure volume and surface area in $\mathbb{R}^n$ (for $n \geq 2$) with n-dimensional Lebesgue measure $\mathcal{L}^n$ and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, respectively. We let $B(p, r)$ and $U(p, r)$ denote, respectively, the closed and open balls in $\mathbb{R}^n$ with center p and radius r, and we set $\alpha(n) = \mathcal{L}^n(B(\mathbf{0}, 1))$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^n$. If $A, B \subset \mathbb{R}^n$ and $0 < m \leq n$, we write $A \subset_m B$ (read "A is $\mathcal{H}^m$-almost contained in B") when $\mathcal{H}^m(A \setminus B) = 0$. If $A, B \subset \mathbb{R}^n$, $A \triangle B$ denotes the symmetric difference of A and B: $A \triangle B = (A \setminus B) \cup (B \setminus A)$. If $A, B \subset \mathbb{R}^n$ we write $A \Subset B$ if $\overline{A}$, the topological closure of A in $\mathbb{R}^n$, is a compact subset of B. Given a point $p \in \mathbb{R}^n$ and a unit vector $u \in \mathbb{R}^n$, we define the open half-spaces $H_+(p, u) = \{x : (x - p) \cdot u > 0\}$ and $H_-(p, u) = \{x : (x - p) \cdot u < 0\}$ and the hyperplane $H(p, u) = \{x : (x - p) \cdot u = 0\}$ through p and orthogonal to u, and for each $r > 0$ we also define the closed disk

$$D(p, u, r) = H(p, u) \cap B(p, r), \tag{2}$$

having area $\mathcal{H}^{n-1}(D(p, u, r)) = \alpha(n-1)r^{n-1}$.

If $K \subset \mathbb{R}^n$, $p \in \mathbb{R}^n$, and $1 \leq m \leq n$, the m-dimensional density of K at p is

$$\Theta^m(K, p) = \lim_{r \to 0} \big(\mathcal{H}^m(K \cap B(p, r)) \,/\, \alpha(m) r^m\big),$$

provided the limit exists.

Let $\mathcal{C}$ denote the collection of all bounded, $\mathcal{L}^n$-measurable subsets $K \subset \mathbb{R}^n$ having finite perimeter, $P(K)$. If $\chi_K \in L^1(\mathbb{R}^n)$, then $P(K) < \infty$ if and only if $\chi_K \in BV(\mathbb{R}^n)$. In particular, if K is a bounded, $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n$, then $K \in \mathcal{C}$ if and only if $\chi_K \in BV(\mathbb{R}^n)$, where χ_K is the characteristic function of K. Whenever $K \in \mathcal{C}$, we let ∂K (the *reduced boundary* of K) be the set of all points p

in $\mathbb{R}^n$ at which K has a measure-theoretic exterior unit normal $n_K(p)$ in the sense of H. Federer [1969, 4.5.5], satisfying

$$\Theta^n\big(H_+(p, n_K(p)) \cap K,\ p\big) = \Theta^n\big(H_-(p, n_K(p)) \setminus K,\ p\big) = 0$$

whenever $p \in \partial K$. Also, if $K \in \mathscr{C}$ then $P(K) = \mathscr{H}^{n-1}(\partial K) < \infty$. If $K \in \mathscr{C}$ and $\mathscr{L}^n(K) > 0$, the isoperimetric inequality implies that we also have $\mathscr{H}^{n-1}(\partial K) > 0$, and so it follows that ∂K has Hausdorff dimension equal to $n-1$. Some excellent references that treat sets of finite perimeter and functions of bounded variation in detail are [Ambrosio et al. 2000; Burago and Zalgaller 1988; Evans and Gariepy 1992; Giusti 1984; Krantz and Parks 1999; Mattila 1995].

The following proposition, which follows quickly from [Federer 1969, 4.5.6], will be useful later.

Proposition 1. *Suppose $K \in \mathscr{C}$. Then for $\mathscr{H}^{n-1}$-almost every $p \in \partial K$ we have $\Theta^{n-1}(\partial K, p) = 1$, and for $\mathscr{H}^{n-1}$-almost every $p \notin \partial K$ we have $\Theta^{n-1}(\partial K, p) = 0$.*

2.2. *Integral currents.* The standard reference for currents is the treatise [Federer 1969]. Other very good references are [Almgren 1986a; Burago and Zalgaller 1988; Federer and Fleming 1960; Hardt and Simon 1986; Krantz and Parks 2008; Morgan 2009; Simon 1983].

An m-dimensional *current*[1] T is any element of $\mathscr{D}_m^*$, the dual space of the real vector space

$$\mathscr{D}^m = \{\varphi : \varphi \text{ is a } C^\infty\ m\text{-form on } \mathbb{R}^n \text{ having compact support}\}.$$

If $m > 0$ the *boundary* of T is defined so as to satisfy Stokes's theorem; i.e., $\partial T(\varphi) = T(d\varphi)$ whenever φ is an $(m-1)$-form. ∂T is therefore an $(m-1)$-current.

A set $X \subset \mathbb{R}^n$ is called *m-rectifiable*[2] if for each $\varepsilon > 0$ there exists a compact C^1 submanifold with boundary M_ε such that $\mathscr{H}^m(X \triangle M_\varepsilon) < \varepsilon$ (cf. [Almgren 1986a]). Perhaps the most interesting m-currents are those associated with rectifiable sets. An m-dimensional current T in $\mathbb{R}^n$ is a *rectifiable m-current* if it can be represented in the form $T = (\mathscr{H}^m \llcorner S) \wedge (\theta\sigma)$ — more compactly, $T = \boldsymbol{t}(S, \theta, \sigma)$ — where

(1) S is a bounded, m-rectifiable subset of $\mathbb{R}^n$,

(2) $\theta : S \to \{1, 2, 3, \dots\}$ is $\mathscr{H}^m \llcorner S$ summable (θ being the multiplicity, or density, function for the set S), and

(3) $\sigma : S \to \Lambda_m \mathbb{R}^n$ is the orientation function: $\sigma(x)$ is a simple m-vector satisfying $|\sigma(x)| = 1$, and, for $\mathscr{H}^m$-a.e. $x \in S$, the linear subspace of $\mathbb{R}^n$ spanned by $\sigma(x)$ is the

[1] Currents were introduced in [Rham 1955]. They are generalizations of Schwartz's distributions.

[2] This differs from the terminology used in [Federer 1969], where such sets are referred to as $\mathscr{H}^m$-measurable and $(\mathscr{H}^m, m)$ rectifiable.

approximate tangent m-subspace to S at p. (See [Federer 1969, 4.1.28; Almgren 1986a; Almgren et al. 1993, 3.1.3].)

We denote the set of rectifiable m-currents in $\mathbb{R}^n$ by $\mathcal{R}_m$. For the remainder of this section, $T \in \mathcal{R}_m$ unless otherwise noted. Whenever $\varphi \in \mathcal{D}^m$, we have

$$T(\varphi) = (\mathcal{H}^m \llcorner S) \wedge (\theta\sigma)(\varphi) = \int_{x\in S} \langle \sigma(x), \varphi(x) \rangle \theta(x)\, d\mathcal{H}^m x.$$

T is called an *integral* m-current if T is a rectifiable m-current, and (for $m > 0$ only) ∂T is a rectifiable $(m-1)$-current. The space of all integral m-currents in $\mathbb{R}^n$ is denoted $\mathbb{I}_m(\mathbb{R}^n)$. It follows from the closure theorem [Federer 1969, 4.2.16] that $\mathbb{I}_m(\mathbb{R}^n) = \{T \in \mathcal{R}_m : \boldsymbol{M}(\partial T) < \infty\}$ whenever $m > 0$.

The *variation measure* associated with T, $\|T\| = \mathcal{H}^m \wedge \theta$, assigns to each subset $X \subset \mathbb{R}^n$ the upper integral of θ over the set $S \cap X$:

$$\|T\|(X) = \int^*_{x\in S\cap X} \theta(x)\, d\mathcal{H}^m x.$$

The *mass* (a semi-norm) of T [Federer 1969, 4.1.7] is the integral of the density function over the set S: $\boldsymbol{M}(T) = \|T\|(\mathbb{R}^n)$. The *support* of T is

$$\operatorname{spt} T = \bigcap \{\Omega \subset \mathbb{R}^n \text{ closed} : \operatorname{spt}(\varphi) \cap \Omega = \varnothing \Rightarrow T(\varphi) = 0\},$$

as with distributions. Since $(\operatorname{spt}\varphi) \cap \overline{S} = \varnothing$ implies $T(\varphi) = 0$, it follows that $S \subset \operatorname{spt} T$. However, the set $\operatorname{spt} T$ can be much bigger than S in general (unless S is known to be regular).

The *flat* (semi-)norm $\mathcal{F}$ is defined by $\mathcal{F}(T) = \inf\{\boldsymbol{M}(Q) + \boldsymbol{M}(R)\}$, where the infimum is taken over all currents $R \in \mathcal{R}_m$ and (if $m < n$) $Q \in \mathcal{R}_{m+1}$ for which $T = R + \partial Q$. In particular, rectifiable $(n-1)$-currents S and T are flat close to each other when $S - T$ can be altered slightly (i.e., the piece R has small mass) so as to bound a crystal Q having small mass. $\mathcal{F}$ is useful for determining how close together surfaces are geometrically.

The *weak topology* on $\mathcal{D}_m$ is specified by asserting $T_i \to T$ weakly if and only if $T_i(\varphi) \to T(\varphi)$ for each $\varphi \in \mathcal{D}^m$. Convergence in the mass norm (strong convergence) implies convergence in the flat norm (flat convergence), which implies convergence on fixed m-forms (weak convergence). Suppose (for some $m > 0$) that $\{T_i\}$ is a sequence in $\mathbb{I}_m(\mathbb{R}^n)$, and that $T \in \mathbb{I}_m(\mathbb{R}^n)$. If $\sup_{i\geq 1} \boldsymbol{M}(T_i) + \boldsymbol{M}(\partial T_i) < \infty$, then weak convergence $T_i \to T$ is equivalent to convergence in the flat norm [Simon 1983, 31.2]. If $m = n$, then mass convergence is implied as well [Almgren et al. 1993, 3.1.5].

2.3. ***Crystals.*** Following [Almgren 1976] and other sources, we define crystals (and polycrystals) using integral currents associated with Lebesgue measurable subsets of $\mathbb{R}^n$ having finite perimeter. We consider both weak and strong convergence for such integral currents. Throughout this section, suppose that $K, L \in \mathscr{C}$. The current

$$E^n = \mathscr{L}^n \wedge (e_1 \wedge \cdots \wedge e_n)$$

is called the *n-dimensional Euclidean current* in $\mathbb{R}^n$. It is a locally integral current with $\partial E^n = 0$. It follows that

$$[K] = E^n \llcorner K = \mathscr{L}^n \llcorner K \wedge (e_1 \wedge \cdots \wedge e_n) = \boldsymbol{t}(K, 1, e_1 \wedge \cdots \wedge e_n)$$

is an integral n-current in $\mathbb{R}^n$. The boundary $(n-1)$-current $\partial[K]$ is given by

$$\partial[K] = \mathscr{H}^{n-1} \llcorner \partial K \wedge {*n_K} = \boldsymbol{t}(\partial K, 1, {*n_K}),$$

where $*n_K$ is the Hodge dual of n_K. Since $\partial \circ \partial[K] = 0$, we have $\partial[K] \in \mathbb{I}_{n-1}(\mathbb{R}^n)$. Also, $\boldsymbol{M}(\partial[K]) = \mathscr{H}^{n-1}(\partial K)$. Next, we define

$$\mathscr{K} = \{T \in \mathbb{I}_n(\mathbb{R}^n) : T = [K] \text{ for some } K \text{ in } \mathscr{C}\}.$$

Any element $[K] \in \mathscr{K}$ will be called a *crystal*. We will also use the term crystal to refer to the underlying set K. If we have several crystals, such as K_1 and K_2, for simplicity we will often refer to these as crystals 1 and 2. Because elements of $\mathscr{K}$ are integrals of differential forms, two sets K and L differing by a set having $\mathscr{L}^n$ measure zero give rise to the same crystal in $\mathscr{K}$.

Suppose $K, L \in \mathscr{C}$ with $\mathscr{L}^n(K \triangle L) = 0$. Then $\partial K = \partial L$. Furthermore,

$$\boldsymbol{M}([K] - [L]) = \boldsymbol{M}([K \setminus L] - [L \setminus K]) = \mathscr{L}^n(K \triangle L) = 0, \tag{3}$$

so $[K] = [L]$. In this way, sets of measure zero are ignored automatically.

Finally, we state some important facts concerning slicing of n-dimensional integral currents by distance functions to a point. The results in the following theorem are specializations of results given for integral currents of any dimension sliced by general Lipschitz functions in [Almgren et al. 1993, 3.1.8], and in a much more general setting in [Federer 1969, 4.1–4.3].

Theorem 2 (slicing of n-dimensional integral currents [Almgren et al. 1993, 3.1.8; Federer 1969, 4.1–4.3]).

Hypotheses:

(1) $T = \boldsymbol{t}(S, \theta, \sigma) \in \mathbb{I}_n(\mathbb{R}^n)$.

(2) $p \in \mathbb{R}^n$.

(3) $f(x) = |x - p|$ *whenever* $x \in \mathbb{R}^n$.

(4) $m(q) = \|T\|(U_q)$ *for each* $q > 0$, *where* $U_q = \{x \in \mathbb{R}^n : f(x) < q\} = U(p, q)$.

Conclusions: *For* $\mathscr{L}^1$*-almost every* $q > 0$ *each of the following is true*:

(1) $\|T\|(\partial U_q) = 0$.

(2) $T \llcorner U_q \in \mathbb{I}_n(\mathbb{R}^n)$.

(3) $(\partial T) \llcorner U_q \in \mathbb{I}_{n-1}(\mathbb{R}^n)$.

(4) *The slice current* $\langle T, f, q\rangle$, *defined as*

$$\langle T, f, q\rangle = \boldsymbol{t}(S \cap \partial U_q, \theta, \sigma')$$

for the right σ' (*see [Federer 1969,* 4.3.1*]*), *is an element of* $\mathbb{I}_{n-1}(\mathbb{R}^n)$.

(5) $\operatorname{spt}\langle T, f, q\rangle \subset (\operatorname{spt} T) \cap \partial U_q$.

(6) $\boldsymbol{M}(\langle T, f, q\rangle) \le m'(q)$.

(7) $\partial(T \llcorner U_q) = (\partial T) \llcorner U_q + \langle T, f, q\rangle$.

2.4. *Surface energy integrands.*

Definition 3. A *surface energy integrand* (or *surface energy density function*) on $\mathbb{R}^n$ is a function $\phi : \mathbb{R}^n \to [0, \infty)$ that satisfies

(a) $\phi(x) \ge 0$ for all $x \in \mathbb{R}^n$, with $\phi(x) = 0$ if and only if $x = \mathbf{0}$,

(b) $\phi(cx) = c\phi(x)$ whenever $c \ge 0$ and $x \in \mathbb{R}^n$, and

(c) $\phi(x + y) \le \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}^n$ (triangle inequality).

Definition 4. Suppose ϕ is a surface energy integrand on $\mathbb{R}^n$. We say that ϕ is of *class 1* provided ϕ is continuously differentiable at each $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

I.e., ϕ is a surface energy integrand provided it is a continuous, positive-valued function on unit vectors in $\mathbb{R}^n$, which, when extended by positive homogeneity of degree one (i.e., $\phi(cx) = c\phi(x)$ if $c \ge 0$) to all of $\mathbb{R}^n$, becomes a convex function. If $\phi(v) = \phi(-v)$ for all unit vectors v in $\mathbb{R}^n$, so that ϕ is even, then ϕ is a norm on $\mathbb{R}^n$. ϕ determines which orientations are cheap, which in turn affects the orientation of any exposed pieces of surface. In the case of evolution problems, ϕ determines which directions are favorable for growth. Perhaps the simplest case is the one where ϕ is taken to be the Euclidean norm: $\phi_E(v) = |v|$ for all $v \in \mathbb{R}^n$. ϕ_E is called the *area integrand* since surface energy is just surface area if $\phi = \phi_E$. If ϕ is a positive constant multiple of ϕ_E, then no directions are preferable, and we call ϕ isotropic; otherwise, ϕ is anisotropic. While there are isotropic surface energy densities in nature (e.g., as in soap bubbles), anisotropic ones are far more common.

The next theorem, stated in greater generality in [Federer 1969, 5.1.2] and proved in detail in [Caraballo 1997, 4.4] using Jensen's inequality [Federer 1969, 2.4.19], asserts that planar interfaces are not more expensive than non-planar ones having the same boundary, provided the surface energy integrand is convex.

Theorem 5 (planar interfaces are cheap). *Suppose $T = \boldsymbol{t}(S, 1, \sigma) \in \mathbb{I}_{n-1}(\mathbb{R}^n)$. Suppose that $D = \boldsymbol{t}(D_0, 1, \tau) \in \mathbb{I}_{n-1}(\mathbb{R}^n)$, where D_0 is a bounded, $\mathcal{H}^{n-1}$-measurable subset of an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, and there is a simple $(n-1)$-vector τ_0 such that $\tau(z) = \tau_0$ for all z in D_0. Suppose further that $\partial D = \partial T$ (i.e., $T - D$ is an integral $(n-1)$-cycle). Let ϕ be a fixed surface energy integrand. Then*

$$\int_{z \in S} \phi(\sigma(z))\, d\mathcal{H}^{n-1} z \geq \int_{z \in D_0} \phi(\tau(z))\, d\mathcal{H}^{n-1} z = \phi(\tau_0)\, \mathcal{H}^{n-1}(D_0). \tag{4}$$

2.5. ***Polycrystals and interfaces between crystals.*** An *admissible partition* of $\mathbb{R}^n$ is any partition $\{K_1, \dots, K_s\}$ of $\mathbb{R}^n$ into s pairwise disjoint, $\mathcal{L}^n$-measurable subsets K_u having finite perimeter, and for which K_u is bounded for each $u < s$, so that $K_s = \mathbb{R}^n \setminus (K_1 \cup \cdots \cup K_{s-1})$ is the only unbounded region. Since $\mathbb{R}^n$ has no boundary, $\partial K_s = \partial(K_1 \cup \cdots \cup K_{s-1})$. The last set K_s can represent the “outside” or “melt” in the case of crystals. We could also consider s different immiscible fluids in a bounded container, in which case K_s would also be bounded.

A *polycrystal* is any s-tuple $([K_1], \dots, [K_s])$, associated with an admissible partition $\{K_1, \dots, K_s\}$ of $\mathbb{R}^n$ through the relations $[K_u] = E^n \llcorner K_u \in \mathcal{K}$ if $1 \leq u \leq s-1$, and $[K_s] = E^n - [K_1 \cup \cdots \cup K_{s-1}]$. For convenience, we write $P = K_{1..s}$ for such a polycrystal P, and we let $\mathcal{P}^s$ denote the space of polycrystals $P = K_{1..s}$, as above. When $s = 2$, a polycrystal can be modeled as a single crystal, since there is only one interface, and we let ϕ denote the surface energy density function. The case $s > 2$ is substantially more complicated, since each interface is allowed to have its own surface energy density function. Let ϕ_{uv} denote the surface energy density function for the u-v interface (with $1 \leq u < v \leq s$). Given a collection of integrands $\{\phi_{uv}\}_{1 \leq u < v \leq s}$, we set

$$\begin{aligned} \phi_0 &= \inf\big\{\phi_{uv}(w) : |w| = 1,\ 1 \leq u < v \leq s\big\}, \\ \phi^0 &= \sup\big\{\phi_{uv}(w) : |w| = 1,\ 1 \leq u < v \leq s\big\}. \end{aligned} \tag{5}$$

A continuity-compactness argument shows that these extrema are attained and that $0 < \phi_0 \leq \phi^0 < \infty$.

Strong (or mass) convergence in $\mathcal{K}$ corresponds to convergence in volume of the underlying sets of finite perimeter, in accordance with (3). More precisely, if $L_i \in \mathcal{C}$ for each $i = 1, 2, \dots$, and if $L \in \mathcal{C}$, then $[L_i] \to [L]$ *strongly* (or, *in mass*) provided $\mathcal{L}^n(L_i \triangle L) \to 0$ as $i \to \infty$, since $\boldsymbol{M}([L_i] - [L]) = \mathcal{L}^n(L_i \triangle L)$. We now extend this notion to polycrystals in a natural way.

Definition 6. If $\{P_i\}$ is a sequence of polycrystals $P_i = K_{1..s,i} \in \mathcal{P}^s$, and $P = K_{1..s} \in \mathcal{P}^s$, we say that $P_i \to P$ in the strong topology provided $[K_{u,i}] \to [K_u]$ strongly as $i \to \infty$ for each $1 \leq u \leq s$.

The u-v *interface (for* $u < v$) between two crystals $[K_u]$ and $[K_v]$ is the rectifiable $(n-1)$-current $\Gamma_{uv} = \mathcal{H}^{n-1} \llcorner (\partial K_u \cap \partial K_v) \wedge *n_{K_u}$, associated with the rectifiable set $\Gamma^*_{uv} = \Gamma^*_{vu} = \partial K_u \cap \partial K_v$. Γ_{vu} is defined in the same manner, and its orientation is opposite to that of Γ_{uv}. The main disadvantage of working in such a general context is that very complicated objects have to be manipulated. For example, sets with excessively wild boundaries may yield currents whose boundary supports have positive, even large n-dimensional volume. Also, the boundaries of crystals in polycrystals may conceivably be so mixed up that the local structure is everywhere complex, so that we cannot work with individual interfaces. We preclude such pathological behavior in Theorem 7, which shows the relationship between the interfaces in a polycrystal and the boundaries of the individual crystals. Measure-theoretic unit normal vectors from one crystal into another exist $\mathcal{H}^{n-1}$ almost everywhere, and boundary currents are the sum of the individual interface currents.

Theorem 7 (boundary structure theorem [Caraballo 1997, Theorem 6; Caraballo 2008, Theorem 2.4]). *Let* $P = K_{1..s}$ *be in* $\mathcal{P}^s$. *Let* $\mathcal{B} = \bigcup_{u=1}^{s} \partial K_u$.

(1) *For* $\mathcal{H}^{n-1}$*-almost every point* $p \in \mathcal{B}$ *there exist distinct integers* u *and* v *so that unit normals* $n_{K_u}(p)$ *and* $n_{K_v}(p)$ *exist, with* $n_{K_u}(p) = -n_{K_v}(p)$.

(2) *For each* $1 \le u \le s$, *the function* n_{K_u} *is* $(\mathcal{H}^{n-1} \llcorner \partial K_u)$*-measurable.*

(3) *For each* $1 \le u \le s$, $\partial[K_u] = \sum_{i \ne u} \Gamma_{ui}$.

(4) $\mathcal{H}^{n-1}(\mathcal{B} \setminus \bigcup_{1 \le u < v \le s} \Gamma^*_{uv}) = 0$.

Remark 8. Property (1) implies that the set of "bad" points (such as typical multiple junctions) at which there is no well-defined interface normal direction has $\mathcal{H}^{n-1}$ measure zero.

The next theorem (see [Federer 1969, 2.8.15; Krantz and Parks 2008, 4.2.13; Morgan 2009, 2.7]) is indispensable because it allows us to carry out our analysis locally on interfaces of polycrystals.

Theorem 9 (Besicovitch–Federer covering theorem). *Suppose* μ *is a Borel measure on* $\mathbb{R}^n$ $(n \ge 1)$, *that* A *is a subset of* $\mathbb{R}^n$ *with* $\mu(A) < \infty$, *and that* $\mathcal{F}$ *is a nonempty family of closed balls* (*having positive radii uniformly bounded from above*) *such that each point* $a \in A$ *is contained in at least one member of* $\mathcal{F}$, *and* $\inf\{R : B(a, R) \in \mathcal{F}\} = 0$ *for each* $a \in A$. *Then there exists a countable, disjointed subcollection of* $\mathcal{F}$ *which covers* μ*-almost all of* A.

2.6. ***Surface energy of a polycrystal.*** Suppose $P = K_{1..s} \in \mathcal{P}^s$, $U \subset \mathbb{R}^n$ is open, and $1 \le u < v \le s$. We define the *surface energy* of the u-v interface of P in U

according to the formula

$$SE^{u,v}(P, U) = \int_{p \in \Gamma^*_{uv} \cap U} \phi_{uv}(n_{K_u}(p))\, d\mathcal{H}^{n-1} p.$$

If $U = \mathbb{R}^n$, we simply write $SE^{u,v}(P)$. We define the surface energy of P in U in the obvious way as $SE(P, U) = \sum_{1 \le u < v \le s} SE^{u,v}(P, U)$. If $U = \mathbb{R}^n$, we simply write $SE(P)$. That is,

$$SE(P) = \sum_{1 \le u < v \le s} SE^{u,v}(P, \mathbb{R}^n) = \sum_{1 \le u < v \le s} \int_{p \in \Gamma^*_{uv}} \phi_{uv}(n_{K_u}(p))\, d\mathcal{H}^{n-1} p. \tag{6}$$

If V is any closed subset of $\mathbb{R}^n$ and $1 \le u < v \le s$, we define $SE^{u,v}(P, V) = SE^{u,v}(P) - SE^{u,v}(P, \mathbb{R}^n \setminus V)$ and $SE(P, V) = SE(P) - SE(P, \mathbb{R}^n \setminus V)$. Without loss of generality, when computing surface energies of interfaces, surface energy density functions are always evaluated at the normals pointing toward the higher-numbered crystal.

2.7. ***A Lebesgue point theorem for partitions.*** In this section, we prove a result concerning some surprising cancellation of oscillations. It is of independent interest, as it will be useful for localizing many arguments involving surface energy of partitions in $\mathbb{R}^n$ (and in other nice manifolds, since the argument is local). In the present paper, we use it to establish 5 in the proof of Theorem 14.

The following theorem shows that surface energy is well-behaved in small balls centered at Lebesgue points of suitable functions, and moreover $\mathcal{H}^{n-1}$ every point in $\bigcup_{u=1}^{s} \partial K_u$ is such a Lebesgue point. This is somewhat surprising since the boundary may be nonsmooth at such points, and since our surface energy density functions are typically non-differentiable and may be extremely sensitive to orientation changes. We are given only (see Theorem 7) that $\mathcal{H}^{n-1}$-almost every $x \in \bigcup_{u=1}^{s} \partial K_u$ has a well-defined measure-theoretic unit normal vector, which means that small balls centered at x consist essentially of two crystals, in the sense that each of two crystals occupies approximately one half of the total volume of the small balls.

Result (4) in the following theorem ensures that

$$SE^{i,j}(P, B_r) \approx \phi_{ij}(n_{K_i}(x))\, \mathcal{H}^{n-1}(D(x, n_{K_i}(x), r)),$$

with an error that becomes arbitrarily small compared to r^{n-1} as $r \to 0^+$, provided x is a Lebesgue point of $u = \phi_{ij} \circ n_{K_i}$.

Theorem 10 (Lebesgue point estimate). *Suppose $P = K_{1..s} \in \mathcal{P}^s$. Fix integers i and j with $1 \le i < j \le s$, let $\varphi = \mathcal{H}^{n-1} \llcorner (\partial K_i \cap \partial K_j)$, and let ϕ_{ij} be any surface energy density function. Let $u(x) = \phi_{ij}(n_{K_i}(x))$ for each x. Then*:

(1) *u is φ-measurable.*

(2) $\int_{z\in\Omega} |u(z)| d\varphi z < \infty$ *for all bounded, φ-measurable $\Omega \subset \mathbb{R}^n$.*

(3) *For φ-almost every x,*

$$\lim_{R\to 0^+} \frac{\int_{z\in B(x,R)} |u(z)-u(x)|\, d\varphi z}{\varphi(B(x,R))} = 0.$$

Those points x satisfying this equation are the Lebesgue points of u.

(4) *For $\mathcal{H}^{n-1}$-almost every $x \in \partial K_i \cap \partial K_j$,*

$$\lim_{R\to 0^+} \frac{SE^{i,j}(P, B(x,R)) - \phi_{ij}(n_{K_i}(x))\, \mathcal{H}^{n-1}\big(D(x, n_{K_i}(x), R)\big)}{\alpha(n-1)R^{n-1}} = 0.$$

Proof. Since ϕ_{ij} is continuous and n_{K_i} is φ-measurable by Theorem 7(2), it follows that u is φ-measurable, so (1) holds. Let $\phi_{\max} = \sup\{\phi_{ij}(v) : |v| = 1\}$. We have $\phi_{\max} \in (0, \infty)$ by a standard continuity/compactness argument, and so $\int_{z\in\Omega} |u(z)| d\varphi z \le \phi_{\max} \cdot \mathcal{H}^{n-1}(\partial K_i \cap \partial K_j) < \infty$, proving (2). Since (1) and (2) are true, we invoke Corollary 2.9.9 of [Federer 1969] with $f = u$, $S = B(x, R)$, and $Y = \mathbb{R}$ to deduce (3). To prove (4), suppose $x \in \partial K_i \cap \partial K_j$ is a Lebesgue point of u, and let B_R denote $B(x, R)$. Since $\varphi = \mathcal{H}^{n-1} \llcorner (\partial K_i \cap \partial K_j)$, $SE^{i,j}(P, B_R) = \int_{B_R} u(z)\, d\varphi z$. Since $u(x) = \phi_{ij}(n_{K_i}(x))$, we have $\int_{z\in B_R} u(x)\, d\varphi z = u(x)\, \varphi(B_R)$.

Fix any arbitrary $\varepsilon > 0$. Conclusion (3) guarantees the existence of an $R' > 0$ such that for $\mathcal{L}^1$-almost every $R \in (0, R')$ we have

$$\left|\frac{SE^{i,j}(P, B_R)}{\varphi(B_R)} - u(x)\right| = \frac{\left|SE^{i,j}(P, B_R) - u(x)\,\varphi(B_R)\right|}{\varphi(B_R)} \le \frac{\int_{z\in B_R} |u(z)-u(x)|\, d\varphi z}{\varphi(B_R)} < \frac{\varepsilon}{2}. \tag{7}$$

Because $x \in \partial K_i \cap \partial K_j$, there exists an $R'' > 0$ such that for $\mathcal{L}^1$-almost every $R \in (0, R'')$

$$\left|1 - \frac{\varphi(B_R)}{\alpha(n-1)\, R^{n-1}}\right| \le \frac{\varepsilon}{2\phi^0}.$$

For $\mathcal{L}^1$-almost every $R \in \big(0, \min(R', R'')\big)$, it follows that

$$\begin{aligned} &\left|\left|\frac{SE^{i,j}(P, B_R)}{\varphi(B_R)} - u(x)\right| - \left|\frac{SE^{i,j}(P, B_R)}{\alpha(n-1)\, R^{n-1}} - u(x)\right|\right| \\ &\le \left|\frac{SE^{i,j}(P, B_R)}{\varphi(B_R)} - \frac{SE^{i,j}(P, B_R)}{\alpha(n-1)\, R^{n-1}}\right| = \frac{SE^{i,j}(P, B_R)}{\varphi(B_R)} \left|1 - \frac{\varphi(B_R)}{\alpha(n-1)\, R^{n-1}}\right| \\ &\le \phi^0 \left|1 - \frac{\varphi(B_R)}{\alpha(n-1)\, R^{n-1}}\right| \le \frac{\varepsilon}{2}. \end{aligned} \tag{8}$$

The bounds (7) and (8), together with the triangle inequality, give

$$\left|\frac{SE^{i,j}(P,B_R)-\phi_{ij}(n_{K_i}(x))\,\mathcal{H}^{n-1}\big(D(x,n_{K_i}(x),R)\big)}{\alpha(n-1)R^{n-1}}\right| = \left|\frac{SE^{i,j}(P,B_R)}{\alpha(n-1)\,R^{n-1}}-u(x)\right| \le \epsilon,$$

so (4) follows since ϵ was arbitrary. □

3. Lower semicontinuity of surface energy

3.1. ***Restrictions on the integrands.*** We begin by defining B2-convexity, introduced in [Morgan 1997]. The sets X and Y need not have positive $\mathcal{L}^n$ measure.

Definition 11. A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfies *B2-convexity* if, for any $K_{1..s}\in\mathcal{P}^s$ and for each pair (u,v) with $u\neq v$, the union $\bigcup_{z\notin\{u,v\}}K_z$ of the other regions in $K_{1..s}$ may be partitioned into two disjoint, $\mathcal{L}^n$-measurable sets X and Y in such a way that, when one is renamed to u and the other is renamed to v, the resulting polycrystal (corresponding to the partition $\{K_u\cup X,\ K_v\cup Y\}$ of $\mathbb{R}^n$ into two sets) has surface energy not exceeding $SE(K_{1..s})$.

B2-convexity is significant for several reasons. Although it was shown in [Caraballo 2010] that B2-convexity is not necessary for lower semicontinuity of (6), many other conditions on surface energy density functions $\{\phi_{uv}\}_{1\le u<v\le s}$ in $\mathbb{R}^n$ imply B2-convexity, most notably partitioning regularity (introduced in [Almgren 1976]), (B)-convexity (introduced in [Ambrosio and Braides 1990b]; cf. [Ambrosio et al. 2000]), LSC1 and LSC3 (introduced in [Caraballo 1997]), and A-convexity, A2-convexity, and directional control (introduced in [Caraballo 2008]). Since B2-convexity is sufficient for lower semicontinuity of the surface energy functional (6), as we will show, each of the other conditions implies lower semicontinuity as well. B2-convexity is quite general, and at the same time it is concrete (because it is a replacement condition) to the extent that proving that B2-convexity holds is one of the easiest ways to establish that a condition on the integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ implies lower semicontinuity.

For the special case $\phi_{uv}=c_{uv}\phi$, for a given fixed, even surface energy integrand ϕ, and for constants $c_{uv}>0$ satisfying the triangle inequalities ($c_{uw}\le c_{uv}+c_{vw}$), Morgan [1997] uses the max-flow/min-cut theorem from graph theory to show that B2-convexity is equivalent to lower semicontinuity of (6).

Remark 12. B2-convexity arose when F. Morgan observed (personal communication) that the author was using more restrictive conditions in order to make a certain replacement construction (the reduction to two crystals in a key step, without increase of energy) work, in the lower semicontinuity proof from [Caraballo

1997]. He suggested making the condition itself be that the replacement construction works, and he named that condition B2-convexity [Morgan 1997].

3.2. ***A local property of lower semicontinuity.*** The following theorem, roughly speaking, shows that if lower semicontinuity holds locally then it holds in general. More precisely, if lower semicontinuity fails then it fails by some percentage (as defined in the theorem statement), and that implies it must fail by at least that percentage in some fixed, closed ball. This powerful result will allow us to work locally in a fixed closed ball when we prove that B2-convexity implies surface energy is lower semicontinuous.

Theorem 13. *Suppose* $P = K_{1..s} \in \mathcal{P}^s$, *and* $P_j = K_{1..s,j} \in \mathcal{P}^s$ *for each* $j = 1, 2, 3, \ldots,$ *and set* $E = SE(P)$ *and* $E_j = SE(P_j)$. *Suppose lower semicontinuity fails*, *so that* $E - \liminf_{j\to\infty} E_j > 0$, *and let*

$$\eta = \frac{E - \liminf_{j\to\infty} E_j}{E} \times 100 \in (0, 100] \tag{9}$$

denote the percentage by which lower semicontinuity fails. Let $\{B^k\}_1^\infty$ *be a countable*, *disjointed collection of closed balls in* $\mathbb{R}^n$ *such that*

$$E = \sum_{k=1}^{\infty} E(B^k).$$

Then there exists a positive integer k such that lower semicontinuity fails by at least η *percent in* B^k. *I.e.*,

$$\frac{E(B^k) - \liminf_{j\to\infty} E_j(B^k)}{E(B^k)} \times 100 \geq \eta \tag{10}$$

for some positive integer k, *where* $E(B^k) = SE(P, B^k)$, *and* $E_j(B^k) = SE(P_j, B^k)$.

Proof. Suppose, to the contrary, that for each positive integer k lower semicontinuity fails by strictly less than η percent in B^k. Then

$$\liminf_{j\to\infty} E_j \geq \liminf_{j\to\infty} \left(\sum_{k=1}^{\infty} E_j(B^k) \right) \geq \sum_{k=1}^{\infty} \liminf_{j\to\infty} E_j(B^k)$$
$$> \sum_{k=1}^{\infty} \left(1 - \frac{\eta}{100}\right) E(B^k) = \left(1 - \frac{\eta}{100}\right) E.$$

The first inequality holds since $E_j \geq \sum_{k=1}^{\infty} E_j(B^k)$ for each positive integer j. The second follows from standard real analysis, and the strict inequality is implied by the assumption that the percentage by which lower semicontinuity fails is strictly less than η in each B^k. Therefore, the percentage by which lower semicontinuity fails equals

$$\frac{E - \liminf_{j\to\infty} E_j}{E} \times 100 < \eta,$$

a contradiction. □

3.3. ***The main lower semicontinuity theorems.*** Here we state our main lower semicontinuity theorems. We prove them in Sections 3.4 and 3.5, respectively. The weak lower semicontinuity is a little surprising, since in general a collection of integral n-currents $T_1, \ldots, T_{s-1}$ does not specify a polycrystal.

Theorem 14 (surface energy is strongly lower semicontinuous). *Suppose the surface energy density functions $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfy B2-convexity. Suppose $P_i = K_{1..s,i}$ is a polycrystal in $\mathcal{P}^s$ for each $i = 1, 2, 3, \ldots$. Suppose $P = K_{1..s} \in \mathcal{P}^s$. If $[K_{u,i}] \to [K_u]$ strongly as $i \to \infty$, for each $u < s$, then*

$$SE(P) \le \liminf_{i\to\infty} SE(P_i).$$

Theorem 15 (surface energy is weakly lower semicontinuous). *Suppose the surface energy density functions $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfy B2-convexity. Suppose that, for each positive integer i, $P_i = K_{1..s,i}$ is a polycrystal in $\mathcal{P}^s$ such that the P_i satisfy*

$$\sup_i\left(\sum_{u=1}^{s-1} \boldsymbol{M}(\partial[K_{u,i}])\right) < \infty \quad \textit{and} \quad \operatorname{diam}\left\{\bigcup_i\left(\bigcup_{u=1}^{s-1} \operatorname{spt}[K_{u,i}]\right)\right\} < \infty. \tag{11}$$

Suppose further that, for each $u < s$, $[K_{u,i}]$ converges weakly to an integral n-current T_u as $i \to \infty$. Then there exists a polycrystal $P = K_{1..s} \in \mathcal{P}^s$ such that $[K_{u,i}] \to [K_u]$ strongly as $i \to \infty$, for each $u < s$, and

$$SE(P) \le \liminf_{i\to\infty} SE(P_i).$$

3.4. ***Proof of the strong lower semicontinuity theorem.*** For $j \ge 1$, $1 \le u < v \le s$, and B a closed ball in $\mathbb{R}^n$, we will use the abbreviations $E^{u,v}(B) = SE^{u,v}(P, B)$, $E(B) = SE(P, B)$, $E = SE(P)$, $E_j^{u,v}(B) = SE^{u,v}(P_j, B)$, $E_j(B) = SE(P_j, B)$, and $E_j = SE(P_j)$.

Suppose that lower semicontinuity fails. Then there exists a positive real number $\eta \in (0, 100]$ such that lower semicontinuity fails by η percent, in the sense of (9) (see Theorem 13). Fix η, let ϕ_0 and ϕ^0 be as in (5), and define the constants $\lambda > 0$ and $\delta \in (0, 1)$ according to

$$\lambda = \eta\,\alpha(n-1)\frac{\phi_0}{200} \quad \text{and} \quad 1-(1-\delta)^{n-1} = \eta\frac{\phi_0}{2000\,\phi^0}. \tag{12}$$

Let $\mathcal{B} = \bigcup_{u=1}^{s} \partial K_u$, and define

$$\mathcal{B}' = \bigcup_{1 \le u < v \le s} \left(\Gamma^*_{uv} \cap \{\text{Lebesgue points of } \phi_{uv} \circ n_{K_u}\} \right).$$

It follows from Theorem 10 that $\mathcal{H}^{n-1}(\mathcal{B} \setminus \mathcal{B}') = 0$. We define the index function

$$I : \mathcal{B}' \to \{(u, v) : 1 \le u < v \le s\}$$

by requiring that $I(x) = (I_1(x), I_2(x)) = (u, v)$ whenever $x \in \Gamma^*_{uv}$. This is well-defined because of the definition of $\mathcal{B}'$.

We will now define a collection $\mathcal{F}$ of closed balls $B(x, R)$, satisfying all the estimates we will need to work locally. Let $\mathcal{F}$ be the collection of all closed balls $B(x, R) \subset \mathbb{R}^n$ such that $x \in \mathcal{B}'$, $\mathcal{H}^{n-1}(\partial B(x, R) \cap \mathcal{B}') = 0$, and $0 < R < R_0(x)$, where $R_0(x)$ is such that whenever $x \in \mathcal{B}'$ and $0 < R < R_0(x)$ the following inequalities hold:

$$\left| E^{I_1(x), I_2(x)}(B(x, R)) - \alpha(n-1)\, R^{n-1}\, \phi_{I_1(x) I_2(x)}(n_{K_{I_1(x)}}(x)) \right| < \frac{\lambda}{10} R^{n-1}, \tag{13}$$

$$E(B(x, R)) - E^{I_1(x), I_2(x)}(B(x, R)) < \frac{\lambda}{10} R^{n-1}, \tag{14}$$

$$\mathcal{L}^n\left((K_{I_1(x)} \triangle H_-(x, n_{K_{I_1(x)}}(x))) \cap B(x, R) \right) < \frac{\delta\lambda}{160\phi^0} R^n, \tag{15}$$

$$\mathcal{L}^n\left((K_{I_2(x)} \triangle H_+\big(x, n_{K_{I_1(x)}}(x)\big)) \cap B(x, R) \right) < \frac{\delta\lambda}{160\phi^0} R^n,$$

$$\mathcal{L}^n\left(\Bigg(\bigcup_{\substack{u \in \{1, \ldots, s\} \\ u \notin \{I_1(x), I_2(x)\}}} K_u \Bigg) \cap B(x, R) \right) < \frac{\delta\lambda}{80\phi^0} R^n. \tag{16}$$

Claim 1. *Each point $x \in \mathcal{B}'$ is contained in at least one member of $\mathcal{F}$, and*

$$\inf\{R : B(x, R) \in \mathcal{F}\} = 0$$

for each $x \in \mathcal{B}'$.

Proof. That (13) is true for all sufficiently small values of R follows from conclusion (4) of Theorem 10, since x is a Lebesgue point of $\phi_{I_1(x) I_2(x)} \circ n_{K_{I_1(x)}}$. Clearly, $E(B(x, R)) - E^{I_1(x), I_2(x)}(B(x, R)) \ge 0$. Equation (14) holds for all small enough values of R because the $(n-1)$-density of the $I_1(x)$-$I_2(x)$ interface is 1 at x, and so the $(n-1)$-density of all of the other interfaces combined must be 0 at x. The remaining inequalities above are true for all sufficiently small values of R because $n_{K_{I_1(x)}}$ is a (measure-theoretic) unit normal vector at x, and so the n-densities of crystals $I_1(x)$ and $I_2(x)$ are each $\frac{1}{2}$ at x, and that means the n-densities of all the other crystals at x must be zero. Finally, $\mathcal{L}^n(\mathcal{B}') = 0$ implies that $\mathcal{H}^{n-1}\big(\partial B(x,R)\cap\mathcal{B}'\big)=0$ for $\mathcal{L}^1$ almost every $R>0$. It follows that $\mathcal{F}$ is fine at each $x\in\mathcal{B}'$. □

We apply the Besicovitch–Federer covering theorem (Theorem 9) with $\mu=\mathcal{H}^{n-1}\llcorner\mathcal{B}'$ and $A=\mathcal{B}'$ there to deduce that there exists a countable, disjointed subcollection $\{B^k\}_1^\infty$ of $\mathcal{F}$ which covers μ-almost all of $\mathcal{B}'$. We therefore have $E=\sum_{k=1}^\infty E(B^k)$. By Theorem 13, there exists a positive integer k such that lower semicontinuity fails by at least η percent in B^k.

Notation 16. We now fix x and R, the center and radius of $B^k=B(x,R)$, for the remainder of the proof. For simplicity, and without loss of generality, we will suppose $I(x)=(1,2)$, so that x is on the 1-2 interface of P. For any $s>0$ we abbreviate $B_s=B(x,s)$, $U_s=U(x,s)$, and $D_s=D\big(x,n_{K_1}(x),s\big)$. We also abbreviate $H_-=H_-(x,n_{K_1}(x))$, $H_+=H_+(x,n_{K_1}(x))$, and $H=H(x,n_{K_1}(x))$. For any $s>0$, if the 1-2 interface inside B_s were planar, we would find the surface energy on D_s by integrating ϕ_{12} over D_s, evaluating ϕ_{12} at the fixed vector $n_{K_1}(x)$ which is orthogonal to D_s. We therefore introduce the convenient notation

$$SE(D_s)=\int_{z\in D_s}\phi_{12}(n_{K_1}(x))\,d\mathcal{H}^{n-1}z=\phi_{12}(n_{K_1}(x))\,\mathcal{H}^{n-1}(D_s)\\ =\phi_{12}(n_{K_1}(x))\,\alpha(n-1)s^{n-1}. \tag{17}$$

Even though x will not be on the 1-2 interface of P_j in general, since $[K_{u,j}]\to[K_u]$ in the mass norm as $j\to\infty$ for each u, (15), (16), and the triangle inequality guarantee that we can select a positive integer N sufficiently large so that $j>N$ implies

$$\mathcal{L}^n\big((K_{1,j}\,\triangle\,H_-)\cap B_R\big)<\frac{\delta\lambda}{80\phi^0}R^n, \tag{18}$$

$$\mathcal{L}^n\big((K_{2,j}\,\triangle\,H_+)\cap B_R\big)<\frac{\delta\lambda}{80\phi^)}R^n, \tag{19}$$

$$\mathcal{L}^n\left(\bigcup_{u=3}^{s}(K_{u,j}\cap B_R)\right)<\frac{\delta\lambda}{40\phi^0}R^n. \tag{20}$$

Fix such an N, and let j be any positive integer for which $j>N$. Let $\mathcal{B}_j=\bigcup_{u=1}^s\partial K_{u,j}$. We would now like to show that the interfaces of P_j in B_R other than the 1-2 interface can essentially be ignored, given the proper construction.

Claim 2. *There exists $r\in(R-(\delta/2)R,R)$ such that $\mathcal{H}^{n-1}(\mathcal{B}_j\cap\partial B_r)=0$ and such that there is a polycrystal $P_j'=K'_{1..s,j}\in\mathcal{P}^s$ satisfying*

(1) $K'_{u,j}\setminus B_r=K_{u,j}\setminus B_r$ *for each* $u=1,\dots,s$;

(2) $U_r\subset K'_{1,j}\cup K'_{2,j}$; *i.e.,* P_j' *consists only of crystals* 1 *and/or* 2 *in the interior of* B_r; *and*

(3) $SE(P_j',B_r)-E_j(B_R)\le(\lambda/10)R^{n-1}$.

Proof. Since $\mathscr{L}^n(\mathscr{B}_j) = 0$, for $\mathscr{L}^1$-almost every r we have $\mathscr{H}^{n-1}(\mathscr{B}_j \cap \partial B_r) = 0$. For any such r in the interval $(R - (\delta/2)R,\ R)$, B2-convexity ensures that it is possible to replace crystals 3, 4, $\dots$, s of P_j in B_r, using only crystals 1 and 2, so that surface energy in the *interior* of B_r does not increase. This procedure eliminates every interface (in the interior of B_r) except the 1-2 interface, which is generally altered. The resulting polycrystal may have new interfaces along ∂B_r, but they will have small surface area if we select r judiciously. Since $j > N$, the set of all $r \in (R - (\delta/2)R,\ R)$ such that

$$\mathscr{H}^{n-1}\Big(\bigcup_{u=3}^{s} K_{u,j} \cap \partial B_r\Big) \le \big[\lambda/(10\phi^0)\big] R^{n-1} \tag{21}$$

has $\mathscr{L}^1$ measure at least $(\delta/4)R$. If not, Federer's coarea formula [1969, 3.2] would imply that

$$\int_{R-(\delta/2)R}^{R} \mathscr{H}^{n-1}\Big(\bigcup_{u=3}^{s} K_{u,j} \cap \partial B_r\Big)\, dr \ge \frac{\delta R}{4}\,\frac{\lambda}{10\phi^0} R^{n-1} = \frac{\delta\lambda}{40\phi^0} R^n,$$

contradicting (20). In particular, there exists an $r \in (R - \delta/2R,\ R)$ for which $\mathscr{H}^{n-1}(\mathscr{B}_j \cap \partial B_r) = 0$ and (21) holds. For this r, the surface energy created along ∂B_r cannot exceed $\phi^0\ \mathscr{H}^{n-1}\big(\bigcup_{u=3}^{s} K_{u,j} \cap \partial B_r\big)$, which in turn is at most $(\lambda/10)R^{n-1}$ because of (21). □

In the next claim, we will alter the 1-2 interface slightly by adding a slice current C having small area and hence little surface energy, so as to produce a new integral current having the same boundary as a disk. It will then be possible to make use of the convexity of ϕ_{12} and invoke Theorem 5 to deduce that the new current has surface energy no less than that of the disk. Since the current C has small area by construction, we will be able to deduce that $SE(P_j', B_r)$ cannot be much less than $SE(D_q)$.

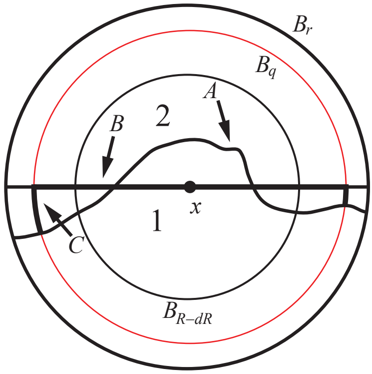
Claim 3. *There exists $q \in (R - \delta R, r)$ for which* $SE(D_q) - SE(P_j', B_r) \le \dfrac{\lambda}{10} R^{n-1}$.

Proof. Let $f(z) = |z - x|$ whenever $z \in \mathbb{R}^n$. Then f is Lipschitz with constant 1. Let $V = [K_{1,j}' \cap B_r]$, $W = [H_- \cap B_r]$, and $T = V - W = \boldsymbol{t}(S, \theta, \sigma)$. Whenever $q > 0$, define $m(q) = \|T\|(U_q)$, as in Theorem 2.

Applying Theorem 2 in turn to the currents V, W, and T, we see that for $\mathscr{L}^1$-almost every $q \in (R - \delta R, r)$ each of the conclusions of that theorem holds simultaneously for the currents V, W, and T. Therefore, for each such q, $A = (\partial V) \llcorner U_q$ and $B = (\partial W) \llcorner U_q$ are $n-1$-dimensional integral currents in $\mathbb{R}^n$, and we can define the slice current $C = \langle T, f, q \rangle$ (see Theorem 2(4)), which is also an $(n-1)$-dimensional integral current. Theorem 2(7) implies

$$\partial(T \llcorner U_q) = (\partial T) \llcorner U_q + \langle T, f, q\rangle = (\partial V) \llcorner U_q - (\partial W) \llcorner U_q + \langle T, f, q\rangle. \tag{22}$$

Applying the boundary operator ∂ to the extreme equality in (22) and using linearity gives $\partial B = \partial(A + C)$.



Since spt B is contained in a hyperplane (i.e., H), we may apply Theorem 5 with T, D, and ϕ there replaced by $A + C$, B, and ϕ_{12} respectively, and make use of the fact that surface energy is no more than ϕ^0 times mass, to conclude that

$$SE(D_q) \le \int_{z \in \partial K'_{1,j} \cap U_q} \phi_{12}(n_{K_1}(z))\, d\mathscr{H}^{n-1} z + \phi^0 \boldsymbol{M}(C) \qquad (23)$$
$$\le SE(P'_j, B_r) + \phi^0 \boldsymbol{M}(C).$$

Now we'll show that q can be chosen so that $\boldsymbol{M}(C)$ will be sufficiently small. By Theorem 2(6), for $\mathscr{L}^1$-almost every $q \in (R - \delta R, r)$ we have $\boldsymbol{M}(\langle T, f, q\rangle) \le m'(q)$, and so we can estimate

$$\int_{R-\delta R}^{r} \boldsymbol{M}(\langle T, f, q\rangle)\, dq \le \int_{R-\delta R}^{r} m'(q)\, dq \le m(r) \qquad (24)$$
$$= \|T\|(U_r) \le \|T\|(\mathbb{R}^n) = \boldsymbol{M}(T).$$

Since $r < R$,

$$\begin{aligned} \boldsymbol{M}(T) &= \mathscr{L}^n\big((K'_{1,j} \triangle H_-) \cap B_r\big) \\ &\le \mathscr{L}^n\big((K'_{1,j} \triangle K_{1,j}) \cap B_r\big) + \mathscr{L}^n\big((K_{1,j} \triangle H_-) \cap B_r\big) \\ &\le \mathscr{L}^n\left(\bigcup_{u=3}^{s} (K_{u,j} \cap B_r)\right) + \mathscr{L}^n\big((K_{1,j} \triangle H_-) \cap B_r\big) \le \frac{\delta\lambda}{20\phi^0} R^n. \end{aligned}$$

The last inequality follows from (18) and (20). The set of numbers q in $(R - \delta R, r)$ for which

$$\boldsymbol{M}(C) = \boldsymbol{M}(\langle T, f, q\rangle) > \frac{\lambda}{10\phi^0} R^{n-1} \qquad (25)$$

has measure strictly less than $\delta R/2$. If (25) were true for a set of points q having measure at least $\delta R/2$, it would follow from Federer's coarea formula [1969, 3.2] that

$$\int_{R-\delta R}^{r} \boldsymbol{M}(\langle T, f, q\rangle)\, dq > \frac{\delta\lambda}{20\phi^0} R^n \geq \boldsymbol{M}(T),$$

contradicting (24). Therefore, q can be chosen in $(R-\delta R, r)$ so as to also satisfy

$$\boldsymbol{M}(C) \leq \frac{\lambda}{10\phi^0} R^{n-1}.$$

This last inequality and (24) immediately yield

$$SE(D_q) - SE(P'_j, B_r) \leq \phi^0 \boldsymbol{M}(C) \leq \frac{\lambda}{10} R^{n-1}. \qquad \square$$

We now fix q as in 3 for the remainder of the proof.

Claim 4. $0 < SE(D_R) - SE(D_q) \leq (\lambda/10) R^{n-1}$.

Proof. Let $f(x) = \alpha(n-1)\phi_{12}(n_{K_1}(x))$. Then $SE(D_R) = f(x)R^{n-1}$ and (since $R(1-\delta) < q < R$) we have $SE(D_q) = f(x)q^{n-1} > f(x)R^{n-1}(1-\delta)^{n-1}$. Using (12), we get

$$\begin{aligned} SE(D_R) - SE(D_q) &< f(x)R^{n-1}\big(1-(1-\delta)^{n-1}\big) \\ &\leq \phi^0\,\alpha(n-1)\,R^{n-1}\frac{\eta\phi_0}{2000\phi^0} = \frac{\lambda}{10}R^{n-1}. \end{aligned} \qquad \square$$

Claim 5. $\big|E^{1,2}(B_R) - SE(D_R)\big| < (\lambda/10)R^{n-1}$.

Proof: This is estimate (13) above. $\square$

Claim 6. $0 \leq E(B_R) - E^{1,2}(B_R) \leq (\lambda/10)R^{n-1}$.

Proof: This follows from (14)). $\square$

Claim 7. $E(B_R) \geq \frac{1}{2} SE(D_R)$.

Proof: Claims 5 and 6 give

$$\begin{aligned} SE(D_R) - E(B_R) = \big(SE(D_R) - E^{1,2}(B_R)\big) + \big(E^{1,2}(B_R) - E(B_R)\big) < \frac{\lambda}{10}R^{n-1} \\ \leq \tfrac{1}{20}\phi_0\,\alpha(n-1)R^{n-1} < \tfrac{1}{2}SE(D_R), \end{aligned}$$

as needed. $\square$

Claim 8. $E_j(B_R) \geq E(B_R) - (\lambda/2)R^{n-1}$.

Proof: This follows by adding the inequalities in Claims 2, 3, 4, 5, and 6. $\square$

Claim 9. *There exists $\varepsilon > 0$ such that for each $j > N$ we have*

$$E_j(B_R) \geq E(B_R) - \big((\eta/100)E(B_R) - \varepsilon\big).$$

Proof. Define ϵ so that $(400/\eta)\epsilon = \phi_0\, \alpha(n-1)\, R^{n-1}$. Claim 7 gives

$$4E(B_R) \geq 2SE(D_R) \geq 2\phi_0\, \alpha(n-1)R^{n-1} = (400/\eta)\epsilon + \phi_0\, \alpha(n-1)R^{n-1},$$

and so $(\eta/100)E(B_R) - \epsilon \geq (\eta/400)\phi_0\, \alpha(n-1)R^{n-1} = (\lambda/2)R^{n-1}$. The result now follows from 8. □

For convenience, we state the following elementary lemma from real analysis:

Lemma 17. *Suppose* $\{x_j\}$ *is a sequence of real numbers for which* $\liminf_{j\to\infty} x_j > -\infty$, *and* $a \in (-\infty, \infty)$. *Then* $\liminf_{j\to\infty} x_j > a$ *if and only if* $\exists\, \varepsilon > 0$ *such that* $\exists\, N > 0$ *with the property that for all* $j > N$ *we have* $x_j \geq a + \varepsilon$.

Claim 10. $E \leq \liminf_{j\to\infty} E_j$.

Proof. Letting $x_j = E_j(B_R)$ and $a = (1-\eta/100)E(B_R)$, 9 implies there exist $\epsilon > 0$ and $N > 0$ such that for all $j > N$ we have $x_j \geq a + \varepsilon$. Lemma 17 implies that

$$\liminf_{j\to\infty} E_j(B_R) > (1-\eta/100)E(B_R). \tag{26}$$

Suppose our claim is false. Then lower semicontinuity fails by η percent for some $\eta \in (0, 100]$. Theorem 13 then implies that lower semicontinuity fails by at least η percent inside B_R, so that

$$\frac{E(B_R) - \liminf_{j\to\infty} E_j(B_R)}{E(B_R)} \times 100 \geq \eta,$$

which contradicts (26). □

3.5. ***Proof of the weak lower semicontinuity theorem.*** Suppose $1 \leq u < s$. For each $\varphi \in \mathscr{D}^n$, we are given that $[K_{u,i}](\varphi) \to T_u(\varphi)$, as $i \to \infty$. Equation (11) implies

$$\sup_i \boldsymbol{M}(\partial[K_{u,i}]) \leq \sup_i \left(\sum_{u=1}^{s-1} \boldsymbol{M}(\partial[K_{u,i}]) \right) < \infty.$$

The isoperimetric inequality [Almgren 1986b; Almgren et al. 1993, 3.1.7] gives an upper bound for $\boldsymbol{M}([K_{u,i}])$ in terms of $\boldsymbol{M}(\partial[K_{u,i}])^{n/(n-1)}$, and so

$$\sup_i (\boldsymbol{M}([K_{u,i}]) + \boldsymbol{M}(\partial[K_{u,i}])) < \infty.$$

As noted in Section 2.2, it follows that the weak convergence of $[K_{u,i}]$ to T_u is equivalent to flat convergence of $[K_{u,i}]$ to T_u [Simon 1983, 31.2]. Since, additionally, these are n-dimensional integral currents in $\mathbb{R}^n$, we have mass convergence as well [Almgren et al. 1993, 3.1.5]. Since $\boldsymbol{M}([K_{u,i}] - T_u) \to 0$, there exists a $K_u \in \mathscr{C}$ such that $T_u = [K_u]$. Repeating the construction above for each $1 \leq u < s$ and defining $K_s = \mathbb{R}^n \setminus (K_1 \cup \cdots \cup K_{s-1})$, we obtain a polycrystal $P = K_{1..s} \in \mathscr{P}^s$

such that $[K_{u,i}] \to [K_u]$ strongly as $i \to \infty$, for each $u < s$. Lower semicontinuity of surface energy now follows from Theorem 14. □

4. Appendix: Restrictions on the integrands

In this appendix we will briefly discuss several sets of conditions, on the surface energy density functions $\{\phi_{uv}\}_{1 \le u < v \le s}$, which are mentioned in the Introduction and in Section 3.1. We will also state some relations among the conditions.

In [Almgren 1976] Chapter VI, F. Almgren considered partitions of $\mathbb{R}^n$ minimizing surface energy subject to volume constraints. It was the first work of its kind in such a general and rigorous setting. His surface energy density functions were allowed to depend on position, not just orientation. However, each was a constant multiple of a fixed norm ϕ of class 1: in our notation, $\phi_{uv} = c_{uv}\phi$ for positive constants c_{uv} $(u \ne v)$. The constants c_{uv} were further restricted so as to satisfy a condition, called *partitioning regularity*, which ensures that for any polycrystal $P = K_{1..s}$ any given crystal K_u may be renamed, by adding it to K_v for some judiciously chosen $v \ne u$, in such a way as to decrease surface energy. This condition sufficed for both lower semicontinuity and regularity and was more restrictive than necessary for lower semicontinuity alone.

In [Caraballo 2008], we introduced a related condition which does not require smoothness and which does not require that the ϕ_{uv}'s all be multiples of a fixed ϕ. However, in this condition and in all other conditions on the integrands $\{\phi_{uv}\}$ in this paper, including B2-convexity, the functions ϕ_{uv} depend on orientation only and not on position.

Definition 18. A family of integrands $\{\phi_{uv}\}_{1 \le u < v \le s}$ satisfies *A-convexity* if, for any $K_{1..s} \in \mathcal{P}^s$ and for each u for which $\mathcal{L}^n(K_u) > 0$, there exists a $v \ne u$ such that $\mathcal{L}^n(K_v) > 0$, and such that $SE(P^{u,v}) \le SE(K_{1..s})$, where $P^{u,v}$ is the polycrystal corresponding to the partition $\{L_1, L_2, \dots, L_s\}$ of $\mathbb{R}^n$, where $L_u = \varnothing$, $L_v = K_v \cup K_u$, and $L_w = K_w$, if $w \notin \{u, v\}$.

I.e., the integrands $\{\phi_{uv}\}$ satisfy A-convexity if any non-trivial region u may be added to a judiciously chosen non-trivial region v without increasing surface energy. A-convexity is clearly implied by the restriction of Almgren's condition to the case where the integrands depend only on orientation. A-convexity is sufficient but not necessary for lower semicontinuity (see [Caraballo 2010]).

More than a decade after Almgren's seminal work [Almgren 1976], but prior to all other work we cite on lower semicontinuity of surface energy in this setting, Ambrosio and Braides gave the first necessary and sufficient condition for strong lower semicontinuity of the surface energy functional (1), BV-ellipticity (see [Ambrosio and Braides 1990b] and [Ambrosio et al. 2000]). Let $K^* = \{K_1, K_2, \dots, K_s\}$ denote a partition of $\mathbb{R}^n$ into $\mathcal{L}^n$-measurable subsets of $\mathbb{R}^n$ (not necessarily bounded or unbounded) having locally finite perimeter. If U is a bounded, open subset of $\mathbb{R}^n$ we can define the surface energy of K^* in U in the obvious way:

$$SE(K^*, U) = \sum_{1 \le u < v \le s} \int_{p \in U \cap \partial K_u \cap \partial K_v} \phi_{uv}(n_{K_u}(p))\, d\mathcal{H}^{n-1} p. \tag{27}$$

Definition 19. A family of integrands $\{\phi_{uv}\}_{1 \le u < v \le s}$ is said to satisfy *BV-ellipticity* if $SE(K^*, Q) \ge \phi_{uv}(w)$ whenever
a) $1 \le u < v \le s$,
b) $w \in \mathbb{R}^n$ is a unit vector,
c) Q is an open unit cube in $\mathbb{R}^n$, centered at the origin and having all faces parallel or perpendicular to w, and
d) K^* is a partition of $\mathbb{R}^n$, as above, with $\bigcup_{h \notin \{u,v\}} K_h \Subset Q$, $H_-(\mathbf{0}, w) \setminus Q \subset K_u$, and $H_+(\mathbf{0}, w) \setminus Q \subset K_v$.

If $\bigcup_{h \notin \{u,v\}} K_h =_n \varnothing$, $K_u =_n H_-(\mathbf{0}, w)$, and $K_v =_n H_+(\mathbf{0}, w)$, then $SE(K^*, Q) = \phi_{uv}(w)$. Thus, we see that, when BV-ellipticity holds, bounded perturbations of a planar interface, possibly involving other regions, are never cheaper than the original planar interface.

(B)-convexity and joint convexity were also introduced in [Ambrosio et al. 2000; Ambrosio and Braides 1990b], and each was shown to be sufficient for strong lower semicontinuity of the surface energy functional (1).

Definition 20. A family of integrands $\{\phi_{uv}\}_{1 \le u < v \le s}$ satisfies (B)-*convexity* if, for any $K_{1..s} \in \mathcal{P}^s$ and for each u for which $\mathcal{L}^n(K_u) > 0$, region u of $K_{1..s}$ may be replaced by some configuration involving the remaining regions in $K_{1..s}$ in such a way that the resulting polycrystal has surface energy not exceeding $SE(K_{1..s})$.

With (B)-convexity, we can remove any single crystal and allow the remaining crystals to flow into the space it formerly occupied, in such a way that surface energy does not increase.

Definition 21. A family of integrands $\{\phi_{uv}\}_{1 \le u < v \le s}$ satisfies *joint convexity* if

$$\phi_{uv}(w) = \sup_{z \in \{1,2,3,\dots\}} \big\langle g_z(u) - g_z(v),\ w \big\rangle \text{ whenever } 1 \le u < v \le s,\ w \in \mathbb{R}^n$$

for some sequence $\big\{ g_1(\cdot), g_2(\cdot), g_3(\cdot), \dots \big\}$ of functions from $\{1, 2, \dots, s\}$ into $\mathbb{R}^n$.

A very basic condition, necessary for the lower semicontinuity of the surface energy functional (1), is the following.

Definition 22. A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfies the *triangle inequalities* if, when we extend it to a family $\{\phi_{uv}\}_{u,v\in\{1,2,\dots,s\}}$ using the conventions

$$\phi_{ji}(w)=\phi_{ij}(-w) \text{ for all } w\in\mathbb{R}^n, \text{ if } 1\le i<j\le s, \tag{28}$$

$$\phi_{ii}(w)=0 \text{ for all } w\in\mathbb{R}^n, \text{ for each } i\in\{1,2,\dots,s\}, \tag{29}$$

we have

$$\phi_{ij}(w)\le\phi_{ik}(w)+\phi_{kj}(w) \text{ whenever } i,j,k\in\{1,2,\dots,s\},\ w\in\mathbb{R}^n. \tag{30}$$

The triangle inequalities are necessary for lower semicontinuity of surface energy since otherwise two planar interfaces could merge, resulting in a sudden increase in surface energy. When $s\le 3$ they imply joint convexity and hence are also sufficient for lower semicontinuity [Ambrosio and Braides 1990b]. When $s>3$, they are not sufficient for lower semicontinuity [Caraballo 2009].

Definition 23 [Caraballo 1997]. A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ is said to satisfy LSC1 if, for any $K_{1..s}\in\mathcal{P}^s$ and for each pair (u,v) with $u\ne v$,

$$\min\bigl(SE(P^u),SE(P^v)\bigr)\le SE(K_{1..s}),$$

where P^u is the polycrystal corresponding to the partition $\{K_u\cup\bigcup_{z\notin\{u,v\}}K_z,\ K_v\}$ of $\mathbb{R}^n$ into two sets, and P^v the one corresponding to the partition $\{K_u,\ K_v\cup\bigcup_{z\notin\{u,v\}}K_z\}$ of $\mathbb{R}^n$ into two sets.

I.e., this condition holds provided for any pair u and v it is possible to replace all of the other regions with just u, or all of the others with just v, in such a way that surface energy does not increase. We can generalize this condition by allowing replacements in which some regions are replaced by u and others by v:

Definition 24. A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfies A2-*convexity* if, for any $K_{1..s}\in\mathcal{P}^s$ and for each pair (u,v) with $u\ne v$, $\mathcal{L}^n(K_u)>0$, and $\mathcal{L}^n(K_v)>0$, the remaining regions of $K_{1..s}$ may be renamed, in each case to either u or v, in such a way that the resulting polycrystal has surface energy not exceeding $SE(K_{1..s})$.

The even weaker condition B2-convexity arises by eliminating the requirement that a given region must be renamed using just u or just v.

Definition 25 (cf. [Caraballo 1997; 2008]). A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ is said to be *pointwise within a factor of* λ $(0<\lambda<\infty)$ if, when we extend it to a family $\{\phi_{uv}\}_{u,v\in\{1,2,\dots,s\}}$ using the conventions (28) and (29), the following condition holds for each unit vector $w\in\mathbb{R}^n$:

$$\frac{\sup\{\phi_{uv}(w):1\le u\ne v\le s\}}{\inf\{\phi_{uv}(w):1\le u\ne v\le s\}}\le\lambda.$$

Definition 26 (cf. [Caraballo 1997; 2008]). A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ satisfies LSC3 if it is pointwise within a factor of 2.

LSC3 places restrictions on each pair of integrands. It turns out that it suffices to control the ratios of only certain pairs of the ϕ_{uv}'s. Thus, we have the following definition from [Caraballo 2008]:

Definition 27. A family of integrands $\{\phi_{uv}\}_{1\le u<v\le s}$ is said to be *directionally controlled* if, when we extend it to a family $\{\phi_{uv}\}_{u,v\in\{1,2,\dots,s\}}$ using the conventions (28) and (29), we have

$$\phi_{ij}(w) \le 2\phi_{ik}(w)$$

whenever $i, j \in \{1, 2, \dots, s\}$, $k \in \{1, \dots, s\} \setminus \{i, j\}$, and $w \in \mathbb{R}^n$.

LSC3 implies directional control; however, examples show the converse to be false. Here are several additional relations among the various restrictions on the integrands $\{\phi_{uv}\}_{1\le u<v\le s}$. For more results and for proofs, see the various papers cited in Section 3.1.

Theorem 28. (1) *A-convexity $\Rightarrow$ A2-convexity $\Rightarrow$ B2-convexity.*

(2) *A-convexity $\Rightarrow$ (B)-convexity $\Rightarrow$ B2-convexity.*

(3) *LSC3 $\Rightarrow$ directional control $\Rightarrow$ A2-convexity $\Rightarrow$ B2-convexity.*

(4) *LSC3 $\Rightarrow$ LSC1 $\Rightarrow$ A2-convexity $\Rightarrow$ B2-convexity.*

Proof. (1) is [Caraballo 2008, Theorem 3.13(a)]. (2) is [Caraballo 2008, Theorem 3.13(b)]. (3) is [Caraballo 2008, Theorem 3.14]. The first assertion in (4) is part of [Caraballo 1997, Theorem 10]. That LSC1 implies A2-convexity follows immediately from the definitions. □

Since BV-ellipticity is equivalent to lower semicontinuity of the surface energy functional (1), it follows that B2-convexity (and hence each of the conditions which implies it) implies BV-ellipticity, which in turn implies the triangle inequalities.

Acknowledgements

The author would like to gratefully acknowledge generous support by Princeton University and the Office of Naval Research during the completion of early drafts of this work.

References

[Almgren 1976] F. J. Almgren, Jr., "Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints", *Mem. Amer. Math. Soc.* **4**:165 (1976), viii+199. MR 54 #8420 Zbl 0327.49043

[Almgren 1986a] F. J. Almgren, Jr., "Deformations and multiple-valued functions", pp. 29–130 in *Geometric measure theory and the calculus of variations* (Arcata, CA, 1984), edited by W. K. Allard and F. J. Almgren, Jr., Proc. Sympos. Pure Math. **44**, AMS, Providence, RI, 1986. MR 87h:49001 Zbl 0595.49028

[Almgren 1986b] F. J. Almgren, Jr., "Optimal isoperimetric inequalities", *Indiana Univ. Math. J.* **35**:3 (1986), 451–547. MR 88c:49032 Zbl 0585.49030

[Almgren and Taylor 1996] F. J. Almgren, Jr. and J. E. Taylor, "Soap bubble clusters", *Forma* **11**:3 (1996), 199–207. MR 99e:49052 Zbl 1017.49502

[Almgren et al. 1993] F. J. Almgren, Jr., J. E. Taylor, and L. Wang, "Curvature-driven flows: a variational approach", *SIAM J. Control Optim.* **31**:2 (1993), 387–438. MR 94h:58067 Zbl 0783.35002

[Ambrosio and Braides 1990a] L. Ambrosio and A. Braides, "Functionals defined on partitions in sets of finite perimeter, I: Integral representation and Γ-convergence", *J. Math. Pures Appl.* (9) **69**:3 (1990), 285–305. MR 91j:49015 Zbl 0676.49028

[Ambrosio and Braides 1990b] L. Ambrosio and A. Braides, "Functionals defined on partitions in sets of finite perimeter, II: Semicontinuity, relaxation and homogenization", *J. Math. Pures Appl.* (9) **69**:3 (1990), 307–333. MR 91j:49016 Zbl 0676.49029

[Ambrosio and Braides 1995] L. Ambrosio and A. Braides, "Energies in SBV and variational models in fracture mechanics", pp. 1–22 in *Homogenization and applications to material sciences* (Nice, 1995), edited by D. Cioranescu et al., GAKUTO Internat. Ser. Math. Sci. Appl. **9**, Gakkōtosho, Tokyo, 1995. MR 98g:73044 Zbl 0904.73045

[Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York, 2000. MR 2003a:49002 Zbl 0957.49001

[Ambrosio et al. 2001] L. Ambrosio, V. Caselles, S. Masnou, and J.-M. Morel, "Connected components of sets of finite perimeter and applications to image processing", *J. Eur. Math. Soc.* **3**:1 (2001), 39–92. MR 2002g:49072 Zbl 0981.49024

[Ambrosio et al. 2003] L. Ambrosio, N. Fusco, and J. E. Hutchinson, "Higher integrability of the gradient and dimension of the singular set for minimisers of the Mumford–Shah functional", *Calc. Var. Partial Differential Equations* **16**:2 (2003), 187–215. MR 2004a:49017 Zbl 1047.49015

[Aubert and Kornprobst 2002] G. Aubert and P. Kornprobst, *Mathematical problems in image processing: partial differential equations and the calculus of variations*, Applied Mathematical Sciences **147**, Springer, New York, 2002. MR 2002m:49001 Zbl 1109.35002

[Bellettini et al. 2002] G. Bellettini, V. Caselles, and M. Novaga, "The total variation flow in $\mathbb{R}^N$", *J. Differential Equations* **184**:2 (2002), 475–525. MR 2003g:35105 Zbl 1036.35099

[Bellettini et al. 2006] G. Bellettini, M. Chermisi, and M. Novaga, "Crystalline curvature flow of planar networks", *Interfaces Free Bound.* **8**:4 (2006), 481–521. MR 2008d:74049 Zbl 1103.74017

[Braides 1998] A. Braides, *Approximation of free-discontinuity problems*, Lecture Notes in Mathematics **1694**, Springer, Berlin, 1998. MR 99j:49001 Zbl 0909.49001

[Brook et al. 2003] A. Brook, R. Kimmel, and N. A. Sochen, "Variational restoration and edge detection for color images", *J. Math. Imaging Vision* **18**:3 (2003), 247–268. MR 2004c:68133 Zbl 1020.68079

[Burago and Zalgaller 1988] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Mathematischen Wissenschaften **285**, Springer, Berlin, 1988. MR 89b:52020 Zbl 0633.53002

[Caraballo 1997] D. G. Caraballo, *A variational scheme for the evolution of polycrystals by curvature*, thesis, Princeton University, 1997, Available at http://tinyurl.com/Caraballo-1997-PhD-thesis. MR 2695091

[Caraballo 2008] D. G. Caraballo, "Crystals and polycrystals in $\mathbb{R}^n$: lower semicontinuity and existence", *J. Geom. Anal.* **18**:1 (2008), 68–88. MR 2009b:49032 Zbl 1191.49011

[Caraballo 2009] D. G. Caraballo, "The triangle inequalities and lower semi-continuity of surface energy of partitions", *Proc. Roy. Soc. Edinburgh Sect. A* **139**:3 (2009), 449–457. MR 2010f:74006 Zbl 1168.49040

[Caraballo 2010] D. G. Caraballo, "B2-convexity and surface energy of space partitions", *J. Math. Pures Appl.* (9) **94**:1 (2010), 58–67. MR 2011f:49067 Zbl 1209.49011

[Evans and Gariepy 1992] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992. MR 93f:28001 Zbl 0804.28001

[Federer 1969] H. Federer, *Geometric measure theory*, Grundlehren der mathematischen Wissenschaften **153**, Springer, New York, 1969. MR 41 #1976 Zbl 0176.00801

[Federer and Fleming 1960] H. Federer and W. H. Fleming, "Normal and integral currents", *Ann. of Math.* (2) **72**:3 (1960), 458–520. MR 23 #A588 Zbl 0187.31301

[Giusti 1984] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics **80**, Birkhäuser, Basel, 1984. MR 87a:58041 Zbl 0545.49018

[Gurtin 1993] M. E. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford University Press, New York, 1993. MR 97k:73001 Zbl 0787.73004

[Gurtin et al. 1986] M. E. Gurtin, W. O. Williams, and W. P. Ziemer, "Geometric measure theory and the axioms of continuum thermodynamics", *Arch. Rational Mech. Anal.* **92**:1 (1986), 1–22. MR 87e:80006 Zbl 0599.73002

[Hardt and Simon 1986] R. Hardt and L. Simon, *Seminar on geometric measure theory*, DMV Seminar **7**, Birkhäuser, Basel, 1986. MR 89c:49031 Zbl 0601.49029

[Krantz and Parks 1999] S. G. Krantz and H. R. Parks, *The geometry of domains in space*, Birkhäuser, Boston, MA, 1999. MR 2000m:28005 Zbl 0929.26001

[Krantz and Parks 2008] S. G. Krantz and H. R. Parks, *Geometric integration theory*, Birkhäuser, Boston, MA, 2008. MR 2009m:49075 Zbl 1149.28001

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995. MR 96h:28006 Zbl 0819.28004

[Morgan 1997] F. Morgan, "Lower semicontinuity of energy clusters", *Proc. Roy. Soc. Edinburgh Sect. A* **127**:4 (1997), 819–822. MR 98e:49033 Zbl 0886.49016

[Morgan 1998] F. Morgan, "Immiscible fluid clusters in $\mathbb{R}^2$ and $\mathbb{R}^3$", *Michigan Math. J.* **45**:3 (1998), 441–450. MR 99k:49093 Zbl 0967.49026

[Morgan 2009] F. Morgan, *Geometric measure theory: a beginner's guide*, 4th ed., Elsevier/Academic Press, Amsterdam, 2009. MR 2009i:49001 Zbl 1179.49050

[Morrey 1952] C. B. Morrey, Jr., "Quasi-convexity and the lower semicontinuity of multiple integrals", *Pacific J. Math.* **2**:1 (1952), 25–53. MR 14,992a Zbl 0046.10803

[Mumford and Shah 1989] D. Mumford and J. Shah, "Optimal approximations by piecewise smooth functions and associated variational problems", *Comm. Pure Appl. Math.* **42**:5 (1989), 577–685. MR 90g:49033 Zbl 0691.49036

[Osher and Fedkiw 2001] S. Osher and R. P. Fedkiw, "Level set methods: an overview and some recent results", *J. Comput. Phys.* **169**:2 (2001), 463–502. MR 2002c:65255 Zbl 0988.65093

[Rham 1955] G. de Rham, *Variétés différentiables: formes, courants, formes harmoniques*, Actualités Sci. Ind./Publ. Inst. Math. Univ. Nancago **1222/III**, Hermann, Paris, 1955. MR 16,957b Zbl 0065.32401

[Sethian 1999] J. A. Sethian, *Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*, 2nd ed., Cambridge Monographs on Applied and Computational Mathematics **3**, Cambridge University Press, Cambridge, 1999. MR 2000c:65015 Zbl 0973.76003

[Simon 1983] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University **3**, Australian National University Centre for Mathematical Analysis, Canberra, 1983. MR 87a:49001 Zbl 0546.49019

[Taylor 1978] J. E. Taylor, "Crystalline variational problems", *Bull. Amer. Math. Soc.* **84**:4 (1978), 568–588. MR 58 #12649 Zbl 0392.49022

[Taylor 1993] J. E. Taylor, "Motion of curves by crystalline curvature, including triple junctions and boundary points", pp. 417–438 in *Differential geometry: partial differential equations on manifolds* (Los Angeles, CA, 1990), edited by R. Greene and S.-T. Yau, Proc. Sympos. Pure Math. **54**, AMS, Providence, RI, 1993. MR 94c:53012 Zbl 0823.49028

[Taylor 1999] J. E. Taylor, "A variational approach to crystalline triple-junction motion", *J. Statist. Phys.* **95**:5-6 (1999), 1221–1244. MR 2000i:74073 Zbl 0952.74015

[Taylor 2003] J. E. Taylor, "Some mathematical challenges in materials science", *Bull. Amer. Math. Soc.* (*N.S.*) **40**:1 (2003), 69–87. MR 2004h:74064 Zbl 1158.74438

[White 1996] B. White, "Existence of least-energy configurations of immiscible fluids", *J. Geom. Anal.* **6**:1 (1996), 151–161. MR 97e:49038 Zbl 0853.49030

Received July 27, 2009. Revised August 6, 2010.

DAVID G. CARABALLO
DEPARTMENT OF MATHEMATICS
GEORGETOWN UNIVERSITY
ST. MARY'S HALL, 3RD FLOOR
GEORGETOWN UNIVERSITY
WASHINGTON, DC 20057-1233
UNITED STATES

dgc3@georgetown.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 253 No. 2 October 2011
