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Consider a body, $\mathcal{B}$, rotating with constant angular velocity ω and fully submerged in a Navier–Stokes liquid that fills the whole space exterior to $\mathcal{B}$. We analyze the flow of the liquid that is steady with respect to a frame attached to $\mathcal{B}$. Our main theorem shows that the velocity field v of any weak solution (v, p) in the sense of Leray has an asymptotic expansion with a suitable Landau solution as leading term and a remainder decaying pointwise like $1/|x|^{1+\alpha}$ as $|x| \to \infty$ for any $\alpha \in (0, 1)$, provided the magnitude of ω is below a positive constant depending on α. We also furnish analogous expansions for ∇v and for the corresponding pressure field p. These results improve and clarify a recent result of R. Farwig and T. Hishida.

1. Introduction

Consider a rigid body rotating with prescribed constant angular velocity $\omega \in \mathbb{R}^3$ in a Navier–Stokes liquid that fills the whole space exterior to the body. We assume that the motion of the liquid with respect to a frame $\mathcal{S}$ attached to the body is steady. Then, after a suitable nondimensionalization, the relevant equations for the liquid in the frame $\mathcal{S}$ become

$$(1\text{-}1)\qquad \begin{cases} -\Delta v + v\cdot\nabla v - \omega\wedge x\cdot\nabla v + \omega\wedge v + \nabla p = 0 & \text{in } \Omega,\\ \operatorname{div} v = 0 & \text{in } \Omega,\\ v = \omega\wedge x & \text{on } \partial\Omega,\\ \lim_{|x|\to\infty} v(x) = 0, \end{cases}$$

where v is the velocity field, p the corresponding pressure, and $\Omega \subset \mathbb{R}^3$ the region exterior to the body. We assume that Ω is an exterior domain with a C^2-smooth (compact) boundary.

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Significant effort has been devoted to the analysis of the fundamental mathematical properties of solutions to (1-1), including existence, uniqueness, asymptotic behavior, and stability. Without pretending to furnish an exhaustive bibliography, we refer the reader to [Borchers 1992; Farwig 1992; 2006; Galdi 2003; Farwig et al. 2004; Galdi and Silvestre 2007a; 2007b; Hishida 2007; Hishida and Shibata 2007; 2009; Farwig and Neustupa 2008; Kračmar et al. 2008; Deuring et al. 2011] and to the references cited therein.

One important question that deserves special attention is the behavior of the velocity and pressure fields at large distances. In particular, the precise asymptotic structure of these fields and the identification of their leading terms have great relevance. Beside its intrinsic mathematical significance, this analysis is also important in several applications, as well as in numerical computations, mainly in the estimation of the error made by approximating the infinite region of flow with a necessarily bounded domain; see, for example, [Deuring and Kračmar 2004].

The problem of the asymptotic structure of solutions to (1-1) appears to be particularly challenging. Even in the simpler case $\omega = 0$ (and a nonzero right-hand side of compact support in $(1\text{-}1)_1$) it has been effectively solved, for small data at least, only lately [Korolev and Šverák 2011].

Farwig and Hishida [2009; 2011b] have recently given a first answer to the velocity field question for smooth solutions to (1-1). More specifically, let $\boldsymbol{T}(v, p) := -pI+\nabla v+(\nabla v)^T$ denote the Cauchy stress tensor with I the identity tensor. They have shown that the velocity field of any (smooth) solution to (1-1) having norm in a suitable Lorentz space sufficiently small and for which the quantity[1]

$$\left(\int_{\partial\Omega} \boldsymbol{T}(v, p)\cdot n \,\mathrm{d}S\right)\cdot \frac{\omega}{|\omega|}$$

is also small can be represented at large distances as

$$v(x) = U(x) + R(x), \tag{1-2}$$

where $U = U(x)$ is the velocity field of a particular Landau solution and R is a "remainder" with $R \in L^q(\Omega)$ for some $q \in (\frac{3}{2}, 3)$. Since $U(x)$ behaves like $1/|x|$ for large $|x|$, the relation (1-2) indicates that U is the leading term in the Lebesgue summability sense. The Landau solution involved in (1-2) is a field $U \in \mathscr{D}'(\mathbb{R}^3)$ solution to the Navier–Stokes system

$$\begin{cases} -\Delta U + U\cdot\nabla U + \nabla P = \left(\left(\displaystyle\int_{\partial\Omega} \boldsymbol{T}(v, p)\cdot n\,\mathrm{d}S\right)\cdot\frac{\omega}{|\omega|}\right)\dfrac{\omega}{|\omega|}\,\delta, \\ \operatorname{div} U = 0, \end{cases} \tag{1-3}$$

[1] This quantity represents the force exerted by the liquid on the "body" (the complement of Ω, that is) in the direction of ω.

where δ denotes the delta distribution supported at $0 \in \mathbb{R}^3$; see, for example, [Farwig and Hishida 2011b] and (3-2) below for an explicit form of (U, P). We only note that U is smooth away from the origin and satisfies $U = O(1/|x|)$ and $\nabla U = O(1/|x|^2)$ as $|x| \to \infty$.

The objective of the present paper is to improve and clarify these results of [Farwig and Hishida 2009; 2011b].

We establish our findings in the class of *Leray solutions*, which are defined as solutions (v, p) to (1-1) such that

$$\nabla v \in L^2(\Omega) \quad \text{and} \quad v \in L^6(\Omega) \tag{1-4}$$

and that satisfy the energy *inequality*

$$2 \int_\Omega |\boldsymbol{D}v|^2 \, \mathrm{d}x \le \int_{\partial\Omega} \big(\boldsymbol{T}(v, p) \cdot n\big) \cdot (\omega \wedge x) \, \mathrm{d}S, \tag{1-5}$$

where $\boldsymbol{D}v := \frac{1}{2}\big(\nabla v + (\nabla v)^T\big)$ is the stretching tensor of the liquid. As is well known, the class of Leray solutions is nonempty for any $\omega \in \mathbb{R}^3$ (see, for example, [Borchers 1992]). Moreover, by classical elliptic regularity, any Leray solution is smooth [Galdi 1994].

We will prove that, for sufficiently small $|\omega|$, the velocity field v of any Leray solution (v, p) to (1-1) must obey an asymptotic expansion of the type (1-2), where, unlike [Farwig and Hishida 2009; 2011b], $R(x)$ is estimated *pointwise*, with $|R(x)| \le O(1/|x|^{1+\alpha})$ for some $\alpha \in (0, 1)$.[2] We also show an analogous (improved) pointwise estimate for ∇v, with ∇U as leading term. As far as the pressure field p is concerned, we furnish a similar asymptotic expansion. However, the leading term in this expansion is *not* the pressure P of the Landau solution, but P plus an additional term that depends on the component orthogonal to ω of the force exerted by the liquid on the body. More precisely, we prove:

Theorem 1.1 (main theorem). *Let $\alpha \in (0, 1)$. There is an $\varepsilon = \varepsilon(\alpha) > 0$ so that if $|\omega| < \varepsilon$, then any Leray solution (v, p) to (1-1) obeys the asymptotic expansion*

$$v(x) = U(x) + O\left(\frac{1}{|x|^{1+\alpha}}\right) \qquad \textit{as } |x| \to \infty, \tag{1-6}$$

$$\nabla v(x) = \nabla U(x) + O\left(\frac{1}{|x|^{2+\alpha}}\right) \quad \textit{as } |x| \to \infty, \tag{1-7}$$

and (*after possibly adding a constant to* p)

$$p(x) = P(x) + \frac{x}{4\pi|x|^3} \cdot \left(I - \frac{\omega \otimes \omega}{|\omega|^2}\right) \cdot \mathscr{F} + O\left(\frac{1}{|x|^{2+\alpha}}\right) \quad \textit{as } |x| \to \infty, \tag{1-8}$$

[2] Clearly, $R \in L^q$ for large $|x|$, with some $q = q(\alpha) \in (\frac{3}{2}, 3)$.

where

$$\mathcal{F} := \int_{\partial\Omega} \big(\boldsymbol{T}(v, p) - v \otimes v\big) \cdot n \,\mathrm{d}S, \tag{1-9}$$

and (U, P) *is the Landau solution* (U^b, P^b) *given by* (3-2) *corresponding to the parameter* $b := (\mathcal{F} \cdot \omega)\,\omega/|\omega|^2$.

Remark 1.2. Note that $\mathcal{F}$ is equal to the (negative) force exerted by the liquid on the body $\mathcal{B}$. We emphasize that the leading terms in the expansions (1-6) and (1-7) of v and ∇v, respectively, depend only on the component of $\mathcal{F}$ directed along ω, whereas the leading term in the expansion (1-8) of p also depends on the component of $\mathcal{F}$ orthogonal to ω.

Remark 1.3. It is not known in general if one can take $\alpha = 1$ in the above estimates. However, if $\mathbb{R}^3 \setminus \Omega$ possesses suitable rotational symmetry, then $\alpha = 1$ is allowed. However, in such a case, the leading term in the asymptotic expansion is no longer a Landau solution; see [Galdi $\geq$ 2011].

Remark 1.4. The formula (1-6) elucidates in a pointwise fashion the result proved in [Farwig and Hishida 2009; 2011b] in Lebesgue spaces. However, in those papers no information was provided on the asymptotic structure of ∇v and p. Therefore, (1-7) and (1-8) are new.

The proof of Theorem 1.1 relies on the following two crucial results concerning the *linearized* version of (1-1) in the whole space, which is obtained by suppressing the nonlinear term $v \cdot \nabla v$ in (1-1) and by adding a suitable (given) function f, say, on its right-hand side. The first result, Lemma 2.1, is the proof of existence of solutions with a suitable decay order, under the assumption that f is of compact support and orthogonal (in the L^2 scalar product) to the direction of ω. This lemma can be viewed as a corollary to a very general result proved in [Farwig and Hishida 2011a]. The second result, Lemma 2.2, concerns the existence, uniqueness, and corresponding estimates of solutions that converge to zero pointwise, with a specific order of decay, under appropriate decay hypotheses on f. This lemma is obtained by using the time-dependent transformation and the associated method introduced in [Galdi 2003].

Before discussing some preliminaries in Section 2, recalling the definition of Landau solution along with its basic properties in Section 3, and presenting the proof of our main results in Section 4, we introduce some basic notation. Let $G \subset \mathbb{R}^3$ be any domain, and denote its exterior normal unit vector by n.

- $\|\cdot\|_{r,G} = \|\cdot\|_r$ is the norm in the Lebesgue space $L^r(G)$, $1 \leq r \leq \infty$; $\|\cdot\|_{k,r,G}$ is the norm in the usual Sobolev space $W^{k,r}(G)$, $k \in \mathbb{N}$, $1 \leq r \leq \infty$.
- $D^{1,2}(G) := \{v \in L^1_{\mathrm{loc}}(G) \mid |v|_{1,2} < \infty\}$ and $|v|_{1,2} := \big(\int_G |\nabla v|^2 \,\mathrm{d}x\big)^{1/2}$.

- For $\beta \in \mathbb{R}$, define $[\![v]\!]_{\beta,G} := \operatorname{ess\,sup}_{x\in G} |v(x)|(1+|x|)^{\beta}$.
- For $\beta \in \mathbb{R}$, $m \in \mathbb{N}\cup\{0\}$ let $[\![v]\!]_{m,\beta,G} := \sum_{0\le k\le m} [\![\nabla^k v]\!]_{\beta+k,G}$.
- $\mathscr{X}^m_\beta(G) := \{v \in L^1_{\mathrm{loc}}(G) \mid [\![v]\!]_{m,\beta,G} < \infty\}$.
- $\mathbb{R}^3_T := \mathbb{R}^3 \times (0, T)$, and $\mathbb{R}^3_\infty := \mathbb{R}^3 \times (0,\infty)$ when $T = \infty$.
- $B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$ and $B^R = \mathbb{R}^3 \setminus \overline{B}_R$, where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^3$.

For functions $u : \mathbb{R}^3_T \to \mathbb{R}$, we set $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$, and so on. That is, unless otherwise indicated, differential operators act in the spatial variables only. Constants in capital letters are global, and constants in small letters are local.

2. Preliminaries

The proof of our main result relies on two crucial observations concerning the whole space linear problem

$$
\begin{cases} -\Delta w - \omega\wedge x\cdot\nabla w + \omega\wedge w + \nabla q = f & \text{in } \mathbb{R}^3,\\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2-1}
$$

The first observation is due to Farwig and Hishida [2011b, Lemma 3.4]:

Lemma 2.1. *If $f \in C_0^\infty(\mathbb{R}^3)^3$ with*

$$
\left(\int_{\mathbb{R}^3} f(x)\,\mathrm{d}x\right)\cdot\omega = 0, \tag{2-2}
$$

then there exists a solution $(w,q) \in \mathscr{X}^1_2(\mathbb{R}^3)^3 \times \mathscr{X}^0_2(\mathbb{R}^3)$ to (2-1).

Proof. We obtain directly from [Farwig and Hishida 2011b, (3.21) and Lemma 3.4] the existence of a solution $(w,q) \in \mathscr{X}^0_2(\mathbb{R}^3)^3 \times \mathscr{X}^0_2(\mathbb{R}^3)$. Moreover, by elliptic regularity theory for the Stokes operator, $w \in C^\infty(\mathbb{R}^3)$. It remains to show that $[\![\nabla w]\!]_{3,\mathbb{R}^3} < \infty$. This, however, follows by the same argument used in Lemma 3.7 of that reference to prove that $|w(x)| \le c_1|x|^{-2}$. This argument relies on the fact that the fundamental solution $\overline{\Gamma}$ to (2-1) (see (3.20) in the same paper for an explicit expression) satisfies, after setting $\omega = \mathrm{e}_3$ without loss of generality, the following expansion for $|y| \le R$ and $|x| \to \infty$:

$$
\overline{\Gamma}(x,y) = \Phi(x) + O\Big(\frac{1}{|x|^2}\Big), \quad \Phi(x) := \frac{1}{8\pi|x|^3}\begin{pmatrix} 0 & 0 & x_1x_3\\ 0 & 0 & x_2x_3\\ 0 & 0 & x_3^2+|x|^2 \end{pmatrix},
$$

and

$$
w(x) = \int_{\mathbb{R}^3} \overline{\Gamma}(x,y) f(y)\,\mathrm{d}y. \tag{2-3}
$$

By analogy to the proof of [Farwig and Hishida 2011a, Propositions 4.1 and 4.2] one can show that for $|y| \le R$ and $|x| \to \infty$,

$$\nabla\overline{\Gamma}(x, y) = \nabla\Phi(x) + O\Big(\frac{1}{|x|^3}\Big).$$

Thus, after differentiating in (2-3) and exploiting (2-2) where we have set $\omega = \mathrm{e}_3$, it follows that $|\nabla w(x)| \le c_2|x|^{-3}$, which implies $[\![\nabla w]\!]_{3,\mathbb{R}^3} < \infty$. □

The second observation concerns the solvability of (2-1) in weighted spaces for more general f:

Lemma 2.2. *Let $\alpha \in (0, 1)$. If*[3] *$f \in C^\infty(\mathbb{R}^3)^3$ and $f = \operatorname{div} F$ with*[4]

$$[\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha} \equiv \sum_{i,j=1}^{3} [\![F_{ij}]\!]_{2+\alpha} + \sum_{i=1}^{3} [\![\partial_k F_{ki}]\!]_{3+\alpha} < \infty, \tag{2-4}$$

then there exists a unique solution $(w, q) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$ to (2-1) that satisfies

$$[\![w]\!]_{1,1+\alpha} + [\![q]\!]_{2+\alpha} \le C_1\,([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}), \tag{2-5}$$

where $C_1 = C_1(\alpha)$ is independent *of ω.*

Proof. The existence of a weak solution

$$(w, q) \in \big(D^{1,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3\big) \times L^2_{\mathrm{loc}}(\mathbb{R}^3) \tag{2-6}$$

to (2-1) can be shown by a standard Galerkin approximation argument; see, for example, [Silvestre 2004]. We will now prove that this weak solution belongs to the space $\mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$. To this aim, for $t > 0$, put

$$Q(t) := \exp(\hat{\omega}t), \quad \text{with } \hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

and set

$$u(x,t) := Q(t)w\big(Q^T(t)x\big), \quad \mathfrak{p}(x,t) := q\big(Q^T(t)x\big),$$
$$G(x,t) := Q(t)F\big(Q^T(t)x\big).$$

In particular, $u(\cdot\,, 0) = w$ in the sense that $\lim_{t\to 0}\|u(\cdot\,,t) - w\|_6 = 0$. Then

$$\begin{cases} \partial_t u - \Delta u + \nabla\mathfrak{p} = \operatorname{div} G & \text{in } \mathbb{R}^3_\infty, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3_\infty, \\ u(\cdot\,,0) = w & \text{in } \mathbb{R}^3, \end{cases} \tag{2-7}$$

and $u \in L^6(\mathbb{R}^3_T)^3$ for all $T > 0$.

[3]We take f smooth for simplicity only; this assumption can be substantially weakened.

[4]Throughout this paper, we shall use the summation convention over repeated indexes.

To get an integral representation of u, recall the fundamental solution to the time-dependent Stokes problem, that is, the solution (in the sense of distributions) to

$$\begin{cases} \partial_t \Gamma_{ij} - \Delta \Gamma_{ij} + \partial_j \gamma_i = \delta_{ij} \delta(t) \delta(x), \\ \partial_k \Gamma_{ik} = 0, \end{cases} \qquad i, j = 1, 2, 3,$$

where δ_{ij} denotes the Kronecker symbol and $\delta(\,\cdot\,)$ the Dirac delta distribution. The fundamental solution takes the form (see [Oseen 1927, Section 5])

$$\Gamma_{ij} := -\delta_{ij} \Delta \Psi + \partial_i \partial_j \Psi, \quad \gamma_i := \partial_i (\Delta - \partial_t) \Psi,$$

with

$$\Psi(x, t) := \frac{1}{4\pi^{3/2} t^{1/2}} \int_0^1 \mathrm{e}^{-|x|^2 r^2/(4t)} \, \mathrm{d}r.$$

Using Γ we can write the unique (in the class $L^6(\mathbb{R}^3_T)^3$, $T > 0$) solution to (2-7) as

$$\begin{aligned} u_i(x, t) = {} & \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y|^2/(4t)} \, w_i(y) \, \mathrm{d}y \\ & - \int_0^t \int_{\mathbb{R}^3} \partial_j \Gamma_{ih}(x - y, t - \tau) \, G_{jh}(y, \tau) \, \mathrm{d}y \, \mathrm{d}\tau \\ =: {} & I_1(x, t) - I_2(x, t); \end{aligned} \tag{2-8}$$

see [Galdi and Kyed 2011b, Section 3]. Then, since $w \in L^6(\mathbb{R}^3)^3$, Hölder's inequality yields

$$|I_1(Q(t)x, t)| = O(t^{-1/4}) \quad \text{as } t \to \infty, \quad \text{uniformly in } x \in \mathbb{R}^3. \tag{2-9}$$

It is easy to verify that the estimate on $\int_0^\infty |\nabla \Gamma(x, t)| \, \mathrm{d}t$ from Lemma 3.1 of that reference also holds in the present case of vanishing velocity at infinity (the case $\mathcal{R} = 0$ there). Thus

$$|I_2(x, t)| \le c_1 \, [\![F]\!]_{2+\alpha} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2 (1+|y|)^{2+\alpha}} \, \mathrm{d}y. \tag{2-10}$$

From [Galdi 1994, Lemma II.7.2] we conclude that

$$|I_2(x, t)| \le [\![F]\!]_{2+\alpha} \frac{c_2}{(1 + |x|)^{1+\alpha}}, \quad \text{uniformly in } t > 0, \tag{2-11}$$

with $c_2 = c_2(\alpha)$. Since $|w(x)| = |u(Q(t)x, t)| \le |I_1(Q(t)x, t)| + |I_2(Q(t)x, t)|$ for all $t > 0$, from (2-9) and (2-11) we obtain

$$[\![w]\!]_{1+\alpha, \mathbb{R}^3} \le c_3 \, [\![F]\!]_{2+\alpha}. \tag{2-12}$$

Differentiating (2-8) gives $\partial_k u(x, t) = \partial_k I_1(x, t) + \partial_k I_2(x, t)$. Then another standard application of Hölder's inequality yields

$$|\partial_k I_1(x, t)| = O(t^{-3/4}) \quad \text{as } t \to \infty, \quad \text{uniformly in } x \in \mathbb{R}^3. \tag{2-13}$$

Moreover, we have

$$\partial_k I_2(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_k \Gamma_{ih}(x-y,t-\tau)\, \partial_j G_{jh}(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau. \tag{2-14}$$

Now fix $0 \neq x \in \mathbb{R}^3$ and let $R = \frac{1}{2}|x|$. Then

$$\begin{aligned}\partial_k I_2(x,t) = &\int_0^t \int_{\mathrm{B}_R} \partial_j \partial_k \Gamma_{ih}(x-y,\tau)\; G_{jh}(y,t-\tau)\, \mathrm{d}y\, \mathrm{d}\tau \\ &+ \int_0^t \int_{\partial\,\mathrm{B}_R} \partial_k \Gamma_{ih}(x-y,\tau)\; G_{jh}(y,t-\tau)\, n_j\, \mathrm{d}S(y)\, \mathrm{d}\tau \\ &+ \int_0^t \int_{\mathrm{B}^R} \partial_k \Gamma_{ih}(x-y,\tau)\; \partial_j G_{jh}(y,t-\tau)\, \mathrm{d}y\, \mathrm{d}\tau \\ =: &\, J_1 + J_2 + J_3.\end{aligned} \tag{2-15}$$

Employing [Galdi and Kyed 2011b, Lemma 3.1] as above, this time to estimate $\int_0^\infty |\nabla^2 \Gamma(x,\tau)|\, \mathrm{d}\tau$, we find

$$\begin{aligned}|J_1| &\leq c_4 \int_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^3(1+|y|)^{2+\alpha}}\, \mathrm{d}y \\ &\leq c_5 \frac{1}{|x|^3} \int_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{(1+|y|)^{2+\alpha}}\, \mathrm{d}y \leq [\![F]\!]_{2+\alpha}\big(c_6|x|^{-(2+\alpha)} + c_7|x|^{-3}\big).\end{aligned} \tag{2-16}$$

Furthermore, by the same lemma, we have

$$|J_2| \leq c_8 \int_{\partial\,\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^2|y|^{2+\alpha}} \mathrm{d}S(y) \leq c_9\, [\![F]\!]_{2+\alpha}|x|^{-(2+\alpha)}. \tag{2-17}$$

Finally, using again the same lemma, as well as [Galdi 1994, Lemma II.7.2], we estimate

$$\begin{aligned}|J_3| &\leq c_{10} \int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{3+\alpha}}\, \mathrm{d}y \\ &\leq c_{10} \frac{1}{R} \int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{2+\alpha}}\, \mathrm{d}y \leq c_{11}\, [\![\operatorname{div} F]\!]_{3+\alpha}|x|^{-(2+\alpha)}.\end{aligned} \tag{2-18}$$

Since $|\nabla w(x)| = |\nabla u(Q(t)x,t)| \leq |\nabla I_1(Q(t)x,t)| + |\nabla I_2(Q(t)x,t)|$, $t > 0$, we deduce from (2-13)–(2-18) that

$$\operatorname*{ess\,sup}_{|x|>1} |\nabla w(x)|(1+|x|)^{2+\alpha} \leq c_{12}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big). \tag{2-19}$$

To complete the estimate for ∇w, recall (2-14) and estimate, using [Galdi and Kyed 2011b, Lemma 3.1],

$$|\partial_k I_2(x,t)| \leq c_{13}\, [\![\operatorname{div} F]\!]_{3+\alpha} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2(1+|y|)^{3+\alpha}}\, \mathrm{d}y.$$

It follows that $|\partial_k I_2(x,t)| \le c_{14}\,[\![\operatorname{div} F]\!]_{3+\alpha}$ for $|x|\le 1$ and all $t>0$. Combining this estimate with (2-13), we conclude that $\operatorname{ess\,sup}_{|x|\le 1} |\nabla w(x)| \le c_{15}\,[\![\operatorname{div} F]\!]_{3+\alpha}$. This, together with (2-19), yields

$$[\![\nabla w]\!]_{2+\alpha,\mathbb{R}^3} \le c_{16}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big). \tag{2-20}$$

We now turn our attention to the pressure term q. Taking div in $(2\text{-}1)_1$ we get

$$\Delta q = -\partial_i\partial_j F_{ij} \quad \text{in } \mathbb{R}^3.$$

From the fact that $F \in L^{3/2}(\mathbb{R}^3)^{3\times 3}$, by standard Calderón–Zygmund estimates, it follows that, after possibly modifying q by adding a constant, $q \in L^{3/2}(\mathbb{R}^3)$. Together with the summability properties of div F, this yields the validity of the representation

$$q(x) = -\int_{\mathbb{R}^3} \partial_j \mathscr{E}(y-x)\,\partial_i F_{ij}(y)\,\mathrm{d}y, \tag{2-21}$$

where $\mathscr{E}$ denotes the fundamental solution to the Laplace equation. Now fix $R = \frac{1}{2}|x| > 0$ and split

$$q(x) = -\int_{\mathrm{B}_R} \partial_i \mathscr{E}(y-x)\,\partial_j F_{ij}(y)\,\mathrm{d}y - \int_{\mathrm{B}^R} \partial_i \mathscr{E}(y-x)\,\partial_j F_{ij}(y)\,\mathrm{d}y =: K_1 + K_2.$$

We can estimate

$$\begin{aligned}
|K_1| &\le \left|\int_{\partial\mathrm{B}_R} \partial_i \mathscr{E}(y-x)\,F_{ij}(y)\,n_j\,\mathrm{d}S(y)\right| + \left|\int_{\mathrm{B}_R} \partial_j\partial_i \mathscr{E}(y-x)\,F_{ij}(y)\,\mathrm{d}y\right| \\
&\le c_{17}\left(\int_{\partial\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^2|y|^{2+\alpha}}\,\mathrm{d}S(y) + \int_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^3(1+|y|)^{2+\alpha}}\,\mathrm{d}y\right) \\
&\le [\![F]\!]_{2+\alpha}\big(c_{18}|x|^{-(2+\alpha)} + c_{19}|x|^{-3}\big).
\end{aligned}$$

Moreover, using again [Galdi 1994, Lemma II.7.2], we obtain

$$\begin{aligned}
|K_2| &\le \int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{3+\alpha}}\,\mathrm{d}y \\
&\le \frac{1}{R}\int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{2+\alpha}}\,\mathrm{d}y \le c_{20}\,[\![\operatorname{div} F]\!]_{3+\alpha}|x|^{-(2+\alpha)}.
\end{aligned}$$

It follows that

$$\operatorname*{ess\,sup}_{|x|>1} |q(x)|(1+|x|)^{2+\alpha} \le c_{21}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big). \tag{2-22}$$

To complete the estimate for q, we estimate directly from (2-21)

$$|q(x)| \le c_{22}\,[\![\operatorname{div} F]\!]_{3+\alpha}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2(1+|y|)^{3+\alpha}}\,\mathrm{d}y,$$

from which it follows that $\operatorname{ess\,sup}_{|x|\le 1} |q(x)| \le c_{23} [\![\operatorname{div} F]\!]_{3+\alpha}$. Combined with (2-22) we thus have

$$[\![q]\!]_{2+\alpha} \le c_{24}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big). \tag{2-23}$$

Summarizing (2-12), (2-20), and (2-23) we get (2-5). It remains to show uniqueness of the solution in the class $\mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$. Since (2-1) is a linear problem, we consider only the case $f = 0$ and a solution $(w, q) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$. Dot-multiplying the first equation in (2-1) by w, integrating over B_R, and then letting $R \to \infty$, we obtain $\nabla w = 0$. Consequently, $(w, q) = (0, 0)$. □

3. Landau solution

The Landau solution (U^b, P^b), corresponding to a parameter $b \in \mathbb{R}^3$, is a solution in $\mathscr{D}'(\mathbb{R}^3)^3 \times \mathscr{D}'(\mathbb{R}^3)$ to

$$\begin{cases} -\Delta U + U \cdot \nabla U + \nabla P = b\,\delta, \\ \operatorname{div} U = 0, \end{cases} \tag{3-1}$$

axially symmetric about the axis $b\mathbb{R}$ and (-1)-homogeneous. Here δ denotes the delta distribution. The Landau solution can be given explicitly. Assume for simplicity that $b = k\,\mathrm{e}_3$, $k \in \mathbb{R}$. Then

$$\begin{aligned} U^b(x) &= \frac{2}{|x|}\left(\frac{c(x_3/|x|) - 1}{(c - x_3/|x|)^2}\,\frac{x}{|x|} + \frac{1}{c - x_3/|x|}\,\mathrm{e}_3\right) && \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \\ P^b &= \frac{4}{|x|^2}\,\frac{(c(x_3/|x|) - 1)}{(c - x_3/|x|)^2} && \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \end{aligned} \tag{3-2}$$

where

$$k = m8\pi c3(c^2 - 1)\Big(2 + 6c^2 - 3c(c^2 - 1)\log\frac{c+1}{c-1}\Big). \tag{3-3}$$

As one may easily verify, for each $k \in \mathbb{R} \setminus \{0\}$, there exists a unique $c \in \mathbb{R}$ with $|c| > 1$ so that (k, c) satisfies (3-3). Hence, for each $b \in \mathbb{R}^3 \setminus \{0\}$, a Landau solution (U^b, P^b) to (3-1) is given. Moreover, we have $b = k\,\mathrm{e}_3 \to 0$ as $|c| \to \infty$. The Landau solution was originally constructed in [Landau 1944]. For the explicit calculation of the expressions above, refer to [Cannone and Karch 2004].

An important observation concerning the rotating body case is that

$$b \wedge x \cdot \nabla U^b - b \wedge U^b = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\},$$

since U^b is symmetric about $b\mathbb{R}$ (see [Farwig and Hishida 2011b]).

We conclude from the above that (U^b, P^b) is a solution to

$$\begin{cases} -\Delta U^b + U^b \cdot \nabla U^b - b \wedge x \cdot \nabla U^b + b \wedge U^b + \nabla P^b = 0 & \text{in } \mathbb{R}^3 \setminus \{0\}, \\ \operatorname{div} U^b = 0 & \text{in } \mathbb{R}^3 \setminus \{0\}, \end{cases} \tag{3-4}$$

satisfying

$$|U^b(x)| \le \frac{\kappa_1(b)}{|x|} \quad \text{and} \quad |\nabla U^b(x)| \le \frac{\kappa_2(b)}{|x|^2} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \tag{3-5}$$

with

$$\lim_{b\to 0} \kappa_1(b) = 0 \quad \text{and} \quad \lim_{b\to 0} \kappa_2(b) = 0. \tag{3-6}$$

Properties (3-4), (3-5), and (3-6) are all we need in order to prove Theorem 1.1.

4. Proof of the main theorem

Before proving Theorem 1.1, we outline the idea behind the proof.

Let (v, p) be a Leray solution to (1-1) satisfying the energy inequality (1-5). If $|\omega|$ is sufficiently small, it was proved in [Galdi and Kyed 2011a] that

$$[\![v]\!]_1 + [\![\nabla v]\!]_2 + [\![p]\!]_2 < \infty. \tag{4-1}$$

Moreover, elliptic regularity implies $v,\ p \in C^\infty(\Omega)$. Now let $R > \operatorname{diam}(\mathbb{R}^3 \setminus \Omega)$ and $\chi_R \in C_0^\infty(\mathbb{R}^3)$ be a "cut-off" function with $\chi_R = 0$ in B_R and $\chi_R = 1$ in $\mathbb{R}^3 \setminus \mathrm{B}_{2R}$. Put

$$w := \chi_R v - \mathfrak{B}(\nabla \chi_R \cdot v), \qquad q := \chi_R p,$$

where

$$\mathfrak{B} : C_0^\infty(\mathrm{B}_{2R}) \to C_0^\infty(\mathrm{B}_{2R})^3$$

is the *Bogovskiĭ operator*, defined by the property that $\operatorname{div} \mathfrak{B}(f) = f$ whenever $\int_{\mathrm{B}_{2R}} f(x)\,\mathrm{d}x = 0$. (See [Galdi 1994, Theorem III.3.2] for details.) In the case above,

$$\int_{\mathrm{B}_{2R}} \nabla \chi_R \cdot v \,\mathrm{d}x = \int_{\partial \mathrm{B}_{2R}} v \cdot n \,\mathrm{d}S = \int_{\partial\Omega} \omega \wedge x \cdot n \,\mathrm{d}S = 0.$$

Hence (w, q) satisfies

$$\begin{cases} -\Delta w + w \cdot \nabla w - \omega \wedge x \cdot \nabla w + \omega \wedge w + \nabla q = G_v & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $G_v \in C_0^\infty(\mathbb{R}^3)$, and

$$[\![w]\!]_1 + [\![\nabla w]\!]_2 + [\![q]\!]_2 < \infty. \tag{4-2}$$

Next we introduce the Landau solution (U, P) corresponding to the parameter $b := (\mathcal{F} \cdot \omega)\omega/|\omega|^2$, that is, $(U, P) := (U^b, P^b)$. As above, put

$$\tilde{U} := \chi_R U - \mathfrak{B}(\nabla \chi_R \cdot U), \qquad \tilde{P} = \chi_R P.$$

Then $(\tilde{U}, \tilde{P})$ satisfies

$$\begin{cases} -\Delta \tilde{U} + \tilde{U} \cdot \nabla \tilde{U} - \omega \wedge x \cdot \nabla \tilde{U} + \omega \wedge \tilde{U} + \nabla \tilde{P} = G_U & \text{in } \mathbb{R}^3, \\ \operatorname{div} \tilde{U} = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $G_{\tilde U} \in C_0^\infty(\mathbb{R}^3)$, and by (3-5),

$$\llbracket \tilde U \rrbracket_1 + \llbracket \nabla \tilde U \rrbracket_2 + \llbracket \tilde P \rrbracket_2 < \infty. \tag{4-3}$$

A crucial observation now is that since $v = \omega \wedge x$ on $\partial\Omega$,

$$\begin{aligned}\int_{\mathbb{R}^3} G_v \,\mathrm{d}x &= \int_{\mathrm{B}_{2R}} \operatorname{div}\big(-\boldsymbol{T}(w,q) + w \otimes w - w \otimes (\omega \wedge x) + (\omega \wedge x) \otimes w\big)\,\mathrm{d}x \\ &= \int_{\partial\, \mathrm{B}_{2R}} \big(-\boldsymbol{T}(v,p) + v \otimes v - v \otimes (\omega \wedge x) + (\omega \wedge x) \otimes v\big) \cdot n \,\mathrm{d}S \\ &= \int_{\partial\Omega} \big(\boldsymbol{T}(v,p) - v \otimes v\big) \cdot n \,\mathrm{d}S,\end{aligned}$$

Similarly, since $(U, P) = (U^b, P^b)$ solves (3-4) with right-hand side $b\delta$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} G_U \,\mathrm{d}x &= \int_{\mathrm{B}_{2R}} \operatorname{div}\big(-\boldsymbol{T}(\tilde U,\tilde P) + \tilde U \otimes \tilde U - \tilde U \otimes (\omega \wedge x) + (\omega \wedge x) \otimes \tilde U\big) \cdot n \,\mathrm{d}S \\ &= \int_{\partial\, \mathrm{B}_{2R}} \big(-\boldsymbol{T}(U,P) + U \otimes U - U \otimes (\omega \wedge x) + (\omega \wedge x) \otimes U\big) \cdot n \,\mathrm{d}S = b,\end{aligned}$$

Consequently, by the definition of b,

$$\left(\int_{\mathbb{R}^3} \big(G_v - G_U\big)\,\mathrm{d}x\right) \cdot \omega = 0.$$

Thus, by Lemma 2.1, there exists a solution (V_0, P_0) to

$$\begin{cases} -\Delta V_0 - \omega \wedge x \cdot \nabla V_0 + \omega \wedge V_0 + \nabla P_0 = G_v - G_U & \text{in } \mathbb{R}^3, \\ \operatorname{div} V_0 = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{4-4}$$

satisfying

$$\llbracket V_0 \rrbracket_2 + \llbracket \nabla V_0 \rrbracket_3 + \llbracket P_0 \rrbracket_2 < \infty. \tag{4-5}$$

As a consequence of (4-4), $\Delta P_0 = \operatorname{div}(G_v - G_U)$, and hence

$$P_0(x) = \nabla \mathscr{E}(x) \cdot \int_{\mathbb{R}^3} \big(G_v(y) - G_U(y)\big)\,\mathrm{d}y + O(|x|^{-3}), \tag{4-6}$$

where $\mathscr{E}$ denotes the fundamental solution to the Laplace equation. Now consider

$$z := w - \tilde U - V_0 \quad \text{and} \quad \pi := q - \tilde P - P_0. \tag{4-7}$$

As can easily be verified, $(z, \pi) \in \mathscr{X}_1^1(\mathbb{R}^3)^3 \times \mathscr{X}_2^0(\mathbb{R}^3)$ satisfies the *linear* problem

$$\begin{cases} -\Delta z - \omega \wedge x \cdot \nabla z + \omega \wedge z + z \cdot \nabla w + \tilde U \cdot \nabla z + \nabla \pi \\ \qquad\qquad = -\operatorname{div}(V_0 \otimes w + \tilde U \otimes V_0) & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4-8}$$

Our main result, namely, the asymptotic expansions (1-6)–(1-8), now follows if we can show $[\![z]\!]_{1,1+\alpha} + [\![\pi]\!]_{2+\alpha} < \infty$. To do this, first, we use Lemma 2.2 in combination with (4-2), (4-3), and (4-5) to establish the existence of a solution to (4-8) with this property, and, second, we show uniqueness of solutions to (4-8) in the class $\mathscr{X}^1_1(\mathbb{R}^3)^3 \times \mathscr{X}^0_2(\mathbb{R}^3)$.

Lemma 4.1. *Let* $\alpha \in (0, 1)$. *There is an* $\varepsilon = \varepsilon(\alpha) > 0$ *so that if* $|\omega| < \varepsilon$ *there exists a solution* $(z, \pi) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$ *to* (4-8).

Proof. We shall use a perturbation argument in the space

$$X := \{(\mathfrak{z}, \mathfrak{p}) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3) \mid \operatorname{div} \mathfrak{z} = 0\},$$
$$\|(\mathfrak{z}, \mathfrak{p})\|_X := [\![\mathfrak{z}]\!]_{1,1+\alpha} + [\![\mathfrak{p}]\!]_{2+\alpha}.$$

Clearly, $(X, \|\cdot\|_X)$ is a Banach space. Let $(\mathfrak{z}, \mathfrak{p}) \in X$. Consider the system

$$\text{(4-9)} \qquad \begin{cases} -\Delta z - \omega \wedge x \cdot \nabla z + \omega \wedge z + \nabla \pi \\ \qquad = -\mathfrak{z} \cdot \nabla w - \tilde{U} \cdot \nabla \mathfrak{z} - \operatorname{div}(V_0 \otimes w + \tilde{U} \otimes V_0) & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Note that $\mathfrak{z} \cdot \nabla w + \tilde{U} \cdot \nabla \mathfrak{z} = \operatorname{div}(\mathfrak{z} \otimes w + \tilde{U} \otimes \mathfrak{z})$, and put

$$F := \mathfrak{z} \otimes w + \tilde{U} \otimes \mathfrak{z} + V_0 \otimes w + \tilde{U} \otimes V_0.$$

Since $[\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha} < \infty$, by Lemma 2.2 there exists a unique solution $(z, \pi) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$ to (4-9). We now define the map $\mathscr{J} : X \to X$ by $\mathscr{J}(\mathfrak{z}, \mathfrak{p}) := (z, \pi)$, and show the existence of a fixed point of $\mathscr{J}$ by the contraction mapping theorem. Therefore, consider $(\mathfrak{z}_1, \mathfrak{p}_1), (\mathfrak{z}_2, \mathfrak{p}_2) \in X$ and put $(z_1, \pi_1) := \mathscr{J}(\mathfrak{z}_1, \mathfrak{p}_1)$ and $(z_2, \pi_2) := \mathscr{J}(\mathfrak{z}_2, \mathfrak{p}_2)$. Clearly, $(z_1 - z_2, \pi = \pi_1 - \pi_2)$ satisfies

$$\text{(4-10)} \quad \begin{cases} -\Delta(z_1 - z_2) - \omega \wedge x \cdot (z_1 - z_2) + \omega \wedge (z_1 - z_2) + \nabla \pi \\ \qquad\qquad = -\operatorname{div}\big((\mathfrak{z}_1 - \mathfrak{z}_2) \otimes w + \tilde{U} \otimes (\mathfrak{z}_1 - \mathfrak{z}_2)\big) & \text{in } \mathbb{R}^3, \\ \quad \operatorname{div}(z_1 - z_2) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Lemma 2.2 implies that

$$[\![z_1 - z_2]\!]_{1,1+\alpha} + [\![\pi_1 - \pi_2]\!]_{2+\alpha} \le C_1(\alpha)[\![\mathfrak{z}_1 - \mathfrak{z}_2]\!]_{1,1+\alpha}\big([\![w]\!]_{1,1} + [\![\tilde{U}]\!]_{1,1}\big).$$

From [Galdi and Kyed 2011a, Theorem 4.1] we obtain $\lim_{|\omega|\to 0}[\![v]\!]_{1,1,\Omega} = 0$. Since $w = \chi_R v - \mathfrak{B}(\nabla \chi_R \cdot v)$, using well-known L^q-estimates for $\mathfrak{B}$ (see [Galdi 1994, Chapter III.3]) and Sobolev embedding, one sees easily that $\lim_{|\omega|\to 0}[\![w]\!]_{1,1}$ vanishes. The theorem just cited also gives $\lim_{|\omega|\to 0} b(\omega, v, p) = 0$, which, together with (3-5), (3-6) implies $\lim_{|\omega|\to 0}[\![\tilde{U}]\!]_{1,1} = 0$. Consequently, for sufficiently small $|\omega|$, $\mathscr{J}$ is a contraction, and, by the contraction mapping theorem, there exists a fixed point $(z, \pi) \in \mathscr{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathscr{X}^0_{2+\alpha}(\mathbb{R}^3)$ of $\mathscr{J}$. Clearly, by the construction of $\mathscr{J}$, this fixed point is a solution to (4-8). □

Lemma 4.2. *There is an $\varepsilon > 0$ so that if $|\omega| < \varepsilon$ then a solution (z, π) to (4-8) in $\mathscr{X}^1_1(\mathbb{R}^3)^3 \times \mathscr{X}^0_2(\mathbb{R}^3)$ is unique in this class.*

Proof. Assume that $(z_1, \pi_1), (z_2, \pi_2) \in \mathscr{X}^1_1(\mathbb{R}^3)^3 \times \mathscr{X}^0_2(\mathbb{R}^3)$ both solve (4-8). Then $(z, \pi) := (z_1 - z_2, \pi_1 - \pi_2)$ solves

$$\begin{cases} -\Delta z - \omega \wedge x \cdot \nabla z + \omega \wedge z + \nabla \pi = -\operatorname{div}(z \otimes w + \tilde{U} \otimes z) & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4-11}$$

Testing (4-11) with z, integrating over B_R, subsequently letting $R \to \infty$, and finally applying the Hardy-type inequality

$$\int_{\mathbb{R}^3} \frac{|z|^2}{(1+|x|)^2}\, \mathrm{d}x \le c_1 \int_{\mathbb{R}^3} |\nabla z|^2\, \mathrm{d}x,$$

we obtain $|z|^2_{1,2} \le c_2 |z|^2_{1,2} [\![w]\!]_1$. As in the proof of Lemma 4.1, we use that $\lim_{|\omega|\to 0} [\![w]\!]_1 = 0$, which in this case yields $|z|_{1,2} = 0$ when ω is sufficiently small. Consequently, $(z_1, \pi_1) = (z_2, \pi_2)$. □

Combining Lemma 4.1 and Lemma 4.2, we can now prove our main result.

Proof of Theorem 1.1. Since $v(x) - U(x) = w(x) - \tilde{U}(x)$ for $|x| \ge 2R$, the expansions (1-6) and (1-7) follow if we can show that $[\![w - \tilde{U}]\!]_{1,1+\alpha} < \infty$. Similarly, since $p(x) - P(x) = q(x) - \tilde{P}(x)$ for $|x| \ge 2R$, and recalling (4-6), the expansion (1-8) follows if we can show that $[\![q - \tilde{P} - P_0]\!]_{2+\alpha} < \infty$. Since $[\![V_0]\!]_{1,2} < \infty$, both of these assertions are consequences of the fact that (z, π) defined by (4-7) satisfies $[\![z]\!]_{1,1+\alpha} + [\![\pi]\!]_{2+\alpha} < \infty$, which follows from Lemma 4.1 and Lemma 4.2, provided $|\omega|$ is sufficiently small. □

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