BRAIDING KNOTS IN CONTACT 3-MANIFOLDS

ELENA PAVELESCU

BRAIDING KNOTS IN CONTACT 3-MANIFOLDS

ELENA PAVELESCU

We show that a transverse link in a contact structure supported by an open book decomposition can be transversely braided, and we generalize Markov's theorem on when the closures of two braids represent (transversely) isotopic links.

1. Introduction

Alexander [1923] proved that any knot in $\mathbb{R}^3$ can be braided about the z-axis. Bennequin, in a paper [1983] that marked the start of modern contact topology and led to the Bennequin inequality and eventually to Eliashberg's definition of tightness, proved the transverse case for $(\mathbb{R}^3, \xi_{std})$. In this paper, we generalize Bennequin's result to any closed oriented 3-dimensional manifold M, by looking at an open book decomposition for M together with its supported contact structure. After we completed the proof of this theorem, it was brought to our attention that the same result was independently obtained through different means by Mitsumatsu and Mori in [Mitsumatsu 2006].

Bennequin used braid theory in $\mathbb{R}^3$ to show the standard contact structure was tight. Since then, powerful analytic machinery such as holomorphic curves and Seiberg–Witten theory has been used to prove tightness. By studying braids in general open books, it is hoped that tight contact structures can be better understood from a purely topological/combinatorial perspective. The results presented in this paper constitute a first step in this program. While these results are purely geometric, it would be nice to have an algebraic description of a generalized "braid group". In particular, one needs to define the stabilization moves with respect to different binding components and account for the fundamental group of the page. We will explore this algebraic perspective in a subsequent paper.

Markov [1936] gave an equivalent condition for the closures of two braids in $\mathbb{R}^3$ to be isotopic as links, if and only if the two braids differ by conjugations in the braid group and positive and negative Markov moves. Orevkov and Shevchishin [2003] proved the transversal case for $(\mathbb{R}^3, \xi_{std})$. Wrinkle [2002] independently obtained a different proof. We generalize Markov's theorem to any closed oriented

3-dimensional manifold. We prove the transverse case and recover the topological case proved in [Skora 1992] and [Sundheim 1993].

In [Kawamuro and Pavelescu 2011] the braiding results presented in this paper are used to find combinatorial self-linking formulae for null-homologous braids in annulus open book decompositions and pants open book decompositions, extending Bennequin's self-linking formula for a braid in the standard contact 3-sphere.

2. Contact structures and open book decompositions

Let M be a compact, oriented 3-manifold, and ξ an oriented 2-plane field on M. We say ξ is a *contact structure* on M if $\xi = \ker \alpha$ for some global 1-form α satisfying $\alpha \wedge d\alpha \neq 0$. In this paper we assume that the manifold M is oriented and that the contact structure ξ is oriented and positive (that is, $\alpha \wedge d\alpha > 0$).

A *contactomorphism* between two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) is a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi_* \xi_1 = \xi_2$.

On $\mathbb{R}^3$, consider the two contact structures ξ_1 and ξ_2 given by the 1-forms $\alpha_1 = dz - y\,dx$ and $\alpha_2 = dz + r^2\,d\theta$ (given in cylindrical coordinates). Then $(\mathbb{R}^3, \xi_1)$ and $(\mathbb{R}^3, \xi_2)$ are contactomorphic and we refer to both ξ_1 and ξ_2 as the standard contact structure on $\mathbb{R}^3$, ξ_{std}.

Theorem 2.1 (Gray's theorem). *Let $\{\xi_t\}_{t \in [0,1]}$ be a family of contact structures on a manifold M that differ on a compact set $C \subset \mathrm{int}(M)$. Then there exists an isotopy $\psi_t : M \to M$ such that*

(i) $(\psi_t)_* \xi_1 = \xi_t$;

(ii) ψ_t *is the identity outside of an open neighborhood of* C.

While the proof of this result is well-known, we sketch it here as we will need elements of it in later arguments. For more details see [Geiges 2008, p. 61].

Proof. We look for ψ_t as the flow of a vector field X_t. If $\xi_t = \ker \alpha_t$, then ψ_t has to satisfy

$$(2\text{-}1) \qquad\qquad \psi_t^* \alpha_t = \lambda_t \alpha_0$$

for some nonvanishing function $\lambda_t : M \to \mathbb{R}$. By taking the derivative with respect to t on both sides and rearranging the terms, we get

$$(2\text{-}2) \qquad \psi_t^* \left(\frac{d\alpha_t}{dt} + \mathscr{L}_{X_t} \alpha_t \right) = \frac{d\lambda_t}{dt} \alpha_0 = \frac{d\lambda_t}{dt} \frac{1}{\lambda_t} \psi^* \alpha_t.$$

This is equivalent to

$$(2\text{-}3) \qquad \psi_t^* \left(\frac{d\alpha_t}{dt} + d(\iota_{X_t} \alpha_t) + \iota_{X_t} d\alpha_t \right) = \psi_t^* (h_t \alpha_t)$$

for
$$h_t = \frac{d}{dt}(\log \lambda_t) \circ \psi_t^{-1}.$$

If X_t is chosen in ξ_t, then $\iota_{X_t}\alpha_t = 0$, and (2-3) becomes

(2-4)
$$\frac{d\alpha_t}{dt} + \iota_{X_t} d\alpha_t = h_t \alpha_t.$$

Applying (2-4) to the Reeb vector field of α_t, v_{α_t} (that is, the unique vector field v_t such that $\alpha_t(v_t) = 1$ and $d\alpha_t(v_t, \cdot) = 0$), we find $h_t = (d\alpha_t/dt)(v_{\alpha_t})$ and X_t given by

(2-5)
$$\iota_{X_t} d\alpha_t = h_t \alpha_t - \frac{d\alpha_t}{dt}.$$

The form $d\alpha_t$ gives an isomorphism

$$\Gamma(\xi_t) \to \Omega^1_{\alpha_t}, \quad v \mapsto \iota_v d\alpha_t,$$

where $\Gamma(\xi_t) = \{v \mid v \in \xi_t\}$ and $\Omega^1_{\alpha_t} = \{1\text{-forms } \beta \mid \beta(v_t) = 0\}$, and thus X_t is uniquely determined by (2-4). By construction, the flow of X_t is the desired ψ_t. For the subset of M where the ξ_t agree we choose the α_t to agree. This implies that $d\alpha_t/dt = 0$, $h_t = 0$ and $X_t = 0$, and all equalities hold. $\qquad \square$

In a contact manifold (M, ξ), an oriented arc $\gamma \subset M$ is called *transverse* if for all $p \in \gamma$ and ξ_p the contact plane at p, $T_p\gamma \pitchfork \xi_p$ and $T_p\gamma$ intersects ξ_p positively. Also, if γ is a closed curve then it is called a *transverse knot*.

An *open book decomposition* of M is a pair (L, π) where

- L is an oriented link in M called the *binding* of the open book, and
- $\pi : M \setminus L \to S^1$ is a fibration whose fiber, $\pi^{-1}(\theta)$, is the interior of a compact surface $\Sigma \subset M$ such that $\partial\Sigma = L$, for all $\theta \in S^1$. The surface Σ is called the *page* of the open book.

Alternatively, an open book decomposition of a 3-manifold M consists of a surface Σ, with boundary, together with a diffeomorphism $\phi : \Sigma \to \Sigma$, with ϕ the identity near $\partial\Sigma$, such that

$$M = (\Sigma \times [0, 1]/\sim) \cup_f \coprod_i S^1 \times D^2,$$

where $(x, 1) \sim (\phi(x), 0)$. Note that

$$\partial(\Sigma \times [0, 1]/\sim) = \coprod_i T_i^2,$$

with each torus T_i^2 having a product structure $S^1 \times [0, 1]/\sim$. Let

$$\lambda_i = \{pt\} \times [0, 1]/\sim, \quad \lambda_i \in T_i^2.$$

The gluing diffeomorphism used to construct M is defined by

$$f : \partial \left(\coprod_i S^1 \times D^2 \right) \to \partial \left(\coprod_i T_i^2 \right), \quad \{pt\} \times \partial D^2 \to \lambda_i.$$

The map ϕ is called the *monodromy* of the open book. See [Etnyre 2006] for more details.

Theorem 2.2 [Alexander 1920]. *Every closed oriented* 3-*manifold has an open book decomposition.*

A contact structure ξ on M is said to be *supported* by an open book decomposition (Σ, ϕ) of M if ξ can be isotoped through contact structures so that there exists a 1-form α for ξ such that

- $d\alpha$ is a positive area form on each page, and

- $\alpha(v) > 0$ for all $v \in TL$ that induce the orientation on L.

Theorem 2.3 [Thurston and Winkelnkemper 1975]. *Every open book decomposition* (Σ, ϕ) *supports a contact structure* ξ_ϕ.

We sketch this well-known proof, as we need the details in later arguments.

Proof. Let

$$M = (\Sigma \times [0, 1]/\sim) \cup_f \coprod_i S^1 \times D^2$$

be given as before. We first construct a contact structure on $\Sigma \times [0, 1]/\sim$, and then we extend it in a neighborhood of the binding.

In the neighborhood $N = S^1 \times D^2$ of each component of the binding, consider coordinates (ψ, x, θ), such that (ψ, x) are coordinates on the page (with ψ the coordinate along the binding) and $d\theta$ and $\pi^* d\theta$ agree, where $\pi^* d\theta$ is the pullback through $\pi : M \setminus L \to S^1$ of the coordinate on S^1. Let λ be a 1-form on the page that is an element of the set (it is easy to check that this set is not empty)

$$S = \{1\text{-forms } \lambda \text{ such that } d\lambda \text{ is a volume form on } \Sigma \text{ and}$$
$$\lambda = (1 + x)\, d\psi \text{ near } \partial \Sigma = L\}.$$

On $\Sigma \times [0, 1]$ take $\tilde{\lambda} = (1 - \theta)\lambda + \theta(\phi^* \lambda)$ and consider the 1-form $\alpha_K = \tilde{\lambda} + K\, d\theta$. For sufficiently large K, α_K is a contact form and it descends to a contact form on $\Sigma \times [0, 1]/\sim$. To extend this form on the solid tori neighborhood of the binding, we pull back α through the gluing map f and get

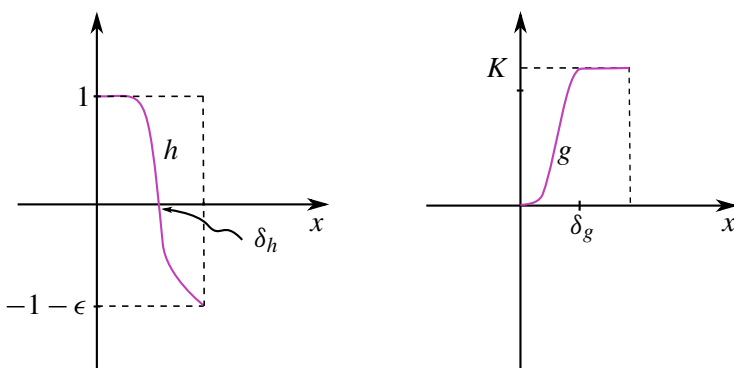$$\alpha_f = K\, d\theta - (x + \epsilon)\, d\psi.$$

Figure 1. The functions h and g.

We are looking to extend this form on the entire $S^1 \times D^2$ to a contact form of the form

$$h(x)\, d\psi + g(x)\, d\theta.$$

This is possible if there exist functions $h, g : [0, 1] \to \mathbb{R}^3$ such that

(i) $h(x)g'(x) - h'(x)g(x) > 0$ (given by the contact condition);

(ii) $h(x) = 1$ near $x = 0$, $h(x) = -(x + \epsilon)$ near $x = 1$; and

(iii) $g(x) = r^2$ near $x = 0$, $g(x) = K$ near $x = 1$.

The two functions h and g described in Figure 1 work for our purpose. The conditions (ii) and (iii) are obviously satisfied, and if δ_h and δ_g are such that $h(x) < 0$ on $[\delta_h, 1]$ and $g(x) = K$ on $[\delta_g, 1]$, then (i) is satisfied as long as $\delta_g < \delta_h$. $\qquad\square$

The following correspondence, which is one of the central results in contact geometry, relates open book decompositions and contact structures.

Theorem 2.4 [Giroux 2002]. *Let M be a closed, oriented 3-manifold. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix} contact\ structures\ \xi\ on\ M^3 \\ up\ to\ contact\ isotopy \end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix} open\ book\ decompositions\ (\Sigma, \phi) \\ of\ M^3\ up\ to\ positive\ stabilization \end{matrix}\right\}.$$

See [Etnyre 2006] for details.

3. Braiding knots in contact 3-manifolds

In this section we generalize the following theorem:

Theorem 3.1 [Bennequin 1983]. *Any transverse link K in $(\mathbb{R}^3, \xi_{\text{std}})$ is transversely isotopic to a link braided about the z-axis.*

Let (L, π) be an open book decomposition for M. A link $K \subset M$ is said to be *braided about* L if K is disjoint from L and there exists a parametrization of K, $f : \coprod S^1 \to M$, such that if θ is the coordinate on each S^1 then $(d/d\theta)(\pi \circ f) > 0$, for any θ. We call those arcs where this condition is not satisfied *bad arcs* of K.

Theorem 3.2. *Suppose (L, π) is an open book decomposition for the 3-manifold M and ξ is supported by (L, π). Let K be a transverse link in M. Then K can be transversely isotoped to a braid.*

Proof. The idea of the proof is to find a family of diffeomorphisms of M keeping each page of the open book setwise fixed and sending the parts of the link where the link is not braided into a neighborhood of the binding. A neighborhood of the binding is contactomorphic to a neighborhood of the z-axis in $(\mathbb{R}^3, \xi_{\text{std}})/_{z \sim z+1}$ and there the link can be braided, according to Theorem 3.1. What we really use is that Theorem 3.1 works not only for links, but also for arcs with good ends.

In the neighborhood $N = S^1 \times D^2$ of each component of the binding, consider coordinates (ψ, x, θ) and let $\lambda \in S$ as described in Theorem 2.3. On $\Sigma \times [0, 1]$ take $\tilde{\lambda} = (1 - \theta)\lambda + \theta(\phi^*\lambda)$ and consider the family of 1-forms given by $\alpha_t = \tilde{\lambda} - (K/t)d\theta$, where $t \in [-1, 0)$ and K is a large constant.

This family of 1-forms descends to a family of 1-forms on $\Sigma \times [0, 1]/\sim$ (since $\phi^*(\alpha_t|_{\Sigma \times \{0\}}) = \alpha_t|_{\Sigma \times \{1\}}$). Both $\xi_{-1} = \ker(\alpha_{-1})$ and ξ are contact structures supported by (L, π), and by Giroux's correspondence they are isotopic. Therefore, without loss of generality, we may assume $\xi = \ker(\alpha_{-1})$.

For large enough K, the family of 1-forms $\{\alpha_t\}_t$ is a family of contact 1-forms:

$$\alpha_t \wedge d\alpha_t = \left(\tilde{\lambda} - K\frac{1}{t}d\theta\right) \wedge (d\tilde{\lambda}) = \tilde{\lambda} \wedge d\tilde{\lambda} - K\frac{1}{t}d\theta \wedge d\tilde{\lambda} > 0.$$

Note that $d\tilde{\lambda}$ is an area form on the page, while $d\theta$ vanishes on the page and is positive on the positive normal to the page. This implies that the term that is subtracted is always negative ($t \in [-1, 0)$), and therefore α_t is a contact form for sufficiently large K. We want to extend this family to the whole M, so we need to patch in the solid tori neighborhood of the binding. In the neighborhood of the binding we have $\alpha_t = (1 + x)\, d\psi - K(1/t)d\theta$. We pull back this family through the map f used to glue the solid tori in the definition of the open book. With our chosen coordinates we have

$$f(\psi, x, \theta) = (x - 1 + \epsilon, -\psi, \theta)$$

and the pullback family

$$\alpha_{f,t} = -(x + \epsilon)\, d\psi - K\frac{1}{t}d\theta.$$

We are looking to extend these forms on the entire $S^1 \times D^2$ to a family of the form

$$h_t(x) \, d\psi + g_t(x) \, d\theta.$$

This is possible if there exist functions $h_t, g_t : [0, 1] \to \mathbb{R}$ for $t \in [-1, 0)$ such that

(i) $h_t(x)g_t'(x) - h_t'(x)g_t(x) > 0$, $t \in [-1, 0)$ (given by the contact condition);

(ii) $h_t(x) = 1$ near $x = 0$, $h_t(x) = -(x + \epsilon)$ near $x = 1$; and
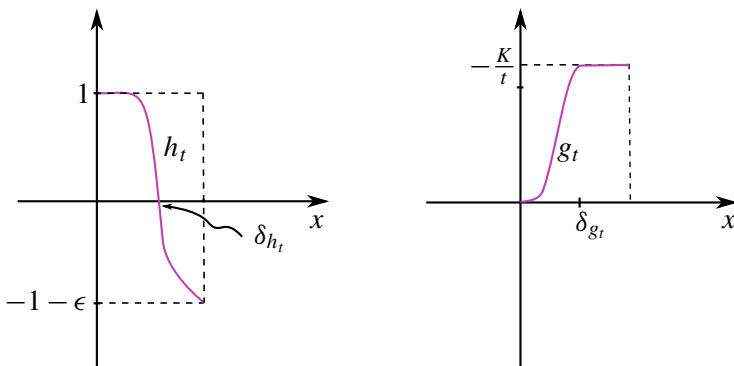
(iii) $g_t(x) = x^2$ near $x = 0$, $g_t(x) = -K/t$ near $x = 1$.

Figure 2. The functions h_t and g_t.

The two families $\{h_t\}_{t \in [-1,0)}$ and $\{g_t\}_{t \in [-1,0)}$ described in Figure 2 work for our purpose. The conditions (ii) and (iii) are satisfied for our choice of h_t and g_t, and if δ_{h_t} and δ_{g_t} are such that $h_t(x) < 0$ for $x \in [\delta_{h_t}, 1]$ and $g_t(x) = -K/t$ for $x \in [\delta_{g_t}, 1]$, then (i) is satisfied as long as $\delta_{g_t} < \delta_{h_t}$.

Denote the extended family of forms also by α_t, and by ξ_t the family of contact structures given by $\xi_t = \ker(\alpha_t)$, with $t \in [-1, 0)$. By Gray's theorem, there exists a family of diffeomorphisms $f_t : M \to M$ such that $(f_t)_* \xi_{-1} = \xi_t$.

As announced, we would like to construct a family of diffeomorphisms $\{f_t\}_t$ that fix the pages setwise. Following the proof of Theorem 2.1, we construct $\{f_t\}_t$ as the flow of a vector field $X_t \in \xi_t$, for which we have the equality (2-5).

We already know that such an X_t exists, but would need it to be tangent to the page. First notice that

$$\frac{d\alpha_t}{dt} = \frac{1}{t^2} K \, d\theta$$

and choose some vector $v \in T\Sigma \cap \xi_t$. Applying both sides of (2-5) to v, we get

(3-1) $$d\alpha_t(X_t, v) = \frac{d\alpha_t}{dt}(v_{\alpha_t})\alpha_t(v) - \frac{d\alpha_t}{dt}(v).$$

As $v \in \xi_t = \ker(\alpha_t)$ and v has no θ-component, the last equality is equivalent to

$$\text{(3-2)} \hspace{4cm} d\alpha_t(X_t, v) = 0.$$

As $d\alpha_t$ is an area form on ξ_t, this implies that X_t and v are linearly dependent and therefore

$$X_t \in T\Sigma \cap \xi_t$$

($X_t = 0$ at singular points, where $T\Sigma$ and ξ_t coincide).

We are now looking at the singularities of X_t, $t \in [-1, 0)$. Because for $t_1 \neq t_2$, α_{t_1} and α_{t_2} differ by a multiple of $d\theta$, the flowlines given by X_{t_1} and X_{t_2} on a page Σ_θ coincide. Therefore, it suffices to look at the singularities of X_{t_0} for some $t_0 \in [-1, 0)$. On Σ_θ there are no negative elliptic singularities away from the binding since the contact planes and the planes tangent to the page almost coincide, as oriented plane fields (a negative elliptic singularity e would require ξ_e and $T_e\Sigma$ to coincide but have different orientations). Thus, for each θ, all points on Σ_θ, except for singularities of X_{t_0} and stable submanifolds of hyperbolic points, flow in finite time into an arbitrarily small neighborhood of the binding. Define S_θ as the set of points on Σ_θ that are either singularities of X_{t_0} or on stable submanifolds of hyperbolic points. Let $\mathscr{S} = \cup S_\theta$ as θ varies from 0 to 2π.

First, note that we can arrange the monodromy map ϕ to fix the singularities on the cutting page, by thinking of ϕ as a composition of Dehn twists away from these points. For isolated values of θ, X_{t_0} might exhibit connections between hyperbolic singularities. With that said, $\mathscr{S}$ has a CW structure with

- as its 1-skeleton, the union of $\{x\} \times [0, 1]$ for singular points x, and connections between hyperbolic singularities; and

- as its 2-skeleton, the union of stable submanifolds of hyperbolic singularities.

We want to arrange K in such a way that the bad arcs of K (where K is not braided) are disjoint from $\mathscr{S}$. Figure 3 depicts a bad arc of K. At the point p, which lies on the bad arc, K intersects the contact plane ξ_p positively, and it intersects the page negatively (the positive normal vectors to the contact plane and page are shown by arrows). We introduce wrinkles along K as in Figure 3. See below for the explicit definition.
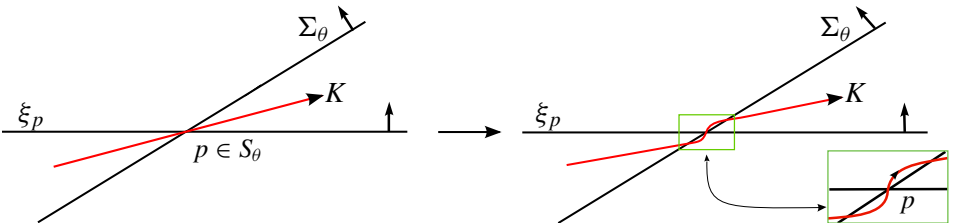
Figure 3. Wrinkling K in order to avoid intersections with $\mathscr{S}$.

The wrinkles may increase the number of arcs where the link is not braided, but this is fine, because these new arcs avoid $\mathcal{S}$.

By general position, we may assume that $K \cap$ (1-skeleton of $\mathcal{S}$) is empty and that $K \pitchfork$ (2-skeleton of $\mathcal{S}$) is a finite number of points. A small neighborhood of $p \in K \pitchfork$ (2-skeleton of $\mathcal{S}$) in Σ_θ is foliated by intervals $(-\epsilon, \epsilon)$, in the same way as a small disk in the xy-plane centered at $(0,1,0)$ in $(\mathbb{R}^3, \xi_{\text{std}})$. It follows that p has a neighborhood in M that is contactomorphic to a neighborhood of $q = (0, 1, 0)$ in $(\mathbb{R}^3, \xi_{\text{std}})$. Consider the standard (x, y, z) coordinate system in such a neighborhood of q. The contact plane at q is given by the equation $z = x$. Because at p the contact plane and the plane tangent to the page almost coincide, we consider the plane $z = (1 + \epsilon)x$ at q to correspond to the plane tangent to the page at p. We identify the bad arc of K with the segment given by

$$ y = 1, \quad z = (1 + \epsilon/2)x, $$

for $x \in [-\delta, \delta]$. We call this segment W. See Figure 4. With this setting, the wrinkle takes W to $\tilde{W}$ with the following properties:

(i) $\tilde{W}$ is given by $z = (1 + 3\epsilon/2)x$, $y = 1$ for $x \in (-\delta/3, \delta/3)$;

(ii) $\tilde{W}$ is given by $z = (1 + \epsilon/2)x$, $y = 1$ for $x \in [-\delta, -2\delta/3) \cup (2\delta/3, \delta]$; and

(iii) $dz/dx > 0$ along $\tilde{W}$ for $x \in (-\delta, \delta)$.
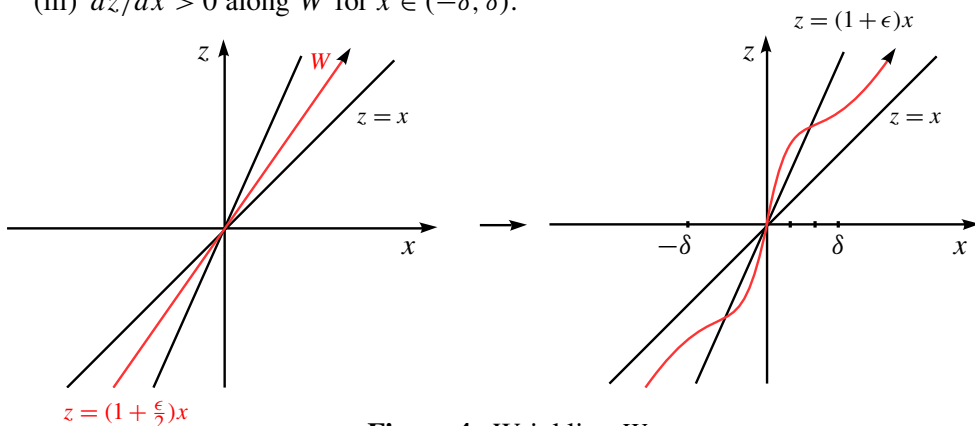
Figure 4. Wrinkling W.

Condition (iii) ensures that the link remains transverse throughout the wrinkling.

The arc $\tilde{W}$ still intersects the set $\mathcal{S}$ at the point $q \in \tilde{W}$, but after wrinkling this intersection no longer sits on a bad arc.

We perform this wrinkling at each point where a bad arc of K intersects $\mathcal{S}$, and obtain a link $\tilde{K}$ that does not intersect $\mathcal{S}$ along any of its bad arcs. We now apply the family of diffeomorphisms $\{f_t\}_{t \in [-1,0)}$. Then $\tilde{K}_\epsilon := f_\epsilon(\tilde{K})$ has all its bad arcs in a neighborhood of the binding. By Theorem 3.1, there is a transverse isotopy taking $\tilde{K}_\epsilon$ to a braid B. This braid B is transversely isotopic to K. $\qquad\square$

4. Markov's theorem in an open book decomposition

The goal of this section is to generalize the following theorem:

Theorem 4.1 [Orevkov and Shevchishin 2003]. *In* $(\mathbb{R}^3, \xi_{std})$, *two braids represent transversely isotopic links if and only if one can pass from one braid to the other by braid isotopies, positive Markov moves and their inverses.*

A *Markov move* near a component of the binding L is given by the introduction of a new loop around L, as in Figure 5. These moves are done near the binding, where the standard coordinates apply. While a positive Markov move (and its inverse) keeps the link transverse, a negative Markov move (and its inverse) fails to do so (see [Bennequin 1983] for details).
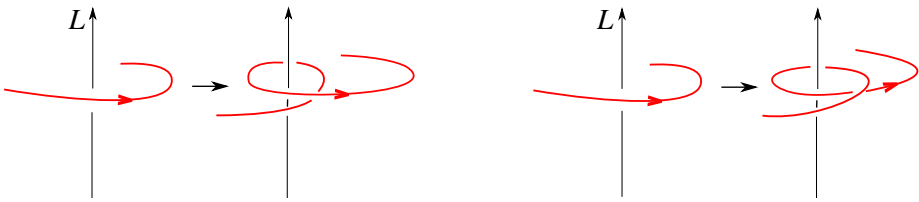
Figure 5. Positive (left) and negative (right) Markov moves.

Theorem 4.2. *Let M be a 3-dimensional, closed, oriented manifold and* (L, π) *be an open book decomposition for M together with its supported contact structure* ξ. *Let* K_0 *and* K_1 *be transverse closed braid representatives of the same topological link. Then* K_0 *and* K_1 *are transversely isotopic if and only if they differ by braid isotopies and positive Markov moves and their inverses.*

The topological version has been previously proven in [Skora 1992] and [Sundheim 1993]. This case immediately follows from the proof of the transverse case, as one does not need to worry about transversality throughout the isotopy. If transversality is not required, both positive and negative Markov moves are allowed.

Proof. The reverse implication is straightforward. An isotopy through braids is done away from the binding. Because the contact planes almost coincide with the planes tangent to the pages, this isotopy stays transverse with respect to the contact structure.

The direct implication takes more work. Let K_0 and K_1 be transverse braid representatives of the same topological link K, and let $\{K_t\}_{t \in [0,1]}$ be a transverse isotopy from K_0 to K_1. We parametrize the isotopy by $\mathcal{K} : \coprod S^1 \times [0, 1] \to M$, such that $\mathcal{K}_t$ defined by $s \to \mathcal{K}(s, t)$ is a parametrization of K_t, where s is the positively oriented coordinate on each S^1. Denote by $\mathfrak{A} \subset M$ the immersed annulus $\mathcal{K}(\coprod S^1 \times [0, 1])$.

Let θ be the positive coordinate normal to the page. A *bad zone* of $\mathcal{K}$ is a connected component of $\mathfrak{A}$ where $(\partial/\partial s)(\pi \circ \mathcal{K}) \leq 0$. We denote by $\mathcal{B}$ the union of all bad zones of $\mathcal{K}$.

We wish to take all the bad zones of $\mathcal{K}$ in a neighborhood of the binding. This way, the proof is reduced to the standard case proved in [Orevkov and Shevchishin 2003]. For this, we use the family of diffeomorphisms $\{f_t\}_{t\in[-1,0)}$ constructed in the proof of Theorem 3.2. Keeping the same notations, we want to arrange the isotopy $\mathcal{K}$ in such a way that $\mathcal{B} \cap \mathcal{S} = \varnothing$.

By general position, $\mathcal{B} \cap \mathcal{S}$ consists of arcs l, along which $\partial(\pi \circ \mathcal{K})/\partial s \leq 0$, and points at which $\partial(\pi \circ \mathcal{K})/\partial s \leq 0$. The arcs l are intersections of the 2-cells of $\mathcal{S}$ with $\mathcal{B}$, while the points are intersections of the 1-cells of $\mathcal{S}$ with $\mathcal{B}$. We modify the isotopy $\mathcal{K}$ in such a way as to replace the arc $l \subset \mathcal{B} \cap \mathcal{S}$ by another arc $l' \subset \mathcal{S}$, along which $\partial(\pi \circ \mathcal{K})/\partial s > 0$, and to eliminate the isolated points of $\mathcal{B} \cap \mathcal{S}$. We describe the process below.

Let $l \subset \mathcal{B} \cap \mathcal{S}$, and identify a small region of $\mathcal{B}$ containing l with the rectangle $R = [\frac{1}{2}, \frac{3}{2}] \times [0, 1]$ (denote by t the coordinate on the first factor) such that l is identified with $[\frac{1}{2}, \frac{3}{2}] \times \{\frac{1}{2}\}$, and let $R_t := \{t\} \times [0, 1] \subset R$, $t \in [\frac{1}{2}, \frac{3}{2}]$. We make the following identifications:

- For $t \in [\frac{1}{2}, \frac{3}{2}]$, identify R_t with the segment $W_t \subset \mathbb{R}^3$ given by $y = t$ and $z = (t + \epsilon/2)x$.
- For $t \in [\frac{1}{2}, \frac{3}{2}]$, identify the contact plane at $P(t, \frac{1}{2}) \in l$ with the contact plane at $(0, t, 0)$, that is, the plane given by $z = tx$.
- For $t \in [\frac{1}{2}, \frac{3}{2}]$, identify the page of the open book at $(t, \frac{1}{2}) \in l$ with the plane at $(0, t, 0)$ given by $z = (t + \epsilon)x$.

With these settings, the wrinkle takes W_t to $\tilde{W}_t$, $t \in [\frac{1}{2}, \frac{3}{2}]$, with the following properties:

- $\tilde{W}_t$ is given by $z = (t + 3\epsilon/2)x$, $y = t$ for $x \in (-\delta/3, \delta/3)$.
- $\tilde{W}$ is given by $z = (t + \epsilon/2)x$, $y = t$ for $x \in [-\delta, -2\delta/3) \cup (2\delta/3, \delta]$.
- $dz/dx > 0$ along $\tilde{W}_t$ for $x \in (-\delta, \delta)$.

See Figure 6.

We perform this wrinkling along every arc in the intersection $\mathcal{B} \cap \mathcal{S}$. If two such arcs l_1 and l_2 intersect each other, there exist two disjoint regions on $\mathfrak{A}(l_1 \cup l_2)$, one containing l_1 and the other containing l_2, that we can identify with rectangles R_1 and R_2 as above. We wrinkle along l_1 and l_2 separately.

The wrinkling translates to the original setup as in Figure 7. The bad zone (colored green in Figure 7) no longer intersects $\mathcal{S}$.

For an isolated point $x \in \mathcal{B} \cap \mathcal{S}$, we pick a small bad arc l containing x on the isotopy annulus and we do the wrinkling along l as above.
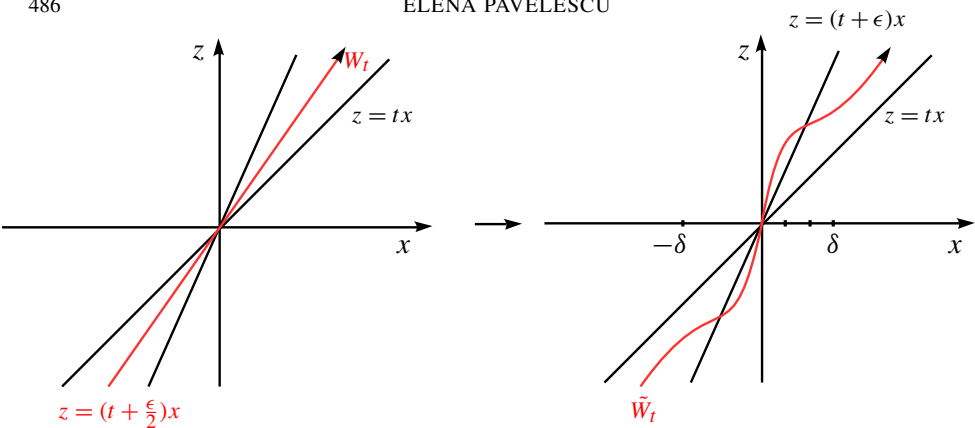
Figure 6. Wrinkling W_t.
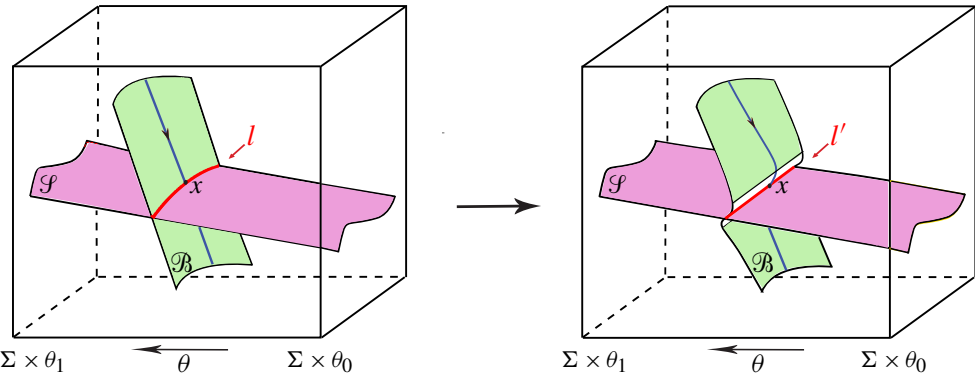
Figure 7. Isotopy modification along an l arc. The new intersection arc l' sits outside of $\mathcal{B}$ (green region).

We obtain an isotopy annulus that does not intersect $\mathcal{S}$ along any bad arcs. □

Acknowledgements

References

[Alexander 1920] J. W. Alexander, "Note on Riemann spaces", *Bull. Amer. Math. Soc.* **26**:8 (1920), 370–372. MR 1560318 JFM 47.0529.02

[Alexander 1923] J. W. Alexander II, "A lemma on systems of knotted curves", *Proc. Nat. Acad. Sci. USA* **9** (1923), 93–95. JFM 49.0408.03

[Bennequin 1983] D. Bennequin, "Entrelacements et équations de Pfaff", pp. 87–161 in *Third Schnepfenried geometry conference* (Schnepfenried, France, 1982), vol. 1, edited by D. Bernard and G. Reeb, Astérisque **107**, Soc. Math. France, Paris, 1983. MR 86e:58070 Zbl 0573.58022

[Etnyre 2006] J. B. Etnyre, "Lectures on open book decompositions and contact structures", pp. 103–141 in *Floer homology, gauge theory, and low-dimensional topology*, edited by D. A. Ellwood et al., Clay Math. Proc. **5**, Amer. Math. Soc., Providence, RI, 2006. MR 2007g:57042 Zbl 1108.53050

[Geiges 2008] H. Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics **109**, Cambridge University Press, 2008. MR 2008m:57064 Zbl 1153.53002

[Giroux 2002] E. Giroux, "Géométrie de contact: de la dimension trois vers les dimensions supérieures", pp. 405–414 in *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), vol. 2, edited by T. Li, Higher Ed. Press, Beijing, 2002. MR 2004c:53144 Zbl 1015.53049

[Kawamuro and Pavelescu 2011] K. Kawamuro and E. Pavelescu, "The self-linking number in annulus and pants open book decompositions", *Algebr. Geom. Topol.* **11**:1 (2011), 553–585. MR 2783238 Zbl 05851997

[Markov 1936] A. Markov, "Über die freie Äquivalenz der geschlossenen Zöpfe", *Recueil Math. Moscou, n. Ser.* **1** (1936), 73–78. Zbl 0014.04202

[Mitsumatsu 2006] Y. Mitsumatsu, "Convergence of contact structures to foliations", pp. 353–371 in *Foliations 2005*, edited by P. Walczak et al., World Sci., Hackensack, NJ, 2006. With an appendix by Mitsumatsu and Atsuhide Mori. MR 2007k:53144 Zbl 1222.53090

[Orevkov and Shevchishin 2003] S. Y. Orevkov and V. V. Shevchishin, "Markov theorem for transversal links", *J. Knot Theory Ramifications* **12**:7 (2003), 905–913. MR 2004j:57011 Zbl 1046.57005

[Skora 1992] R. K. Skora, "Closed braids in 3-manifolds", *Math. Z.* **211**:2 (1992), 173–187. MR 93m:57009 Zbl 0758.57007

[Sundheim 1993] P. A. Sundheim, "The Alexander and Markov theorems via diagrams for links in 3-manifolds", *Trans. Amer. Math. Soc.* **337**:2 (1993), 591–607. MR 93h:57016 Zbl 0795.57007

[Thurston and Winkelnkemper 1975] W. P. Thurston and H. E. Winkelnkemper, "On the existence of contact forms", *Proc. Amer. Math. Soc.* **52** (1975), 345–347. MR 51 #11561 Zbl 0312.53028

[Wrinkle 2002] N. C. Wrinkle, *The Markov theorem for transverse knots*, thesis, Columbia University, Ann Arbor, MI, 2002, available at http://tinyurl.com/8xh77sd. MR 2703285

ELENA PAVELESCU
DEPARTMENT OF MATHEMATICS
OCCIDENTAL COLLEGE
1600 CAMPUS ROAD
LOS ANGELES 90041
UNITED STATES

pavelescu@oxy.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org