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We establish first- and second-order gradient estimates for positive solutions of the heat equations under general geometric flows. Our results generalize the recent work of S. Liu, who established similar results for the Ricci flow. Both results can also be considered as the generalization of P. Li, S. T. Yau, and J. Li's gradient estimates under geometric flow setting. We also give an application to the mean curvature flow.

1. Introduction

Starting with the pioneering work of P. Li and S. T. Yau [1986], gradient estimates are also called differential Harnack inequalities, because we can obtain the classical Harnack inequality after integrating along the space-time curve. They are very powerful tools in geometric analysis. For example, R. Hamilton [1993; 1995b] established differential Harnack inequalities for the scalar curvature along the Ricci flow and for the mean curvature along the mean curvature flow. Both have important applications in the singularity analysis.

In Perelman's breakthrough work [2002] on the Poincaré conjecture and the geometrization conjecture, an important role was played by a differential Harnack inequality. Since then, there have been many works on gradient estimates along the Ricci flow or the conjugate Ricci flow for the solution of the heat equation or the conjugate heat equation; examples include [Cao 2008; Cao and Hamilton 2009; Kuang and Zhang 2008; Zhang 2006].

Under some curvature constraints, Guenther [2002] has established gradient estimates for positive solutions of the heat equation under general geometric flow on a closed manifold. Using this result, she derived a Harnack-type inequality and found a lower bound for the heat kernel under Ricci flow. As mentioned in [Liu 2009] (see also Section 4 of this paper), we can weaken the assumption of Guenther's results by removing the bound on the gradient of scalar curvature when restricting to the Ricci flow case. We can also obtain a local gradient estimate for complete case.

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While most of the works deal with the first-order case, higher-order gradient estimates have their own interest. Indeed, they are closely related to the boundedness of the Riesz transform and the Sobolev inequality. J. Li [1991] obtained second-order gradient estimates for heat kernels on complete noncompact Riemannian manifolds. In [Li 1994], he used the boundedness of Riesz transform to prove the Sobolev inequality on Riemannian manifolds with some constraints.

S. Liu [2009] obtained the first and the second order gradient estimates for positive solutions of the heat equations under Ricci flow. His work generalized [Li and Yau 1986] and [Li 1991].

In this paper, we generalize Liu's work to general geometric flow. Of course, we need impose stronger conditions on the flow and the curvature. Compared to general geometric flow, there are two advantages to the Ricci flow: the contracted second Bianchi identity, which gives a nice expression for the commuting formula (Section 4), and the fact that the Ricci curvature arises when we use the Bochner formula to compute the Laplacian of $|\nabla u|^2$. Sometimes, the Ricci curvature will be canceled with the time derivative of the metric under the Ricci flow.

K. Ecker, D. Knopf, L. Ni and P. Topping [Ecker et al. 2008] recently obtained a local gradient estimate for bounded positive solution of the conjugate heat equation for general geometric flow.

Our paper is organized as follows: We prove first-order gradient estimates in Section 2 and second-order gradient estimates in Section 3. We give two applications to the Ricci flow and the mean curvature flow in Section 4.

2. First-order gradient estimates

For a function f on $M \times [0, T]$, where T is a positive constant, we write

$$f_t = \partial_t f = \frac{\partial f(x,t)}{\partial t}.$$

Theorem 1 (gradient estimate: local version). *Let $(M, g(t))$ be a smooth one-parameter family of complete Riemannian manifolds evolving by*

$$\frac{\partial}{\partial t} g = 2h, \tag{2-1}$$

for t in some time interval $[0, T]$. Let M be complete under the initial metric $g(0)$. Given $x_0 \in M$ and $R > 0$, let u be a positive solution to the equation

$$(\Delta - \partial_t) u(x,t) = 0$$

in the cube $Q_{2R,T} := \{(x,t) \mid d(x, x_0, t) \le 2R,\ 0 \le t \le T\}$. Suppose that there exist constants $K_1, K_2, K_3, K_4 \ge 0$ such that

$$\mathrm{Ric} \ge -K_1 g, \quad -K_2 g \le h \le K_3 g, \quad |\nabla h| \le K_4,$$

on $Q_{2R,T}$. Then for $(x,t)\in Q_{R,T}$, we have

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)}-\alpha\frac{u_t(x,t)}{u(x,t)}\le C\left(K_1+K_2+K_3+K_4+\sqrt{K_4}+\frac{1}{t}+\frac{1}{R^2}\right) \tag{2-2}$$

for any $\alpha>1$, where C depends on n,α only.

More explicitly, we have

$$\begin{aligned}\frac{|\nabla u(x,t)|^2}{u^2(x,t)}-\alpha\frac{u_t(x,t)}{u(x,t)}\le \frac{n\alpha^2}{t}+\frac{C\alpha^2}{R^2}\left(R\sqrt{K_1}+\frac{\alpha^2}{\alpha-1}\right)+C\alpha^2K_2\\ +\frac{n\alpha^2}{\alpha-1}(K_1+(\alpha-1)K_3+K_4)+n\alpha^2(K_2+K_3+\sqrt{2K_4}),\end{aligned} \tag{2-3}$$

for any $\alpha>1$, where C depends only on n.

Remark 2. When $h=-\mathrm{Ric}$, (2-1) is the Ricci flow equation. In this case our results reduce to [Liu 2009]. Note that for Ricci flow the assumption $|\nabla \mathrm{Ric}|\le K_4$ is not needed because of the contracted second Bianchi identity (see Section 4).

As in [Li and Yau 1986], let $f=\log u$; then

$$(\Delta-\partial_t)f=-|\nabla f|^2. \tag{2-4}$$

Set

$$F=t\left(\frac{|\nabla u(x,t)|^2}{u^2(x,t)}-\alpha\frac{u_t(x,t)}{u(x,t)}\right)=t(|\nabla f|^2-\alpha f_t).$$

To prepare the ground for the proof of the theorem we need some lemmas.

Lemma 3. *Suppose the metric evolves by (2-1). Then, for any smooth function f, we have*

$$\frac{\partial}{\partial t}|\nabla f|^2=-2h(\nabla f,\nabla f)+2\langle\nabla f,\nabla f_t\rangle$$

and

$$\frac{\partial}{\partial t}\Delta f=\Delta\frac{\partial}{\partial t}f-2\langle h,\nabla^2 f\rangle-2\langle \operatorname{div} h-\tfrac{1}{2}\nabla(\mathrm{tr}_g h),\nabla f\rangle. \tag{2-5}$$

Here, div h *is the divergence of* h.

Proof. To prove the first equation, write

$$\frac{\partial}{\partial t}|\nabla f|^2=\frac{\partial}{\partial t}\left(g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right)=-2h(\nabla f,\nabla f)+2\langle\nabla f,\nabla f_t\rangle.$$

For the second, recall that

$$\frac{\partial}{\partial t}\Gamma_{ij}^k=g^{kl}\{\nabla_i h_{jl}+\nabla_j h_{il}-\nabla_l h_{ij}\}.$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\Delta f &= \frac{\partial}{\partial t}\left\{g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right)\right\} \\ &= -2h^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f}{\partial x_k}\right) + \Delta \frac{\partial f}{\partial t} - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\frac{\partial f}{\partial x_k} \\ &= -2\langle h, \nabla^2 f\rangle + \Delta\frac{\partial f}{\partial t} - g^{ij}g^{kl}\{\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\}\nabla_k f \\ &= \Delta \frac{\partial f}{\partial t} - 2\langle h, \nabla^2 f\rangle - 2g^{kl}\{g^{ij}\nabla_i h_{jl} - \frac{1}{2}\nabla_l(\mathrm{tr}_g\, h)\}\nabla_k f \\ &= \Delta\frac{\partial}{\partial t} f - 2\langle h, \nabla^2 f\rangle - 2\langle \mathrm{div}\, h - \frac{1}{2}\nabla(\mathrm{tr}_g\, h), \nabla f\rangle.\end{aligned}$$

□

Lemma 4. *Suppose $(M, g(t))$ satisfies the hypotheses of Theorem 1. We have*

$$\begin{aligned}(\Delta - \partial_t)F \geq{}& -2\langle \nabla f, \nabla F\rangle + \frac{t}{n}(|\nabla f|^2 - f_t)^2 - (|\nabla f|^2 - \alpha f_t) \\ &- 2(K_1 + (\alpha-1)K_3)t|\nabla f|^2 - 3\sqrt{n}\alpha K_4 t|\nabla f| - \alpha^2 nt(K_2+K_3)^2.\end{aligned}$$

Proof. For a given time t, choose $\{x_1, x_2, \dots, x_n\}$ to be a normal coordinate system at a fixed point. Subscripts i, j will denote covariant derivatives in the x_i, x_j directions. We will compute at the fixed point.

Using the Bochner formula, (2-4) and Lemma 3, we calculate

$$\begin{aligned}\Delta F &= t\left(2|\nabla^2 f|^2 + 2\,\mathrm{Ric}(\nabla f, \nabla f) + 2\langle \nabla f, \nabla\Delta f\rangle - \alpha\Delta(f_t)\right) \\ &= t\left(2|\nabla^2 f|^2 + 2\,\mathrm{Ric}(\nabla f, \nabla f) + 2\langle\nabla f, \nabla(f_t - |\nabla f|^2)\rangle\right) \\ &\qquad - \alpha t\left((\Delta f)_t + 2\langle h, \nabla^2 f\rangle + 2\langle \mathrm{div}\, h - \tfrac{1}{2}\nabla(\mathrm{tr}_g\, h), \nabla f\rangle\right) \\ &= -2\langle \nabla f, \nabla F\rangle + 2t(|\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + (1-\alpha)h(\nabla f, \nabla f) \\ &\qquad - \alpha\langle h, \nabla^2 f\rangle - \alpha\langle \mathrm{div}\, h - \tfrac{1}{2}\nabla(\mathrm{tr}_g\, h), \nabla f\rangle) + t(|\nabla f|^2)_t - \alpha t f_{tt}.\end{aligned}$$

On the other hand, we have

$$F_t = (|\nabla f|^2 - \alpha f_t) + t\left((|\nabla f|^2)_t - \alpha f_{tt}\right).$$

Therefore, we arrive at

$$\begin{aligned}(\Delta - \partial_t)F ={}& -2\langle \nabla f, \nabla F\rangle + 2t(|\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + (1-\alpha)h(\nabla f, \nabla f) \\ &- \alpha\langle h, \nabla^2 f\rangle - \alpha\langle \mathrm{div}\, h - \tfrac{1}{2}\nabla(\mathrm{tr}_g\, h), \nabla f\rangle) - (|\nabla f|^2 - \alpha f_t).\end{aligned}$$

By our assumption, we have

$$-(K_2 + K_3)g \leq h \leq (K_2 + K_3),$$

which implies that

$$|h|^2 \le (K_2+K_3)^2|g|^2 = n(K_2+K_3)^2.$$

Applying those bounds and Young's inequality yields

$$|\alpha\langle h, \nabla^2 f\rangle| \le \tfrac{1}{2}|\nabla^2 f|^2 + \tfrac{1}{2}\alpha^2|h|^2 \le \tfrac{1}{2}|\nabla^2 f|^2 + \tfrac{1}{2}\alpha^2 n(K_2+K_3)^2.$$

On the other hand,

$$|\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)| = |g^{ij}\nabla_i h_{jl} - \tfrac{1}{2}g^{ij}\nabla_l h_{ij}| \le \tfrac{3}{2}|g||\nabla h| \le \tfrac{3}{2}\sqrt{n}K_4.$$

We conclude by our assumptions that

$$\begin{aligned}(\Delta-\partial_t)F \ge{}& -2\langle\nabla f, \nabla F\rangle + t|\nabla^2 f|^2 - (|\nabla f|^2 - \alpha f_t)\\ &- 2(K_1+(\alpha-1)K_3)t|\nabla f|^2 - 3\sqrt{n}\alpha K_4 t|\nabla f| - \alpha^2 nt(K_2+K_3)^2.\end{aligned}$$

Finally, with the help of the inequality

$$|\nabla^2 f|^2 \ge \frac{1}{n}(\operatorname{tr}\nabla^2 f)^2 = \frac{1}{n}(\Delta f)^2 = \frac{1}{n}(|\nabla f|^2 - f_t)^2,$$

we complete the proof of the lemma. □

Proof of Theorem 1. By our assumption of the bounds of h and the evolution of the metric, we know that $g(t)$ is uniformly equivalent to the initial metric $g(0)$, that is,

$$e^{-2K_2T}g(0) \le g(t) \le e^{2K_3T}g(0).$$

Thus we know that $(M, g(t))$ is also complete for $t\in[0,T]$.

Now let $\psi(r)$ be a C^2 function on $[0,+\infty)$ such that

$$\psi(r) = \begin{cases}1 & \text{if } r\in[0,1],\\ 0 & \text{if } r\in[2,+\infty),\end{cases} \tag{2-6}$$

$$0\le\psi(r)\le 1, \quad \psi'(r)\le 0, \quad \psi''(r)\ge -C, \quad \frac{|\psi'(r)|^2}{\psi(r)} \le C, \tag{2-7}$$

where C is an absolute constant. Define

$$\varphi(x,t) = \varphi(d(x,x_0,t)) = \psi\left(\frac{d(x,x_0,t)}{R}\right) = \psi\left(\frac{\rho(x,t)}{R}\right),$$

where $\rho(x,t) = d(x,x_0,t)$. For the purpose of applying the maximum principle, the argument of [Calabi 1958] allows us to assume that the function $\varphi(x,t)$, with support in $Q_{2R,T}$, is C^2 at the maximum point.

For any $0 < T_1 \le T$, let (x_1,t_1) be the point in Q_{2R,T_1}, at which φF achieves its maximum value. We can assume that this value is positive, because in the other case the proof is trivial. As $F(x,0)=0$, we know that $t_1>0$. Then at the point (x_1,t_1), we have

$$\nabla(\varphi F)=F\nabla\varphi+\varphi\nabla F=0,\quad \Delta(\varphi F)\le 0,\quad \frac{\partial}{\partial t}(\varphi F)\ge 0. \tag{2-8}$$

Therefore,

$$0\ge(\Delta-\partial_t)(\varphi F)=(\Delta\varphi)F-\varphi_t F+\varphi(\Delta-\partial_t)F+2\nabla\varphi\cdot\nabla F. \tag{2-9}$$

Using the Laplacian comparison theorem, we have

$$\Delta\varphi=\psi'\frac{\Delta\rho}{R}+\psi''\frac{|\nabla\rho|^2}{R^2}\ge-\frac{C}{R^2}-\frac{C}{R}\sqrt{K_1}.$$

Furthermore, we have

$$\frac{|\nabla\varphi|^2}{\varphi}=\frac{(\psi')^2}{\psi}\frac{|\nabla\rho|^2}{R^2}\le\frac{C}{R^2}.$$

By our assumption, $F(x_1,t_1)>0$. By the evolution formula of the geodesic length under geometric flow [Hamilton 1995a], we calculate at the point (x_1,t_1)

$$\begin{aligned}-\varphi_t F&=-\psi'\Big(\frac{\rho}{R}\Big)\frac{1}{R}\frac{d\rho}{dt}F=-\psi'\Big(\frac{\rho}{R}\Big)\frac{1}{R}\int_{\gamma_{t_1}}h(S,S)\,ds\,F\\&\ge\psi'\Big(\frac{\rho}{R}\Big)\frac{1}{R}K_2\rho\,F\ge-\sqrt{C}K_2F,\end{aligned}$$

where γ_{t_1} is the geodesic connecting x and x_0 under the metric $g(t_1)$, S is the unite tangent vector to γ_{t_1} and ds is the element of arc length. Substituting the three inequalities above into (2-9) and using (2-8), we obtain

$$0\ge\Big(-\frac{C}{R^2}-\frac{C}{R}\sqrt{K_1}\Big)F-\sqrt{C}K_2F+\varphi(\Delta-\partial_t)F.$$

Applying Lemma 4 and Young's inequality to this inequality yields

$$\begin{aligned}0\ge\Big(-\frac{C}{R^2}-\frac{C}{R}\sqrt{K_1}\Big)F-\sqrt{C}K_2F-2\frac{\sqrt{C}}{R}\sqrt{\varphi}|\nabla f|F+\frac{t_1}{n}\varphi(|\nabla f|^2-f_t)^2\\-\varphi(|\nabla f|^2-\alpha f_t)-2(K_1+(\alpha-1)K_3+K_4)\varphi t_1|\nabla f|^2\\-\alpha^2nt_1\varphi[(K_2+K_3)^2+2K_4].\end{aligned} \tag{2-10}$$

Multiplying through by φt_1 and setting $y=\varphi|\nabla f|^2$ and $z=\varphi f_t$, (2-10) becomes

$$\begin{aligned}0\ge t_1\Big(-\frac{C}{R^2}-\frac{C}{R}\sqrt{K_1}\Big)(\varphi F)-\sqrt{C}K_2t_1(\varphi F)-2\frac{\sqrt{C}}{R}t_1^2y^{1/2}(y-\alpha z)\\+\frac{t_1^2}{n}(y-z)^2-\varphi^2F-2(K_1+(\alpha-1)K_3+K_4)t_1^2y-\alpha^2nt_1^2\varphi^2[(K_2+K_3)^2+2K_4].\end{aligned} \tag{2-11}$$

Using the inequality $ax^2 - bx \geq -b^2/(4a)$, valid for $a, b > 0$, one obtains

$$\begin{aligned}
&\frac{t_1^2}{n}(y-z)^2 - 2\frac{\sqrt{C}}{R}t_1^2 y^{1/2}(y-\alpha z) - 2(K_1+(\alpha-1)K_3+K_4)t_1^2 y \\
&\quad = \frac{t_1^2}{n}\biggl[\frac{1}{\alpha^2}(y-\alpha z)^2 + \Bigl(\frac{\alpha-1}{\alpha}\Bigr)^2 y^2 - 2n(K_1+(\alpha-1)K_3+K_4)y \\
&\qquad\qquad + \Bigl(2\frac{\alpha-1}{\alpha^2}y - \frac{2n\sqrt{C}}{R}y^{1/2}\Bigr)(y-\alpha z)\biggr] \\
&\quad \geq \frac{t_1^2}{n}\biggl[\frac{1}{\alpha^2}(y-\alpha z)^2 - \frac{\alpha^2 n^2(K_1+(\alpha-1)K_3+K_4)^2}{(\alpha-1)^2} - \frac{\alpha^2 n^2 C}{2(\alpha-1)R^2}(y-\alpha z)\biggr].
\end{aligned}$$

Hence (2-11) becomes

$$\begin{aligned}
&\frac{1}{n\alpha^2}(\varphi F)^2 - (\varphi F)\Bigl(1 + \frac{C}{R^2}t_1 + \frac{C}{R}\sqrt{K_1}t_1 + \frac{Cn\alpha^2 t_1}{2(\alpha-1)R^2} + \sqrt{C}K_2 t_1\Bigr) \\
&\qquad - \Bigl(\frac{n(K_1+(\alpha-1)K_3+K_4)^2\alpha^2 t_1^2}{(\alpha-1)^2} + t_1^2\alpha^2 n\varphi^2[(K_2+K_3)^2+2K_4]\Bigr) \leq 0.
\end{aligned}$$

We apply the quadratic formula and then arrive at

$$\begin{aligned}
\varphi F(x_1, t_1) \leq n\alpha^2 &+ \frac{Cn\alpha^2}{R^2}\Bigl(R\sqrt{K_1} + \frac{\alpha^2}{\alpha-1}\Bigr)t_1 + \sqrt{C}n\alpha^2 K_2 t_1 \\
&+ \frac{n\alpha^2 t_1}{\alpha-1}(K_1+(\alpha-1)K_3+K_4) + n(K_2+K_3+\sqrt{2K_4})\alpha^2 t_1.
\end{aligned}$$

If $d(x, x_0, T_1) < R$, we have $\varphi(x, T_1) = 1$. Then

$$\begin{aligned}
F(x, T_1) &= T_1(|\nabla f|^2 - \alpha f_t) \leq \varphi F(x_1, t_1) \\
&\leq n\alpha^2 + \frac{Cn\alpha^2}{R^2}\Bigl(R\sqrt{K_1} + \frac{\alpha^2}{\alpha-1}\Bigr)T_1 + \sqrt{C}n\alpha^2 K_2 T_1 \\
&\qquad + \frac{n\alpha^2 T_1}{\alpha-1}(K_1+(\alpha-1)K_3+K_4) + n(K_2+K_3+\sqrt{2K_4})\alpha^2 T_1.
\end{aligned}$$

As T_1 is arbitrary, we obtain the result. □

From the local result above, we get a global one:

Corollary 5. *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary, and let $g(t)$ evolves by (2-1) for $t \in [0, T]$ and satisfy*

$$\mathrm{Ric} \geq -K_1 g, \quad -K_2 g \leq h \leq K_3 g, \quad |\nabla h| \leq K_4.$$

If u is a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$, then for $(x,t) \in M \times (0, T]$, we have

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \le \frac{n\alpha^2}{t} + C\big(K_1+K_2+K_3+K_4+\sqrt{K_4}\big), \tag{2-12}$$

for any $\alpha > 1$, *where* C *depends only on* n, α.

Proof. By the uniform equivalence of $g(t)$, we know that $(M, g(t))$ is complete noncompact for $t \in [0, T]$. Letting $R \to +\infty$ in (2-3) completes the proof. □

Using Lemma 3, we can also derive a similar gradient estimate on a closed Riemannian manifold.

Theorem 6. *Let* $(M, g(t))$ *be a closed Riemannian manifold, where* $g(t)$ *evolves by* (2-1) *for* $t \in [0, T]$ *and satisfies*

$$\mathrm{Ric} \ge -K_1 g, \quad -K_2 g \le h \le K_3 g, \quad |\nabla h| \le K_4.$$

If u *is a positive solution to the equation* $(\Delta - \partial_t)u(x,t) = 0$, *then for* $(x,t) \in M \times (0, T]$, *we have*

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \le \frac{n\alpha^2}{t} + \frac{n\alpha^2}{\alpha-1}\,(K_1+(\alpha-1)K_3+K_4) + n\alpha^2(K_2+K_3+\sqrt{2K_4}), \tag{2-13}$$

for any $\alpha > 1$, *where* C *depends only on* n, α.

Proof. We use the same symbols F, f as above. Set

$$\bar F(x,t) = F(x,t) - \frac{n\alpha^2}{\alpha-1}\,(K_1+(\alpha-1)K_3+K_4)\,t - n\alpha^2(K_2+K_3+\sqrt{2K_4})t.$$

If $\bar F(x,t) \le n\alpha^2$ for any $(x,t) \in M \times (0, T]$, the proof is complete.

If (2-13) doesn't hold, then at the maximal point (x_0, t_0) of $\bar F(x,t)$, we have

$$\bar F(x_0, t_0) > n\alpha^2.$$

Since $\bar F(x, 0) = 0$, we know that $t_0 > 0$ here. Then applying the maximum principle, we have at the point (x_0, t_0),

$$\nabla \bar F(x_0,t_0) = 0, \quad \Delta \bar F(x_0,t_0) \le 0, \quad \frac{\partial}{\partial t}\bar F(x_0,t_0) \ge 0.$$

Therefore we obtain

$$0 \ge (\Delta - \partial_t)\bar F \ge (\Delta - \partial_t)F.$$

Using Lemma 4 and the fact that

$$\begin{aligned}(|\nabla f|^2 - f_t)^2 &= \Big(\frac{1}{\alpha}(|\nabla f|^2 - \alpha f_t) + \frac{\alpha-1}{\alpha}|\nabla f|^2\Big)^2 \\ &= \frac{1}{\alpha^2}\Big(\frac{F}{t_0}\Big)^2 + 2\frac{\alpha-1}{\alpha^2}|\nabla f|^2\Big(\frac{F}{t_0}\Big) + \frac{(\alpha-1)^2}{\alpha^2}|\nabla f|^4,\end{aligned}$$

we get that

$$0 \geq \frac{t_0}{n\alpha^2}\left(\frac{F}{t_0}\right)^2 - \left(\frac{F}{t_0}\right) - \frac{n\alpha^2}{(\alpha-1)^2}(K_1+(\alpha-1)K_3+K_4)^2 t_0 \\ - n\alpha^2\big((K_2+K_3)^2+2K_4\big)t_0 + \frac{2t_0}{n}\frac{\alpha-1}{\alpha^2}|\nabla f|^2\frac{F}{t_0}.$$

Since

$$\frac{F}{t_0} = \frac{\bar{F}}{t_0} + \frac{n\alpha^2}{\alpha-1}(K_1+(\alpha-1)K_3+K_4) + n\alpha^2(K_2+K_3+\sqrt{2K_4}) > 0,$$

we get

$$\frac{t_0}{n\alpha^2}\left(\frac{F}{t_0}\right) - \frac{F}{t_0} - \frac{n\alpha^2}{(\alpha-1)^2}(K_1+(\alpha-1)K_3+K_4)^2 t_0 - n\alpha^2\big((K_2+K_3)^2+2K_4\big)t_0 \leq 0.$$

Solving this quadratic inequality yields

$$\frac{F}{t_0} \leq \frac{n\alpha^2}{t_0} + \frac{n\alpha^2}{\alpha-1}(K_1+(\alpha-1)K_3+K_4) + n\alpha^2(K_2+K_3+\sqrt{2K_4}).$$

This implies that $\bar{F}(x_0, t_0) \leq n\alpha^2$, in contradiction with our assumption. So (2-13) holds. □

Remark 7. In Corollary 5 and Theorem 9, if $K_1 = K_4 = 0$, we can let $\alpha \to 1$.

Integrating the gradient estimate in space-time as in [Li and Yau 1986] or [Guenther 2002], we can derive the following Harnack-type inequality.

Corollary 8. *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary or a closed Riemannian manifold. Assume $g(t)$ evolves by (2-1) for $t \in [0, T]$ and satisfies*

$$\operatorname{Ric} \geq -K_1 g, \quad -K_2 g \leq h \leq K_3 g, \quad |\nabla h| \leq K_4.$$

If u is a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$, then for any pair of points (x, t_1), (y, t_2) in $M \times (0, T]$ such that $t_1 < t_2$ we have

$$u(x, t_1) \leq u(y, t_2)\left(\frac{t_2}{t_1}\right)^{2n\varepsilon} \exp\left(\frac{\varepsilon\Lambda}{2(t_2-t_1)} + C\frac{t_2-t_1}{2\varepsilon}K\right), \quad \text{for any } \varepsilon > \tfrac{1}{2},$$

where $K = K_1+K_2+K_3+K_4+\sqrt{K_4}$, the constant C depends only on n and ε, and

$$\Lambda = \inf_{\gamma} \int_0^1 |\gamma'(s)|^2_{\sigma(s)}\, ds$$

is the infimum over smooth curves γ joining y to x ($\gamma(0) = y$, $\gamma(1) = x$) of the averaged square velocity of γ measured at time $\sigma(s) = (1-s)t_2 + st_1$.

Proof. The gradient estimates in Corollary 8 and Theorem 9 can both be written as

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \le \frac{n\alpha^2}{t} + C_{n,\alpha}K.$$

Fix $\varepsilon > \frac{1}{2}$, take any curve γ satisfying the assumption and set

$$l(s) = \ln u(\gamma(s), \sigma(s)).$$

Then $l(0) = \ln u(y, t_2)$ and $l(1) = \ln u(x, t_1)$. Direct calculation shows that

$$\frac{\partial l(s)}{\partial s} = (t_2 - t_1)\left(\frac{\nabla u}{u}\frac{\gamma'(s)}{t_2 - t_1} - \frac{u_t}{u}\right) \le \frac{\varepsilon|\gamma'(s)|^2_\sigma}{2(t_2 - t_1)} + \frac{t_2 - t_1}{2\varepsilon}\left(CK + \frac{4\varepsilon^2 n}{\sigma(s)}\right).$$

Integrating this inequality over $\gamma(s)$, we have

$$\text{(2-14)}\quad \ln\frac{u(x,t_1)}{u(y,t_2)} = \int_0^1 \frac{\partial l(s)}{\partial s}ds \le \int_0^1 \frac{\varepsilon|\gamma'(s)|^2_\sigma}{2(t_2 - t_1)}ds + C\frac{t_2 - t_1}{2\varepsilon}K + 2\varepsilon n\ln\frac{t_2}{t_1},$$

which implies the corollary. □

3. Second-order gradient estimates

In this section we derive the second order gradient estimate for the positive solution of the heat equation along a general geometric flow, which generalizes the results in [Li 1991; Liu 2009].

Theorem 9. *Let $g(t)$ be a solution to (2-1) on a Riemannian manifold M^n for t in some time interval $[0, T]$. Assume that $(M, g(0))$ is a complete noncompact manifold without boundary. Suppose that $(M, g(t))$ satisfies*

$$|\operatorname{Rm}| \le k_1, \quad |\nabla \operatorname{Rm}| \le k_2, \quad -k_3 g \le h \le k_3 g, \quad |\nabla h| \le k_4,$$

for some nonnegative constants k_1, k_2, k_3, k_4. Let u be a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$. Then, for any $(x,t) \in M \times (0, T]$ and $\alpha > 1$, we have

$$\frac{|\nabla^2 u(x,t)|}{u(x,t)} + \alpha\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - 5\alpha\frac{u_t(x,t)}{u(x,t)} \le C\left(k_1 + k_2^{2/3} + k_3 + k_4 + \sqrt{k_4} + \frac{1}{t}\right),$$

where C depends only on n and α.

Before proving the theorem, we need a lemma. Set

$$F(x,y,t) = tF_1 = t\left(\frac{|\nabla^2 u(x,t)|}{u(x,t)} + \alpha\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \beta\frac{u_t(x,t)}{u(x,t)}\right),$$

where β is a constant to be fixed.

Lemma 10. *Suppose $(M, g(t))$ satisfies the hypotheses of Theorem 9. Then for sufficiently small $\delta > 0$, $\gamma - 1 > 0$, and $\varepsilon > 0$, we have, with $\beta = 5\alpha$,*

$$(\Delta - \partial_t)F \geq -2\langle \nabla F, \nabla \log u\rangle + \frac{\delta\alpha}{t}F^2 + 2\delta\alpha\beta F\frac{u_t}{u} - 2\delta\alpha^2 F\frac{|\nabla u|^2}{u^2} - C(k_1+k_3)F - \frac{C(k_1+k_3)^2}{4(\gamma-1)^2}t - \varepsilon n k_3^2\beta t - \frac{F}{t} - \frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2 t - \frac{C\beta^{4/3}}{\delta^{1/3}\alpha}k_4^{4/3}t - \frac{C}{\delta^{1/3}\alpha}(k_2+k_4)^{4/3}t - 2Ct\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{1}{t}+k_1+k_3+k_4+\sqrt{k_4}\right)^2,$$

where C depends on n and α.

Proof. As in the proof of Lemma 4, choose $\{x_1, x_2, \ldots, x_n\}$ to be a normal coordinate system at a fixed point. Subscript i, j, k will denote covariant derivatives in the x_i, x_j, x_k directions.

We will first calculate the evolution equation for F_1 and divide it into three parts. In the calculation, we will use the following formula a few times:

$$(\Delta - \partial_t)\frac{f}{g} = \frac{1}{g}(\Delta - \partial_t)f - \frac{f}{g^2}(\Delta - \partial_t)g - \frac{2}{g}\left\langle \nabla\frac{f}{g}, \nabla g\right\rangle. \tag{3-1}$$

<u>Part 1.</u> We first calculate a parabolic inequality for $|\nabla^2 u|/u$.

Using (3-1) and the fact that $(\Delta - \partial_t)u = 0$, we obtain that

$$(\Delta - \partial_t)\left(\frac{|\nabla^2 u|}{u}\right) = -2\left\langle \nabla\frac{|\nabla^2 u|}{u}, \nabla u\right\rangle + \frac{1}{u}(\Delta - \partial_t)|\nabla^2 u|.$$

Note that

$$\Delta|\nabla^2 u|^2 = 2|\nabla^2 u|\,\Delta|\nabla^2 u| + 2\big|\nabla|\nabla^2 u|\big|^2$$

and

$$\Delta|\nabla^2 u|^2 = \sum_{ijk}(u_{ij}^2)_{kk} = 2|\nabla^3 u|^2 + 2\sum_{ijk} u_{ij}u_{ijkk}.$$

We get that

$$\Delta|\nabla^2 u| = \frac{|\nabla^3 u|^2 + \sum_{ijk} u_{ij}u_{ijkk} - \big|\nabla|\nabla^2 u|\big|^2}{|\nabla^2 u|} \geq \frac{\sum_{ijk} u_{ij}u_{ijkk}}{|\nabla^2 u|}.$$

Here we have used the fact that $|\nabla^3 u| \geq \big|\nabla|\nabla^2 u|\big|$.

The Ricci identity gives

$$u_{ijkk} = u_{kkij} + \sum_l R_{kjkl,i}u_l + \sum_l R_{kijl,k}u_l + \sum_l R_{kikl}u_{lj} + \sum_l R_{kjkl}u_{li} + 2\sum_l R_{kijl}u_{kl}.$$

By our assumption on the curvature, we have

$$\Delta|\nabla^2 u| \geq \frac{\langle \nabla^2 u, \nabla^2(\Delta u)\rangle}{|\nabla^2 u|} - Ck_1|\nabla^2 u| - Ck_2|\nabla u|.$$

Noting that the metric evolves by (2-1), we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\nabla_j u = \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^p \frac{\partial u}{\partial x_p}\right) &= \nabla_i\nabla_j u_t - \left(\frac{\partial}{\partial t}\Gamma_{ij}^p\right)\nabla_p u \\ &= \nabla_i\nabla_j u_t - (\nabla_i h_{jp} + \nabla_j h_{ip} - \nabla_p h_{ij})\nabla_p u.\end{aligned}$$

This leads to

$$\begin{aligned}\partial_t(|\nabla^2 u|^2) &= \frac{\partial}{\partial t}(g^{ik}g^{jl}\nabla_i\nabla_j u\nabla_k\nabla_l u) \\ &= -4h^{ik}\nabla_i\nabla_j u\nabla_k\nabla_j u + 2\langle\nabla_i\nabla_j u_t, \nabla_i\nabla_j u\rangle \\ &\qquad - 2(\nabla_i h_{jp} + \nabla_j h_{ip} - \nabla_p h_{ij})\nabla_p u\nabla_i\nabla_j u,\end{aligned}$$

$$\partial_t|\nabla^2 u| \geq \frac{\langle \nabla^2 u, \nabla^2(\partial_t u)\rangle}{|\nabla^2 u|} - Ck_3|\nabla^2 u| - Ck_4|\nabla u|.$$

Combining together all of the above, we conclude that

$$\begin{aligned}(3\text{-}2)\quad &(\Delta - \partial_t)\left(\frac{|\nabla^2 u|}{u}\right) \\ &\geq -2\left\langle\nabla\left(\frac{|\nabla^2 u|}{u}\right), \nabla\log u\right\rangle - C(k_1+k_3)\frac{|\nabla^2 u|}{u} - C(k_2+k_4)\frac{|\nabla u|}{u}.\end{aligned}$$

Part 2. We next calculate a parabolic inequality for $|\nabla u|^2/u^2$. Using (3-1) and the fact that $(\Delta - \partial_t)u = 0$, we obtain

$$\begin{aligned}&(\Delta - \partial_t)\left(\frac{|\nabla u|^2}{u^2}\right) \\ &= -2\left\langle\nabla\left(\frac{|\nabla u|^2}{u^2}\right), \nabla\log u\right\rangle + \frac{(\Delta - \partial_t)|\nabla u|^2}{u^2} + 2\frac{|\nabla u|^4}{u^4} - \frac{2}{u^3}\langle\nabla|\nabla u|^2, \nabla u\rangle.\end{aligned}$$

Using Bochner's formula and Lemma 3, we obtain

$$(\Delta - \partial_t)|\nabla u|^2 = 2|\nabla^2 u|^2 + 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2h(\nabla u, \nabla u).$$

Therefore,

$$
\begin{aligned}
(\Delta-\partial_t)\left(\frac{|\nabla u|^2}{u^2}\right)
&\geq -2\langle\nabla\left(\frac{|\nabla u|^2}{u^2}\right), \nabla\log u\rangle + 2\frac{|\nabla^2 u|^2}{u^2} + 2\frac{|\nabla u|^4}{u^4} \\
&\qquad - \frac{4}{u^3}|\nabla^2 u|\,|\nabla u|^2 + \frac{2}{u^2}(\mathrm{Ric}+h)(\nabla u, \nabla u) \\
&\geq -2\langle\nabla\left(\frac{|\nabla u|^2}{u^2}\right), \nabla\log u\rangle + 2\delta\frac{|\nabla^2 u|^2}{u^2} - \frac{2\delta}{1-\delta}\frac{|\nabla u|^4}{u^4} - 2(n-1)(k_1+k_3)\frac{|\nabla u|^2}{u^2}.
\end{aligned}
$$

Here we have used Young's inequality to obtain

$$
\frac{4}{u^3}|\nabla^2 u|\,|\nabla u|^2 \leq 2(1-\delta)\frac{|\nabla^2 u|^2}{u^2} + \frac{2}{1-\delta}\frac{|\nabla u|^4}{u^4}.
$$

Thus we have

$$
\begin{aligned}
(\Delta-\partial_t)\left(\alpha\frac{|\nabla u|^2}{u^2}\right) &\geq -2\left\langle\nabla\left(\alpha\frac{|\nabla u|^2}{u^2}\right), \nabla\log u\right\rangle \\
&\qquad +2\delta\alpha\frac{|\nabla^2 u|^2}{u^2} - \frac{2\delta\alpha}{1-\delta}\frac{|\nabla u|^4}{u^4} - 2(n-1)(k_1+k_3)\alpha\frac{|\nabla u|^2}{u^2}.
\end{aligned}
\tag{3-3}
$$

Part 3. Finally, using (3-1) and Lemma 3, we get, for any $\varepsilon > 0$,

$$
\begin{aligned}
(\Delta-\partial_t)\left(\frac{u_t}{u}\right)
&= \frac{(\Delta-\partial_t)u_t}{u} - 2\left\langle\nabla\left(\frac{u_t}{u}\right), \nabla\log u\right\rangle \\
&= \frac{1}{u}[(\Delta u)_t + 2\langle h, \nabla^2 u\rangle + 2\langle \operatorname{div} h - \tfrac{1}{2}\nabla(\mathrm{tr}_g h), \nabla u\rangle - u_{tt}] - 2\left\langle\nabla\left(\frac{u_t}{u}\right), \nabla\log u\right\rangle \\
&= -2\left\langle\nabla\left(\frac{u_t}{u}\right), \nabla\log u\right\rangle + \frac{2}{u}\langle h, \nabla^2 u\rangle + \frac{2}{u}\langle \operatorname{div} h - \tfrac{1}{2}\nabla(\mathrm{tr}_g h), \nabla u\rangle \\
&\leq -2\left\langle\nabla\left(\frac{u_t}{u}\right), \nabla\log u\right\rangle + 2\sqrt{n}k_3\frac{|\nabla^2 u|}{u} + 3\sqrt{n}k_4\frac{|\nabla u|}{u} \\
&\leq -2\left\langle\nabla\left(\frac{u_t}{u}\right), \nabla\log u\right\rangle + \frac{1}{\varepsilon}\frac{|\nabla^2 u|^2}{u^2} + \varepsilon n k_3^2 + 3\sqrt{n}k_4\frac{|\nabla u|}{u}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(\Delta-\partial_t)\left(-\beta\frac{u_t}{u}\right) \\
\geq -2\left\langle\nabla\left(-\beta\frac{u_t}{u}\right), \nabla\log u\right\rangle - \frac{\beta}{\varepsilon}\frac{|\nabla^2 u|^2}{u^2} - \varepsilon n k_3^2\beta - 3\sqrt{n}k_4\beta\frac{|\nabla u|}{u}.
\end{aligned}
\tag{3-4}
$$

Combining the results from parts 1, 2, and 3, we obtain that for any $0 < \delta < 1$ and $\varepsilon > 0$,

$$(\Delta - \partial_t)F_1 \geq -2\langle \nabla F_1, \nabla \log u\rangle - C(k_1+k_3)\frac{|\nabla^2 u|}{u} + \left(\delta\alpha - \frac{\beta}{\varepsilon}\right)\frac{|\nabla^2 u|^2}{u^2} + \delta\alpha\frac{|\nabla^2 u|^2}{u^2} - \frac{2\delta\alpha}{1-\delta}\frac{|\nabla u|^4}{u^4} - 2(n-1)(k_1+k_3)\alpha\frac{|\nabla u|^2}{u^2} - 3\sqrt{n}k_4\beta\frac{|\nabla u|}{u} - \varepsilon n k_3^2\beta - C(k_2+k_4)\frac{|\nabla u|}{u}. \tag{3-5}$$

By the definition of F_1, we have

$$\frac{|\nabla^2 u|}{u} \leq F_1 + \beta\frac{u_t}{u}, \tag{3-6}$$

and

$$\begin{aligned}\frac{|\nabla^2 u|^2}{u^2} &= \left(F_1 - \alpha\frac{|\nabla u|^2}{u^2} + \beta\frac{u_t}{u}\right)^2 \\ &= F_1^2 + \alpha^2\frac{|\nabla u|^4}{u^4} + \beta^2\frac{u_t^2}{u^2} + 2\beta F_1\frac{u_t}{u} - 2\alpha F_1\frac{|\nabla u|^2}{u^2} - 2\alpha\beta\frac{|\nabla u|^2}{u^2}\frac{u_t}{u}.\end{aligned} \tag{3-7}$$

Inserting (3-6) and (3-7) into (3-5) and applying Young's inequality, we arrive at

$$\begin{aligned}(\Delta - \partial_t)F_1 \geq{}& -2\langle \nabla F_1, \nabla \log u\rangle - C(k_1+k_3)F_1 - \frac{C(k_1+k_3)^2}{4(\gamma-1)^2} \\ &+ \left[\tfrac{1}{2}\delta\alpha\beta^2 - C\beta^2(\gamma-1)^2\right]\frac{u_t^2}{u^2} + \left(\delta\alpha - \frac{\beta}{\varepsilon}\right)\frac{|\nabla^2 u|^2}{u^2} \\ &- \left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\frac{|\nabla u|^4}{u^4} + \delta\alpha F_1^2 + 2\delta\alpha\beta F_1\frac{u_t}{u} \\ &- 2\delta\alpha^2 F_1\frac{|\nabla u|^2}{u^2} - \varepsilon n k_3^2\beta - \frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2 \\ &- \frac{C\beta^{4/3}}{\delta^{1/3}\alpha}k_4^{4/3} - \frac{C}{\delta^{1/3}\alpha}(k_2+k_4)^{4/3},\end{aligned} \tag{3-8}$$

for any $\gamma - 1 > 0$.

Using the inequality

$$\left(\frac{|\nabla u|^2}{u^2}\right)^2 \leq 2\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)^2 + 2\gamma^2\frac{u_t^2}{u^2},$$

we calculate

$$\left[\tfrac{1}{2}\delta\alpha\beta^2 - C\beta^2(\gamma-1)^2\right]\frac{u_t^2}{u^2} - \left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{|\nabla u|^2}{u^2}\right)^2$$
$$\geq \left[\tfrac{1}{2}\delta\alpha\beta^2 - 2\gamma^2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right) - C\beta^2(\gamma-1)^2\right]\frac{u_t^2}{u^2}$$
$$- 2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)^2.$$

Setting $\beta = 5\alpha$, we check that

$$\tfrac{1}{2}\delta\alpha\beta^2 - 2\gamma^2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right) - C\beta^2(\gamma-1)^2$$
$$= 8\delta\alpha^3\left(\tfrac{25}{16} - \gamma^2\right) - \frac{\delta}{\alpha(1-\delta)^2}\gamma^2 - C\beta^2(\gamma-1)^2,$$

which is nonnegative when $\delta > 0$, $\gamma - 1 > 0$ are sufficiently small.

Now we take $\varepsilon \geq 5/\delta$ such that $\delta\alpha - \beta/\varepsilon \geq 0$. Then (3-8) becomes

$$(\Delta - \partial_t)F_1 \geq -2\langle\nabla F_1, \nabla\log u\rangle - C(k_1+k_3)F_1 - \frac{C(k_1+k_3)^2}{4(\gamma-1)^2}$$
$$- 2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)^2 + \delta\alpha F_1^2$$
$$+ 2\delta\alpha\beta F_1\frac{u_t}{u} - 2\delta\alpha^2 F_1\frac{|\nabla u|^2}{u^2} - \varepsilon n k_3^2\beta - \frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2$$
$$- \frac{C\beta^{4/3}}{\delta^{1/3}\alpha}k_4^{4/3} - \frac{C}{\delta^{1/3}\alpha}(k_2+k_4)^{4/3}.$$

Applying Corollary 5 and noting that

$$(\Delta - \partial_t)F = t(\Delta - \partial_t)F_1 - F_1, \tag{3-9}$$

we complete the proof of the lemma. □

Proof of Theorem 9. As in the proof of Theorem 1, we see that $(M, g(t))$ is complete for $t \in [0, T]$. Let $\rho(x,t) = d(x, x_0, t)$ and

$$\varphi(x,t) = \psi\left(\frac{\rho(x,t)}{R}\right).$$

Set

$$\varphi F(x,t) := \psi\left(\frac{\rho(x,t)}{R}\right)F(x,t),$$

where $(x,t) \in Q_{2R,T}$. Suppose (x_1, t_1) is the point where φF achieves its maximum in Q_{2R,T_1}, where $0 < T_1 \leq T$.

If $|\nabla^2 u(x_1,t_1)|=0$, Corollary 5 yields

$$(\varphi F)(x_1,t_1)=\varphi t_1\left(\alpha\frac{|\nabla u|^2}{u^2}-\beta\frac{u_t}{u}\right)\le C_{n,\alpha,\beta}\big((k_1+k_3+k_4+\sqrt{k_4})t_1+1\big), \tag{3-10}$$

which implies the result.

Using arguments from [Calabi 1958; Li 1991], we can assume φF to be smooth at (x_1,t_1) and $\varphi F(x_1,t_1)>0$.

As in the proof of Theorem 1, using Lemma 10, we obtain at the point (x_1,t_1)

$$\begin{aligned}0&\ge(\Delta-\partial_t)(\varphi F)\\&\ge\left(-\frac{C}{R^2}-\frac{C}{R}\sqrt{k_1}\right)F-Ck_1F+\varphi(\Delta-\partial_t)F\\&\ge\left(-\frac{C}{R^2}-\frac{C}{R}\sqrt{k_1}\right)F-Ck_1F-\frac{F|\nabla\varphi|^2}{2s\varphi}-2Fs\varphi\frac{|\nabla u|^2}{u^2}\\&\quad+\frac{\delta\alpha}{t_1}\varphi F^2+2\delta\alpha\beta\varphi F\frac{u_t}{u}-2\delta\alpha^2\varphi F\frac{|\nabla u|^2}{u^2}-C(k_1+k_3)\varphi F\\&\quad-\frac{C(k_1+k_3)^2}{4(\gamma-1)^2}\varphi t_1-\varepsilon nk_3^2\beta\varphi t_1-\frac{\varphi F}{t_1}-\frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2\varphi t_1\\&\quad-\frac{C\beta^{4/3}}{\delta^{1/3}\alpha}k_4^{4/3}\varphi t_1-\frac{C}{\delta^{1/3}\alpha}(k_2+k_4)^{4/3}\varphi t_1\\&\quad-2C\varphi t_1\left(4\delta\alpha^3+\frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\right)^2.\end{aligned}\tag{3-11}$$

Using Corollary 5, we have

$$\begin{aligned}&2\delta\alpha\beta\varphi F\frac{u_t}{u}-2\delta\alpha^2\varphi F\frac{|\nabla u|^2}{u^2}-2Fs\varphi\frac{|\nabla u|^2}{u^2}\\&\quad\ge(2\delta\alpha\beta-2s\gamma-2\delta\alpha^2\gamma)\varphi F\frac{u_t}{u}-C(2\delta\alpha^2+2s)\varphi F\left(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\right).\end{aligned}$$

Observe that $2\delta\alpha\beta-2s\gamma-2\delta\alpha^2\gamma=0$ when we set $s=\delta\alpha^2\left(\frac{5}{\gamma}-1\right)$. Then (3-11) becomes

$$\begin{aligned}0&\ge\frac{\delta\alpha}{t_1}\varphi F^2-\varphi FC(2s+2\delta\alpha^2)\left(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\right)\\&\quad+\left(-\frac{C}{R^2}-\frac{C}{R}\sqrt{k_1}\right)F-\frac{CF}{2sR^2}-Ck_1F-C(k_1+k_3)\varphi F-\frac{\varphi F}{t_1}\\&\quad-\frac{C(k_1+k_3)^2}{4(\gamma-1)^2}\varphi t_1-\varepsilon nk_3^2\beta\varphi t_1-\frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2\varphi t_1-\frac{C\beta^{4/3}}{\delta^{1/3}\alpha}\\&\quad-2C\varphi t_1\left(4\delta\alpha^3+\frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\right)^2.\end{aligned}\tag{3-12}$$

Multiplying through by φt_1 and using $0 \le \varphi \le 1$, we have

$$\begin{aligned}0 \ge \delta\alpha(\varphi F)^2 &- (\varphi F)\biggl(\Bigl(\frac{C}{R^2}+\frac{C}{R}\sqrt{k_1}\Bigr)t_1+\frac{10C\delta\alpha^2}{\gamma}\Bigl(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\Bigr)t_1\biggr)\\ &-(\varphi F)\biggl(\frac{Ct_1}{2\delta\alpha^2\bigl(\frac{5}{\gamma}-1\bigr)R^2}+(1+C(k_1+k_3)t_1)\biggr)-\frac{C(k_1+k_3)^2}{4(\gamma-1)^2}t_1^2-\varepsilon n k_3^2\beta t_1^2\\ &-\frac{2(n-1)^2}{\delta\alpha}(k_1+k_3)^2t_1^2-\frac{C\beta^{4/3}}{\delta^{1/3}\alpha}k_4^{4/3}t_1^2-\frac{C}{\delta^{1/3}\alpha}(k_2+k_4)^{4/3}t_1^2\\ &-2C\Bigl(4\delta\alpha^3+\frac{\delta}{2\alpha(1-\delta)^2}\Bigr)\Bigl(\frac{1}{t_1}+k_1+k_3+k_4+\sqrt{k_4}\Bigr)^2t_1^2.\end{aligned}$$

Solving this quadratic inequality, one obtains

$$\varphi F(x_1,t_1) \le C\left(1+(k_1+k_2^{2/3}+k_3+k_4+\sqrt{k_4})t_1+\frac{1}{R^2}t_1\right).$$

By a similar argument as that in the proof of Theorem 1, we conclude in $Q_{R,T}$,

$$F_1(x,t) \le C\left(k_1+k_2^{2/3}+k_3+k_4+\sqrt{k_4}+\frac{1}{t}+\frac{1}{R^2}\right).$$

where C depends on n and α. Because M is noncompact, we can let $R \to +\infty$. This completes the proof of Theorem 9. □

4. Applications to Ricci flow and mean curvature flow

In this section, we apply our results to the special cases of the Ricci flow and the mean curvature flow.

4.1. ***The Ricci flow.*** When $h = -\mathrm{Ric}$, (2-1) is the Ricci flow equation introduced in [Hamilton 1982]. In this situation our results reduced to those in [Liu 2009]. In this case our results in Section 2 do not need the assumption $|\nabla \mathrm{Ric}| \le K_4$, because of the second contracted Bianchi identity. Indeed, checking the proof of Theorem 1 carefully, we find that we need the bound on $|\nabla h|$ because we want to control the term

$$\operatorname{div} h - \tfrac{1}{2}\nabla(\mathrm{tr}_g h)$$

in (2-5). But the contracted second Bianchi identity says that when $h = -\mathrm{Ric}$,

$$\operatorname{div}\mathrm{Ric} - \tfrac{1}{2}\nabla R = 0.$$

Now (2-5) becomes

$$\frac{\partial}{\partial t}\Delta f = \Delta\frac{\partial}{\partial t} f + 2\langle \mathrm{Ric}, \nabla^2 f\rangle.$$

4.2. ***The mean curvature flow.*** Let M be an n-dimensional closed smooth submanifold in N^{n+p}. Given an embedding $F_0 : M \to N$, we consider a one-parameter family of smooth maps $F_t = F(\cdot, t) : M \to N^{n+p}$ with corresponding images $M_t = F_t(M)$, where F satisfies the mean curvature flow equation

$$\frac{d}{dt} F(x,t) = \boldsymbol{H}(x,t), \quad F(x,0) = F_0(x). \tag{4-1}$$

Here $\boldsymbol{H}(x,t)$ is the mean curvature vector of $M_t = F_t(M)$ at $F(x,t)$ in N.

It is easy to check (or see [Huisken 1984; Chen and Li 2001]) that the induced metric on M_t evolves by

$$\frac{\partial}{\partial t} g_{ij} = -2H^\alpha A^\alpha_{ij},$$

where $\{A^\alpha_{ij}\}$ is the second fundamental form of M_t in N. In this case, the tensor h in (2-1) becomes

$$h_{ij} = -H^\alpha A^\alpha_{ij}. \tag{4-2}$$

In this section, we will always assume that

$$|K_N| + |\nabla K_N| + |\nabla^2 K_N| \le L,$$

for some constant L. Here, K_N is the curvature tensor of N.

Using the evolution of the second fundamental form and the standard maximum principle, we can obtain a derivative estimate:

Proposition 11. *Let $\{M_t\}_{0 \le t \le T}$ be a closed smooth solution of mean curvature flow in a Riemannian manifold N. Suppose that there exist constants Λ_0 and Λ_1 such that*

$$|A| \le \Lambda_0 \quad \text{on } M \times [0,T],$$
$$|\nabla A| \le \Lambda_1 \quad \text{on } M_0.$$

Then there is a constant K depending only on n, Λ_0, Λ_1 and L such that

$$|\nabla A| \le K \quad \text{on } M \times [0,T].$$

Remark 12. Another version of derivative estimate for the second fundamental form along the mean curvature flow, similar to the derivative estimate along the Ricci flow given in [Shi 1989], is proved in [Han and Sun 2012].

Proof of Proposition 11. Our proof follows [Huisken 1990]. C_i will denote various constants depending only on n, Λ_0 and L. By our assumption and the evolution equations of the second fundamental form and its derivative [Han and Sun 2012, Corollary 3.5], we have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)|A|^2 \le -2|\nabla A|^2 + C_1 \tag{4-3}$$

and

(4-4) $$\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla A|^2 \le -2|\nabla^2 A|^2 + C_2(|\nabla A|^2 + 1).$$

Therefore,

(4-5) $$\Big(\frac{\partial}{\partial t} - \Delta\Big)\big(|\nabla A|^2 + C_2|A|^2\big) \le -C_2\big(|\nabla A|^2 + C_2|A|^2\big) + C_3.$$

As M_t is closed for each t, we obtain by the maximum principle that

$$\big(|\nabla A|^2 + C_2|A|^2\big)(x,t) \le e^{-C_2 t}\sup_{M_0}\big(|\nabla A|^2 + C_2|A|^2\big) + \frac{C_3}{C_2}(1 - e^{-C_2 t}) \le K,$$

where K depends on n, Λ_0, Λ_1 and L. This proves the proposition. □

Theorem 13. *Let $\{M_t\}_{0\le t\le T}$ be a closed smooth solution of mean curvature flow in a Riemannian manifold N. Suppose that there exist constants Λ_0 and Λ_1 such that*

$$|A| \le \Lambda_0 \quad \text{on } M \times [0,T],$$
$$|\nabla A| \le \Lambda_1 \quad \text{on } M_0.$$

If u is a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$, then for $(x,t) \in M \times (0,T]$, we have

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \le \frac{n\alpha^2}{t} + CK,$$

for any $\alpha > 1$. Here C depends on n, α only and K depends Λ_0, Λ_1 and L.

Proof. During the proof of this theorem, the constant K will denote a constant depending only on Λ_0, Λ_1 and L which may vary from one line to the next.

By Proposition 11 and (4-2), we see that $|h| \le K$ and $|\nabla h| \le K$. On the other hand, using the Gauss equation, we have

$$R_{ijkl} - K_{ijkl} = A^\alpha_{ik}A^\alpha_{jl} - A^\alpha_{il}A^\alpha_{jk},$$

where K_{ijkl} is the curvature tensor on N. Hence our assumption and Proposition 11 imply that $|Rm| \le K$ and $|\nabla Rm| \le K$. This shows that all the assumptions of Theorem 6 are satisfied, and the conclusion follows. □

Remark 14. K. Smoczyk [1999] proved a similar gradient estimate for the positive solution of the heat equation along the Lagrangian mean curvature flow, and obtained a Harnack inequality for the Lagrangian angle. Indeed, under the Lagrangian mean curvature flow, the Lagrangian angle evolves by $(\Delta - \partial_t)\theta = 0$.

Next we deal with the complete noncompact case.

Proposition 15. *Let $\{M_t\}_{0\le t\le T}$ be a smooth solution of mean curvature flow in a Riemannian manifold N. Assume that M_0 is complete noncompact without boundary. Suppose that there exist two constants Λ_0 and Λ_1 such that*

$$|A|\le\Lambda_0 \quad on\ M\times[0,T],$$
$$|\nabla A|\le\Lambda_1 \quad on\ M_0.$$

Then there is a constant K depending only on n, Λ_0, Λ_1 and L such that

$$|\nabla A|\le K \quad on\ M\times[0,T].$$

Proof. Recall that in the proof of Proposition 11, we obtained

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\big(|\nabla A|^2+C_2|A|^2\big)\le -C_2\big(|\nabla A|^2+C_2|A|^2\big)+C_3.$$

Set $F=|\nabla A|^2+C_2|A|^2$. Then

$$(\Delta-\frac{\partial}{\partial t})F\ge C_2F-C_3. \tag{4-6}$$

Let $Q_{R,T}$ and φ be defined as in the proof of Theorem 1. We consider φF. Suppose

$$(\varphi F)(x_1,t_1)=\sup_{M\times[0,T]}(\varphi F).$$

Then $(x_1,t_1)\in Q_{2R,T}$. We consider two cases.

- If $t_1=0$, then $(\varphi F)(x,t)\le(\varphi F)(x_1,0)\le F(x_1,0)\le\sup_{M_0}F\le\Lambda_1+C_2\Lambda_0\le K$. In particular, for any $(x,t)\in Q_{R,T}$, we have

$$F(x,t)\le K.$$

- If, on the contrary, $t_1>0$, the maximum principle gives

$$\Big(\Delta-\frac{\partial}{\partial t}\Big)(\varphi F)(x_1,t_1)\le 0.$$

By the Gauss equation and our assumption, the Ricci curvature is bounded. Similarly to the proof of Theorem 1 and using (4-6), we get that, at (x_1,t_1),

$$0\ge\Big(\Delta-\frac{\partial}{\partial t}\Big)(\varphi F)(x_1,t_1)\ge-\frac{C}{R^2}-\frac{C\sqrt{K}}{R}+C_2\varphi F-C_3.$$

Thus we obtain that

$$(\varphi F)(x_1,t_1)\le\frac{C_3}{C_2}+\frac{C}{C_2R^2}+\frac{C\sqrt{K}}{C_2R}.$$

In particular, for any $(x,t)\in Q_{R,T}$, we have

$$F(x,t)\le\frac{C_3}{C_2}+\frac{C}{C_2R^2}+\frac{C\sqrt{K}}{C_2R}.$$

Combining the two cases above and letting $R \to \infty$ we get $F(x, t) \le K$, for some constant K depends on n, Λ_0, Λ_1 and L. This proves the proposition. □

Arguing as in the proof of Theorem 13 and using Corollary 5 and Theorem 9, we obtain

Theorem 16. *Let $\{M_t\}_{0\le t\le T}$ be a smooth solution of mean curvature flow in a Riemannian manifold N. Assume that M_0 is complete noncompact without boundary. Suppose that there exist two constants Λ_0 and Λ_1 such that*

$$|A| \le \Lambda_0 \quad \text{on } M \times [0, T],$$
$$|\nabla A| \le \Lambda_1 \quad \text{on } M_0.$$

If u is a positive solution to the equation $(\Delta - \partial_t)u(x, t) = 0$, then for $(x, t) \in M \times (0, T]$ we have

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} \le \frac{n\alpha^2}{t} + CK$$

and

$$\frac{|\nabla^2 u(x,t)|}{u(x,t)} + \alpha \frac{|\nabla u(x,t)|^2}{u^2(x,t)} - 5\alpha \frac{u_t(x,t)}{u(x,t)} \le C\Big(K + \frac{1}{t}\Big),$$

for any $\alpha > 1$. Here C depends only on n, α and K depends on n, Λ_0, Λ_1 and L.

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