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A NOTE ON p-HARMONIC l-FORMS ON COMPLETE MANIFOLDS

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Let (M^m, g) be an m-dimensional complete noncompact manifold. We show that for all $p > 1$ and $l > 1$, any bounded set of p-harmonic l-forms in $L^q(M)$, with $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology if the curvature operator of M is asymptotically nonnegative.

1. Introduction

Let (M^m, g) be an m-dimensional complete oriented Riemannian manifold with associated Riemannian metric g. Let d be the exterior differential operator and let

$$\delta \equiv * d *$$

be the codifferential operator, where the linear operator $*$ is defined pointwise by

$$*(\omega_1 \wedge \cdots \wedge \omega_l) \equiv \omega_{l+1} \wedge \cdots \wedge \omega_m,$$

for a positively oriented orthonormal coframe $\{\omega_1, \omega_2, \ldots, \omega_m\}$ at the point. The Hodge–Laplace–Beltrami operator $\triangle$ acting on the space of smooth l-forms $\Lambda^l(M)$ is defined by

$$\triangle \equiv -(d\delta + \delta d).$$

Definition 1.1. An l-form ω on M is a p-harmonic l-form if ω satisfies $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$ for all $p > 1$.

When $p = 2$, the p-harmonic l-form $\omega \in \Lambda^l(M)$ is called a harmonic l-form on (M, g), that is,

$$\triangle_g \omega = 0.$$

When $l = 0$, let Ω be a compact domain on the Riemannian manifold (M, g), and let ω be a real smooth function on M. For $p > 1$, the p-energy of ω on Ω is

$$E_p(\Omega, \omega) \equiv \frac{1}{p} \int_\Omega |\nabla\omega|^p \, dV_g.$$

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The function ω is said to be p-harmonic on M if ω is a critical point of $E_p(\Omega, \cdot)$ for all $\Omega \subset M$, that is, if ω satisfies the Euler–Lagrange equation

$$\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega) = 0.$$

A curvature operator K_l on manifold M^m is defined as follows:

$$K_l = \begin{cases} \text{lower bound of the curvature operator on } M & \text{for } l > 1; \\ (m-1)^{-1} \times (\text{lower bound of the Ricci curvature}) & \text{for } l = 1. \end{cases}$$

We call this curvature operator K_l of M asymptotically nonnegative if $K_l \geq -K(r)$, where

$$K(r) : [0, \infty) \to [0, \infty)$$

is a nonnegative and nonincreasing continuous function of distance r to a fixed point $z \in M$, with

$$\int_0^\infty r K(r) < \infty.$$

Yau [1975] proved that any positive harmonic function on a manifold with nonnegative Ricci curvature must be constant. Much work has been done in the finite dimension of space of polynomial growth harmonic functions of growth order at most d [Li 1997; Colding and Minicozzi 1997; Li and Tam 1995; Li and Wang 1999]. Concerning general harmonic l-forms, Li [1980] established a dimension estimate of the space of polynomial growth harmonic forms. In this paper, we study general p-harmonic l-forms and p-harmonic maps on complete noncompact manifolds, for $p > 1$ and $l \neq 0$. For $p = 2$, Chen and Sung [2007] considered the space consisting of all harmonic l-forms of polynomial growth for all $l \geq 1$, and gave a dimension estimate of such a space when M has asymptotically nonnegative curvature. Since the set of p-harmonic l-forms is no longer linear, it is interesting to study the set of p-harmonic l-forms and to seek topological and geometrical links. Interestingly, Zhang [2001] proved that any $L^q(M)$ p-harmonic 1-forms must be zero on a manifold with nonnegative Ricci curvature for $p > 1$ and $0 < q < \infty$. Chang et al. [2010] generalized Zhang's result to a complete manifold M with asymptotically nonnegative curvature and finite first Betti number. They proved that a bounded set of $L^q(M)$ p-harmonic 1-forms on (M, g) has a uniformly convergent subsequence.

Next we introduce the Sobolev inequality. A geodesic ball $B_x(r)$ in a complete manifold M is said to admit a Sobolev inequality $S(C, \nu)$ if there exist constants $C > 0$ and $\nu > 2$ such that for all $f \in C_0^\infty(B_x(r))$, we have

$$\left(\int_{B_x(r)} |f|^{2\nu/(\nu-2)}\right)^{(\nu-2)/\nu} \leq C r^2 V_x^{-2/\nu}(r) \int_{B_x(r)} (|\nabla f|^2 + r^{-2} f^2),$$

where $V_x(r)$ is the volume of geodesic ball $B_x(r)$. Using the Bochner formula, the Moser iteration [1961] and the Sobolev inequality, Chang et al. [2010] showed that any bounded set of p-harmonic 1-forms in $L^q(M)$, with $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology if M has asymptotically nonnegative Ricci curvature and finite first Betti number. However, the Bochner formula does not work for p-harmonic l-forms for $l > 1$. We derive a new type of Bochner formula to overcome this obstacle. We study the set of p-harmonic l-forms, for $l > 1$, on a complete noncompact manifold M, and then study the set of p-harmonic maps from a complete manifold M to a complete manifold N. In Section 2, we derive a different type of Bochner formula for p-harmonic l-forms and prove that any bounded set of p-harmonic l-forms in $L^q(M)$, with $0 < q < \infty$, must be relatively compact with respect to the uniform convergence topology if the curvature operator of M is asymptotically nonnegative. Of course, this implies that the linear space of harmonic l-forms must be finite-dimensional when $p = 2$ and $l \geq 0$. Also, there is no nonzero p-harmonic l-form on M in $L^q(M)$ if the curvature operator of M is nonnegative. In Section 3, we also derive a different type of Bochner formula for p-harmonic maps from M with asymptotically nonnegative Ricci curvature to N with nonpositive sectional curvature. We prove that the set of such p-harmonic maps with finite p-energy on M has a uniformly convergent subsequence. The p-harmonic map is constant if M is compact with nonnegative Ricci curvature, which is an extension of the fact in the harmonic map case ($p = 2$).

2. p-harmonic l-forms

Any smooth l-form on an m-dimensional manifold M satisfies the Kato inequality:

Lemma 2.1 [Wan and Xin 2004; Calderbank et al. 2000; Herzlich 2000]. *Let ω be a differentiable l-form on M. Then*

$$\big|\nabla|\omega|^2\big| \leq 2|\omega|\,|\nabla\omega|.$$

Lemma 2.2 [Bochner 1946]. *Let $\omega = \sum_I a_I\, \omega_I$ be an l-form on M. Then*

$$\Delta|\omega|^2 = 2\langle\Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2K_l\langle\omega, \omega\rangle.$$

Let (M, g) be a complete noncompact manifold. We wish to study the set of L^q p-harmonic l-forms on M for $l > 1$ and $0 < q < \infty$. To prove the main theorem for all $l > 1$, we show a different type of Bochner formula for p-harmonic l-forms:

Lemma 2.3 (Bochner-type formula for p-harmonic forms). *Let ω be a p-harmonic l-form on an m-dimensional complete Riemannian M^m. Then*

$$|\omega|\Delta|\omega|^{p-1} = \big\langle\Delta(|\omega|^{p-2}\omega), \omega\big\rangle + |\omega|^{2-p}\Big(\big|\nabla(|\omega|^{p-2}\omega)\big|^2 - \big|\nabla|\omega|^{p-1}\big|^2\Big) + K_l|\omega|^p,$$

in the sense of distributions.

Proof. The Bochner–Weitzenböck formula for $|\omega|^{p-2}\omega$ asserts that

$$\tag{2-1} \tfrac{1}{2}\Delta\big||\omega|^{p-2}\omega\big|^2 = \big\langle\Delta(|\omega|^{p-2}\omega),\, |\omega|^{p-2}\omega\big\rangle + \big|\nabla(|\omega|^{p-2}\omega)\big|^2 + K_l\big||\omega|^{p-2}\omega\big|^2.$$

The left side of (2-1) is given by

$$\tfrac{1}{2}\Delta\big||\omega|^{p-2}\omega\big|^2 = \tfrac{1}{2}\Delta|\omega|^{2p-2} = \tfrac{1}{2}\Delta\big(|\omega|^{p-1}\big)^2 = |\omega|^{p-1}\,\Delta|\omega|^{p-1} + \big|\nabla|\omega|^{p-1}\big|^2.$$

Hence,

$$|\omega|^{p-1}\,\Delta|\omega|^{p-1} + \big|\nabla|\omega|^{p-1}\big|^2 = \big\langle\Delta(|\omega|^{p-2}\omega),\, |\omega|^{p-2}\omega\big\rangle + \big|\nabla(|\omega|^{p-2}\omega)\big|^2 + K_l\,|\omega|^{2p-4}\,|\omega|^2.$$

It follows that

$$|\omega|^{p-1}\,\Delta|\omega|^{p-1} = |\omega|^{p-2}\big\langle\Delta(|\omega|^{p-2}\omega),\, \omega\big\rangle + \Big(\big|\nabla(|\omega|^{p-2}\omega)\big|^2 - \big|\nabla|\omega|^{p-1}\big|^2\Big) + K_l\,|\omega|^{2p-2}. \qquad \square$$

For l-forms with $l > 1$, the volume comparison property holds on M with asymptotically nonnegative curvature operator [Li and Tam 1995]. Therefore, inside geodesic ball $B_x(R)$ with $r(x) = 2R$, the volume doubling property holds [Li and Tam 1995]. Also, by [Saloff-Coste 1992], a local weak Poincaré inequality holds on geodesic ball $B_x(R)$, and hence we have the Sobolev inequality $S(C, \nu)$ on $B_x(R)$ [Hajłasz and Koskela 1995]; that is, there exists a real number $\nu > 2$ such that

$$\bigg(\int_{B_x(R)} |f|^{2\nu/(\nu-2)}\,dV\bigg)^{(\nu-2)/\nu} \le C\cdot r^2\cdot V^{-2/\nu}(B)\int_{B_x(R)} |\nabla f|^2\,dV,$$

for all $f \in C_0^\infty(B_x(r))$, where $r \le R$.

Theorem 2.4 (main theorem). *Let M^m be an m-dimensional complete Riemannian manifold with asymptotically nonnegative curvature operator K_l, for $l > 1$. Then a bounded set of $L^q(M)$ p-harmonic l-forms on (M^m, g) has a uniformly convergent subsequence, for $1 < p < \infty$ and $0 < q < \infty$.*

Proof. Let ω be a p-harmonic l-form on M^m. Lemma 2.3 asserts that

$$|\omega|^{p-1}\,\Delta|\omega|^{p-1} = |\omega|^{p-2}\big\langle\Delta(|\omega|^{p-2}\omega),\, \omega\big\rangle + \Big(\big|\nabla(|\omega|^{p-2}\omega)\big|^2 - \big|\nabla|\omega|^{p-1}\big|^2\Big) + K_l|\omega|^{2p-2}.$$

By the Kato inequality, we have

$$\big|\nabla|\omega|^{p-1}\big| = \big|\nabla\big||\omega|^{p-2}\omega\big|\big| \le \big|\nabla(|\omega|^{p-2}\omega)\big|.$$

Therefore,

$$|\omega|^{p-1}\Delta|\omega|^{p-1} \ge |\omega|^{p-2}\big\langle\Delta(|\omega|^{p-2}\omega),\,\omega\big\rangle - K(R)|\omega|^{2p-2},$$

where $-K(R)$ is the pointwise lower bound of the curvature operator. Let η be a compactly supported nonnegative smooth function on M.

$$\begin{aligned}\int_M \eta^2|\omega|^{p-1}\Delta|\omega|^{p-1} &\ge \int_M \eta^2\,|\omega|^{p-2}\big\langle\Delta(|\omega|^{p-2}\omega),\,\omega\big\rangle - K(R)\int_M \eta^2|\omega|^{2p-2}\\ &= \int_M \eta^2|\omega|^{p-2}\big\langle\delta d(|\omega|^{p-2}\omega),\,\omega\big\rangle - K(R)\int_M \eta^2|\omega|^{2p-2}\\ &= -K(R)\int_M \eta^2|\omega|^{2p-2}.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}K(R)\int_M \eta^2|\omega|^{2p-2}&\\ &\ge \int_M \nabla(\eta^2|\omega|^{p-1})\cdot\nabla|\omega|^{p-1}\\ &\ge \frac{(p-1)^2}{4}\int_M \eta^2|\omega|^{2p-6}\big|\nabla|\omega|^2\big|^2 - (p-1)\int_M \eta|\nabla\eta||\omega|^{2p-4}\big|\nabla|\omega|^2\big|.\end{aligned}$$

It follows that

$$\begin{aligned}\text{(2-2)}\qquad \frac{(p-1)^2}{4}\int_M \eta^2|\omega|^{2p-6}\big|\nabla|\omega|^2\big|^2&\\ &\le (p-1)\int_M \eta|\nabla\eta||\omega|^{2p-4}\big|\nabla|\omega|^2\big| + K(R)\int_M \eta^2|\omega|^{2p-2},\end{aligned}$$

for all $p > 1$.

By Young's inequality, we have

$$(p-1)\eta|\nabla\eta||\omega|^{2p-4}\big|\nabla|\omega|^2\big| \le \frac{(p-1)^2}{8}\eta^2|\omega|^{2p-6}\big|\nabla|\omega|^2\big|^2 + 2|\nabla\eta|^2|\omega|^{2p-2}.$$

Since

$$|\omega|^{2p-6}\big|\nabla|\omega|^2\big|^2 = \frac{4}{(p-1)^2}\big|\nabla|\omega|^{p-1}\big|^2,$$

then (2-2) can be written as

$$\text{(2-3)}\qquad \int_M \eta^2\big|\nabla|\omega|^{p-1}\big|^2 \le 4\int_M |\nabla\eta|^2|\omega|^{2p-2} + 2K(R)\int_M \eta^2|\omega|^{2p-2},$$

for all $p > 1$.

For $R > 0$ and $x \in \partial B_z(2R)$, let $\eta \in \mathscr{C}_0^\infty(B_x(R))$ be a cut-off function satisfying

$$\eta(y) = \begin{cases}1 & \text{if } y \in B_x(\rho R),\\ 0 & \text{if } y \in M\setminus B_x(\gamma R).\end{cases}$$

Note that $\eta \in [0, 1]$ on M and $|\nabla\eta| \le 2/((\gamma-\rho)R)$, for $0<\rho<\gamma\le 1$.

By the Sobolev inequality and (2-3),

$$\Big(\int_{B_x(\rho R)} (|\omega|^{p-1})^{2\alpha}\Big)^{1/\alpha} \le \Big(\int_{B_x(\gamma R)} (\eta|\omega|^{p-1})^{2\alpha}\Big)^{1/\alpha}$$
$$\le c_s(\nu) V_x(R)^{-2/\nu} R^2\, 16\Big(\frac{1}{(\gamma-\rho)^2 R^2} + K(R)\Big)\int_{B_x(\gamma R)} |\omega|^{2p-2},$$

where $\alpha = \nu/(\nu-2)$, and $c_s(\nu)$ is the Sobolev constant.

By the assumption on function $K(R)$, it is easy to see that

$$K(R) \le \frac{c}{R^2}$$

on ball $B_x(R)$. Therefore,

$$\Big(\int_{B_x(\rho R)} |\omega|^{2(p-1)\cdot\alpha}\Big)^{1/\alpha} \le c_s(\nu) V_x(R)^{-2/\nu} 4^2\Big(\frac{1}{(\gamma-\rho)^2}\Big)\int_{B_x(\gamma R)} |\omega|^{2(p-1)}, \tag{2-4}$$

where $\alpha = \nu/(\nu-2)$.

Define

$$p = q_0\,\alpha^i + 1 \quad \text{and} \quad R_i = (\rho + 2^{-i}(\gamma-\rho))R,$$

for $i = 0, 1, 2, 3, \ldots$. Observe that $\lim_{i\to\infty} R_i = \rho R$. Let $\rho R = R_{i+1}$ and $\gamma R = R_i$ in inequality (2-4) and iterate the inequality; then

$$\sup_{B_x(\rho R)} |\omega|^{2q_0} \le C V_x(R)^{-1}\Big(\frac{1}{\gamma-\rho}\Big)^{\nu}\int_{B_x(\gamma R)} |\omega|^{2q_0}. \tag{2-5}$$

When $q \ge 2q_0$, by (2-5), we have

$$|\omega|(x) \le C\Big(V_x(R)^{-1}\int_{B_x(R)} |\omega|^q\Big)^{1/q},$$

for some constant C.

When $0 < q < 2q_0$, let $h_i = \sum_{j=1}^{i+1} 2^{-j}$, $\rho = h_i$, and $\gamma = h_{i+1}$, for all $i = 0, 1, 2, 3 \ldots$. By (2-5), we have

$$\sup_{B_x(h_i R)} |\omega|^{2q_0} \le C V_x(R)^{-1} 2^{(i+2)\nu}\int_{B_x(h_{i+1}R)} |\omega|^q \cdot \sup_{B_x(h_{i+1}R)} |\omega|^{2q_0-q}. \tag{2-6}$$

Write $M(i) = \sup_{B_x(h_i R)} |\omega|^{2q_0}$. Inequality (2-6) becomes

$$M(i) \le C V_x(R)^{-1} 2^{(i+2)\nu}\int_{B_x(R)} |\omega|^q M(i+1)^{(2q_0-q)/2q_0}. \tag{2-7}$$

Let $\lambda = 1 - q/2q_0 \in (0, 1)$; iterating inequality (2-7), we have

$$M(0) \le \prod_{i=0}^{j-1} \tilde{c}^{\lambda^i} M^{\lambda^j}(j) = \prod_{i=0}^{j-1} \left(C V_x(R)^{-1} 2^{\nu(i+1)} \int_{B_x(R)} |\omega|^q \right)^{\lambda^i} M^{\lambda^j}(j).$$

Let $j \to \infty$; we have

$$M(0) \le (C)^{2q_0/q} V_x(R)^{-2q_0/q} \left(\int_{B_x(R)} |\omega|^q \right)^{2q_0/q}.$$

Hence,

$$|\omega|(x) \le (C)^{1/q} V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q} \le C V_x(R)^{-1/q} \left(\int_{B_x(R)} |\omega|^q \right)^{1/q},$$

for some constant C.

For ω a p-harmonic l-form on M, and $x \in \partial B_z(2R)$, we have

$$|\omega|(x) \le C \left(V_x(R)^{-1} \int_{B_x(R)} |\omega|^q \right)^{1/q}.$$

When the $L^q(M)$ norm of ω is assumed to be bounded by a fixed constant, since we also have $V_x(R) \ge cR$, we conclude that for any given $\epsilon > 0$, by taking R to be sufficiently large, $|\omega| < \epsilon$ on $M \setminus B_z(R)$. On the other hand, using the standard elliptic PDE theory, on ball $B_z(R)$, the length of ω and all its covariant derivatives can be bounded by the $L^q(M)$ norm of ω. In particular, we conclude that any bounded sequence of such ω admits a uniformly convergent subsequence on M. This finishes the proof of the theorem. □

An immediate corollary is obtained from the proof of Theorem 2.4.

Corollary 2.5. *Let (M^m, g) be a complete noncompact manifold with nonnegative curvature operator. Then any bounded $L^q(M)$ p-harmonic l-forms on (M, g) must be zero.*

3. p-Harmonic maps

Here we derive a different type of Bochner formula for p-harmonic maps and study the set of p-harmonic maps with finite p-energy. Let (M^m, g) be a complete Riemannian manifold (without boundary) of dimension m with metric g, and let (N^n, g') be a complete manifold of dimension n with metric g'. For any smooth map $f : M \to N$ and compact domain $\Omega \subset M$, we define the p-energy of f on Ω:

$$E_p(\Omega, f) \equiv \frac{1}{p} \int_\Omega |df(x)|^p \, dV_g,$$

where $|df(x)|$ is the norm of the differential $df(x)$ of f at $x \in \Omega$, dV_g is the volume element of M, and $1 < p < \infty$ is a fixed number. Let $f^{-1}TN$ be the induced vector bundle by f over M. Then df can be viewed as a section of the bundle $\Lambda^1(f^{-1}TN) = T^*M \otimes f^{-1}TN$. We denote by $|df(x)|$ its norm at a point x of M, induced by the metrics g and g'.

A map f is called p-harmonic if it is a critical point of p-energy functional $E_p(\Omega, \cdot)$ for any compact domain $\Omega \subset M$. That is, f is a p-harmonic map if and only if

$$\frac{dE_p(f_s)}{ds} = 0$$

at $s = 0$ for any one-parameter family of maps $f_s : M \to N$ with $f_0 = f$ and $f_s(x) = f(x)$ if $x \in M \setminus \Omega$. We define the p-tension field $\tau_p(f)$ of f by

$$\tau_p(f) = -\delta(|df|^{p-2} df),$$

where $\delta : \Lambda^1(f^{-1}TN) \to \Lambda^0(f^{-1}TN)$ is the codifferential operator. Equivalently, a smooth map $f : M \to N$ is p-harmonic if and only if $\tau_p(f) = 0$.

Assume that (M, g) is a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and that (N, g') is a complete manifold with non-positive sectional curvature. We denote the Ricci tensor of (M, g) by Ricci_M, and the curvature tensor of (N, g') by R_N. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame on M; by the Weitzenböck formula [Eells and Lemaire 1983], we have

$$\begin{aligned}(3\text{-}1)\quad \tfrac{1}{2}\Delta|df|^2 &= \langle \Delta df, df\rangle + |\nabla df|^2 + \sum_{i=1}^{m} \big\langle df(\mathrm{Ricci}_M(e_i)) \cdot df(e_i)\big\rangle \\ &\qquad - \sum_{i,j=1}^{m} \big\langle R_N\big(df(e_j), df(e_i)\big) df(e_i), df(e_j)\big\rangle \\ &\geq \langle \Delta df, df\rangle + |\nabla df|^2 - K|df|^2.\end{aligned}$$

Lemma 3.1 (Bochner-type formula for p-harmonic maps). *Let $u : M \to N$ be a smooth p-harmonic map and $\{e_i\}_{i=1}^m$ be an orthonormal basis of the tangent space of M. Then*

$$\begin{aligned}(3\text{-}2)\quad |du|^{p-1}\Delta|du|^{p-1} &= |du|^{p-2}\big\langle \Delta(|du|^{p-2}du), du\big\rangle \\ &\quad + \Big(\big|\nabla(|du|^{p-2}du)\big|^2 - \big|\nabla|du|^{p-1}\big|^2\Big) \\ &\quad + |du|^{2p-4} \sum_{i}^{m} \big\langle \mathrm{Ricci}_M(du(e_i)), du(e_i)\big\rangle \\ &\quad - |du|^{2p-4} \sum_{i,j=1}^{n} \big\langle R_N(du(e_i), du(e_j)) du(e_i), du(e_j)\big\rangle,\end{aligned}$$

in the sense of distributions. Also, if $\mathrm{Ricci}_M \geq 0$ *and* $K_N \leq 0$, *then*

$$|du|^{p-1}\Delta|du|^{p-1} \geq |du|^{p-2}\big\langle\Delta(|du|^{p-2}du),\, du\big\rangle + \big(\big|\nabla(|du|^{p-2}du)\big|^2 - \big|\nabla|du|^{p-1}\big|^2\big).$$

Proof. The Bochner–Weitzenböck formula for $|du|^{p-1}$ asserts that

$$\begin{aligned}
\tfrac{1}{2}\Delta|du|^{2p-2} &= \tfrac{1}{2}\Delta\big||du|^{p-2}du\big|^2 \\
&= \big\langle\Delta(|du|^{p-2}du),\, |du|^{p-2}du\big\rangle + \big|\nabla(|du|^{p-2}du)\big|^2 \\
&\qquad + \sum_i^m \big\langle|du|^{p-2}(\mathrm{Ricci}_M(du(e_i)),\, |du|^{p-2}du(e_i)\big\rangle \\
&\qquad - \sum_{i,j=1}^n \big\langle|du|^{p-2}R_N(du(e_i), du(e_j))du(e_i),\, |du|^{p-2}du(e_j)\big\rangle \\
&= \big\langle\Delta(|du|^{p-2}du), |du|^{p-2}du\big\rangle + \big|\nabla(|du|^{p-2}\, du)\big|^2 \\
&\qquad + |du|^{2p-4} \sum_i^m \big\langle\mathrm{Ricci}_M(du(e_i)), du(e_i)\big\rangle \\
&\qquad - |du|^{2p-4} \sum_{i,j=1}^n \big\langle R_N(du(e_i), du(e_j))du(e_i), du(e_j)\big\rangle.
\end{aligned}$$

On the other hand,

$$\tfrac{1}{2}\Delta|du|^{2p-2} = \tfrac{1}{2}\Delta(|du|^{p-1})^2 = |du|^{p-1}\Delta|du|^{p-1} + \big|\nabla|du|^{p-1}\big|^2.$$

Hence,

$$\begin{aligned}
|du|^{p-1}\Delta|du|^{p-1} + \big|\nabla|du|^{p-1}\big|^2 &= \big\langle\Delta(|du|^{p-2}du), |du|^{p-2}du\big\rangle + \big|\nabla(|du|^{p-2}du)\big|^2 \\
&\qquad + |du|^{2p-4} \sum_i^m \big\langle(\mathrm{Ricci}_M(du(e_i)), du(e_i)\big\rangle \\
&\qquad - |du|^{2p-4} \sum_{i,j=1}^n \big\langle R_N(du(e_i), du(e_j))du(e_i), du(e_j)\big\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
|du|^{p-1}\Delta|du|^{p-1} &= |du|^{p-2}\big\langle\Delta(|du|^{p-2}du), du\big\rangle + \big(\big|\nabla(|du|^{p-2}du)\big|^2 - \big|\nabla|du|^{p-1}\big|^2\big) \\
&\qquad + |du|^{2p-4} \sum_i^m \big\langle(\mathrm{Ricci}_M(du(e_i)), du(e_i)\big\rangle \\
&\qquad - |du|^{2p-4} \sum_{i,j=1}^n \big\langle R_N(du(e_i),du(e_j))du(e_i), du(e_j)\big\rangle.
\end{aligned}$$

If $\mathrm{Ricci}_M \geq 0$ and $K_N \leq 0$, then

$$|du|^{p-1}\Delta|du|^{p-1} \geq |du|^{p-2}\big\langle\Delta(|du|^{p-2}du), du\big\rangle + \Big(\big|\nabla(|du|^{p-2}du)\big|^2 - \big|\nabla|du|^{p-1}\big|^2\Big). \qquad \square$$

Theorem 3.2. *Let (M, g) be a complete noncompact manifold with asymptotically nonnegative Ricci curvature, and let (N, g') be a complete Riemannian manifold with nonpositive sectional curvature. Then the set of p-harmonic maps u from M to N with $\int_M |du|^p dV_g \leq C$, for some $C > 0$ and $1 < p < \infty$, has a uniformly convergent subsequence.*

Proof. Let u be a p-harmonic map; if $K_N < 0$, the Bochner type formula (3-2) asserts that

$$|du|^{p-1}\,\Delta|du|^{p-1} \geq du|^{p-2}\big\langle\Delta(|du|^{p-2}du), du\big\rangle + \Big(\big|\nabla(|du|^{p-2}du)\big|^2 - \big|\nabla|du|^{p-1}\big|^2\Big) - |du|^{2p-2}K(R).$$

By the Kato inequality, we have

$$\big|\nabla|du|^{p-1}\big| = \big|\nabla\big||du|^{p-2}du\big|\big| \leq \big|\nabla(|du|^{p-2}du)\big|.$$

Thus,

$$|du|^{p-1}\Delta|du|^{p-1} \geq |du|^{p-2}\big\langle\Delta(|du|^{p-2}du), du\big\rangle - |du|^{2p-2}K(R). \tag{3-3}$$

Dividing both sides of (3-3) by $|du|^{p-2}$, we get

$$|du|\Delta|du|^{p-1} \geq \big\langle\Delta(|du|^{p-2}du), du\big\rangle - |du|^p K(R).$$

Let η be a compactly supported nonnegative smooth function on M; then

$$\begin{aligned}\int_M \eta^2|du|\Delta|du|^{p-1} &\geq \int_M \eta^2\langle(d\delta+\delta d)|du|^{p-2}du, du\rangle - \int_M \eta^2|du|^p K(R)\\ &= \int_M \eta^2\langle d|du|^{p-2}du, d(du)\rangle - \int_M \eta^2|du|^p K(R)\\ &= -\int_M \eta^2|du|^p K(R).\end{aligned}$$

On the other hand, by integration by parts,

(3-4)
$$
\begin{aligned}
-\int_M \eta^2|du|^p K(R) &\le \int_M \eta^2 |du|\Delta|du|^{p-1}\\
&= -\int_M \nabla(\eta^2|du|)\cdot\nabla|du|^{p-1}\\
&= -\int_M (\eta^2\nabla|du| + |du|2\eta\cdot\nabla\eta)\cdot\big((p-1)|du|^{p-2}\nabla|du|\big)\\
&= -(p-1)\int_M \eta^2|du|^{p-2}\big|\nabla|du|\big|^2\\
&\quad -2(p-1)\int_M \eta\cdot\nabla\eta|du|^{p-1}\cdot\nabla|du|.
\end{aligned}
$$

Since
$$
\frac{4}{p^2}\big|\nabla|du|^{p/2}\big|^2 = \frac{4}{p^2}\Big|\frac{p}{2}|du|^{(p/2)-1}\nabla|du|\Big|^2 = \big|du|^{p-2}|\nabla|du|\big|^2
$$
and
$$
\frac{2}{p}|du|^{p/2}\nabla|du|^{p/2} = \frac{2}{p}|du|^{p/2}\frac{p}{2}|du|^{(p/2)-1}\nabla|du| = |du|^{p-1}\nabla|du|,
$$
inequality (3-4) can be rewritten as
$$
\begin{aligned}
-\int_M &\eta^2|du|^p K(R)\\
&\le -\frac{4(p-1)}{p^2}\int_M \eta^2|\nabla|du|^{p/2}|^2 - \frac{4(p-1)}{p}\int_M |du|^{p/2}\cdot\nabla\eta\cdot\eta\cdot\nabla|du|^{p/2}.
\end{aligned}
$$
By Young's inequality,
$$
\begin{aligned}
-\int_M &\eta^2|du|^p K(R)\\
&\le -\frac{4(p-1)}{p^2}\int_M \eta^2\big|\nabla|du|^{p/2}\big|^2 + \Big(\zeta\int_M \eta^2\big|\nabla|du|^{p/2}\big|^2 + \frac{c_1}{\zeta}\int_M |\nabla\eta|^2|du|^p\Big),
\end{aligned}
$$
for some positive constants c_1 and $0<\zeta<1$. Therefore,

(3-5)
$$
\begin{aligned}
\Big(\frac{4(p-1)}{p^2} - 2\zeta\Big)\int_M \eta^2\big|\nabla|du|^{p/2}\big|^2 \le \frac{c_2}{\zeta}\Big(\int_M |\nabla\eta|^2|du|^p + \int_M \eta^2|du|^p K(R)\Big).
\end{aligned}
$$

For $R>0$ and $x\in\partial B_z(2R)$, let $\eta\in C_0^\infty(B_x(R))$ be a cut-off function such that
$$
\eta(y) = \begin{cases} 1 & \text{if } y\in B_x(\rho R),\\ 0 & \text{if } y\in M\setminus B_x(\gamma R).\end{cases}
$$

Note that $\eta \in [0, 1]$ on M and $|\nabla\eta| \le c_3/R$, for $0 < \rho < \gamma \le 1$ and some positive constant c_3.

By the curvature assumption on function $K(R)$, we have

$$K(R) \le \frac{c_4}{R^2},$$

for some constant c_4. Let $\zeta = (p-1)/p^2$; then inequality (3-5) becomes

$$\int_{B_x(R)} \left|\nabla |du|^{p/2}\right|^2 \le \frac{c_5}{R^2}\int_{B_x(R)} |du|^p + \int_{B_x(R)} \frac{c_6}{R^2}|du|^p \le \frac{C}{R^2}\int_{B_x(R)} |du|^p.$$

Therefore, for u a p-harmonic map from M to N and $x \in \partial B_z(2R)$, we have

$$\int_{B_x(R)} \left|\nabla |du|^{p/2}\right|^2 \le \frac{C}{R^2}\int_M |du|^p.$$

When $\int_M |du|^p$ is assumed to be bounded by a fixed constant, by taking R to be sufficiently large, for any $\epsilon > 0$, we have $|\nabla |du|^{p/2}| < \epsilon$ on $M\backslash B_z(R)$. On the other hand, $\left|\nabla |du|^{p/2}\right|$ can be bounded by the finite energy of u on ball $B_z(R)$. We conclude that the set of such p-harmonic maps admits a uniformly convergent subsequence. If M is a compact manifold with nonnegative Ricci curvature, then the p-harmonic map is constant, which is an extension of the fact in the harmonic map case ($p = 2$) [Eells and Sampson 1964]. □

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