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We establish a sharp relative volume comparison theorem for small balls on Kähler manifolds with lower bound on Ricci curvature, assuming real analyticity of the metric. The model spaces being compared to are complex space forms, that is, Kähler manifolds with constant holomorphic sectional curvature. Moreover, we give an example showing that on Kähler manifolds, the pointwise Laplacian comparison theorem does not hold when the Ricci curvature is bounded from below.

1. Introduction

Comparison theorems are fundamental tools in geometric analysis. They are vital in estimates of spectra, heat kernels and the Sobolev constants. The classical Bishop–Gromov relative volume comparison theorem [Bishop and Crittenden 1964; Gromov 1981; Li 1993] in Riemannian geometry is this:

Theorem 1.1. *Let M^n be a complete Riemannian manifold of dimension n such that* $\mathrm{Ric} \geq (n-1)K$. *For any $p \in M$ and $0 < a < b$, the volume of geodesic balls satisfies*

$$\frac{\mathrm{Vol}\, B_p(b)}{\mathrm{Vol}\, B_p(a)} \leq \frac{\mathrm{Vol}\, B_{M_K}(b)}{\mathrm{Vol}\, B_{M_K}(a)},$$

where M_K is the simply connected real space form with sectional curvature K and $\mathrm{Vol}\, B_{M_K}(r)$ *is the volume of the geodesic ball in M_K with radius r. Equality holds if and only if $B_p(b)$ is isometric to $B_{M_K}(b)$.*

The key ingredient in Theorem 1.1 is the Laplacian comparison theorem [Cheeger and Ebin 2008; Schoen and Yau 1994]:

Theorem 1.2. *Let M^n be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)K$. *Let M_k be the simply connected real space form with sectional curvature K. Denote by $r_M(x)$ the distance function from p to x in M. Let r_{M_k} be the distance function on M_k. Then for any $x \in M$ and $y \in M_k$ with $r_M(x) = r_{M_k}(y)$,*

$$\Delta r_M(x) \leq \Delta r_{M_k}(y).$$

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The model spaces in the theorems above are real space forms. In the Kähler category, a natural question is whether we can replace the model spaces by Kähler models, that is, complex space forms which are Kähler manifolds with constant holomorphic sectional curvature. Li and Wang [2005] showed that when the bisectional curvature has a lower bound, both of the theorems above hold with Kähler models. So the question left is: what can we get if we only assume the lower bound of the Ricci curvature? This note addresses the local case. Our main theorem is:

Theorem 1.3. *Let M^n be a Kähler manifold of complex dimension n with a real analytic metric. Assume* $\mathrm{Ric} \geq K$, *where K is any real number. Given any point $p \in M$, there exists $r = r(p, M) > 0$ such that for any $0 < a < b < r$, the volume of geodesic balls satisfies*

$$\frac{\mathrm{Vol}\, B_{M^n}(p, b)}{\mathrm{Vol}\, B_{M^n}(p, a)} \leq \frac{\mathrm{Vol}\, B_{N_K}(b)}{\mathrm{Vol}\, B_{N_K}(a)},$$

where N_K denotes the rescaled complex space form with $\mathrm{Ric} = K$ *and $\Delta_{N_K} r$ is the Laplacian of distance function on N_K. Equality holds if and only if M is locally isometric to N_K.*

Remark. Theorem 1.3 is a local version of the Bishop–Gromov relative volume comparison theorem on Kähler manifolds. However, one cannot directly extend the result to any radius. A simple counterexample is a product of several $\mathbb{P}^1$ with the standard product metric: the diameter is greater than that of the complex space form. Thus, when r is large, the inequality in Theorem 1.3 does not hold.

We will prove a slightly stronger result:

Theorem 1.4. *Under the hypotheses of Theorem 1.3, there exists $r_0 = r_0(p, M) > 0$ such that for any $r < r_0$, the average Laplacian comparison holds,*

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} \leq \Delta_{N_K} r(r),$$

where $\Delta_{N_K} r$ is the Laplacian of distance function on N_K. Moreover, the equality holds if and only if M is locally isometric to N_K.

Remark. Theorem 1.4 is a local version of Theorem 1.2 in the average sense. However, on Kähler manifolds with lower bound on Ricci curvature, the pointwise Laplacian comparison does not even hold locally (see Section 6).

The idea of the proof of Theorem 1.4 is very simple. We shall expand the area of the geodesic sphere $A(\partial B_p(r))$ as a power series, then compare the coefficients with those of the rescaled complex space form. The computation is complicated since it involves the covariant derivatives of the curvature tensor of arbitrary order.

This note is organized as follows:

In Section 2, we state two propositions which demonstrate the relation between the derivatives of $A(\partial B_p(r))$ and the covariant derivatives of the curvature tensor at p. Section 3 is the first part of the proof of Proposition 2.1. We shall estimate the derivatives of $A(\partial B_p(r))$ up to order 4. In the estimate of the 4-th derivative, the Kähler condition is employed. The most important part is Section 4. We use induction to prove Proposition 2.1. Besides the routine computation, there are two technical lemmas (Lemma 4.4 and Lemma 4.6) which simplify the computation of higher order covariant derivatives of the curvature tensor significantly. One should note that the Kähler condition is essential in these two lemmas. We complete the proof of Proposition 2.2 and Theorem 1.4 in Section 5. The last section is devoted to giving an example showing that the pointwise Laplacian comparison with the complex space form does not necessarily hold if the complex dimension is greater or equal to 2.

2. Basic set up

Throughout this note, we implicitly evaluate derivatives of functions of r at $r = 0$. Given a point p on a Kähler manifold M^n, fix a unit vector $e_0 \in T_pM$. Along the geodesic l from p with initial direction e_0, consider the Jacobian equation $J'' = R(e_0, J)\, e_0$. Set up an orthonormal frame $\{e_k\}$ at p such that

$$Je_{2i} = e_{2i+1} \quad \text{and} \quad Je_{2i+1} = -e_{2i}$$

for $0 \le i \le n-1$. Parallel transport the frame along the geodesic l. Consider the Jacobian field J_u with initial value $J_u(0) = 0$, $J_u'(0) = e_u$.

We may write

$$J_u = J_u(r, e_0) = \sum_{i=1}^{\infty} \sum_{v=0}^{2n-1} r^i C_{u,i}^v\, e_v \tag{2-1}$$

where $C_{u,i}^v$ are constants independent of r. Denote $R_{e_0 e_u e_0 e_v}$ by R_{uv} when e_0 is fixed. Plugging (2-1) into the Jacobian equation, we get

$$\sum_i \sum_v i(i-1) r^{i-2} C_{u,i}^v\, e_v = \sum_k \sum_w r^k C_{u,k}^w R(e_0, e_w)\, e_0. \tag{2-2}$$

Along the geodesic l,

$$R(e_0, e_w)\, e_0 = \sum_{s=0}^{2n-1} \sum_{j=0}^{\infty} \frac{R_{sw}^{(j)}}{j!} e_s r^j$$

where $R_{sw}^{(j)}$ denotes the j-th covariant derivative of R_{sw} along e_0 at p.

Inserting this into (2-2), we get

$$\sum_{i,v} i(i-1)r^{i-2}C^v_{u,i}\, e_v = \sum_{k,j,w,s} r^{k+j} C^w_{u,k} \frac{R^{(j)}_{sw}}{j!} e_s.$$

Comparing coefficients, we obtain

$$C^v_{u,i} = \sum_{k+j=i-2,\, w} C^w_{u,k} \frac{R^{(j)}_{vw}}{j!\, i(i-1)}. \tag{2-3}$$

A simple iteration now gives the constants $C^v_{u,i}$. First we have $C^v_{u,1} = \delta^v_u$ and $C^w_{u,2} = 0$. Then we get

$$C^v_{u,3} = \sum_w C^w_{u,1} \frac{R_{vw}}{6} = \frac{R_{uv}}{6}, \qquad C^v_{u,4} = \sum_w C^w_{u,1} \frac{R'_{vw}}{12} = \frac{R'_{vu}}{12},$$
$$C^v_{u,5} = \sum_w \Big(C^w_{u,1} \frac{R''_{vw}}{40} + C^w_{u,3} \frac{R_{vw}}{20} \Big) = \frac{1}{120} \Big(\sum_s R_{us} R_{sv} + 3R''_{uv} \Big).$$

Plugging these values into (2-1), we have

$$J_u = re_u + \frac{r^3}{6} R_{uv} e_v + \frac{r^4}{12} R'_{uv} e_v + \frac{r^5}{120} \Big(\sum_s R_{us} R_{sv} + 3R''_{uv} \Big) e_v + O(r^6). \tag{2-4}$$

We write dA for the standard measure of the unit tangent bundle $UT_p(M)$ at p, and we write $\int_{\partial B(p,r)} dA$ as $\int$. We define

$$W = \frac{\int \sqrt{\det\langle J_u, J_v\rangle}}{r^{2n-1}},$$

and introduce two propositions:

Proposition 2.1. *Assume the hypotheses of Theorem 1.3. Let the derivatives of W of order from 1 to $2m-1$ for $m \geq 1$ be the same as that of the complex space form.*

(1) *If $m = 1, 2$, then $\mathrm{Ric} = K$ at p.*
If $m \geq 3$, then

$$R_{i\bar{j}k\bar{l}} = \frac{K}{n+1} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

at p. Moreover, if $u, v, e_0 \in UT_p(M)$ are any unit vectors, then $R^{(\lambda)}_{uv} = 0$ for $1 \leq \lambda \leq m-3$ and $\mathrm{Ric}^{(l)}(e_0, e_0) = 0$ for $1 \leq l \leq 2m-4$. The superscripts are the orders of covariant derivatives along the direction e_0.

(2) *In either case, $W^{(2m)}$ is less than or equal to that of the complex space form.*

Proposition 2.2. *Under the same conditions as in Theorem 1.3, if the derivatives of W of order 1 to $2m$ for $m \geq 1$ are the same as the complex space form, then $W^{(2m+1)} = 0$.*

We divide the proof of Proposition 2.1 into two parts: $m = 1, 2$ and $m \geq 3$.

3. The proof of Proposition 2.1, case $m = 1, 2$

By (2-1), we have

$$\frac{\langle J_u, J_v\rangle}{r^2} = \sum_{i,j,w} r^{i+j-2} C^w_{u,i} C^w_{v,j}. \tag{3-1}$$

By (2-4),

$$\frac{\langle J_u, J_u\rangle}{r^2} = 1 + \tfrac{1}{3} R_{uu} r^2 + \tfrac{1}{6} R'_{uu} r^3 + \Big(\tfrac{2}{45}\sum_s R^2_{us} + \tfrac{1}{20} R''_{uu}\Big) r^4 + O(r^5).$$

If $u \neq v$,

$$\frac{\langle J_u, J_v\rangle}{r^2} = \tfrac{1}{3} R_{uv} r^2 + \tfrac{1}{6} R'_{uv} r^3 + \Big(\tfrac{2}{45}\sum_s R_{us} R_{vs} + \tfrac{1}{20} R''_{uv}\Big) r^4 + O(r^5).$$

Now use the above two expressions to see that

$$\begin{aligned}\frac{\det\langle J_u, J_v\rangle}{r^{4n-2}} &= 1 + \tfrac{1}{3}\sum_u R_{uu} r^2 + \tfrac{1}{6}\sum_u R'_{uu} r^3 \\ &\quad + \Big(\tfrac{2}{45}\sum_{u,s} R^2_{us} + \tfrac{1}{20}\sum_u R''_{uu} + \tfrac{1}{9}\sum_{u<v}\big(R_{uu}R_{vv} - R^2_{uv}\big)\Big) r^4 + O(r^5).\end{aligned} \tag{3-2}$$

Considering the identity $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$, we get

$$\begin{aligned}\frac{\sqrt{\det\langle J_u, J_v\rangle}}{r^{2n-1}} &= 1 + \tfrac{1}{6}\sum_u R_{uu} r^2 + \tfrac{1}{12}\sum_u R'_{uu} r^3 \\ &\quad + \Big(\tfrac{1}{45}\sum_{u,s} R^2_{us} + \tfrac{1}{40}\sum_u R''_{uu} + \tfrac{1}{18}\sum_{u<v}\big(R_{uu}R_{vv} - R^2_{uv}\big) - \tfrac{1}{72}\Big(\sum_u R_{uu}\Big)^2\Big) r^4 + O(r^5).\end{aligned} \tag{3-3}$$

Since $W = \frac{1}{r^{2n-1}} \int \sqrt{\det\langle J_u, J_v\rangle}$, we find

$$W'(0) = 0 \quad \text{and} \quad W''(0) = -c\, s,$$

where c is a positive constant depending only on n, and s is the scalar curvature at p. Therefore $W''(0)$ is less than or equal to that of the complex space form. This proves Proposition 2.1 for $m = 1$.

Now we consider $m = 2$. According to the assumption of Proposition 2.1, W'' is the same as that of the complex space form. Therefore $s = nK$ at p. Since the Ricci curvature is bounded from below by K, $\mathrm{Ric} = Kg$ at p. By (3-3), it is simple to see that the r^3 coefficient of W is 0 by symmetry. Thus to complete the proof for $m = 2$, we just need to show that the 4th derivative of W is less than or equal to that of the complex space form.

We keep in mind that $\mathrm{Ric} = Kg$ at p. The r^4 coefficient of W is

$$\begin{aligned}
c_4 &= \int \Big(\tfrac{1}{45} \sum_{u,s} R_{us}^2 + \tfrac{1}{40} \sum_u R''_{uu} + \tfrac{1}{18} \sum_{u<v} \big(R_{uu}R_{vv} - R_{uv}^2\big) - \tfrac{1}{72} \Big(\sum_u R_{uu}\Big)^2 \Big) \\
&= \frac{1}{360} \int \Big(8 \sum_u R_{uu}^2 + 16 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} \\
&\qquad\qquad + 20 \sum_{u<v} R_{uu}R_{vv} - 20 \sum_{u<v} R_{uv}^2 - 5 \Big(\sum_u R_{uu}\Big)^2 \Big) \\
&= \frac{1}{360} \int \Big(-2 \sum_u R_{uu}^2 + 10 (\sum_u R_{uu})^2 - 4 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} - 5 \Big(\sum_u R_{uu}\Big)^2 \Big) \\
&= \frac{1}{360} \int \Big(9 \sum_u R''_{uu} - 4 \sum_{u<v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5 \Big(\sum_u R_{uu}\Big)^2 \Big).
\end{aligned}$$

Note that the Ricci curvature attains the minimum K at p, so

$$\sum_u R''_{uu} = -\mathrm{Ric}''(e_0, e_0) \le 0.$$

Therefore we have

$$\begin{aligned}
c_4 &= \frac{1}{360} \int \Big(9 \sum_u R''_{uu} - 4 \sum_{u<v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5K^2 \Big) \\
&\le -\frac{1}{360} \int \Big(2 \sum_u R_{uu}^2 - 5K^2 \Big) \\
&= -\frac{1}{360} \int \Big(2 \sum_{u \ne 1} R_{uu}^2 + 2R_{11}^2 - 5K^2 \Big) \\
&\le -\frac{1}{360} \int \Big(\frac{1}{n-1} \Big(\sum_{u \ne 1} R_{uu} \Big)^2 + 2R_{11}^2 - 5K^2 \Big) \\
&= -\frac{1}{360} \int \Big(\frac{1}{n-1} (\mathrm{Ric}(e_0, e_0) + R_{11})^2 + 2R_{11}^2 - 5K^2 \Big) \\
&= -\frac{1}{360} \int \Big(\frac{1}{n-1} K^2 + \frac{2}{n-1} K R_{11} + \Big(\frac{1}{n-1} + 2 \Big) R_{11}^2 - 5K^2 \Big) \\
&\le -\frac{1}{360} \Big(\int \frac{1}{n-1} K^2 + \frac{2}{n-1} K \int R_{11} + C_1 \Big(\int R_{11} \Big)^2 - \int 5K^2 \Big) \\
&= C_2 K^2,
\end{aligned} \tag{3-4}$$

where C_1, C_2 are constants depending only on n.

We explain the inequalities above. In the first inequality, we drop the two terms $\sum_{u<v} R_{uv}^2$ and $\sum_u R''_{uu}$. In the second inequality, we apply the Schwartz inequality for directions e_u that are perpendicular to e_1, e_0. In the third inequality we use the Schwartz inequality $\int R_{11}^2 \ge C \big(\int R_{11}\big)^2$. We make use of the Kähler condition to

obtain $\int R_{11} = C_3\, s = nC_3 K$, where C_3 is a constant depending only on n. This explains the last equality.

The right hand side of (3-4) is exactly the case of the complex space form. Therefore when W' and W'' are the same as the complex space form, $W^{(3)} = 0$ and $W^{(4)}$ is less than or equal to that of the complex space form. Equation (3-4) becomes an equality if and only if the holomorphic sectional curvature is constant at p and $\mathrm{Ric}''(e_0, e_0) = 0$ for any $e_0 \in UT_pM$. This completes the proof for $m = 2$.

4. The proof of Proposition 2.1, case $m \geq 3$

Denote $\mathrm{Ric}^{(l)}(e_0, e_0)$ by $\mathrm{Ric}^{(l)}$. According to the assumption of Proposition 2.1, the derivatives of W of order 1 to $2m-1$ are the same as the complex space form. From results in the last section, the holomorphic sectional curvature is constant at p and $\mathrm{Ric}'' = 0$ for any e_0. That is to say,

$$R_{i\bar{j}k\bar{l}} = \frac{K}{n+1}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \quad \text{and} \quad \mathrm{Ric}'' = 0$$

at p. Therefore, we have proved part (1) of Proposition 2.1 for $m = 3$.

Now we use induction. Assuming that part (1) of Proposition 2.1 holds for $k = m$, we shall prove that it holds for $k = m+1$.

Claim 4.1. *Let $C^v_{u,i}$ be the coefficients defined in (2-1) for $i \leq m$. Under the hypothesis of the induction above, $C^v_{u,i}$ are constants independent of the direction e_0. In fact, they are the same as that of the complex space form.*

Proof. The claim follows if we insert the induction hypothesis into (2-3). □

Let us write

$$\frac{\det\langle J_u, J_v\rangle}{r^{4n-2}} = 1 + \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j + O(r^{2m+1}). \tag{4-1}$$

Combining Claim 4.1 with (3-1), we find that a_i are constants independent of the direction e_0. Equation (3-1) also yields $C^v_{u,m+1} = C^u_{v,m+1}$ for all u, v. Direct expansion of the determinant via (3-1) gives

$$\begin{aligned} b_{2m} = \sum_{u,v}\big(C^v_{u,m+1}\big)^2 + 4\sum_{u<v} C^u_{u,m+1}C^v_{v.m+1} + 2\sum_u C^u_{u,2m+1} \\ -4\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1} + \sum_{i=1}^{m} C^v_{u,m+i}C_{i,m,u,v} + C_{0,m} \end{aligned} \tag{4-2}$$

where $C_{i,m,u,v}$ and $C_{0,m}$ are all constants independent of the direction e_0.

Note also

$$b_m = 2\sum_u C^u_{u,m+1} + \text{Constant}\,. \tag{4-3}$$

Let us set $\gamma_m = \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j$ for $m \geq 1$. Applying the Taylor series expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \sum_{k=3}^{\infty} \lambda_k x^k$$

for $|x| < 1$, we obtain

$$\frac{\sqrt{\det\langle J_u, J_v\rangle}}{r^{2n-1}} = 1 + \tfrac{1}{2}\gamma_m - \tfrac{1}{8}\gamma_m^2 + \sum_{k=3}^{\infty} \lambda_k \gamma_m^k + O(r^{2m+1}). \tag{4-4}$$

Lemma 4.2. *The $2m$-th order coefficient of the expansion of W is*

$$\begin{aligned} c_{2m} = \int \Bigl(\tfrac{1}{2} \sum_{u,v} (C^v_{u,m+1})^2 + 2 \sum_{u<v} C^u_{u,m+1} C^v_{v,m+1} + \sum_u C^u_{u,2m+1} \\ -2 \sum_{u<v} C^v_{u,m+1} C^u_{v,m+1} - \tfrac{1}{2}\Bigl(\sum_u C^u_{u,m+1}\Bigr)^2 + \sum_{i=1}^{m} C^v_{u,m+i} \widetilde{C}_{i,m,u,v} \Bigr) + \widetilde{C}_{0,m} \end{aligned} \tag{4-5}$$

where $\widetilde{C}_{i,m,u,v}$ and $\widetilde{C}_{0,m}$ are constants independent of the direction e_0.

Proof. It suffices to find out the contribution of each term in (4-4) to c_{2m}. We keep in mind that coefficients a_i in (4-1) are independent of e_0.

By (4-2), the contribution of the term $1 + \frac{1}{2}\gamma_m$ to c_{2m} is

$$\begin{aligned} \int \tfrac{1}{2} \sum_{u,v} (C^v_{u,m+1})^2 + 2 \sum_{u<v} C^u_{u,m+1} C^v_{v,m+1} + \sum_u C^u_{u,2m+1} \\ -2 \sum_{u<v} C^v_{u,m+1} C^u_{v,m+1} + \tfrac{1}{2}\Bigl(\sum_{i=1}^{m} C^v_{u,m+i} C_{i,m,u,v} + C_{0,m} \Bigr). \end{aligned} \tag{4-6}$$

The contribution of the term $-\frac{1}{8}\gamma_m^2$ to c_{2m} is

$$-\int \Bigl(\tfrac{1}{8} b_m^2 + \sum_{i=1}^{m} C^v_{u,m+i} p_{i,m,u,v} \Bigr) + p_{0,m}. \tag{4-7}$$

By (4-3), it could be written as

$$-\int \Bigl(\tfrac{1}{2}\Bigl(\sum_u C^u_{u,m+1} \Bigr)^2 + \sum_{i=1}^{m} C^v_{u,m+i} p_{i,m,u,v} \Bigr) + p_{0,m}. \tag{4-8}$$

The contribution of $\sum_{k=3}^{\infty} \lambda_k \gamma_m^k$ to c_{2m} is

$$\int \sum_{i=1}^{m} C^v_{u,m+i} q_{i,m,u,v} + q_{0,m}. \tag{4-9}$$

In (4-7), (4-8), (4-9), $p_{i,m,u,v}$, $q_{i,m,u,v}$, $p_{0,m}$ and $q_{0,m}$ are all constants independent of the direction e_0. Lemma 4.2 follows if we combine (4-6), (4-7), (4-8) and (4-9). □

Lemma 4.3. *There is a negative definite quadratic form Q, constants $h_{m,i}$ and C and a negative constant C_m such that*

$$c_{2m} = \int Q\big(R_{uv}^{(m-2)}\big) + \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int \mathrm{Ric}^{(2m-2)} + C. \tag{4-10}$$

Proof. By the induction hypothesis and (2-3), we have

$$\begin{aligned} C^u_{u,2m+1} &= \sum_{k+j=2m-1,w} \frac{C^w_{u,k} R^{(j)}_{uw}}{j!\,(2m+1)2m} \\ &= \frac{1}{(2m+1)2m}\bigg(\sum_w \bigg(\frac{R^{(m-2)}_{uw} C^w_{u,m+1}}{(m-2)!} + \sum_{j=m-1}^{2m-2} B_{j,m,w,u} R^{(j)}_{uw}\bigg) + R_{uu} C^u_{u,2m-1}\bigg) \end{aligned} \tag{4-11}$$

where $B_{j,m,w,u}$ are constants. For $i \le m$, we have

$$C^v_{u,m+i} = \sum_{j=m-2}^{m+i-3} d_{m,i,j,w,u} R^{(j)}_{uw} + C \tag{4-12}$$

where C and $d_{m,i,j,w,u}$ are constants. In particular, we have

$$C^v_{u,m+1} = \sum_{k+j=m-1,w} C^w_{u,k} \frac{R^{(j)}_{vw}}{j!\,m(m+1)} = \frac{1}{m(m+1)}\bigg(\frac{R^{(m-2)}_{vu}}{(m-2)!} + C^v_{u,m-1} R_{vv}\bigg). \tag{4-13}$$

By the induction hypothesis,

$$\sum_u R^{(m-2)}_{uu} = -\mathrm{Ric}^{(m-2)} = 0. \tag{4-14}$$

Therefore

$$\begin{aligned} \sum_u \big(R^{(m-2)}_{uu}\big)^2 &= \Big(\sum_u R^{(m-2)}_{uu}\Big)^2 - 2\sum_{u<v} R^{(m-2)}_{uu} R^{(m-2)}_{vv} \\ &= -2\sum_{u<v} R^{(m-2)}_{uu} R^{(m-2)}_{vv}. \end{aligned} \tag{4-15}$$

Inserting (4-11), (4-12), (4-13) into (4-5), we find

$$c_{2m} = \int Q\big(R^{(m-2)}_{uv}\big) + \sum_{i=-2}^{m-2} \int \sum_{u,v} h_{m,i,u,v} R^{(m+i)}_{uv} + C. \tag{4-16}$$

Now we prove that Q is negative definite. Let us check each term in (4-5). By (4-13), the term $\frac{1}{2}\sum_{u,v}\big(C^v_{u,m+1}\big)^2$ in (4-5) contributes to the quadratic term

$$\sum_{u,v} \frac{1}{2m^2(m+1)^2((m-2)!)^2}\big(R^{(m-2)}_{uv}\big)^2. \tag{4-17}$$

The term $2\sum_{u<v} C^u_{u,m+1}C^v_{v,m+1}$ contributes to the quadratic term

$$\sum_{u<v}\frac{2}{m^2(m+1)^2((m-2)!)^2}R^{(m-2)}_{uu}R^{(m-2)}_{vv}. \tag{4-18}$$

By (4-15), it could be written as

$$-\frac{1}{m^2(m+1)^2((m-2)!)^2}\sum_u\bigl(R^{(m-2)}_{uu}\bigr)^2. \tag{4-19}$$

By (4-11) and (4-13), the term $\sum_u C^u_{u,2m+1}$ contributes to the quadratic term

$$\sum_{u,v}\frac{1}{2m^2(m+1)(2m+1)((m-2)!)^2}\bigl(R^{(m-2)}_{uv}\bigr)^2. \tag{4-20}$$

The term $-2\sum_{u<v} C^v_{u,m+1}C^u_{v,m+1}$ contributes to the quadratic term

$$-\sum_{u<v}\frac{2}{m^2(m+1)^2((m-2)!)^2}\bigl(R^{(m-2)}_{uv}\bigr)^2. \tag{4-21}$$

The term $-\frac{1}{2}\bigl(\sum_u C^u_{u,m+1}\bigr)^2$ is obviously negative semidefinite.

By combining (4-17), (4-18), (4-19), (4-20) and (4-21), it follows that the quadratic form in (4-10) is negative definite.

Consider the linear terms in (4-16). By the induction hypothesis, the coefficients $h_{m,i,u,v}$ are unchanged if we take a unitary transformation keeping the direction e_0 fixed. Comparing the coefficients of the linear order terms, we see that $h_{m,i,u,v}=0$ if $u\neq v$, and $h_{m,i,u,u}=h_{m,i,v,v}$ if $u\neq e_1$ and $v\neq e_1$. Therefore, the linear terms $h_{m,i,u,u}R^{(m+i)}_{uu}$ could be absorbed into $\mathrm{Ric}^{(m+i)}$ with the terms $-h_{m,i}R^{(m+i)}_{11}$ left. Also note that by induction hypothesis, $\mathrm{Ric}^{(l)}=0$ for $0<l\leq 2m-3$ (the term $\mathrm{Ric}^{(2m-3)}$ vanishes as the Ricci curvature attains its minimum at p). Finally, one verifies that $\sum_u C^u_{u,2m+1}$ is the only term in (4-5) that has contribution to $R^{(2m-2)}_{uv}$. Therefore the linear terms in (4-16) could be written as

$$\sum_{i=-2}^{m-4}h_{m,i}\int R^{(m+i)}_{11}+C_m\int \mathrm{Ric}^{(2m-2)}.$$

From (4-11), it is simple to check that C_m is negative. □

By the induction hypothesis and that the Ricci curvature attains its minimum at p, we have $\mathrm{Ric}^{(2m-2)}\geq 0$. It follows from Lemma 4.3 that

$$c_{2m}\leq\sum_{i=-2}^{m-4}h_{m,i}\int R^{(m+i)}_{11}+\text{Constant}. \tag{4-22}$$

We would like to prove that the linear terms $\int R^{(m+i)}_{11}$ vanish for $-2\leq i\leq m-4$. Note that by symmetry, if $m+i$ is odd, the integral equals 0. Let us deal with case when $m+i$ is even. We shall check when $i=m-4$. The other cases are similar. Let

$$A = -\frac{1}{4}\int R_{11}^{(2m-4)}. \tag{4-23}$$

Set up an orthonormal frame $\{f_i\}$ at p such that $Jf_{2j}=f_{2j+1}$ and $Jf_{2j+1}=-f_{2j}$ for $0\le j\le n-1$. Letting $\beta_j=\frac{1}{2}(f_{2j}-\sqrt{-1}f_{2j+1})$, in a small neighborhood of p, we parallel transport the frame along each geodesic through p. Suppose that

$$e_0=\sum_{j=0}^{n-1}\bigl(z_j\beta_j+\overline{z_j\beta_j}\bigr). \tag{4-24}$$

Lemma 4.4. *Under the assumption of the induction in Proposition 2.1, $Rm^{(\lambda)}=0$ at p for $1\le\lambda\le m-3$, where $Rm^{(\lambda)}$ denotes any covariant derivative of the curvature tensor of order λ at p.*

Proof. We use induction. If $\lambda=0$, the result automatically holds since there is nothing to prove. Suppose the result holds for $k<\lambda$. For $k=\lambda$, we plug (4-24) in $R_{uv}^{(\lambda)}$.

Claim 4.5. *We can commute the covariant derivatives of $R_{uv}^{(\lambda)}$.*

Proof. To prove the claim, we only need to consider the case $\lambda\ge 2$. By the induction hypothesis of Lemma 4.4, the covariant derivatives of the curvature tensor vanish up to order $\lambda-1$ at p. If $\lambda>3$, the claim follows from the Ricci identity. Now suppose $\lambda=2$. By the Ricci identity, the difference of commuting the covariant derivatives is a function of the curvature tensor. Note that the curvature tensor at p is the same as of the complex space form. This completes the proof for $\lambda=2$. ☐

We insert (4-24) into $R_{Je_0Je_0}^{(\lambda)}$. By Claim 4.5 and the Bianchi identities, $R_{Je_0Je_0}^{(\lambda)}$ becomes a polynomial in the variables $z_j,\bar z_j$. The coefficients of the polynomial are exactly all the covariant derivatives of Rm at p of order λ. According to the assumption of Lemma 4.4, $R_{Je_0Je_0}^{(\lambda)}$ is identically 0 for all e_0. Therefore, the coefficients of the polynomial are all 0. This completes the induction of Lemma 4.4. ☐

Lemma 4.6. *Under the assumption of the induction in Proposition 2.1, A could be written as $\sum_{i=1}^{m-2} g_{i,m}\Delta^i s$, where s denotes the scalar curvature, and $g_{i,m}$ are constants depending only on n, m and i.*

Proof. Define $X=\frac{1}{2}(e_0-\sqrt{-1}Je_0)$, then $A=\int R_{X\overline{X}X\overline{X},e_0e_0\dots e_0}$, where the number of e_0 is $2m-4$. Integrating and plugging (4-24) into the result, we find

$$A=\sum_{\alpha_1\alpha_2\dots\alpha_{2m}}\left(\int\alpha_1\alpha_2\dots\alpha_{2m}\right)R_{\alpha_1\alpha_2\dots\alpha_{2m}} \tag{4-25}$$

where each α_i is either z_j or $\overline{z}_k$ for $0 \le j, k \le n-1$, with the further condition that $\alpha_1, \alpha_3 \in \{z_j\}$, and $\alpha_2, \alpha_4 \in \{\overline{z_k}\}$. Under the subscript of R, z_j stands for β_j, and $\overline{z}_k$ stands for $\overline{\beta}_k$.

From the expression of (4-25), we see that $z_i, \overline{z}_i$ must all go in pairs in the sequence $\alpha_1\alpha_2 \ldots \alpha_{2m}$, otherwise the integral $\int \alpha_1\alpha_2 \ldots \alpha_{2m}$ would equal 0. Using the Kähler identities, we can switch the covariant derivatives in (4-25) and rearrange it as

$$A = \sum_{I_1, I_2, \ldots I_n} C_{I_1 I_2 \ldots I_n} R_{I_1 I_2 \ldots I_n} + B. \tag{4-26}$$

Here the symbol I_j denotes $z_j\overline{z}_j \ldots z_j\overline{z}_j$; we have $\sum_j |I_j| = 2m$; subscripts after the fourth subscript of R denote covariant derivatives; $C_{I_1 I_2 \ldots I_n}$ are the coefficients in (4-25); and B is a combination of covariant derivatives of Rm of lower order.

From (4-23), we see that the coefficients $C_{I_1 I_2 \ldots I_n}$ in (4-26) are unitary invariants. For fixed $I_3, I_4, \ldots, I_n$, let $d = |I_1| + |I_2|$. Denote the coefficient $C_{I_1 I_2 \ldots I_n}$ by C_p, where $0 \le |I_1| = p \le d$. We want to find relations between the different C_p. Define a unitary transformation by setting $\widetilde{\beta_i} = \beta_i$ for $i \neq 1, 2$ and let

$$\beta_1 = \cos\theta \widetilde{\beta_1} + \sin\theta \widetilde{\beta_2} \quad \text{and} \quad \beta_2 = -\sin\theta \widetilde{\beta_1} + \cos\theta \widetilde{\beta_2}.$$

Insert the unitary transformation above in (4-26). Then the new coefficient $\tilde{C}_d$ becomes $\sum_{p=0}^{d} C_p \cos^{2p}\theta \sin^{2(d-p)}\theta$. Therefore we have:

$$\sum_{p=0}^{d} C_p \cos^{2p}\theta \sin^{2(d-p)}\theta = C_d = C_d(\cos^2\theta + \sin^2\theta)^d. \tag{4-27}$$

Claim 4.7. $C_p = C_d\binom{d}{p}$.

Proof. Divide by $cos^{2d}\theta$ on both sides, then (4-27) becomes

$$\sum_{p=0}^{d} C_p \tan^{2(d-p)}\theta = C_d = C_d(1 + \tan^2\theta)^d.$$

Since θ is arbitrary, the claim follows. □

By Claim 4.7, $C_p/C_d = \binom{d}{p}$. Since we can substitute any index u, v for $1, 2$, the ratios of all coefficients in (4-26) are determined. Note that to get the relations between C_p, we only use the condition that the form (4-23) is unitary invariant. Since $\Delta^{m-2}s$ is also unitary invariant with respect to the frame, we can write it in the same form as (4-26). By the same argument, the ratios of coefficients of $\Delta^{m-2}s$ are the same as of coefficients in (4-26). It follows that the term $\sum_{I_1, I_2, \ldots I_n} C_{I_1 I_2 \ldots I_n} R_{I_1 I_2 \ldots I_n}$ in (4-26) equals $C(m,n)\Delta^{(m-2)}s$ modulo lower order covariant derivatives, where $C(m,n)$ is a constant depending only on m, n.

Now we make an important observation. From the Ricci identity,

$$R_{i_1\bar{i}_2\dots i_p\alpha\beta i_{p+3}\dots i_{2m}} - R_{i_1\bar{i}_2\dots i_p\beta\alpha i_{p+3}\dots i_{2m}}$$

is the sum of $(RmRm^{(p-4)})_{,i_{p+3}\dots i_{2m}}$. By Lemma 4.4, $Rm^{(\lambda)} = 0$ for $1 \le \lambda \le m-3$. It follows that $(RmRm_{,i_5\dots i_p})_{,i_{p+3}\dots i_{2m}}$ can be expanded as a linear combination of the covariant derivatives of curvature tensor. Therefore $A - C(m,n)\Delta^{(m-2)}s$ can be written as a linear combination of the covariant derivatives of the curvature tensor with the highest order $2m-6$. Furthermore it is unitary invariant since the curvature tensor is unitary invariant at p. By induction, we have completed the proof of Lemma 4.6. □

From the induction in Proposition 2.1, $\mathrm{Ric}^{(l)} = 0$ for $1 \le l \le 2m-4$. Integrating over the unit sphere in T_pM we find, by similar arguments as in the proof of Lemma 4.6, that for l even

$$0 = \int \mathrm{Ric}_{e_0e_0,e_0e_0\dots e_0} = \sum_{k=1}^{l/2} C_{l,k}\Delta^k s \tag{4-28}$$

where the order of the covariant derivative above is l. It is straightforward to check that the highest order coefficient $C_{l,l/2}$ is not equal to 0. Then, by induction, $\Delta^k s = 0$ at p for $1 \le k \le m-2$. Combining this with Lemma 4.6, it follows that $A = 0$. Similarly all linear terms in (4-10) vanish. Therefore, under the induction hypothesis in Proposition 2.1, in order that c_{2m} in (4-10) achieves the maximum, we must have $\mathrm{Ric}^{(2m-2)} = 0$ and $R^{(\lambda)}_{uv} = 0$ for $1 \le \lambda \le m-2$. This is exactly the case of the complex space form. Therefore we have completed the induction step for part (1) in Proposition 2.1 and, as a byproduct, we have proved part (2) as well. The proof is thus complete. □

5. The proof of Theorem 1.4

Proof of Proposition 2.2. Using the same argument as in the last section, we find that $W^{(2m+1)}$ is a linear combination of $\int R_{11}^{(m+i)}$ for $1 \le i \le m-3$ (the terms of order greater than $2m-3$ can be absorbed into $\mathrm{Ric}^{(m+i)}$). Then $W^{(2m+1)}$ is equal to 0 by similar arguments as in the proof of Lemma 4.6. □

Proof of Theorem 1.4. Consider the two cases below:

1. All coefficients of the power series of W are equal to that of the complex space form. From Proposition 2.1, all covariant derivatives of the curvature tensor at p are the same as the complex space form. Since the metric is real analytic, we conclude that near p, the manifold is isometric to the complex space form.

2. There is a $i_0 \ge 1$ such that for all $i < i_0$, the coefficients of the power series of W are equal to that of the complex space form, but the i_0-th coefficient is less than that of the complex space form. Checking the power series of W'/W at p,

we find that for sufficiently small r, W'/W is less than that of the complex space form. From the definition of W we have, for small r,

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} = \frac{\int \sqrt{\det\langle J_u, J_v\rangle}\,'}{\int \sqrt{\det\langle J_u, J_v\rangle}} < \Delta_{N_K} r(r). \qquad \square$$

6. An example

In this section we give an example showing that the analogous Laplacian comparison theorem is not true on Kähler manifolds when the Ricci curvature is bounded from below by a nonzero constant. The example is in dimension 2. For higher dimensions, the construction is similar.

Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in the usual way. The corresponding almost complex structure J is given by

$$J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}, \quad J\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}, \quad J\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4} \quad \text{and} \quad J\frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}.$$

Given a small ball near the origin of $\mathbb{C}^2$, define a function f to be

$$f = |z_1|^2 + |z_2|^2 + a|z_1|^4 + 8a|z_1|^2|z_2|^2 + a|z_2|^4 + \tfrac{8}{3}a^2|z_1|^6 \\ +28a^2|z_1|^4|z_2|^2 + 28a^2|z_1|^2|z_2|^4 + \tfrac{8}{3}a^2|z_2|^6 + p(|z_1|, |z_2|),$$

where a is a nonzero constant and p is a homogeneous polynomial of degree 8 to be determined later. We define

$$\omega = \tfrac{\sqrt{-1}}{2}\partial\bar{\partial} f = \tfrac{\sqrt{-1}}{2}\sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j.$$

It is straightforward to check that ω defines a Kähler metric g if the ball is sufficiently small (note that the metric is not complete).

Direct computation gives

$$g_{1\bar{1}} = 1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 \\ +112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4 + O((|z_1| + |z_2|)^6),$$

$$g_{1\bar{2}} = 8a\bar{z}_1 z_2 + 56a^2 z_1\bar{z}_1^2 z_2 + 56a^2\bar{z}_1 z_2^2\bar{z}_2 + O((|z_1| + |z_2|)^6),$$

and $g_{2\bar{2}} = g_{1\bar{1}}$. Therefore

$$\begin{aligned}\det(g_{i\bar{j}}) &= g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 \\ &= \big(1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4\big)^2 \\ &\qquad -\big|8a\bar{z}_1 z_2 + 56a^2 z_1\bar{z}_1^2 z_2 + 56a^2\bar{z}_1 z_2^2\bar{z}_2\big|^2 + O((|z_1| + |z_2|)^6) \\ &= 1 + 12a(|z_1|^2 + |z_2|^2) + 84a^2(|z_1|^4 + |z_2|^4) \\ &\qquad +240a^2|z_1|^2|z_2|^2 + O((|z_1| + |z_2|)^6).\end{aligned}$$

Using $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$, we have

$$\mathrm{Ric} + 12ag = \partial\bar{\partial}(-\log(\det g_{i\bar{j}}) + 12af) = \partial\bar{\partial}\, O((|z_1| + |z_2|)^6).$$

Therefore $\mathrm{Ric}+12ag$ vanishes up to order 3 at the origin. If we choose the function p to be $-\lambda(|z_1|^8 + |z_2|^8 + 8(|z_1|^6|z_2|^2 + |z_1|^2|z_2|^6))$, a direct computation gives

$$\mathrm{Ric} + 12ag = \partial\bar{\partial}(24\lambda(|z_1|^2 + |z_2|^2)^3 + O((|z_1| + |z_2|)^6)$$

where the term $O((|z_1| + |z_2|)^6)$ does not depend on λ. If λ is sufficiently large, $\mathrm{Ric} + 12ag \geq 0$ near the origin. Set $K = -12a$. Thus, near the origin, $\mathrm{Ric} \geq K$. By direct computation $R_{1212} = R_{1313} = R_{1414} = 4a$ and $R_{1u1v} = 0$ at the origin if $u \neq v$. Combining this with the fact that the second derivatives of the Ricci tensor vanish at the origin we find, after a slight computation, that the fourth order term of (3-3) is greater than that of the complex space form if $e_0 = \partial/\partial x_1$. So when r is very small, $\sqrt{\det\langle J_u, J_v\rangle}$ is greater than that of the complex space form along the geodesic with initial direction $\partial/\partial x_1$ at the origin. Since

$$\Delta r = \frac{\partial \log \sqrt{\det\langle J_u, J_v\rangle}}{\partial r},$$

it follows that the pointwise Laplacian comparison with the complex space forms is not true for Kähler manifolds.

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Volume 254 No. 2 December 2011

Curvatures of spheres in Hilbert geometry 257

ALEXANDER BORISENKO and EUGENE OLIN

A formula equating open and closed Gromov–Witten invariants and its applications to mirror symmetry 275

KWOKWAI CHAN

A note on p-harmonic l-forms on complete manifolds 295

LIANG-CHU CHANG and CHIUNG-JUE ANNA SUNG

The Cheeger constant of curved strips 309

DAVID KREJČIŘÍK and ALDO PRATELLI

Structure of solutions of 3D axisymmetric Navier–Stokes equations near maximal points 335

ZHEN LEI and QI S. ZHANG

Local comparison theorems for Kähler manifolds 345

GANG LIU

Structurable algebras of skew-rank 1 over the affine plane 361

SUSANNE PUMPLÜN

An analogue of Krein's theorem for semisimple Lie groups 381

SANJOY PUSTI

Une remarque de dynamique sur les variétés semi-abéliennes 397

GAËL RÉMOND

Fourier transforms of semisimple orbital integrals on the Lie algebra of SL_2 407

LOREN SPICE

On noncompact τ-quasi-Einstein metrics 449

LIN FENG WANG

Decomposition of de Rham complexes with smooth horizontal coefficients for semistable reductions 465

QIHONG XIE

A differentiable sphere theorem inspired by rigidity of minimal submanifolds 499

HONG-WEI XU and LING TIAN