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We give an integral representation of K-positive definite functions on a real rank n connected, noncompact, semisimple Lie group with finite centre. Moreover, we characterize the λ's for which the τ-spherical function $\phi^{\tau}_{\sigma,\lambda}$ is positive definite for the group $G = \mathrm{Spin}_e(n, 1)$ and the complex spin representation τ.

1. Introduction

A continuous function f on $\mathbb{R}$ is said to be *positive definite* if for any real numbers $x_1, \dots, x_m$ and complex numbers $\xi_1, \dots, \xi_m$ the following holds:

$$\sum_{k,j=1}^{m} f(x_j - x_k)\xi_k\overline{\xi_j} \geq 0.$$

This definition is equivalent to

$$\int_{\mathbb{R}} f(x)(\phi * \phi^*)(-x)\,dx \geq 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}),$$

where $\phi^*(x) = \overline{\phi(-x)}$. Also, an even continuous function f on $\mathbb{R}$ is said to be *evenly positive definite* if

$$\int_{\mathbb{R}} f(x)(\phi * \phi^*)(-x)\,dx = \int_{\mathbb{R}} f(x)(\phi * \phi^*)(x)\,dx \geq 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R})_e,$$

where $C_c^\infty(\mathbb{R})_e$ denotes the set of infinitely differentiable compactly supported even functions on $\mathbb{R}$. Then it is clear that the set of even positive definite functions is a subset of the set of evenly positive definite functions. Bochner's theorem and M. G. Krein's theorem respectively give integral representations of positive definite functions and evenly positive definite functions. Precisely, for a positive definite function f on $\mathbb{R}$, there exists a finite positive measure μ on $\mathbb{R}$ such that

$$f(x) = \int_{\mathbb{R}} e^{i\lambda x}\,d\mu(\lambda).$$

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Also for an evenly positive definite function f on $\mathbb{R}$, there exists a finite positive even measure σ on $\mathbb{R} \cup i\mathbb{R}$ such that

$$f(x) = \int_{\mathbb{R}\cup i\mathbb{R}} e^{i\lambda x}\, d\sigma(\lambda).$$

From this integral representation it follows that a bounded evenly positive definite function is a positive definite function. We note that the measure σ in the integral representation of an evenly positive definite function is not unique, whereas the measure μ in the integral representation of a positive definite function is unique. However, if an evenly positive definite function satisfies a certain restriction on its growth for $|x| \to \infty$, then the integral representation becomes unique [Gelfand and Vilenkin 1964].

Let G be a connected, noncompact semisimple Lie group with finite centre, and let K be a fixed maximal compact subgroup of G. Integral representations of K-positive definite distributions and K-positive definite functions have been derived for real rank-one semisimple Lie groups with finite centre in [Sitaram 1978] and [Pusti 2011], respectively.

An analogue of Krein's theorem on $\mathbb{R}^n$ has been obtained by N. Bopp [1979]. In this case, instead of evenly positive definite functions, one considers functions which are positive definite relative to the action of a finite subgroup of $\mathrm{O}(n)$. Here too, if we impose a certain growth condition, then the integral representation of these functions is unique. In this paper, using Bopp's result, we derive an integral representation for the K-positive definite functions on a real rank n connected, noncompact, semisimple Lie group with finite centre. We observe that the set of positive definite functions is a proper subset of the set of K-positive definite functions. Next, we consider the τ-positive definite functions, $\tau \in \widehat{K}$. The K-positive definite functions are a special instance of the τ-positive definite functions (for τ equals the trivial representation). We give an example in which the set of τ-positive definite functions is same as the set of positive definite functions. That is, the same conclusion (as in K-positive definite function) is not true for τ-positive definite functions. Finally we characterize the λ's for which the τ-spherical function $\phi^{\tau}_{\sigma,\lambda}$ is a positive definite function for the group $G = \mathrm{Spin}_e(n, 1)$ and the complex spin representation τ. We note that G. van Dijk and A. Pasquale [1999] studied positive definiteness of $\phi^{\tau}_{\sigma,\lambda}$ for the group $G = \mathrm{Sp}(1, n)$.

2. Preliminaries

Most of our notations are standard and can be found in [Anker 1991]. Let G be a real rank n connected, noncompact, semisimple Lie group with finite centre with Lie algebra $\mathfrak{g}$, and let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition of $\mathfrak{g}$, and let K be the maximal compact subgroup of G

with Lie algebra $\mathfrak{k}$. We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$. Since G is of real rank n, we have $\dim \mathfrak{a} = n$. We denote the real dual of $\mathfrak{a}$ by $\mathfrak{a}^*$ and its complex dual by $\mathfrak{a}^*_{\mathbb{C}}$. The Killing form of $\mathfrak{g}$ induces an Ad K-invariant scalar product on $\mathfrak{s}$ and hence a G-invariant Riemannian metric on G/K (or $K\backslash G$). With this structure, G/K is a Riemannian globally symmetric space of the noncompact type. Also, the Killing form of $\mathfrak{g}$ induces a scalar product on $\mathfrak{a}$ and hence on $\mathfrak{a}^*$. We denote by $\langle \cdot, \cdot \rangle$ its $\mathbb{C}$-bilinear extension to $\mathfrak{a}^*_{\mathbb{C}}$.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$. Here, $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\Sigma \subseteq \mathfrak{a}^*$ is the root system of $(\mathfrak{g}, \mathfrak{a})$. Let W be the Weyl group associated to Σ. We choose a set Σ^+ of positive roots. Let $\mathfrak{a}^+ \subseteq \mathfrak{a}$ be the corresponding positive Weyl chamber and let $\overline{\mathfrak{a}^+}$ be its closure. We denote by $(\mathfrak{a}^*)^+$ and $\overline{(\mathfrak{a}^*)^+}$ the similar cones in $\mathfrak{a}^*$. Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. The element $\rho \in \mathfrak{a}^*$ is defined by

$$\rho(H) = \tfrac{1}{2} \Sigma_{\alpha \in \Sigma^+} m_\alpha \alpha(H),$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$. Let A be the analytic subgroup of G with Lie algebra $\mathfrak{a}$. Then A is a closed subgroup of G and the exponential map is an isomorphism from $\mathfrak{a}$ onto A. We set $A^+ = \exp \mathfrak{a}^+$. Its closure is $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$. Let N be the analytic subgroup of G with Lie algebra $\mathfrak{n}$, and let M be the centralizer of A in K.

The group G can be decomposed as $G = K\overline{A^+}K$. It is called the Cartan decomposition of G and every element x of G can be decomposed as $x = k_1 a k_2$ with $k_1, k_2 \in K$ and $a \in \overline{A^+}$. We let x^+ be the $\overline{\mathfrak{a}^+}$-component of $x \in G$ in the decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$ and let $|x| = \|x^+\|$. Viewed on G/K, $|\cdot|$ is the distance to the origin $0 = \{K\}$. Also, the group G has Iwasawa decomposition $G = KAN$. Let $k(x)$ and $H(x)$ be the components of $x \in G$ in K and $\mathfrak{a}$. Then any element $x \in G$ can be expressed as $x = k(x) \exp H(x) n$ for some $n \in N$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the elementary spherical function ϕ_λ on G is given by

$$\phi_\lambda(x) = \int_K e^{-(i\lambda+\rho)H(x^{-1}k)}\, dk.$$

It satisfies the following properties:

(1) It is K-biinvariant, that is, $\phi_\lambda(k_1 x k_2) = \phi_\lambda(x)$ for all $k_1, k_2 \in K$ and $x \in G$. Also, it is W-invariant in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, that is, $\phi_{w\cdot\lambda}(x) = \phi_\lambda(x)$ for all $w \in W$ and $x \in G$.

(2) The function $\phi_\lambda(x)$ is C^∞ in x and holomorphic in λ.

(3) It is a joint eigenfunction for all G-invariant differential operators on G/K; in particular for the Laplacian Δ on G/K,

$$\Delta \phi_\lambda = -(\langle \lambda, \lambda \rangle + \|\rho\|^2)\phi_\lambda.$$

A function f on G is called K-biinvariant if $f(k_1xk_2)=f(x)$ for all $k_1,k_2\in K$ and $x\in G$. For a K-biinvariant function f on G, its spherical Fourier transform is defined by

$$\hat{f}(\lambda)=\int_G f(x)\phi_\lambda(x^{-1})\,dx$$

for suitable $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$.

The set of infinitely differentiable compactly supported K-biinvariant functions and infinitely differentiable K-biinvariant functions are denoted by $C_c^\infty(G/\!/K)$ and $C^\infty(G/\!/K)$, respectively. For $0<p\le 2$ the L^p-Schwartz space $\mathscr{C}^p(G/\!/K)$ is the set of all functions $f\in C^\infty(G/\!/K)$ such that

$$\sup_{x\in G}(1+|x|)^s\phi_0(x)^{-2/p}\,|f(D;x;E)|<\infty$$

for any $D,E\in\mathscr{U}(\mathfrak{g})$ and any integer $s\ge 0$. The Schwartz space $\mathscr{C}^p(G/\!/K)$ is topologized by the seminorms

$$\sigma^p_{D,E,s}(f)=\sup_{x\in G}(1+|x|)^s\phi_0(x)^{-2/p}\,|f(D;x;E)|\,.$$

Then it follows that $C_c^\infty(G/\!/K)$ is dense in $\mathscr{C}^p(G/\!/K)$ and $\mathscr{C}^p(G/\!/K)$ is dense in $L^p(G/\!/K)$.

Let $\mathscr{P}(\mathfrak{a}^*_{\mathbb{C}})$ be the space of entire functions on $\mathfrak{a}^*_{\mathbb{C}}$, which are of exponential type and rapidly decreasing. The set of W-invariant elements in $\mathscr{P}(\mathfrak{a}^*_{\mathbb{C}})$ is denoted by $\mathscr{P}(\mathfrak{a}^*_{\mathbb{C}})^W$.

For fixed $\epsilon>0$, let $C^{\epsilon\rho}$ be the convex hull of the set $W\cdot\epsilon\rho$ in $\mathfrak{a}^*$, and let $\mathfrak{a}^*_\epsilon=\mathfrak{a}^*+iC^{\epsilon\rho}$ be the tube in $\mathfrak{a}^*_{\mathbb{C}}$ with base $C^{\epsilon\rho}$. For $\epsilon=0$, $\mathfrak{a}^*_\epsilon$ reduces to $\mathfrak{a}^*$. Let $\mathbf{S}(\mathfrak{a}^*)$ be the symmetric algebra over $\mathfrak{a}^*$. We define the Schwartz space $\mathscr{S}(\mathfrak{a}^*_\epsilon)$ as the space of all complex valued functions h such that the following hold true.

(1) h is holomorphic in the interior of $\mathfrak{a}^*_\epsilon$.

(2) h and all its derivatives extend continuously to $\mathfrak{a}^*_\epsilon$.

(3) for any polynomial $P\in\mathbf{S}(\mathfrak{a}^*)$ and any (integer) $t\ge 0$,

$$\sup_{\lambda\in\mathfrak{a}^*_\epsilon}(1+\|\lambda\|)^t\left|P\Big(\frac{\partial}{\partial\lambda}\Big)h(\lambda)\right|<\infty.$$

The space $\mathscr{S}(\mathfrak{a}^*_\epsilon)$ is topologized by the seminorms

$$\tau^\epsilon_{P,t}(h)=\sup_{\lambda\in\mathfrak{a}^*_\epsilon}(1+\|\lambda\|)^t\left|P\Big(\frac{\partial}{\partial\lambda}\Big)h(\lambda)\right|.$$

We denote by $\mathscr{S}(\mathfrak{a}^*_\epsilon)^W$ the subspace of W-invariant functions in $\mathscr{S}(\mathfrak{a}^*_\epsilon)$. For $\epsilon=0$, $\mathscr{S}(\mathfrak{a}^*_\epsilon)$ becomes the classical Schwartz space on $\mathfrak{a}^*$. Then for $\epsilon\ge 0$, $\mathscr{S}(\mathfrak{a}^*_\epsilon)^W$ is a Fréchet algebra (under pointwise multiplication) and $\mathscr{P}(\mathfrak{a}^*_{\mathbb{C}})^W$ is a dense subalgebra of $\mathscr{S}(\mathfrak{a}^*_\epsilon)^W$.

We consider the function

$$\cosh_{\epsilon\rho}(H) = \frac{1}{|W|} \sum_{w \in W} e^{w \cdot \epsilon\rho(H)} \quad \text{on } \mathfrak{a}.$$

Then we define the space $\mathcal{S}_{\epsilon\rho}(\mathfrak{a})$ consisting of all functions $g \in C^\infty(\mathfrak{a})$ such that

$$\sup_{H \in \mathfrak{a}} (1 + \|H\|)^s \cosh_{\epsilon\rho}(H) \left| P\left(\frac{\partial}{\partial H}\right) g(H) \right| < \infty$$

for any polynomial $P \in \mathbf{S}(\mathfrak{a})$ (the symmetric algebra over $\mathfrak{a}$) and any $s \geq 0$.

Theorem 2.1 [Anker 1991]. (1) *The spherical Fourier transform $f \mapsto \hat{f}$ is a topological isomorphism between $C_c^\infty(G/\!/K)$ and $\mathcal{P}(\mathfrak{a}_{\mathbb{C}}^*)^W$ and also between $\mathcal{C}^p(G/\!/K)$ and $\mathcal{S}(\mathfrak{a}_\epsilon^*)^W$, where $\epsilon = 2/p - 1$.*

(2) *The Euclidean Fourier transform $f \mapsto \tilde{f}$ is a topological isomorphism between $\mathcal{S}_{\epsilon\rho}(\mathfrak{a})^W$ and $\mathcal{S}(\mathfrak{a}_\epsilon^*)^W$, where $\tilde{f}(\lambda) = \int_{\mathfrak{a}} f(H) e^{-i\lambda(H)} dH$, $\lambda \in \mathfrak{a}^*$.*

For a suitable K-biinvariant function f on G, the Abel transform is defined by

$$\mathcal{A}f(H) = e^{\rho(H)} \int_N f(\exp Hn)\, dn.$$

It satisfies the relation $\hat{f}(\lambda) = \widetilde{\mathcal{A}f}(\lambda)$ for a suitable K-biinvariant function f on G. Therefore it follows from Theorem 2.1 that the Abel transform $f \mapsto \mathcal{A}f$ is a topological isomorphism between $\mathcal{C}^p(G/\!/K)$ and $\mathcal{S}_{\epsilon\rho}(\mathfrak{a})^W$ for $\epsilon = 2/p - 1$.

3. M. G. Krein's theorem

For $\alpha \geq 0$, we define

$$\mathcal{S}_\alpha(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \|\phi\|_p < \infty \text{for any nonnegative integer } p\},$$

where

$$\|\phi\|_p = \max_{|q| \leq p} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^p e^{\alpha|x|} \left| D^q \phi(x) \right|.$$

Then $S_\alpha(\mathbb{R}^n)$ becomes a Fréchet space and $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $S_\alpha(\mathbb{R}^n)$. For a finite subgroup E of $\mathrm{O}(n)$, let $\mathcal{S}_\alpha(\mathbb{R}^n)^E$ be the subspace of E-invariant functions in $\mathcal{S}_\alpha(\mathbb{R}^n)$.

Theorem 3.1 [Bopp 1979]. *Let E be a finite subgroup of* $\mathrm{O}(n)$ *and let*

$$T : \mathcal{S}_\alpha(\mathbb{R}^n) \to \mathbb{C}$$

be a continuous, linear functional such that

(1) *$T(\eta \cdot \phi) = T(\phi)$ for all $\eta \in E$, $\phi \in \mathcal{S}_\alpha(\mathbb{R}^n)$.*

(2) *$T(\phi * \phi^*) \geq 0$ for all $\phi \in \mathcal{S}_\alpha(\mathbb{R}^n)^E$.*

Then there exists a unique positive tempered measure σ, invariant under the E-action, such that for all $\phi \in \mathcal{S}_\alpha(\mathbb{R}^n)$,

$$T(\phi) = \int_{M \cap T_\alpha} \widetilde{\phi}(\xi)\, d\sigma(\xi),$$

where $M = \{\xi \in \mathbb{C}^n : \text{there exists } \eta \in E \text{ such that } \eta.\xi = \overline{\xi}\}$ *and*

$$T_\alpha = \{\xi \in \mathbb{C}^n : |\operatorname{Im} \xi| \leq \alpha\}.$$

Since we have an isomorphism between $\mathcal{S}_{\epsilon\rho}(\mathbb{R}^n)$ and $\mathcal{S}_{\epsilon\rho}(\mathfrak{a})$ we can rewrite the theorem above in the following way:

Theorem 3.2. *Let $T : \mathcal{S}_{\epsilon\rho}(\mathfrak{a})^W \to \mathbb{C}$ be a continuous, linear functional such that*

$$T(\phi * \phi^*) \geq 0 \quad \textit{for all } \phi \in \mathcal{S}_{\epsilon\rho}(\mathfrak{a})^W.$$

Then there exists a unique positive tempered measure σ, invariant under the W-action, such that for all $\phi \in \mathcal{S}_{\epsilon\rho}(\mathfrak{a})$,

$$T(\phi) = \int_{\mathcal{M} \cap \mathfrak{a}_\epsilon^*} \widetilde{\phi}(\lambda)\, d\sigma(\lambda),$$

where $\mathcal{M} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \text{ there exists } w \in W \text{ such that } w.\lambda = \overline{\lambda}\}$.

We call a K-biinvariant continuous function f on G K-positive definite if for all $g \in C_c^\infty(G/\!/K)$,

$$\int_G f(x)(g * g^*)(x^{-1})\, dx \geq 0,$$

where $g^*(x) = \overline{g(x^{-1})}$ for all $x \in G$. If the equation above is true for every $g \in C_c^\infty(G)$ we say that f is a positive definite function. We prove the following analogue of M. G. Krein's theorem for K-positive definite functions on semisimple Lie groups.

Theorem 3.3. *For a K-positive definite function $f \in \mathcal{C}^p(G/\!/K)'$ $(0 < p \leq 2)$, there exists a unique finite positive measure σ on $\mathcal{M} \cap \mathfrak{a}_\epsilon^*$, invariant under the Weyl group action, such that for all $x \in G$*

$$f(x) = \int_{\mathcal{M} \cap \mathfrak{a}_\epsilon^*} \phi_\lambda(x)\, d\sigma(\lambda),$$

where $\epsilon = 2/p - 1$.

Proof. We define a linear functional $T_f : \mathcal{S}_{\epsilon\rho}(\mathfrak{a})^W \to \mathbb{C}$ by

$$T_f(h) = \int_G f(x)(\mathcal{A}^{-1}h)(x^{-1})\, dx.$$

The integral exists and is continuous by the given condition on f and the isomorphism of the Abel transform on $\mathscr{C}^p(G/\!/K)$. Since $\hat{f}(\lambda)=\widetilde{\mathscr{A}f}(\lambda)$ for all $f\in\mathscr{C}^p(G/\!/K)$, it follows that

$$\widehat{\mathscr{A}^{-1}h}=\widetilde{h}\quad\text{for all } h\in\mathscr{S}_{\epsilon\rho}(\mathfrak{a})^W.$$

Using this, we easily check that

$$\mathscr{A}^{-1}(h_1*h_2)=\mathscr{A}^{-1}h_1*\mathscr{A}^{-1}h_2\quad\text{and}\quad\mathscr{A}^{-1}h_1^*=\big(\mathscr{A}^{-1}h_1\big)^*$$

for all $h_1,h_2\in\mathscr{S}_{\epsilon\rho}(\mathfrak{a})^W$. Then

$$T_f(h*h^*)=\int_G f(x)\left(\mathscr{A}^{-1}h*(\mathscr{A}^{-1}h)^*\right)(x^{-1})\,dx\geq 0,$$

since f is K-positive definite. Therefore, by Theorem 3.2, there exists a unique positive tempered measure σ on $\mathcal{M}\cap\mathfrak{a}_\epsilon^*$, invariant under the Weyl group action, such that

$$T_f(h)=\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}\widetilde{h}(\lambda)\,d\sigma(\lambda)\quad\text{for all } h\in\mathscr{S}_{\epsilon\rho}(\mathfrak{a})^W.$$

This shows that

$$\text{(3-1)}\qquad\int_G f(x)g(x^{-1})\,dx=\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}\widehat{g}(\lambda)\,d\sigma(\lambda)\quad\text{for all } g\in\mathscr{C}^p(G/\!/K).$$

Now we show that the measure σ is finite. For this let $\{g_n\}$ be a Dirac-delta sequence in $\mathscr{C}^p(G/\!/K)$. Then $\{g_n*g_n^*\}$ is also a Dirac-delta sequence in $\mathscr{C}^p(G/\!/K)$. Applying this sequence to the previous equation we get

$$\int_G f(x)(g_n*g_n^*)(x^{-1})\,dx=\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}|\widehat{g_n}(\lambda)|^2\,d\sigma(\lambda).$$

Now we take the limit as $n\to\infty$ on both sides of the equation and apply Fatou's lemma to get $\sigma(\mathcal{M}\cap\mathfrak{a}_\epsilon^*)\leq f(e)$. Therefore the measure σ is finite. From (3-1) we get, using Fubini's theorem,

$$\int_G f(x)g(x^{-1})\,dx=\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}\widehat{g}(\lambda)\,d\sigma(\lambda)=\int_G g(x^{-1})\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}\phi_\lambda(x)\,d\sigma(\lambda)\,dx.$$

This is true for every $g\in\mathscr{C}^p(G/\!/K)$. Hence

$$f(x)=\int_{\mathcal{M}\cap\mathfrak{a}_\epsilon^*}\phi_\lambda(x)\,d\sigma(\lambda)\quad\text{with } \epsilon=\frac{2}{p}-1.\qquad\square$$

We can easily check that a function f on G which has an integral representation as in Theorem 3.3 is a K-positive definite function. That is, the converse of the Theorem 3.3 holds.

A K-biinvariant distribution T on G is called a K-positive definite distribution if $T(\phi * \phi^*) \geq 0$ for all $\phi \in C_c^\infty(G/\!/K)$. It is a positive definite distribution if the inequality above holds for all $\phi \in C_c^\infty(G)$. Barker [1975, p. 201] raised the question whether a K-positive definite distribution is a positive definite distribution. We shall see that the answer is negative, that is, the set of positive definite distributions is a proper subset of the set of K-positive definite distributions. For this let us consider $\lambda_0 \in \mathcal{M} \setminus \mathfrak{a}_1^*$. Our claim is that ϕ_{λ_0} is a K-positive definite distribution but not a positive definite distribution. By the Helgason–Johnson theorem ϕ_λ is bounded if and only if $\lambda \in \mathfrak{a}_1^*$. Since $\lambda_0 \notin \mathfrak{a}_1^*$, ϕ_{λ_0} is not bounded. Therefore, ϕ_{λ_0} is not a positive definite function. Hence ϕ_{λ_0} is not a positive definite distribution. Now $\lambda_0 \in \mathcal{M}$ implies that there exists $w \in W$ such that $w.\lambda_0 = \overline{\lambda}_0$. This shows that $\phi_{\overline{\lambda}_0} = \phi_{\lambda_0}$. Therefore, for a suitable K-biinvariant function f on G,

$$\int_G \phi_{\lambda_0}(x)(f * f^*)(x^{-1})\,dx = \hat{f}(\lambda_0)\widehat{f^*}(\lambda_0) = |\hat{f}(\lambda_0)|^2 \geq 0.$$

This proves our claim.

The same example also shows that the set of positive definite functions is a proper subset of the set of K-positive definite functions.

We now see in the real rank-one case that if we restrict our attention to certain classes of functions, then the set of positive definite functions is same as the set of K-positive definite functions. Any real rank-one connected, noncompact, semisimple Lie group G with finite centre can be classified (up to coverings) as

(1) $G = \mathrm{SO}_e(1, n)$, for which $m_\alpha = n - 1$ and $m_{2\alpha} = 0$,

(2) $G = \mathrm{SU}(1, n)$, for which $m_\alpha = 2n - 2$ and $m_{2\alpha} = 1$,

(3) $G = \mathrm{Sp}(1, n)$, for which $m_\alpha = 4n - 4$ and $m_{2\alpha} = 3$, or

(4) $G = \mathrm{F}_{4(-20)}$, for which $m_\alpha = 8$ and $m_{2\alpha} = 7$.

Let $\mathcal{P}_K$ and $\mathcal{P}$ be the set of K-positive definite functions and the set of positive definite functions on G respectively.

Proposition 3.4. (1) *For the groups* $G = \mathrm{SO}_e(1, n)$ *and* $G = \mathrm{SU}(1, n)$, *we have* $\mathcal{P}_K \cap L^\infty(G/\!/K) = \mathcal{P} \cap L^\infty(G/\!/K)$.

(2) *For the group* $G = \mathrm{Sp}(1, n)$, *we have* $\mathcal{P}_K \cap L^r(G/\!/K) = \mathcal{P} \cap L^r(G/\!/K)$, *for any* $2 < r \leq (2n + 1)$.

(3) *For the group* $G = \mathrm{F}_{4(-20)}$, *we have* $\mathcal{P}_K \cap L^r(G/\!/K) = \mathcal{P} \cap L^r(G/\!/K)$, *for any* $2 < r \leq \frac{11}{3}$.

Proof. It is known [Flensted-Jensen and Koornwinder 1979] that ϕ_λ is positive definite if and only if $\lambda \in \mathbb{R}$ or $\lambda = i\eta$ for $\eta \in [-s_0, s_0] \cup \{\pm\rho\}$, where $s_0 = \rho$ if $m_{2\alpha} = 0$, otherwise $s_0 = \frac{1}{2}m_\alpha + 1$. Therefore:

(a) For the groups $G = \mathrm{SO}_e(1, n)$ and $G = \mathrm{SU}(1, n)$, ϕ_λ is positive definite if and only if $\lambda \in \mathbb{R}$ or $\lambda = i\eta$ with $\eta \in [-\rho, \rho]$.

(b) For the group $G = \mathrm{Sp}(1, n)$, ϕ_λ is positive definite if and only if $\lambda \in \mathbb{R}$ or $\lambda = i\eta$ with $\eta \in [-(2n-1), (2n-1)] \cup \{\pm(2n+1)\}$.

(c) For the group $G = \mathrm{F}_{4(-20)}$, ϕ_λ is positive definite if and only if $\lambda \in \mathbb{R}$ or $\lambda = i\eta$ with $\eta \in [-5, 5] \cup \{\pm 11\}$.

We know that

$$\phi_\lambda \in L^\infty(G/\!/K) \text{ if and only if } \lambda \in S_1, \tag{3-2}$$

where $S_r = \{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \le (2/r - 1)\rho\}$, $r > 0$. Also, for $r > 1$,

$$\phi_\lambda \in L^{r'}(G/\!/K) \text{ if and only if } \lambda \in S_r^\circ, \tag{3-3}$$

where $1/r + 1/r' = 1$ [Pusti et al. 2011]. Now, from Theorem 3.3, a K-positive definite function can be expressed as

$$f(x) = \int_{\mathbb{R}} \phi_\lambda(x)\, d\mu_1(\lambda) + \int_{i\mathbb{R}} \phi_\lambda(x)\, d\mu_2(\lambda),$$

where μ_1 is a finite positive measure and μ_2 is a positive measure such that the integral $\int_{\mathbb{R}} \phi_{i\lambda}(x)\, d\mu_2(\lambda)$ exists for every $x \in G$.

If the K-positive definite function f is in $L^\infty(G/\!/K)$, the measure μ_2 must be supported in $i[-\rho, \rho]$ by (3-2). Therefore a K-positive definite function f which is in $L^\infty(G/\!/K)$ has an integral form:

$$f(x) = \int_{\mathbb{R}} \phi_\lambda(x)\, d\mu_1(\lambda) + \int_{i[-\rho,\rho]} \phi_\lambda(x)\, d\mu_2(\lambda).$$

However, this is a positive definite function for the groups $G = \mathrm{SO}_e(1, n)$ and $G = \mathrm{SU}(1, n)$, by (a). This proves (1).

To prove (2), note that if the K-positive definite function f is in $L^r(G/\!/K)$, $2 < r \le (2n+1)$ by (3-3) it follows that the measure μ_2 must be supported in

$$i\mathbb{R} \cap S_{r'}^\circ \subseteq i(-(2n-1), (2n-1)).$$

This proves that the function f is positive definite.

The proof of (3) is similar. □

Remark 3.5. For the groups $G = \mathrm{SO}_e(1, n)$ and $G = \mathrm{SU}(1, n)$, it follows from the above that a K-positive definite function is a positive definite function if and only if $f \in L^\infty(G/\!/K)$. However, a similar statement is not true for the groups $G = \mathrm{Sp}(1, n)$ and $G = \mathrm{F}_{4(-20)}$. In fact, for the group $G = \mathrm{Sp}(1, n)$, the function $\phi_{(2n+1)i}$ is K-positive definite as well as positive definite, but it does not belong to any $L^r(G/\!/K)$, $2 < r \le (2n+1)$. Similarly, for the group $G = \mathrm{F}_{4(-20)}$, the function

ϕ_{11i} is K-positive definite as well as positive definite but it does not belong to any $L^r(G/\!/K)$, $2 < r \leq \frac{11}{3}$.

Let $f \in \mathscr{C}^2(G/\!/K)$ be a K-positive definite function. Then by Theorem 3.3 there exists a finite positive measure σ, invariant under the Weyl group action such that

$$f(x) = \int_{\mathcal{M}\cap\mathfrak{a}^*} \phi_\lambda(x)\, d\sigma(\lambda).$$

However, this is a positive definite function on G because ϕ_λ is positive definite for $\lambda \in \mathfrak{a}^*$. Hence $\mathscr{P}_K \cap \mathscr{C}^2(G/\!/K) = \mathscr{P} \cap \mathscr{C}^2(G/\!/K)$ (cf. [Bopp 1979] for distributions).

4. τ-positive definite functions

In this section we give an example in which the set of τ-positive definite functions is same as the set of positive definite functions (without imposing any decay condition on functions). For defining the τ-positive definite functions we recall some basic facts [Camporesi 1997; Camporesi and Pedon 2001].

Definition 4.1. For $\tau \in \widehat{K}$ a scalar valued function f on G is said to be τ-radial if $f(kxk^{-1}) = f(x)$ for all $k \in K$, $x \in G$ and if $d_\tau \overline{\chi}_\tau * f = f = f * d_\tau \overline{\chi}_\tau$, where χ_τ and d_τ are respectively the character and dimension of τ.

When τ is the trivial representation of K, a τ-radial function is a K-biinvariant function. We note that the τ-radial functions are radial sections of the homogeneous vector bundle over G/K associated with the representation $\tau \in \widehat{K}$. The set of all compactly supported τ-radial infinitely differentiable functions and infinitely differentiable τ-radial functions are denoted by $C^\infty_{c,\tau}(G)$ and $C^\infty_\tau(G)$, respectively.

Definition 4.2. A τ-radial continuous function f on G is called *τ-positive definite* if

$$\int_G f(x)(g * g^*)(x^{-1})\, dx \geq 0, \quad \text{for all } g \in C^\infty_{c,\tau}(G).$$

Let $G = \mathrm{Spin}_e(n, 1)$, the identity component of $\mathrm{Spin}(n, 1)$. Then, in the notation of the previous section, $K = \mathrm{Spin}(n)$ and $M = \mathrm{Spin}(n-1)$. In the rest of the section we fix these meanings for G, K, M.

Let τ_n be the complex spin representation of K. The following proposition gives information about the irreducibility of τ_n.

Proposition 4.3 [Camporesi and Pedon 2001]. (1) *If n is even, then τ_n splits into two irreducible components given by the positive and negative half-spin representations $\tau_n = \tau_n^+ \oplus \tau_n^-$ and $\tau_n^\pm|_M = \sigma_{n-1}$, where σ_{n-1} is the spin representation of M.*

(2) *If n is odd, then τ_n is irreducible and $\tau_n|_M = \sigma_{n-1}^+ \oplus \sigma_{n-1}^-$, where $\sigma_{n-1}^\pm$, are irreducible components of the spin representation σ_{n-1} of M.*

It is known that (G, K, τ) is a Gelfand triple, that is, the convolution algebra $C^\infty_{c,\tau}(G)$ is commutative when $\tau \in \widehat{K}$ is either τ_n^+ or τ_n^- if n is even and τ_n if n is odd. For n even the $\tau_n^\pm$-spherical function is given by

$$\phi_\lambda^{\tau_n^\pm}(x) = \int_K e^{-(i\lambda+\rho)H(xk)} \chi_{\tau_n^\pm}(kK(xk)^{-1})\, dk.$$

Also, it satisfies $\phi_{-\lambda}^{\tau_n^\pm}(x) = \phi_\lambda^{\tau_n^\pm}(x)$.

For n odd the τ_n-spherical functions are denoted by $\phi^{\tau_n}_{\sigma^+_{n-1},\lambda}$ and $\phi^{\tau_n}_{\sigma^-_{n-1},\lambda}$. They are given by the integral formula

$$\phi^{\tau_n}_{\sigma^\pm_{n-1},\lambda}(x) = 2d_{\sigma^\pm_{n-1}} \int_K \int_M e^{-(i\lambda+\rho)H(xk)} \chi_{\tau_n}(km^{-1}K(xk)^{-1}) \chi_{\sigma^\pm_{n-1}}(m)\, dm\, dk.$$

They satisfy $\phi^{\tau_n}_{\sigma^+_{n-1},-\lambda}(x) = \phi^{\tau_n}_{\sigma^-_{n-1},\lambda}(x)$.

From now on by $\tau \in \widehat{K}$ we will mean either $\tau = \tau_n^+$ or $\tau = \tau_n^-$ if n is even and $\tau = \tau_n$ if n is odd. For n even we shall write the $\tau_n^\pm$-spherical functions $\phi^\tau_{\sigma,\lambda}$ instead of $\phi^{\tau_n^\pm}_{\sigma_{n-1},\lambda}$. Also for n odd we write the τ_n-spherical functions

$$\phi^\tau_{\sigma^\pm,\lambda} \quad \text{instead of} \quad \phi^{\tau_n}_{\sigma^\pm_{n-1},\lambda}.$$

Henceforth while dealing with $G = \mathrm{Spin}_e(n, 1)$ and τ as above we shall simply say *when n is even* and *when n is odd* to distinguish between these two cases.

For a τ-radial function f its spherical Fourier transform is defined by

$$\hat{f}(\sigma, \lambda) = \int_G f(x)\phi^\tau_{\sigma,\lambda}(x^{-1})\, dx$$

when n is even. For n odd it is defined by

$$\hat{f}(\sigma^\pm, \lambda) = \int_G f(x)\phi^\tau_{\sigma^\pm,\lambda}(x^{-1})\, dx.$$

Theorem 4.4 [Gelfand and Vilenkin 1964, Theorem 3, p. 157, Theorem 5, p. 226].

(a) *Let T be a positive definite distribution on $\mathbb{R}$, that is,*

$$T(\phi * \phi^*) \geq 0 \quad \textit{for all } \phi \in C_c^\infty(\mathbb{R}).$$

Then there exists a positive tempered measure μ on $\mathbb{R}$ such that

$$T(\phi) = \int_{\mathbb{R}} \widetilde{\phi}(\lambda)\, d\mu(\lambda) \quad \textit{for all } \phi \in C_c^\infty(\mathbb{R}).$$

(b) *Let T be an evenly positive definite distribution on $\mathbb{R}$, that is, $T(\phi * \phi^*) \geq 0$ for all $\phi \in C_c^\infty(\mathbb{R})_e$. Then there exists positive even measures μ_1 and μ_2 such that*

$$T(\phi) = \int_{\mathbb{R}} \widetilde{\phi}(\lambda)\, d\mu_1(\lambda) + \int_{\mathbb{R}} \widetilde{\phi}(i\lambda)\, d\mu_2(\lambda) \quad \textit{for all } \phi \in C_c^\infty(\mathbb{R})_e,$$

where μ_1 is a tempered measure and μ_2 is such that

$$\int_{\mathbb{R}} e^{a|\lambda|}\, d\mu_2(\lambda) < \infty \quad \textit{for all } a > 0.$$

The next theorem gives integral representations of τ-positive definite functions on $G = \mathrm{Spin}_e(n, 1)$.

Theorem 4.5. *Let $G = \mathrm{Spin}_e(n, 1)$ and let τ denote one of $\{\tau_n^+, \tau_n^-\}$ when n is even and τ_n when n is odd.*

(a) *Let n be even and let f be a τ-positive definite function on G. Then there exists even positive measures μ_1 and μ_2 such that for all $x \in G$*

$$f(x) = \int_{\mathbb{R}} \phi^{\tau}_{\sigma,\lambda}(x)\, d\mu_1(\lambda) + \int_{\mathbb{R}} \phi^{\tau}_{\sigma,i\lambda}(x)\, d\mu_2(\lambda),$$

where μ_1 is finite measure and μ_2 is such that

$$\int_{\mathbb{R}} e^{a|\lambda|}\, d\mu_2(\lambda) < \infty \quad \textit{for all } a > 0.$$

(b) *Let n be odd and let f be a τ-positive definite function on G. Then there exists a finite positive measure μ such that for all $x \in G$*

$$f(x) = \int_{\mathbb{R}} \phi^{\tau}_{\sigma^+,\lambda}(x)\, d\mu(\lambda).$$

Proof. We shall prove (b). The proof of (a) is similar. Let n be odd and let f be a τ-positive definite function on G. We define the linear functional T_f on $C_c^\infty(\mathbb{R})$ as follows:

$$T_f(h) = \int_G f(x)(\mathcal{A}^{-1}h)(x^{-1})\, dx \quad \text{for all } h \in C_c^\infty(\mathbb{R}).$$

Here $\mathcal{A}$ is the Abel transform, which is a topological isomorphism between $C^\infty_{c,\tau}(G)$ and $C_c^\infty(\mathbb{R})$. We also have $\hat{f}(-\lambda) = \widetilde{\mathcal{A}f}(\lambda)$ for all $f \in C^\infty_{c,\tau}(G)$. Then it follows that $\widehat{\mathcal{A}^{-1}h}(-\lambda) = \widetilde{h}(\lambda)$ for all $h \in C_c^\infty(\mathbb{R})$. Using this, we easily check that

$$\mathcal{A}^{-1}(h_1 * h_2) = \mathcal{A}^{-1}h_1 * \mathcal{A}^{-1}h_2$$

and $\mathcal{A}^{-1}h_1^* = (\mathcal{A}^{-1}h_1)^*$ for all $h_1, h_2 \in C_c^\infty(\mathbb{R})$. Then

$$T_f(h * h^*) = \int_G f(x)(\mathcal{A}^{-1}h * (\mathcal{A}^{-1}h)^*)(x^{-1})\, dx \geq 0$$

as f is τ-positive definite. Therefore, by (a), there exists a positive tempered measure μ on $\mathbb{R}$ such that for all $h \in C_c^\infty(\mathbb{R})$

$$T_f(h) = \int_{\mathbb{R}} \widetilde{h}(\lambda)\, d\mu(\lambda).$$

This shows that for all $g \in C^\infty_{c,\tau}(G)$

$$\int_G f(x)g(x^{-1})\,dx = \int_{\mathbb{R}} \widehat{g}_+(\lambda)\,d\mu(\lambda). \tag{4-1}$$

Using approximate identity techniques we can easily prove that the measure μ is finite. Then from Equation (4-1), using Fubini's theorem we get

$$\begin{aligned}\int_G f(x)g(x^{-1})\,dx &= \int_{\mathbb{R}}\int_G g(x)\phi^\tau_{\sigma^+,\lambda}(x^{-1})\,dx\,d\lambda\\ &= \int_G g(x^{-1})\int_{\mathbb{R}} \phi^\tau_{\sigma^+,\lambda}(x)\,d\mu(\lambda)\,dx.\end{aligned}$$

Since this is true for every $g \in C^\infty_{c,\tau}(G)$, it follows that

$$f(x) = \int_{\mathbb{R}} \phi^\tau_{\sigma^+,\lambda}(x)\,d\mu(\lambda).$$ □

It is easy to check that the converse of Theorem 4.5 holds true. We get the following corollary from Theorem 4.5(b):

Corollary 4.6. *The set of τ-positive definite functions is same as the set of positive definite functions when $\tau = \tau_n$ and n is odd.*

Remark 4.7. We saw after Theorem 3.3 that the function ϕ_{λ_0}, $\lambda_0 \in \mathcal{M} \setminus \mathfrak{a}_1^*$ is K-positive definite but not positive definite. When $\tau = \tau_n$ and n is odd, we could try to find a similar example by considering the function $\phi^\tau_{\sigma^+,\lambda_0}$, $\lambda_0 \in i\mathbb{R} \setminus i[-1, 1]$. But $\phi^\tau_{\sigma^+,\lambda_0}$ is neither a τ-positive definite function nor a positive definite function. The argument used in the K-positive definite case does not work here. Indeed, unlike the case of the spherical functions ϕ_λ, which are W-invariant in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, there is no relation between $\phi^\tau_{\sigma^+,\lambda}$ and $\phi^\tau_{\sigma^+,-\lambda}$ when $\tau = \tau_n$ and n is odd.

We Now characterize the λ's for which $\phi^\tau_{\sigma,\lambda}$ is positive definite for

$$G = \mathrm{Spin}_{\mathrm{e}}(n, 1)$$

when τ is the irreducible component of the complex spin representation.

Theorem 4.8. *Let $G = \mathrm{Spin}_{\mathrm{e}}(n, 1)$ and let τ denote one of $\{\tau_n^+, \tau_n^-\}$ when n is even and τ_n when n is odd. Then*

(a) *$\phi^\tau_{\sigma,\lambda}$ is positive definite if and only if $\lambda \in \mathbb{R}$ when $n(\geq 4)$ is even, and*

(b) *$\phi^\tau_{\sigma^\pm,\lambda}$ are positive definite if and only if $\lambda \in \mathbb{R}$ when n is odd.*

Proof. (a) Let n be even and $n \geq 4$. The τ-spherical function $\phi^\tau_{\sigma,\lambda}$ is positive definite if and only if τ is contained in the unitary principal, discrete or complementary series representations. It is well-known that there is no discrete series representation which contains τ. Also, by [Knapp and Stein 1971, Proposition 55] and the Frobenius reciprocity theorem there is no complementary series containing τ. Hence $\phi^{\tau}_{\sigma,\lambda}$ is positive definite if and only if $\lambda \in \mathbb{R}$.

(b) For the case n odd we prove the result without using representation theory. By Corollary 4.6 the τ-spherical function $\phi^{\tau}_{\sigma^+,\lambda}$ is positive definite if and only if it is a τ-positive definite function. That is equivalent to

$$\int_G (f * f^*)(x)\phi^{\tau}_{\sigma^+,\lambda}(x^{-1})\,dx \geq 0 \quad \text{for all } f \in \mathscr{C}^2_\tau(G),$$

where $\mathscr{C}^2_\tau(G)$ is the set of τ-radial L^2-Schwartz class functions on G. That is,

$$\hat{f}(\sigma^+,\lambda)\overline{\hat{f}(\sigma^+,\bar{\lambda})} \geq 0 \quad \text{for all } f \in \mathscr{C}^2_\tau(G), \tag{4-2}$$

since $\overline{\phi^{\tau}_{\sigma,\lambda}(x)} = \phi^{\tau}_{\sigma,\bar{\lambda}}(x^{-1})$. Let us consider a function

$$f \in \mathscr{C}^2_\tau(G)$$

such that $\hat{f}(\sigma^+,\lambda) = \lambda e^{-\lambda^2}$. Such a function exists by the Schwartz space isomorphism theorem [Camporesi and Pedon 2001, Theorem 6.3]. Then (4-2) is true if and only if $\lambda \in \mathbb{R}$. □

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