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FOURIER TRANSFORMS OF SEMISIMPLE ORBITAL INTEGRALS ON THE LIE ALGEBRA OF SL_2

LOREN SPICE

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The Harish-Chandra–Howe local character expansion expresses the characters of reductive, *p*-adic groups in terms of Fourier transforms of nilpotent orbital integrals on their Lie algebras, and Murnaghan–Kirillov theory expresses many characters of reductive, *p*-adic groups in terms of Fourier transforms of semisimple orbital integrals (also on their Lie algebras). In many cases, the evaluation of these Fourier transforms seems intractable, but for SL_2, the nilpotent orbital integrals have already been computed. We compute Fourier transforms of semisimple orbital integrals using a variant of Huntsinger's integral formula and the theory of *p*-adic special functions.

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1. Introduction

1A. ***History.*** Harish-Chandra's p-adic Lefschetz principle suggests that results in real harmonic analysis should have analogues in p-adic harmonic analysis. This principle has had too many successes to list, but it is interesting that the paths to

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results in the Archimedean and non-Archimedean settings are often different. One striking manifestation of this is that the characters for the discrete series of real groups were found *before* the representations to which they were associated were constructed (see [Harish-Chandra 1966, Theorem 16; Schmid 1968, Theorem 4]), whereas, in the p-adic setting, we now have explicit constructions of many representations (see [Howe 1971; Corwin 1989; Moy 1986; Morris 1992; Bushnell and Kutzko 1993a; 1993b; 1994; Moy and Prasad 1994; Adler 1998; Yu 2001; Stevens 2008], among many others), but explicit character tables are still very rare.

This scarcity is of particular concern because, as suggested by Sally, it should be the case that "characters tell all" [Sally and Spice 2009, p. 104]. Note, for example, the recent work of Langlands [2011], which uses in a crucial way (see [ibid., Section 1.d]) the character formulae of [Sally and Shalika 1968] to show the existence of a transfer map dual to the transfer of stable characters, but only for SL_2. It seems likely that one of the main obstacles to extending the results of [Langlands 2011] to other groups is the absence of explicit character formulae for them.

The good news here is that much *is* known about the behaviour of characters in general. For example, the Harish-Chandra–Howe local character expansion [Howe 1973; Harish-Chandra 1999; DeBacker 2002] and Murnaghan–Kirillov theory [Murnaghan 1995a; 1995b; 1996a; 1996b; 2000; Kim and Murnaghan 2003; 2006] give information about the asymptotics (near the identity element) of characters of p-adic groups in terms of Fourier transforms of orbital integrals (nilpotent or semisimple) on the Lie algebra, and many existing character formulae are stated in terms of such orbital integrals; see, for example, [DeBacker 1997, Theorem 5.3.2; Spice 2005, Theorems 6.6 and 7.18; Adler and Spice 2009, Theorem 7.1; DeBacker and Reeder 2009, Lemma 10.0.4]. See also [Adler and Spice 2009, Section 0.1] for a more exhaustive description of what is known in the supercuspidal case.

The bad news is that many applications require completely explicit character tables — in particular, the evaluation of Fourier transforms of orbital integrals when they appear — but Hales [1994] has shown that the orbital integrals may themselves be "nonelementary". This term has a technical meaning, but, for our purposes, it suffices to regard it informally as meaning "difficult to evaluate". (Note, though, that the asymptotic behaviour of orbital integrals "near ∞" is understood in all cases; see [Waldspurger 1995, Proposition VIII.1].) Since SL_2 is both simple enough for many explicit computations to be tractable (for example, the Fourier transforms of nilpotent orbital integrals have already been computed [DeBacker and Sally 2000, Appendix A.3–A.4]), and complicated enough for interesting phenomena to be apparent (for example, unlike GL_2 and PGL_2, it admits nonstable characters), it is a natural focus for our investigations.

Another perspective on the behaviour of characters in the range where Murnaghan–Kirillov theory holds is offered in [Corwin et al. 1995, Theorem 4.2(d); Takahashi

2003, Proposition 2.9(2); 2005, Theorem 2.5], where explicit mention of orbital integrals is replaced (on the "bad shell" — see Section 10B) by arithmetically interesting sums, identified in [Takahashi 2003; 2005] as Kloosterman sums. In fact, exponential sums — specifically, Gauss sums — have long been observed in p-adic harmonic analysis; see, for example, [Corwin et al. 1995, Proposition 3.7; Waldspurger 1995, Section VIII.1; DeBacker 1997, p. 55; Shalika 2004, Section 1.3; Adler and Spice 2009, Section 5.2].

The work recorded here was carried out while preparing [Adler et al. 2011], which provides a proof of the aforementioned SL_2 character formulae [Sally and Shalika 1968] by specialising the results of [Adler and Spice 2009; DeBacker and Reeder 2009]. As discussed above, these general results are stated in terms of Fourier transforms of orbital integrals (see Definition 5.5); so, in order to obtain completely explicit formulae, it was necessary to evaluate those Fourier transforms. The author of the present paper was surprised to discover that this latter evaluation reduced to the computation of *Bessel functions* (see Section 7 and Proposition 8.11). In retrospect, by the p-adic Lefschetz principle mentioned on the first page, it seems natural that the "special functions" described in [Sally and Taibleson 1966] will play some important role in p-adic harmonic analysis, since their classical analogues are so integral to real harmonic analysis (see, for just one example, [Gindikin and Karpelevič 1962, Theorem 2], where Harish-Chandra's $\boldsymbol{c}$-function is calculated in terms of Γ-functions). Relationships between a different sort of Bessel function and a different sort of orbital integral (adapted to the Jacquet–Ye relative trace formula) have already been demonstrated by Baruch [1997; 2001; 2003; 2004; 2005]. We will investigate further applications of complex-valued p-adic special functions in future work.

See also [Cunningham and Gordon 2009, Section 4] for a motivic approach to the calculation of Fourier transforms of semisimple orbital integrals.

1B. ***Outline of the paper.*** We need a lot of notation in order to be completely explicit; we describe it in Sections 2–7. Specifically, Sections 2–4 describe the basic notation for working with groups over p-adic fields, adapted to the particular setting of the group SL_2. Since our formulae will be written "torus-by-torus" (à la [Harish-Chandra 1970, Theorem 12]), we need to describe the tori in SL_2. This can be done very concretely; see Definition 4.1.

In Section 5, we define the functions $\hat{\mu}^G_{X^*}$ (Fourier transforms of orbital integrals) that we want to compute as representing functions for certain invariant distributions on $\mathfrak{sl}_2$ (see Definition 5.5 and Notation 5.7). Since these functions are defined only up to scalar multiples, it is important to be aware of the normalisations involved in their construction. We specify the (Haar) measures that we are using in Definition 2.1 and Proposition 11.2.

As mentioned in Section 1A, p-adic harmonic analysis tends to involve Gauss sums and other fourth roots of unity, and our calculations are no exception; we define and compare some of the relevant constants in Section 6. Finally, with these ingredients in place, we can follow [Sally and Taibleson 1966] in defining the Bessel functions that we will use to evaluate $\hat{\mu}^G_{X^*}$. Already, [Sally and Taibleson 1966] offers considerable information about the values of these functions, but we need to carry the calculations further, especially far from the identity (see Proposition 7.5) and on the "bad shell" (see Proposition 7.7), where (twisted) Kloosterman sums make an appearance.

In Section 8, we define a function $M^G_{X^*}$ (see Definition 8.4), which we will spend most of the rest of the paper computing. This is a reasonable focus because, once the computations are completed, Proposition 11.2 will show that we have actually been computing $\hat{\mu}^G_{X^*}$. The definition of $M^G_{X^*}$ involves a rather remarkable function φ_θ (see Definition 8.2 and Lemma 8.3); it seems likely that generalising our techniques will require understanding the proper replacement for φ_θ.

Proposition 8.11 describes $M^G_{X^*}$ in terms of Bessel functions, and Proposition 8.13 uses Theorem 7.4 to describe their behaviour near 0.

We now proceed according to the "type" of X^* (as in Definition 4.4). The calculations when X^* is split, and when it is unramified, are quite similar; we combine them in Section 9. We split into cases depending on whether the argument to $M^G_{X^*}$ is far from (as in Section 9A) or close to (as in Section 9B) zero; there are qualitative differences in the behaviour, as can be seen by comparing, for example, Theorems 9.5 and 9.7. When X^* is ramified, it turns out that, in addition to the behaviour far from (as in Section 10A) and close to (as in Section 10C) zero, there is a third range of interest in the middle. This is the so called "bad shell" (see Section 10B), and it seems likely that the particularly complicated nature of the formulae here is a reflection of the "nonelementary" behaviour of orbital integrals (hence, by Murnaghan–Kirillov theory, also of characters) described in [Hales 1994].

Finally, we show in Section 11 that the function we have been evaluating actually does represent the desired distribution, that is, equals $\hat{\mu}^G_{X^*}$. (See Proposition 11.2.) We close with some observations (see Theorem 11.3) about the qualitative behaviour of orbital integrals that does not depend (much) on the "type" of X^*.

2. Notation

Suppose that k is a nondiscrete, non-Archimedean local field. We do not make any assumptions on its characteristic, but we assume that its residual characteristic p is not 2. (We occasionally cite [Shalika 2004], which works only with characteristic-0 fields, but we shall not use any results from there that require this restriction.) Let

R denote the ring of integers in k, $\wp$ the prime ideal of R, and ord the valuation on k with value group $\mathbb{Z}$.

Let $\mathfrak{f}$ denote the residue field $R/\wp$ of k. Write $q = |\mathfrak{f}|$ for the number of elements in $\mathfrak{f}$, and put $|x| = q^{-\operatorname{ord}(x)}$ for $x \in k$. If $\alpha \in \mathbb{C}$, then write ν^α for the (multiplicative) character $x \mapsto |x|^\alpha$ of $k^\times$.

Put $\boldsymbol{G} = \mathrm{SL}_2$ and $G = \boldsymbol{G}(k)$, and let $\mathfrak{g}$ and $\mathfrak{g}^*$ denote the Lie algebra and dual Lie algebra of G, respectively.

It is important for our calculations to be quite specific about the Haar measures that we are using. For convenience, we fix the ones used in [Sally and Taibleson 1966, p. 280].

Definition 2.1. Throughout, we shall use the (additive) Haar measure $\mathrm{d}x$ on k that assigns measure 1 to R, and the associated (multiplicative) Haar measure $\mathrm{d}^\times x = |x|^{-1}\,\mathrm{d}x$ on $k^\times$ that assigns measure $1 - q^{-1}$ to $R^\times$. When convenient, we shall write $\mathrm{d}t$ instead of $\mathrm{d}x$.

Definition 2.2. If Φ is an (additive) character of k, then define $\Phi_b : x \mapsto \Phi(bx)$ for $b \in k$. The *depth* of Φ is

$$\mathrm{d}(\Phi) := \begin{cases} \min\{i \in \mathbb{Z} : \Phi \text{ is trivial on } \wp^{i+1}\} & \text{if } \Phi \text{ is nontrivial,} \\ -\infty & \text{otherwise.} \end{cases}$$

The depth of a character is related to what is often called its *conductor* by $\mathrm{d}(\Phi) = \omega(\Phi) - 1$ (in the notation of [Shalika 2004, Section 1.3]). We have that

$$\mathrm{d}(\Phi_b) = \mathrm{d}(\Phi) - \operatorname{ord}(b). \tag{2.3}$$

The notion of depth and the symbol d will be used in multiple contexts (see Definition 4.9); we rely on the context to disambiguate them.

Notation 2.4. Φ is a nontrivial (additive) character of k.

One of the crucial tools of Harish-Chandra's approach to harmonic analysis is the reduction, whenever possible, of questions about a group to questions about its Lie algebra. The exponential map often allows one to effect this reduction, but, since it might converge only in a very small neighbourhood of 0, we replace it with a "mock-exponential map" (see [Adler 1998, Section 1.5]) which has many of the same properties (see Lemma 2.6).

Definition 2.5. The *Cayley map* $\mathrm{c} : k \setminus \{1\} \to k \setminus \{-1\}$ is defined by

$$\mathrm{c}(X) = (1+X)(1-X)^{-1} \quad \text{for } X \in k \setminus \{1\}.$$

The Cayley function is available in many settings; we are using it only as a function defined almost everywhere on k.

Lemma 2.6.

- *The map* c *is a bijection.*
- $\mathrm{c}(-X) = \mathrm{c}(X)^{-1} = \mathrm{c}^{-1}(X)$ *for* $X \in k \setminus \{\pm 1\}$.
- *The map* c *carries* $\wp^i$ *to* $1 + \wp^i$ *for all* $i \in \mathbb{Z}_{>0}$.
- *In the notation of Definition 2.1, the pullback along* c *of the measure* $\mathrm{d}^\times x$ *on* $1 + \wp$ *is the measure* $\mathrm{d}x$ *on* $\wp$.
- *If* $X \in \wp^i$ *and* $Y \in \wp^j$, *with* $i, j \in \mathbb{Z}_{>0}$, *then*
$$\mathrm{c}(X + Y) \equiv \mathrm{c}(X) + 2Y \pmod{1 + \wp^n},$$
where $n = j + \min\{2i, j\}$.

Proof. It is easy to check that $x \mapsto (1 - x)(1 + x)^{-1}$ is inverse to c and satisfies the desired equalities and check that $\mathrm{c}(\wp^i) \subseteq 1 + \wp^i$ and $\mathrm{c}^{-1}(1 + \wp^i) \subseteq \wp^i$. If $f \in C^\infty(1 + \wp)$, then there is some $i \in \mathbb{Z}_{>0}$ such that $f \in C(1 + \wp/1 + \wp^i)$. Since $\mathrm{meas}_{\mathrm{d}x}(\wp^i) = q^{-i} = \mathrm{meas}_{\mathrm{d}^\times x}(1 + \wp^i)$, we see that
$$\begin{aligned}\int_{1+\wp} f(x)\,\mathrm{d}^\times x &= \sum_{x \in 1+\wp/1+\wp^i} f(x)\,\mathrm{meas}_{\mathrm{d}^\times x}(1 + \wp^i)\\ &= \sum_{x \in \wp/\wp^i} (f \circ \mathrm{c})(x) q^{-i}\,\mathrm{meas}_{\mathrm{d}x}(\wp^i) = \int_{\wp} (f \circ \mathrm{c})(x)\,\mathrm{d}x.\end{aligned}$$

Finally, under the stated conditions on X and Y,
$$\begin{aligned}\bigl(\mathrm{c}(X) + 2Y\bigr)\bigl(1 - (X + Y)\bigr) &= \mathrm{c}(X) \cdot (1 - X) + Y\bigl(2(1 - (X + Y)) - \mathrm{c}(X)\bigr)\\ &= (1 + X + Y) + Y\bigl((1 - 2X - \mathrm{c}(X)) - 2Y\bigr).\end{aligned}$$

Since $\mathrm{c}(X) = 1 + 2X(1 - X)^{-1}$, we have that $1 - 2X - \mathrm{c}(X) \in \wp^{2i}$. The result follows. □

3. Fields and algebras

Definition 3.1. For $\theta \in k^\times$, write k_θ for the k-algebra that is $k \oplus k$ (as a vector space), equipped with multiplication $(a, b) \cdot (c, d) = (ac + bd\theta, ad + bc)$. Write $\sqrt{\theta}$ for the element $(0, 1) \in k_\theta$, so that $(a, b) = a + b\sqrt{\theta}$.

We also use the notation $\sqrt{\theta}$ for a matrix (see Definition 4.1); we shall rely on context to make the meaning clear.

If $\theta \notin (k^\times)^2$, then k_θ is isomorphic to $k(\sqrt{\theta})$ (as k-algebras) via the map $(a, b) \mapsto a + b\sqrt{\theta}$, and we shall not distinguish between them.

If $\theta = x^2$, with $x \in k$, then k_θ is isomorphic to $k \oplus k$ (as k-algebras) via the map $(a, b) \mapsto (a + bx, a - bx)$.

Definition 3.2. Define

$$N_\theta(a+b\sqrt{\theta}) = a^2 - b^2\theta, \quad \mathrm{tr}_\theta(a+b\sqrt{\theta}) = 2a,$$
$$\mathrm{Re}_\theta(a+b\sqrt{\theta}) = a, \qquad \mathrm{Im}_\theta(a+b\sqrt{\theta}) = b,$$
$$\mathrm{ord}_\theta(a+b\sqrt{\theta}) = \tfrac{1}{2}\,\mathrm{ord}\big(N_\theta(a+b\sqrt{\theta})\big)$$

for $a+b\sqrt{\theta} \in k_\theta$. Write $C_\theta = \ker N_\theta$ and $V_\theta = \ker \mathrm{tr}_\theta$, and let sgn_θ be the unique (multiplicative) character of $k^\times$ with kernel precisely $N_\theta(k_\theta^\times)$.

If $\theta \notin (k^\times)^2$, then N_θ and tr_θ are the usual norm and trace maps associated to the quadratic extension of fields k_θ/k, and ord_θ is the valuation on k_θ extending ord. In any case, $k_\theta^\times = \{z \in k_\theta : N_\theta(z) \neq 0\}$.

We can describe the signum character explicitly by

$$\mathrm{sgn}_\theta(x) = \begin{cases} 1 & \theta \text{ split,} \\ (-1)^{\mathrm{ord}(x)} & \theta \text{ unramified,} \end{cases} \tag{3.3}$$

$$\begin{cases} \mathrm{sgn}_\theta(\theta) = \mathrm{sgn}_{\mathfrak{f}}(-1) \\ \mathrm{sgn}_\theta(x) = \mathrm{sgn}_{\mathfrak{f}}(\bar{x}) \quad \text{for } x \in R^\times, \end{cases} \tag{3.4}$$

where $\mathrm{sgn}_{\mathfrak{f}}$ is the quadratic character of $\mathfrak{f}^\times$ and $x \mapsto \bar{x}$ the reduction map $R \to \mathfrak{f}$.

4. Tori and filtrations

We begin by defining a few model tori.

Definition 4.1. For $\theta \in k$, put

$$\boldsymbol{T}_\theta = \left\{ \begin{pmatrix} a & b \\ b\theta & a \end{pmatrix} : a^2 - b^2\theta = 1 \right\}.$$

Then

$$\mathfrak{t}_\theta := \mathrm{Lie}(\boldsymbol{T}_\theta) = \left\{ \begin{pmatrix} 0 & b \\ b\theta & 0 \end{pmatrix} \right\}.$$

Write $\sqrt{\theta}$ for the element

$$\begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix}$$

so that $\mathfrak{t}_\theta = \mathrm{Span}_k \sqrt{\theta}$. Call a maximal k-torus in $\boldsymbol{G}$ *standard* exactly when it is of the form $\boldsymbol{T}_\theta$ for some $\theta \in k$.

We also denote by $\sqrt{\theta}$ a specific element of an extension of k (see Definition 4.1); we shall rely on context to make the meaning clear.

Remark 4.2. The group T_θ is isomorphic to $C_\theta = \ker N_\theta$ and the Lie algebra $\mathfrak{t}_\theta$ to $V_\theta = \ker \mathrm{tr}_\theta$ in each case via the map

$$\begin{pmatrix} a & b \\ b\theta & a \end{pmatrix} \mapsto (a, b).$$

We shall use the terms “split”, “unramified”, and “ramified” in many different contexts.

Remark 4.3. If $\boldsymbol{T}$ is a maximal k-torus in $\boldsymbol{G}$ and $\mathfrak{t} = \mathrm{Lie}(T)$, then we shall identify $\mathfrak{t}$ (respectively, $\mathfrak{t}^*$) with the spaces of fixed points for the adjoint (respectively, coadjoint) action on $\mathfrak{g}$ (respectively, $\mathfrak{g}^*$). By abuse of language, we shall sometimes say that $X^* \in \mathfrak{g}^*$ or $Y \in \mathfrak{g}$ lies in, or belongs to, the torus $\boldsymbol{T}$ to mean that $X^* \in \mathfrak{t}^*$ and $Y \in \mathfrak{t}$; equivalently, that $C_{\boldsymbol{G}}(X^*) = \boldsymbol{T} = C_{\boldsymbol{G}}(Y)$. In particular, “$X^*$ and Y belong to a common torus” is shorthand for “$C_{\boldsymbol{G}}(X^*) = C_{\boldsymbol{G}}(Y)$”.

Definition 4.4. A maximal k-torus in $\boldsymbol{G}$ is called (un)ramified if it is elliptic and splits over an (un)ramified extension of k. An element $\theta \in k$ is called split, unramified, or ramified if $\boldsymbol{T}_\theta$ has that property. A regular, semisimple element of $\mathfrak{g}$ or $\mathfrak{g}^*$ is called split, unramified, or ramified if the torus to which it belongs has that property.

Remark 4.5. To be explicit, squares in $k^\times$ are split, and a nonsquare $\theta \in k$ is unramified or ramified if $\max\{\mathrm{ord}(x^2\theta) : x \in k\}$ is even or odd, respectively.

Notation 4.6. If $\boldsymbol{T}$ is a maximal k-torus in $\boldsymbol{G}$ with $T = \boldsymbol{T}(k)$, then write $W(\boldsymbol{G}, \boldsymbol{T}) = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$ for the absolute and $W(G, T) = N_G(T)/T$ for the relative Weyl group of $\boldsymbol{T}$ in $\boldsymbol{G}$.

Every maximal k-torus in $\boldsymbol{G}$ is G-conjugate to some $\boldsymbol{T}_\theta$. (See, for example, [DeBacker and Sally 2000, Section A.2].) In particular,

$$\mathrm{Int}\begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : ad = 1\right\} = \boldsymbol{T}_1.$$

Remark 4.7. For all $\theta \in k$, the group $W(\boldsymbol{G}, \boldsymbol{T}_\theta)$ has order 2, with the nontrivial element acting on $\boldsymbol{T}_\theta$ by inversion. If $\mathrm{sgn}_\theta(-1) = 1$ (in particular, if θ is split or unramified), say, with $N_\theta(a + b\sqrt{\theta}) = -1$, then $W(G, T_\theta)$ also has order 2, with the nontrivial element represented by

$$\begin{pmatrix} a & b \\ -b\theta & -a \end{pmatrix}.$$

If $\theta = 1$, then we may take $(a, b) = (0, 1)$ to recover the familiar Weyl group element. Otherwise (that is, if $\mathrm{sgn}_\theta(-1) = -1$), $W(G, T_\theta)$ is trivial.

The concept of *stable conjugacy* was introduced by Langlands [1979, pp. 2–3] as part of the foundation of the Langlands conjectures.

Definition 4.8. Two

- maximal k-tori $\boldsymbol{T}_i$ in $\boldsymbol{G}$,
- regular semisimple elements $X_i^* \in \mathfrak{g}^*$, or
- regular semisimple elements $Y_i \in \mathfrak{g}$,

with $i = 1, 2$, are called *stably conjugate* exactly when there are a field extension E/k and an element $g \in \boldsymbol{G}(E)$ such that

- $\mathrm{Int}(g)T_1 = T_2$,
- $\mathrm{Ad}^*(g)X_1^* = X_2^*$, or
- $\mathrm{Ad}(g)X_1 = X_2$,

where $T_i = \boldsymbol{T}_i(k)$ for $i = 1, 2$. If the conjugacy can be carried out without passing to an extension field (that is, if we may take $g \in G$), then we will sometimes emphasise this by saying that the tori or elements are *rationally conjugate*.

The Zariski-density of T_i in $\boldsymbol{T}_i$ implies that $\mathrm{Int}(g)\boldsymbol{T}_1 = \boldsymbol{T}_2$, but that this is a strictly weaker condition; indeed, given *any* two maximal tori, there is an element g, defined over some extension field of k, satisfying this condition. In our special case (of $\boldsymbol{G} = \mathrm{SL}_2$), two tori or elements are stably conjugate if and only if they are conjugate in $\mathrm{GL}_2(k)$.

More concretely, two tori $\boldsymbol{T}_\theta$ and $\boldsymbol{T}_{\theta'}$ are stably conjugate if and only if $\theta \equiv \theta' \pmod{(k^\times)^2}$. The stable conjugacy class of the split torus $\boldsymbol{T}_1$ is also a rational conjugacy class.

Suppose that ϵ is an unramified and ϖ a ramified, nonsquare. Then the stable conjugacy class of $\boldsymbol{T}_\epsilon$ splits into 2 rational conjugacy classes, represented by $\boldsymbol{T}_\epsilon$ and $\boldsymbol{T}_{\varpi^2\epsilon}$. The stable conjugacy class of $\boldsymbol{T}_\varpi$ is also a rational conjugacy class if $\mathrm{sgn}_\varpi(-1) = -1$, but it splits into 2 rational conjugacy classes, represented by $\boldsymbol{T}_\varpi$ and $\boldsymbol{T}_{\epsilon^2\varpi}$, if $\mathrm{sgn}_\varpi(-1) = 1$.

We also need filtrations on the Lie algebra and dual Lie algebra of a torus. These definitions are standard (see, for example, [Adler 1998, Section 1.4]) and can be made in far more generality (see [Moy and Prasad 1994, Section 3; 1996, Section 3.3]); we give only simple definitions adapted to $\boldsymbol{G} = \mathrm{SL}_2$.

Definition 4.9. Let $\boldsymbol{T}$ be a maximal k-torus in $\boldsymbol{G}$. Recall that $\boldsymbol{T}$ is G-conjugate to $\boldsymbol{T}_\theta$ for some $\theta \in k$, so that $\mathfrak{t} = \mathrm{Lie}(T)$ is isomorphic to $V_\theta = \ker \mathrm{tr}_\theta \subseteq k_\theta$. For $r \in \mathbb{R}$, write $\mathfrak{t}_r$ for the preimage of $\{Y \in V_\theta : \mathrm{ord}_\theta(Y) \geq r\}$ and $\mathfrak{t}_{r+}$ for the preimage of $\{Y \in V_\theta : \mathrm{ord}_\theta(Y) > r\}$; then write $\mathfrak{t}_r^* = \{X^* \in \mathfrak{t}^* : \Phi(\langle X^*, Y\rangle) = 1 \text{ for all } Y \in \mathfrak{t}_{(-r)+}\}$ (where Φ is the additive character of Notation 2.4).

If $X^* \in \mathfrak{t}^*$ and $Y \in \mathfrak{t}$, then define $\mathrm{d}(X^*) = \max\{r \in \mathbb{R} : X^* \in \mathfrak{t}_r^*\}$ and $\mathrm{d}(Y) = \max\{r \in \mathbb{R} : Y \in \mathfrak{t}_r\}$.

One can define a notion of depth in more generality (see, for example, [Adler and DeBacker 2002, Section 3.3 and Example 3.4.6; Kim and Murnaghan 2003, Section 2.1 and Lemma 2.1.5]), but we only need the special case above. (The only remaining case to consider for $\mathfrak{g} = \mathfrak{sl}_2(k)$ is the depth of a nilpotent element, which is ∞.)

5. Orbital integrals

Our goal in this paper is to compute Fourier transforms of regular, semisimple orbital integrals on $\mathfrak{g}$ (see Definition 5.5 below). Since the Fourier transforms of nilpotent orbital integrals were computed in [DeBacker and Sally 2000, Appendix A], this covers all Fourier transforms of orbital integrals on $\mathfrak{g}$ (for our particular case $\boldsymbol{G} = \mathrm{SL}_2$). The case of orbital integrals on G was discussed in [Sally and Shalika 1984], as the culmination of the series of papers that began with [Sally and Shalika 1968; 1969].

We begin by choosing a representative for the regular, semisimple orbit of interest. By Section 4, we may choose this representative in a standard torus (in the sense of Definition 4.1).

Notation 5.1. $\beta, \theta \in k^\times$, and $X^* = \beta \cdot \sqrt{\theta} \in \mathfrak{t}_\theta^*$.

Here, we are implicitly using the identification of $\mathfrak{t}_\theta$ with $\mathfrak{t}_\theta^*$ via the trace form; what we really mean is that $\langle X^*, Y\rangle = \operatorname{tr} \beta \cdot \sqrt{\theta} \cdot Y$ for $Y \in \mathfrak{t}_\theta$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual pairing between $\mathfrak{t}_\theta^*$ and $\mathfrak{t}_\theta$.

As in Definition 2.2, we may define a new character Φ_β of k, which we use often in our calculations.

Notation 5.2. $-r = \mathrm{d}(X^*)$, $\Phi' = \Phi_\beta$, and $r' = \mathrm{d}(\Phi')$.

By Definition 4.9, $Y \mapsto \Phi(\langle X^*, Y\rangle)$ is trivial on $(\mathfrak{t}_\theta)_{r+}$, but not on $(\mathfrak{t}_\theta)_r$. Therefore, $r' = r + \frac{1}{2}\operatorname{ord}(\theta)$.

Since $C_G(X^*) = T_\theta$ is Abelian, it is unimodular, so there exists a measure on $G/C_G(X^*)$ invariant under the action of G by left translation.

Notation 5.3. Let $\mathrm{d}\dot{g}$ be a translation-invariant measure on $G/C_G(X^*)$.

Since the orbit, $\mathcal{O}^G_{X^*}$, of X^* under the coadjoint action of G is isomorphic as a G-set to $G/C_G(X^*)$, we could transport to it the measure on the latter space; but we do not find it convenient to do so.

Since X^* is semisimple, $\mathcal{O}^G_{X^*}$ is closed in $\mathfrak{g}^*$; see, for example, [Tauvel and Yu 2005, Proposition 34.3.2]. Therefore, the restriction to $\mathcal{O}^G_{X^*}$ of a locally constant, compactly supported function on $\mathfrak{g}^*$ remains locally constant and compactly supported, so that the following definition makes sense.

Definition 5.4. The orbital integral of X^* is the distribution $\mu^G_{X^*}$ on $\mathfrak{g}^*$ defined by

$$\mu^G_{X^*}(f^*) = \int_{G/C_G(X^*)} f^*(\mathrm{Ad}^*(g)X^*)\,\mathrm{d}\dot{g} \quad \text{for all } f^* \in C_c^\infty(\mathfrak{g}^*).$$

We are interested in the Fourier transform of $\mu^G_{X^*}$. The definition of the Fourier transform (of distributions or of functions) requires, in addition to a choice of additive character (see Notation 2.4), also a choice of Haar measure $\mathrm{d}Y$ on $\mathfrak{g}^*$; but we shall build this choice into our representing function (see Notation 5.7), so that it will not show up in our final answer.

Definition 5.5. The Fourier transform of the orbital integral of X^* is the distribution $\hat{\mu}^G_{X^*}$ on $\mathfrak{g}$ defined for all $f \in C_c^\infty(\mathfrak{g})$ by

$$\hat{\mu}^G_{X^*}(f) = \mu^G_{X^*}(\hat{f}),$$

where

$$\hat{f}(Y^*) = \int_{\mathfrak{g}} f(Y)\Phi(\langle Y^*, Y\rangle)\,\mathrm{d}Y \quad \text{for all } Y^* \in \mathfrak{g}^*.$$

It is a result of Harish-Chandra [1999, Theorem 1.1] that $\hat{\mu}^G_{X^*}$ is *representable* on $\mathfrak{g}$, that is, there exists a locally integrable function F on $\mathfrak{g}$ such that

$$\hat{\mu}^G_{X^*}(f) = \int_G f(Y)F(Y)\,\mathrm{d}Y \quad \text{for all } f \in C_c^\infty(\mathfrak{g}).$$

One can say more about the behaviour and asymptotics of the function F. For example, it blows up as Y approaches 0, but its blow-up is controlled by a power of a discriminant function.

Definition 5.6. The *Weyl discriminant* on $\mathfrak{g}$ is the function $D_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{C}$ such that, for all $Y \in \mathfrak{g}$, $D_{\mathfrak{g}}(Y)$ is the coefficient of the degree-1 term in the characteristic polynomial of $\operatorname{ad}(Y)$. Concretely,

$$D_{\mathfrak{g}}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 4(a^2 + bc).$$

Our main interest, however, is in the restriction of the function F above to the set $\mathfrak{g}^{\mathrm{rss}}$ of regular, semisimple elements, where it is locally constant.

Notation 5.7. By abuse of notation, write again $\hat{\mu}^G_{X^*}$ for the function that represents the restriction to $\mathfrak{g}^{\mathrm{rss}}$ of $\hat{\mu}^G_{X^*}$.

When we refer to the computation of the Fourier transform of an orbital integral, it is actually the (scalar) function of Notation 5.7 that we are trying to compute. The main tool in this direction is a general integral formula of Huntsinger (see [Adler and DeBacker 2004, Theorem A.1.2]), but we find it easier to evaluate an integral adapted to our current setting (see Definition 8.4). The computation of this integral will occupy most of the paper; we finally prove it actually represents the distribution $\hat{\mu}^G_{X^*}$ in Proposition 11.2.

Finally, we fix an element at which to evaluate the functions of interest. Since $\hat{\mu}^G_{X^*}$, as just defined, and $M^G_{X^*}$ in Definition 8.4 are G-invariant functions on $\mathfrak{g}^{\mathrm{rss}}$, we may again consider only elements of standard tori.

Notation 5.8. $s, \theta' \in k^\times$, and $Y = s \cdot \sqrt{\theta'} \in \mathfrak{t}_{\theta'}$.

We phrase our computations in terms of the values of two "basic" functions at Y.

Lemma 5.9. $\mathrm{d}(Y) = \frac{1}{2}\operatorname{ord}(s^2\theta')$ *and* $D_{\mathfrak{g}}(Y) = 4s^2\theta'$.

Proof. This is a straightforward consequence of Definitions 4.9 and 5.6. □

6. Roots of unity and other constants

The computation of Fourier transforms of orbital integrals on $\mathfrak{g}$ via Murnaghan–Kirillov theory [Murnaghan 1995a; Kim and Murnaghan 2003; 2006; Adler and DeBacker 2004; Adler and Spice 2009] and also of the values near the identity of characters of G (see [Sally and Shalika 1968; Adler et al. 2011]) involves a somewhat bewildering array of 4-th roots of unity, for each of which there is a variety of notation available. All of these can be expressed in terms of a single "basic" quantity, the Gauss sum, denoted by $G(\Phi)$ in [Shalika 2004, Lemma 1.3.2]. The definition there implicitly depends on a choice of uniformiser, denoted there by π. Although the choice is arbitrary, for later convenience we denote it by $-\varpi$. Recall from Notation 2.4 that Φ is a nontrivial (additive) character of k.

Definition 6.1. If ϖ is a uniformiser of k, then

$$G_\varpi(\Phi) := q^{-1/2} \sum_{X \in R/\wp} \Phi_{(-\varpi)^{\mathrm{d}(\Phi)}}(X^2).$$

It is possible to compute these values exactly (see, for example, [Lidl and Niederreiter 1997, Theorem 5.15]), but we only require a few transformation laws.

Lemma 6.2. *If ϖ is a uniformiser of k, then*

$$\begin{aligned} G_{b\varpi}(\Phi) &= \operatorname{sgn}_\varpi(b)^{\mathrm{d}(\Phi)} G_\varpi(\Phi) && \text{for } b \in R^\times, \\ G_\varpi(\Phi_b) &= \operatorname{sgn}_\varpi(b) G_\varpi(\Phi) && \text{for } b \in k^\times, \\ G_\varpi(\Phi)^2 &= \operatorname{sgn}_\varpi(-1), \\ G_\varpi(\Phi) &= q^{-1/2} \operatorname{sgn}_\varpi(-1)^{\mathrm{d}(\Phi)} \textstyle\sum_{X \in \mathfrak{f}^\times} \overline{\Phi}(X) \operatorname{sgn}_{\mathfrak{f}}(X), \end{aligned}$$

where $\operatorname{sgn}_{\mathfrak{f}}$ *is the quadratic character of* $\mathfrak{f}^\times$, *and* $\overline{\Phi}$ *the* (*additive*) *character of* $\mathfrak{f} = R/\wp$ *arising from the restriction to R of the depth-0 character* $\Phi_{\varpi^{\mathrm{d}(\Phi)}}$ *of k.*

Proof. Since $\sum_{X \in \mathfrak{f}} \overline{\Phi}(X) = 0$, we have that

$$\begin{aligned} \sum_{X \in \mathfrak{f}^\times} \overline{\Phi}(X) \operatorname{sgn}_{\mathfrak{f}}(X) &= \overline{\Phi}(0) + \sum_{X \in \mathfrak{f}^\times} \overline{\Phi}(X)\bigl(1 + \operatorname{sgn}_{\mathfrak{f}}(X)\bigr) \\ &= \overline{\Phi}(0) + 2\sum_{X \in (\mathfrak{f}^\times)^2} \overline{\Phi}(X) = \sum_{X \in \mathfrak{f}} \overline{\Phi}(X^2) = q^{1/2} G_\varpi(\Phi_{(-1)^{\mathrm{d}(\Phi)}}). \end{aligned}$$

In other words,

$$(*) \qquad G_\varpi(\Phi_{(-1)^{\mathrm{d}(\Phi)}}) = q^{-1/2} G(\operatorname{sgn}_{\mathfrak{f}}, \overline{\Phi}),$$

where the notation on the right is as in [Lidl and Niederreiter 1997, Section 5.2] (except that their ψ is our $\operatorname{sgn}_{\mathfrak{f}}$, the quadratic character of $\mathfrak{f}^\times$, and their χ is our $\overline{\Phi}$). The third equality, and the second equality for $b \in R^\times$, now follow from [ibid., Theorem 5.12]. The first equality follows from the second since $G_{b\varpi}(\Phi) = G_\varpi(\Phi_{b^{\mathrm{d}(\Phi)}})$. Taking $b = (-1)^{\mathrm{d}(\Phi)}$ and combining with $(*)$ gives the fourth equality. Finally, by definition, $G_\varpi(\Phi_{(-\varpi)^n}) = G_\varpi(\Phi) = \operatorname{sgn}_\varpi(-\varpi)^n G_\varpi(\Phi)$ for all $n \in \mathbb{Z}$. □

By Proposition 8.11 and Theorem 7.4, our calculations will involve the Γ-factors defined in [Sally and Taibleson 1966, Section 3]. The factor $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)$ is of particular interest. By [ibid., Theorem 3.1(iii)], $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)^2 = \operatorname{sgn}_\varpi(-1)$, so by Lemma 6.2, $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) = \pm G_\varpi(\Phi)$. It will be useful to identify the sign.

Lemma 6.3. *If ϖ is a uniformiser of k, then*

$$\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) = \operatorname{sgn}_\varpi(-1)^{\mathrm{d}(\Phi)+1} G_\varpi(\Phi).$$

Proof. Write $\overline{\Phi} = \Phi_{\varpi^{\mathrm{d}(\Phi)}}$; this is a depth-0 character of k. The definitions of [Sally and Taibleson 1966] depend on a depth-(-1) additive character χ; we take it to be $\overline{\Phi}_\varpi$. The definition of $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)$ involves a principal-value integral (see Definition 8.4), but, as pointed out in the proof of [Sally and Taibleson 1966, Theorem 3.1], by [ibid., Lemma 3.1] and (3.4) it simplifies to

$$\begin{aligned}\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) &= \int_{\operatorname{ord}(x)=-1} \overline{\Phi}_\varpi(x)|x|^{1/2}\operatorname{sgn}_\varpi(x)\,\mathrm{d}^\times x \\ &= \int_{R^\times} \overline{\Phi}_\varpi(\varpi^{-1}x)|\varpi^{-1}x|^{1/2}\operatorname{sgn}_\varpi(\varpi^{-1}x)\,\mathrm{d}^\times x \\ &= q^{1/2}\operatorname{sgn}_\varpi(-1)\operatorname{meas}_{\mathrm{d}^\times x}(1+\wp)\sum_{x\in R^\times/1+\wp}\overline{\Phi}(x)\operatorname{sgn}_{\mathfrak{f}}(x),\end{aligned}$$

where $\mathrm{d}^\times x$ is the Haar measure on $k^\times$ giving $R^\times$ measure $1-q^{-1}$ (see Definition 2.1). Since $\operatorname{meas}_{\mathrm{d}^\times x}(1+\wp) = q^{-1}$, the result now follows from Lemma 6.2. □

We will also need some constants associated to specific elements.

Waldspurger [1995, Proposition VIII.1] describes the "behaviour at ∞" of Fourier transforms of semisimple orbital integrals on general reductive, p-adic Lie algebras. His description involves a 4-th root of unity $\gamma_\psi(X^*, Y)$ (see [ibid., p. 79]); since his ψ is our Φ (Notation 2.4), we denote it by $\gamma_\Phi(X^*, Y)$. See Theorem 11.3 for our quantitative analogues (for the special case of $\mathfrak{sl}_2$) of his result.

We would like to avoid choosing "standard" representatives for $k^\times/(k^\times)^2$ (see Remark 6.9), but doing this is notationally unwieldy. Although our proofs will make use of these choices, none of the statements of the main results (except Theorems 10.8 and 10.9, via Remark 10.7) relies on them.

Notation 6.4. Let ϵ be a lift to $R^\times$ of a nonsquare in $\mathfrak{f}^\times$ and ϖ a uniformiser of k.

Definition 6.5. Recall Notations 5.1 and 5.8. If X^* and Y lie in stably conjugate tori, so that $\theta \equiv \theta' \pmod{(k^\times)^2}$, then

$$\gamma_\Phi(X^*, Y) = \begin{cases} 1 & \theta \equiv 1, \\ \gamma_{\mathrm{un}}(s) & \theta \equiv \epsilon, \\ \gamma_{\mathrm{ram}}(s) & \theta \equiv \varpi, \\ -\gamma_{\mathrm{un}}(s)\gamma_{\mathrm{ram}}(s) & \theta \equiv \epsilon\varpi, \end{cases}$$

where all congruences are taken modulo $(k^\times)^2$ and where

$$\gamma_{\mathrm{un}}(s) := (-1)^{r'+1} \operatorname{sgn}_\epsilon(s) \quad \text{and} \quad \gamma_{\mathrm{ram}}(s) := \operatorname{sgn}_\varpi(-s) G_\varpi(\Phi')$$

(with notation as in Notation 5.2 and Definition 6.1). It simplifies our notation considerably also to put $\gamma_\Phi(X^*, Y) = 1$ if X^* is elliptic and Y is split, and otherwise put $\gamma_\Phi(X^*, Y) = 0$ if X^* and Y do not lie in stably conjugate tori.

Remark 6.6. The dependence of $\gamma_\Phi(X^*, Y)$ on X^* is via r' and Φ' (see Notation 5.2). Expanding these definitions shows that $\gamma_\Phi(X^*, Y) = c_{\theta,\phi} \cdot \operatorname{sgn}_\theta(\beta s)$ when X^* and Y lie in stably conjugate tori, using Notations 5.1 and 5.8.

We have defined $\gamma_\Phi(X^*, Y)$ only when X^* and Y belong to (possibly different) standard tori, in the sense of Definition 4.1. A direct computation shows that, if we replace X^* or Y by a rational conjugate, or replace the pair (X^*, Y) by a stable conjugate, such that X^* and Y still lie in standard tori, then the constant $\gamma_\Phi(X^*, Y)$ does not change. (In the notation of Definition 8.2, $\mathrm{Ad}^*(g)X^*$ lies in a standard torus if and only if $\varphi_\theta(g) = (\alpha, 0)$, in which case $\mathrm{Ad}^*(g)X^* = \beta N_\theta(\alpha) \cdot \sqrt{N_\theta(\alpha)^{-2}\theta}$; and similarly for Y.) This allows us to define $\gamma_\Phi(X^*, Y)$ for all pairs of regular, semisimple elements, if desired.

By Lemma 6.2,

$$\gamma_{\mathrm{ram}}(s)^2 = \operatorname{sgn}_\varpi(-1). \tag{6.7}$$

To make use of Propositions 7.5 and 7.7 below, we need the computation

$$\begin{aligned} \operatorname{sgn}_\varpi(v) G_\varpi(\Phi'_{\varpi^{r'+1}}) &= \operatorname{sgn}_\varpi(\varpi^{-(r'+1)} s\theta) \cdot \operatorname{sgn}_\varpi(\varpi^{r'+1}) G_\varpi(\Phi') \\ &= \operatorname{sgn}_\varpi(-\theta)\gamma_{\mathrm{ram}}(s). \end{aligned} \tag{6.8}$$

Remark 6.9. We will be interested exclusively in the case when $\theta \in \{1, \epsilon, \varpi\}$. This means we seem to be omitting the cases when $\theta \in \{\varpi^2\epsilon, \epsilon^2\varpi, \epsilon^{\pm 1}\varpi\}$, but, actually, this problem is not serious. Indeed, for $b \in k$, write

$$g_b := \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(k).$$

Then

$$\mathrm{Ad}^*(g_b)X^* = \mathrm{Ad}^*(g_b)(\beta \cdot \sqrt{\theta}) = \beta b^{-1} \cdot \sqrt{b^2\theta}$$

(where we identify $\mathfrak{t}_\theta^*$ with $\mathfrak{t}_\theta$ via the trace pairing, as in Notation 5.1); and $\hat{\mu}^G_{X^*} = \hat{\mu}^G_{\mathrm{Ad}^*(g_b)X^*} \circ \mathrm{Ad}(g_b)$. This covers $\theta = \varpi^2\epsilon$ (by taking $b = \varpi^{-1}$) and $\theta = \epsilon^2\varpi$ (by taking $b = \epsilon^{-1}$). Handling $\theta \in \{\epsilon^{\pm 1}\varpi\}$ requires a different observation: since our choice of uniformiser was arbitrary, it could as well have been $\epsilon^{\pm 1}\varpi$ (or, for that matter, $\epsilon^2\varpi$) as ϖ itself. Thus, the formulae for the cases $\theta = \epsilon^n\varpi$ can be obtained by simple substitution.

The definition of $\gamma_\Phi(X^*, Y)$ when $\theta \equiv \epsilon\varpi \pmod{(k^\times)^2}$ is an instance of this; namely, by Lemma 6.2,

$$\begin{aligned}-\gamma_{\mathrm{un}}(s)\gamma_{\mathrm{ram}}(s) &= (-1)^{r'}\operatorname{sgn}_\epsilon(s)\cdot\operatorname{sgn}_\varpi(-s)G_\varpi(\Phi') \\ &= \operatorname{sgn}_{\epsilon\varpi}(-s)\cdot\operatorname{sgn}_\varpi(\epsilon)^{r'}G_\varpi(\Phi') = \operatorname{sgn}_{\epsilon\varpi}(-s)G_{\epsilon\varpi}(\Phi'),\end{aligned}$$

where we have used that $\operatorname{sgn}_\epsilon(-1) = 1$ and $\operatorname{sgn}_\varpi(\epsilon) = -1$.

We next define a constant $c_0(X^*)$ for use in Theorems 9.7 and 10.10. Those theorems (and Proposition 11.2) show that, as the notation suggests, it is the coefficient of the trivial orbit in the expansion of the germ of $\hat{\mu}^G_{X^*}$ in terms of Fourier transforms of nilpotent orbital integrals (see [Harish-Chandra 1999, Theorem 5.11]).

Definition 6.10.

$$c_0(X^*) = \begin{cases} -2q^{-1} & X^* \text{ split}, \\ -q^{-1} & X^* \text{ unramified}, \\ -\frac{1}{2}q^{-2}(q+1) & X^* \text{ ramified}. \end{cases}$$

Recall that $\hat{\mu}^G_{X^*}$ is defined in terms of the measure $\mathrm{d}\dot{g}$ of Proposition 11.2; in the notation of that proposition, whenever X^* is elliptic,

$$c_0(X^*) = (q-1)^{-1}\operatorname{meas}_{\mathrm{d}\dot{g}}(\dot{K}).$$

7. Bessel functions

Our strategy for computing Fourier transforms of orbital integrals is to reduce them to p-adic Bessel functions (see Proposition 8.11, (9.3), and (10.2)). In this context, we are referring to the complex-valued Bessel functions defined in [Sally and Taibleson 1966, Section 4], not the p-adic-valued ones defined in [Dwork 1974].

The definition of these functions depends on an additive character, denoted by χ in [Sally and Taibleson 1966], and a multiplicative character π of k. For internal consistency, we will instead denote the additive character by Φ and the multiplicative character by χ; but, for consistency with their work, we require throughout this section that $\mathrm{d}(\Phi) = -1$, that is, Φ is trivial on R but not on $\wp^{-1}$.

Definition 7.1 [Sally and Taibleson 1966, (4.1)]. For $\chi \in \widehat{k^\times}$, the *p-adic Bessel function of order* χ is given by

$$J_\chi(u, v) = \int\!\!\!\!\!\!P_{k^\times} \Phi(ux + vx^{-1})\chi(x)\,\mathrm{d}^\times x \quad \text{for } u, v \in k^\times,$$

where $\mathrm{d}^\times x$ is the Haar measure on $k^\times$ fixed in Definition 2.1. Also put $J_\chi^\theta = \frac{1}{2}(J_\chi + J_{\chi\,\mathrm{sgn}_\theta})$, with notation as in Definition 3.2.

The locally constant K-Bessel function $K(z|\chi)$ of [Trimble 1994, Definition 3.2] is $J_\chi(\varpi^t, \varpi^t)$ (in the notation of that definition), where ϖ is a uniformiser.

For $\chi \neq 1$, it is natural to extend the Bessel function by putting $J_\chi(u, 0) = \chi(u)^{-1}\Gamma(\chi)$ and $J_\chi(0, v) = \chi(v)\Gamma(\chi^{-1})$, where the Γ-factors are as in [Sally and Taibleson 1966, Section 3]. Under some conditions on χ, we can even define $J_\chi(0, 0)$ (either as 0 or the sum of a geometric series), but we do not need to do this.

The notation J_χ^θ arises naturally in our computations; see Proposition 8.11.

Definition 7.2. We say that a character $\chi \in \widehat{k^\times}$ is *mildly ramified* if χ is trivial on $1 + \wp$, but nontrivial on $k^\times$.

Since our orbital-integral calculations require information about J_χ only for χ mildly ramified, and since more precise information in that case is available in general, we focus our attention there.

Notation 7.3. Fix the following notation for the remainder of the section.

- $u, v \in k^\times$,
- $m = -\operatorname{ord}(uv)$, and
- $\chi \in \widehat{k^\times}$.

This is consistent with Notation 8.6. After Proposition 7.5, we will assume that χ is mildly ramified.

Of particular interest to us later will be the cases where χ is an unramified twist of one of the characters $\mathrm{sgn}_{\theta'}$ of Definition 3.2 (that is, is of the form $\nu^\alpha \,\mathrm{sgn}_\theta$ for some $\alpha \in \mathbb{C}$). Note that $\mathrm{sgn}_\epsilon = \nu^{\pi i/\ln(q)}$.

Theorem 7.4 [Sally and Taibleson 1966, Theorems 4.8 and 4.9].

$$J_\chi(u, v) = \begin{cases} \chi(v)\Gamma(\chi^{-1}) + \chi(u)^{-1}\Gamma(\chi) & m \le 1, \\ \chi(u)^{-1} F_\chi(m/2, uv) & m \ge 2 \text{ and } m \text{ even}, \\ 0 & m > 2 \text{ and } m \text{ odd}, \end{cases}$$

where the Γ*-factors are as in* [Sally and Taibleson 1966, Section 3], *and*

$$F_\chi(m/2, uv) := \int_{\operatorname{ord}(x) = -m/2} \Phi(x + uvx^{-1})\chi(x)\,\mathrm{d}^\times x.$$

The Γ-factor tables of [Sally and Taibleson 1966, Theorem 3.1], together with Lemma 6.3, mean that we understand $J_\chi(u, v)$ completely when $m < 2$, but further calculation is necessary in the remaining cases.

Proposition 7.5. *If*

- $h \in \mathbb{Z}_{>0}$,
- χ *is trivial on* $1 + \wp^h$, *and*
- $m \geq 4h - 1$,

then $J_\chi(uv) = 0$ *if* $uv \notin (k^\times)^2$; *and, if* $w \in k^\times$ *satisfies* $uv = w^2$, *then*

$$J_\chi(u, v) = q^{-m/4}\chi(u^{-1}w) \times \begin{cases} \Phi(2w) + \chi(-1)\Phi(-2w) & 4 \mid m, \\ \operatorname{sgn}_\varpi(w)G_\varpi(\Phi)\bigl(\Phi(2w) + (\chi\operatorname{sgn}_\varpi)(-1)\Phi(-2w)\bigr) & 4 \nmid m. \end{cases}$$

Proof. If m is odd, then the vanishing result follows from Theorem 7.4, so we assume that m is even. In this case, $m \geq 4h$, and, by Theorem 7.4, $J_\chi(u, v) = \chi(u)^{-1}F_\chi(m/2, uv)$.

We evaluate the integral defining $F_\chi(m/2, uv)$ by splitting it into pieces. Write

$$S_{uv} = \bigl\{x \in k : \operatorname{ord}(x) = -m/2 \text{ and } \operatorname{ord}(x - uvx^{-1}) < -m/2 + h\bigr\},$$
$$T_{uv} = \bigl\{x \in k : \operatorname{ord}(x) = -m/2 \text{ and } \operatorname{ord}(x - uvx^{-1}) \geq -m/2 + h\bigr\}.$$

Both S_{uv} and T_{uv} are invariant under multiplication by $1 + \wp$, and if $x \in T_{uv}$, then $uv \in x^2(1 + \wp^h) \subseteq (k^\times)^2$. We claim that the relevant integral may be taken over only T_{uv}.

If $X \in \wp^{m/2-h}$, then by Lemma 2.6 and the fact that $2(m/2 - h) \geq m/2$, we have

$$\mathrm{c}(X) \equiv 1 + 2X \pmod{\wp^{m/2}} \quad\text{and}\quad \mathrm{c}(X)^{-1} \equiv 1 - 2X \pmod{\wp^{m/2}},$$

so

$$\begin{aligned}\int_{S_{uv}} &\Phi(x + uvx^{-1})\chi(x)\,\mathrm{d}^\times x \\ &= (\star)\int_{\wp^{m/2-h}}\int_{S_{uv}} \Phi\bigl(x\cdot\mathrm{c}(X) + uvx^{-1}\cdot\mathrm{c}(X)^{-1}\bigr)\chi\bigl(x\cdot\mathrm{c}(X)\bigr)\,\mathrm{d}^\times x\,\mathrm{d}X \\ &= (\star)\int_{S_{uv}} \Phi(x + uvx^{-1})\chi(x)\int_{\wp^{m/2-h}} \Phi_{2(x-uvx^{-1})}(X)\,\mathrm{d}X\,\mathrm{d}^\times x,\end{aligned}$$

where $(\star) = \operatorname{meas}_{\mathrm{d}X}(\wp^{m/2-h})^{-1}$ is a constant and we used that Φ is trivial on $x\wp^{m/2} \cup uvx^{-1}\wp^{m/2} \subseteq R$ and χ is trivial on $\mathrm{c}(\wp^{m/2-h}) = 1 + \wp^{m/2-h} \subseteq 1 + \wp^h$. By (2.3), we have that $\mathrm{d}(\Phi_{2(x-uvx^{-1})}) > m/2 - h + 1$ (that is, $\Phi_{2(x-uvx^{-1})}$ is a nontrivial character on $\wp^{m/2-h}$) whenever $x \in S_{uv}$, so the inner integral is 0. This shows that, as desired, the integral defining $F_\chi(m/2, uv)$ may be taken over only T_{uv}.

If $uv \notin (k^\times)^2$, then $T_{uv} = \varnothing$, so $J_\chi(u, v) = \chi(u)^{-1} F_\chi(m/2, uv) = 0$; whereas, if $w \in k^\times$ satisfies $w^2 = uv$, then $T_{uv} = w(1 + \wp^h) \sqcup -w(1 + \wp^h)$, so

$$(*)\quad J_\chi(u, v) = \chi(u)^{-1}\biggl(\int_{w(1+\wp^h)} \Phi(x + uvx^{-1})\chi(x)\,\mathrm{d}^\times x \\ + \int_{-w(1+\wp^h)} \Phi(x + uvx^{-1})\chi(x)\,\mathrm{d}^\times x\biggr).$$

Note that $\operatorname{ord}(w) = -m/2$.

We show a detailed calculation of the first integral; of course, that of the second is identical. Note that the integral no longer involves χ. By Lemma 2.6 again, $X \mapsto w \cdot \mathrm{c}(X)$ is a measure-preserving bijection from $\wp^h$ to $w(1 + \wp^h)$, so

$$\int_{w(1+\wp^h)} \Phi(x + uvx^{-1})\chi(x)\,\mathrm{d}^\times x = \chi(w) \int_{\wp^h} \Phi_w\bigl(\mathrm{c}(X) + \mathrm{c}(X)^{-1}\bigr)\,\mathrm{d}X,$$

where we used $uvw^{-1} = w$ and again that χ is trivial on $\mathrm{c}(\wp^h) = 1 + \wp^h$. We will evaluate the latter integral by breaking it into "shells" on which $\operatorname{ord}(X)$ is constant, using the following facts. By direct computation (and Definition 2.5),

$$\mathrm{c}(X) + \mathrm{c}(X)^{-1} = 2\mathrm{c}(X^2)$$

for $X \in k \setminus \{1\}$. If $\operatorname{ord}(X) = i$ and $\operatorname{ord}(Y) = j$, then Lemma 2.6 implies

$$\mathrm{c}\bigl((X + Y)^2\bigr) \equiv \mathrm{c}(X^2 + 2XY) \pmod{\wp^{2j}},$$
$$\mathrm{c}(X^2 + 2XY) \equiv \mathrm{c}(X^2) + 4XY \pmod{\wp^{2j}}.$$

(The second congruence could be made much finer, but we do not need this.)

In particular, fix $i \geq h$ with $2i < m/2-1$, so that $\mathrm{d}(\Phi) = m/2-1 < 2(m/2-1-i)$ (that is, Φ is trivial on $\wp^{2(m/2-1-i)}$). Then

$$\begin{aligned}\int_{\operatorname{ord}(X)=i} &\Phi_w\bigl(\mathrm{c}(X) + \mathrm{c}(X)^{-1}\bigr)\,\mathrm{d}X \\ &= (\star) \int_{\wp^{m/2-1-i}} \int_{\operatorname{ord}(X)=i} (\Phi_{2w} \circ \mathrm{c})\bigl((X + Y)^2\bigr)\,\mathrm{d}X\,\mathrm{d}Y \\ &= (\star) \int_{\operatorname{ord}(X)=i} (\Phi_{2w} \circ \mathrm{c})(X^2) \int_{\wp^{m/2-1-i}} \Phi_{8wX}(Y)\,\mathrm{d}Y\,\mathrm{d}X,\end{aligned}$$

where $(\star) = \operatorname{meas}(\wp^{m/2-1-i})^{-1}$ is a constant. Since $\mathrm{d}(\Phi_{8wX}) = \mathrm{d}(\Phi_w) - \operatorname{ord}(8X) = m/2 - 1 - i$, the inner integral is 0.

Note that $\lceil (m/2 - 1)/2 \rceil \geq h$. Thus

$$J_\chi(u, v) = \int_{\wp^{\lceil (m/2-1)/2 \rceil}} (\Phi_{2w} \circ \mathrm{c})(X^2)\,\mathrm{d}X.$$

If $m/2$ is even, then the integral is over $\wp^{m/4}$, and $\mathrm{c}(X^2) \equiv 1 \pmod{\wp^{m/2} \subseteq \ker \Phi_{2w}}$ for all $X \in \wp^{m/4}$. Thus, in that case,

$$J_\chi(u, v) = \mathrm{meas}_{\mathrm{d}X}(\wp^{m/4})\Phi_{2w}(1) = q^{-m/4}\Phi(2w).$$

If $m/2$ is odd, then the integral is over $\wp^{m/4-1/2}$, and $\mathrm{c}(X^2) \equiv 1+2X^2 \pmod{\wp^{m/2}}$ for all $X \in \wp^{m/4-1/2}$. So, in that case,

$$\begin{aligned} J_\chi(u, v) &= \mathrm{meas}_{\mathrm{d}X}(\wp^{m/4+1/2})\Phi_{2w}(1) \sum_{X \in \wp^{m/4-1/2}/\wp^{m/4+1/2}} \Phi_{4w}(X^2) \\ &= q^{-m/4}\Phi(2w)q^{-1/2} \sum_{X \in R/\wp} \Phi_{4w\varpi^{m/2-1}}(X^2). \end{aligned}$$

By Lemma 6.2, and the fact that $m/2$ is odd, this can be rewritten as

$$q^{-m/4}\Phi(2w)\,\mathrm{sgn}_\varpi(-1)^{m/2-1}G_\varpi(\Phi_{4w}) = q^{-m/4}\Phi(2w)\,\mathrm{sgn}_\varpi(w)G_\varpi(\Phi).$$

The result now follows from $(*)$. $\square$

From now on, we assume that χ is mildly ramified. In particular, we may take $h = 1$, so that Proposition 7.5 holds whenever $m > 2$.

Definition 7.6. For

- $\xi \in \mathfrak{f}^\times$,
- $\overline{\Phi}$ an (additive) character of $\mathfrak{f}$, and
- $\overline{\chi}$ a (multiplicative) character of $\mathfrak{f}^\times$,

define the corresponding *twisted Kloosterman sum* by

$$K(\overline{\chi}, \overline{\Phi}; \xi) := \sum_{x \in \mathfrak{f}^\times} \overline{\Phi}(x + \xi x^{-1})\overline{\chi}(x).$$

Proposition 7.7. *If $m = 2$, then*

$$J_\chi(u, v) = q^{-1}\chi(u\varpi)^{-1}K(\overline{\chi}, \overline{\Phi}; \xi).$$

Here,

- *ξ is the image in $\mathfrak{f}^\times$ of $\varpi^2 uv \in R^\times$*,
- *$\overline{\Phi}$ is the (additive) character of $\mathfrak{f} = R/\wp$ arising from the restriction to R of the depth-0 character $\Phi_{\varpi^{-1}}$ of k, and*
- *$\overline{\chi}$ is the (multiplicative) character of $\mathfrak{f}^\times \cong R^\times/1+\wp$ arising from the restriction to $R^\times$ of χ.*

Proof. By Theorem 7.4,

$$\begin{aligned}\chi(u\varpi)J_\chi(u,v) &= \chi(\varpi)\int_{\operatorname{ord}(x)=-1}\Phi(x+uvx^{-1})\chi(x)\,\mathrm{d}^\times x\\ &= \int_{R^\times}\Phi(\varpi^{-1}x+uv\cdot\varpi x^{-1})\chi(x)\,\mathrm{d}^\times x\\ &= \operatorname{meas}_{\mathrm{d}^\times x}(1+\wp)\sum_{x\in R^\times/1+\wp}\Phi_{\varpi^{-1}}(x+\varpi^2uvx^{-1})\chi(x)\,\mathrm{d}^\times x.\end{aligned}$$

Since $\operatorname{meas}_{\mathrm{d}^\times x}(1+\wp)=q^{-1}$, the result follows. □

Corollary 7.8. *Suppose $m=2$. Then for $\alpha\in\mathbb{C}$,*

$$J_{\nu^\alpha}(u,v)=q^{\alpha-1}|u|^{-\alpha}\sum_{\substack{c\in\wp^{-1}/R\\ c^2\neq uv}}\Phi(2c)\operatorname{sgn}_\varpi(c^2-uv),$$

$$J_{\nu^\alpha\operatorname{sgn}_\varpi}(u,v)=q^{\alpha-1/2}|u|^{-\alpha}\operatorname{sgn}_\varpi(v)G_\varpi(\Phi)\sum_{\substack{c\in\wp^{-1}/R\\ c^2=uv}}\Phi(2c).$$

Proof. If $\chi=\nu^\alpha$, then $\overline{\chi}=1$, so [Lidl and Niederreiter 1997, Theorem 5.47] gives

$$\begin{aligned}K(\overline{\chi},\overline{\Phi};\xi) &= \sum_{\substack{c\in\mathfrak{f}\\ c^2\neq\xi}}\overline{\Phi}(2c)\operatorname{sgn}_{\mathfrak{f}}(c^2-\alpha)\\ &= \sum_{\substack{c\in R/\wp\\ c^2\neq\varpi^2uv}}\overline{\Phi}(2c)\operatorname{sgn}_\varpi(c^2-\varpi^2uv)\\ &= \sum_{\substack{c\in\wp^{-1}/R\\ c^2\neq uv}}\Phi(2c)\operatorname{sgn}_\varpi(c^2-uv).\end{aligned}$$

(Note that our $\overline{\Phi}$ is their χ, and they write $K(\chi;a,b)$ where we write $K(\overline{\Phi},1;ab)$.)

If $\chi=\nu^\alpha\operatorname{sgn}_\varpi$, then $\overline{\chi}=\operatorname{sgn}_{\mathfrak{f}}$, therefore [Lidl and Niederreiter 1997, Exercises 5.84–85] gives that

$$K(\overline{\chi},\overline{\Phi};\xi)=\operatorname{sgn}_{\mathfrak{f}}(\xi)G(\operatorname{sgn}_{\mathfrak{f}},\overline{\Phi})\sum_{\substack{c\in\mathfrak{f}\\ c^2=\xi}}\overline{\Phi}(2c)=\operatorname{sgn}_\varpi(uv)G(\operatorname{sgn}_{\mathfrak{f}},\overline{\Phi})\sum_{\substack{c\in\wp^{-1}/R\\ c^2=uv}}\Phi(2c),$$

where $G(\operatorname{sgn}_{\mathfrak{f}},\overline{\Phi})=\sum_{X\in\mathfrak{f}^\times}\overline{\Phi}(X)\operatorname{sgn}_{\mathfrak{f}}(X)$. (Note that our $\overline{\Phi}$ is their χ and our $\overline{\chi}$ their η, and our $K(\overline{\chi},\overline{\Phi};\xi)$ is their $K(\eta,\chi;1,\xi)$.) Since $\mathrm{d}(\Phi)=-1$, Lemma 6.2 gives that $G(\operatorname{sgn}_{\mathfrak{f}},\overline{\Phi})=q^{1/2}\operatorname{sgn}_\varpi(-1)G_\varpi(\Phi)$.

The result now follows from Proposition 7.7. □

The following apparently specialised corollary allows simplification of many of our "shallow" computations (see Section 9A and Section 10A).

Corollary 7.9. *If* $m \geq 2$ *and* $\operatorname{ord}(u) = \operatorname{ord}(v)$, *then* $J_{\nu^\alpha\chi}(u, v)$ *is independent of* $\alpha \in \mathbb{C}$; *in particular*,

$$J^\epsilon_\chi(u, v) = J_\chi(u, v) \quad \textit{and} \quad J^\varpi_\chi(u, v) = J^\varpi_{\chi\,\mathrm{sgn}_\epsilon}(u, v).$$

If $m \geq 2$ *and* $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, *then* $J_{\nu^\alpha\chi}(u, v) = q^\alpha J_\chi(u, v)$; *in particular*,

$$J^\epsilon_\chi(u, v) = 0 \quad \textit{and} \quad J^\varpi_\chi(u, v) = -J^\varpi_{\chi\,\mathrm{sgn}_\epsilon}(u, v).$$

Proof. Suppose that $m > 2$. If $uv \notin (k^\times)^2$, then $J_{\nu^\alpha\chi}(u, v) = 0$ for all $\alpha \in \mathbb{C}$. If $uv = w^2$, then the only dependence on α in Proposition 7.5 is via the factor $\chi(u^{-1}w)$. If $\operatorname{ord}(u) = \operatorname{ord}(v)$, then also $\operatorname{ord}(w) = \operatorname{ord}(u)$, so $\nu^\alpha(u^{-1}w) = 1$. If $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, then $\operatorname{ord}(w) = \operatorname{ord}(u) - 1$, so $\nu^\alpha(u^{-1}w) = q^\alpha$.

Now suppose that $m = 2$, that is, that $\operatorname{ord}(uv) = -2$. Since $\overline{\nu^\alpha\chi} = \overline{\chi}$, the only dependence on α in Proposition 7.7 is via the factor $\chi(u\varpi)^{-1}$. If $\operatorname{ord}(u) = \operatorname{ord}(v)$, then $\operatorname{ord}(u) = -1$, so $\nu^\alpha(u\varpi) = 1$. If $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, then $\operatorname{ord}(u) = 0$, so $\nu^\alpha(u\varpi) = q^{-\alpha}$. □

8. A mock Fourier transform

We introduce a function $M^G_{X^*}$ specified by an integral formula (see Definition 8.4) reminiscent of the usual one for (the function representing) $\hat{\mu}^G_{X^*}$ (see [Adler and DeBacker 2004, Theorem A.1.2]). We will eventually show (see Proposition 11.2) that it is actually *equal* to $\hat{\mu}^G_{X^*}$, but first we spend some time computing it.

In the notation of Definition 4.1, we have

$$\operatorname{tr} g \cdot \sqrt{\theta} \cdot g^{-1} \cdot \sqrt{\theta'} = N_\theta(\alpha) \cdot \theta' + N_\theta(\gamma), \tag{8.1}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$\alpha = a + b\sqrt{\theta}$, and $\gamma = c + d\sqrt{\theta}$. Since $1 = ad - bc = \operatorname{Im}_\theta(\overline{\alpha} \cdot \gamma)$, we have that $\gamma = \overline{\alpha}^{-1} \cdot (t + \sqrt{\theta})$ for some $t \in k$; specifically, $t = \operatorname{Re}_\theta(\overline{\alpha} \cdot \gamma) = ac - bd\theta$. This calculation motivates the definition of the following map.

Definition 8.2. Define $\varphi_\theta : G \to k^\times_\theta \times k$ by

$$\varphi_\theta(g) = (a + b\sqrt{\theta}, ac - bd\theta)$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Note that φ_θ is a bianalytic map (of k-manifolds), with inverse

$$(\alpha, t) \mapsto \begin{pmatrix} \mathrm{Re}_\theta(\alpha) & \mathrm{Im}_\theta(\alpha) \\ N_\theta(\alpha)^{-1}\bigl(t\cdot\mathrm{Re}_\theta(\alpha)+\theta\cdot\mathrm{Im}_\theta(\alpha)\bigr) & N_\theta(\alpha)^{-1}\bigl(\mathrm{Re}_\theta(\alpha)+t\cdot\mathrm{Im}_\theta(\alpha)\bigr)\end{pmatrix}.$$

It is *not* an isomorphism, but its restrictions to T_θ, A, and

$$\left\{\begin{pmatrix} 1 & 0 \\ b & 1\end{pmatrix} : b \in k\right\}$$

are isomorphisms onto $C_\theta \times \{0\}$, $k^\times \times \{0\}$, and $\{1\} \times k$, respectively. In fact, the next lemma says a bit more.

Lemma 8.3. *If $g \in G$ satisfies $\varphi_\theta(g) = (\alpha, t)$, and*

- *$h \in T_\theta$ is identified with $\eta \in C_\theta$,*
- *$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda\end{pmatrix}$ with $\lambda \in k^\times$, and*
- *$\bar{u} = \begin{pmatrix} 1 & 0 \\ b & 1\end{pmatrix}$ with $b \in k$,*

then

$$\begin{aligned}\varphi_\theta(gh) &= (\alpha\eta, t),\\ \varphi_\theta(ag) &= (\lambda\alpha, t),\\ \varphi_\theta(\bar{u}g) &= (\alpha, t + N_\theta(\alpha)b).\end{aligned}$$

Proof. This is a straightforward computation. □

We can now define our "mock orbital integral". Again, Proposition 11.2 will eventually show that it is actually equal to the function in which we are interested.

Definition 8.4. For $\alpha \in k_\theta^\times$ and $t \in k$, put

$$\langle X^*, Y\rangle_{\alpha,t} := \beta s\bigl(N_\theta(\alpha)\cdot\theta' + N_\theta(\alpha)^{-1}\cdot\theta - N_\theta(\alpha)^{-1}\cdot t^2\bigr).$$

The dependence on α is only via $N_\theta(\alpha)$. Thus, we may define

$$M^G_{X^*}(Y) := \fint_{k_\theta^\times/C_\theta}\fint_k \Phi(\langle X^*, Y\rangle_{\alpha,t})\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha},$$

where

$$\begin{aligned}\fint_k f(x)\,\mathrm{d}t &:= \sum_{n\in\mathbb{Z}}\int_{\mathrm{ord}(x)=n} f(x)\,\mathrm{d}t,\\ \fint_{k^\times} f(x)\,\mathrm{d}^\times x &:= \sum_{n\in\mathbb{Z}}\int_{\mathrm{ord}(x)=n} f(x)\,\mathrm{d}^\times x,\\ \fint_{k_\theta^\times/C_\theta}(f\circ N_\theta)(\alpha)\,\mathrm{d}^\times\dot{\alpha} &:= \fint_{k^\times}[N_\theta(k^\times)](x) f(x)\,\mathrm{d}^\times x\end{aligned}$$

(for those $f \in C^\infty(k)$ for which the sum converges) are "principal-value" integrals, as in [Sally and Taibleson 1966, p. 282]. Here, $\mathrm{d}t$ and $\mathrm{d}^\times x$ are the measures of Definition 2.1, and $[S]$ denotes the characteristic function of S.

By (8.1) (and Notations 5.1 and 5.8), we have that

$$\langle X^*, Y\rangle_{\alpha,t} = \langle \mathrm{Ad}^*(g)X^*, Y\rangle \quad \text{when } \varphi_\theta(g) = (\alpha, t), \tag{8.5}$$

where the pairing $\langle\,\cdot\,,\cdot\,\rangle$ on the right is the usual pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Notation 8.6. $u = \varpi^{-(r'+1)}s\theta'$, $v = \varpi^{-(r'+1)}s\theta$, and $m = -\operatorname{ord}(uv)$.

This is a special case of Notation 7.3. These particular values of u and v will be fixed for the remainder of the paper. It follows that

$$uv = (\varpi^{-(r'+1)}s)^2 \cdot \theta\theta', \tag{8.7}$$

therefore

$$uv \in (k^\times)^2 \quad \Leftrightarrow \quad \theta\theta' \in (k^\times)^2. \tag{8.8}$$

We use Lemma 5.9 to compute

$$\operatorname{ord}(u) = -(r'+1) + \operatorname{ord}(s\theta') = -\bigl(r' + 1 + \tfrac{1}{2}\operatorname{ord}(\theta')\bigr) + \mathrm{d}(Y), \tag{8.9}$$

$$m = 2(r'+1) - \operatorname{ord}(s^2\theta') - \operatorname{ord}(\theta) = 2(r'+1-\mathrm{d}(Y)) - \operatorname{ord}(\theta). \tag{8.10}$$

8A. *Mock Fourier transforms and Bessel functions.* We can now evaluate the integral occurring in Definition 8.4 in terms of Bessel functions — or, rather, the sums J^θ_χ of Definition 7.1.

Proposition 8.11. *Let J^θ_χ be as in Definition 7.1 and $\gamma_{\mathrm{un}}(s)$ and $\gamma_{\mathrm{ram}}(s)$ be as in Definition 6.5. Then*

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2}q^{-(r'+1)/2} \\ \times \Bigl(\bigl(J^\theta_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{un}}(s)J^\theta_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u,v)\bigr) \\ + \gamma_{\mathrm{ram}}(s)\bigl(J^\theta_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u,v) - \gamma_{\mathrm{un}}(s)J^\theta_{\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi}}(u,v)\bigr)\Bigr), \end{aligned}$$

Proof. Recall the notation $\Phi' = \Phi_\beta$ from Notation 5.2. By Definition 8.4,

$$\begin{aligned} (*)\quad M^G_{X^*}(Y) &= ⨍_{k^\times_\theta/C_\theta} \Phi'_s\bigl(N_\theta(\alpha)\cdot\theta' + N_\theta(\alpha)^{-1}\cdot\theta\bigr) \cdot ⨍_k \Phi'(-sN_\theta(\alpha)^{-1}t^2)\,\mathrm{d}t\,\mathrm{d}^\times\dot\alpha \\ &= q^{-(r'+1)/2} ⨍_{k^\times} [N_\theta(k^\times_\theta)](x) j(\theta',\theta;x)\mathscr{H}(\Phi', -sx^{-1})\,\mathrm{d}^\times x, \end{aligned}$$

where

- $j(\theta',\theta;x) := \Phi'_s(\theta'x + \theta x^{-1}) = \Phi\bigl(\beta s(\theta'x+\theta x^{-1})\bigr)$ for $x \in k^\times$, and
- $\mathscr{H}(\Phi', b) = ⨍_k \Phi'(bt^2)\,\mathrm{d}_{\Phi'}t$ for $b \in k^\times$ is as in [Shalika 2004, p. 6].

In particular, $\mathrm{d}_{\Phi'}t$ is the Φ'-self-dual Haar measure on k; by [Shalika 2004, p. 5], it satisfies $\mathrm{d}t = q^{-(r'+1)/2}\,\mathrm{d}_{\Phi'}t$. This is the reason for the appearance of $q^{-(r'+1)/2}$ on the last line of the computation.

The significance of j is that integrating it against a (multiplicative) character χ of $k^\times$ corresponds to evaluating a Bessel function of order χ, in the sense of Definition 7.1. To be precise, our character Φ' has depth r', not -1, so we must work instead with $\Phi'_{\varpi^{r'+1}}$. Then

$$j(\theta', \theta; x) = \Phi'_{\varpi^{r'+1}}\bigl((\varpi^{-(r'+1)}s\theta')x + (\varpi^{-(r'+1)}s\theta)x^{-1}\bigr) = \Phi'_{\varpi^{r'+1}}(ux + vx^{-1}),$$

where (u, v) is as in Notation 8.6, so

$$(\dagger) \qquad \int\!\!\!\!\!\!\!\!\;\;\mathcal{P}_{k^\times} j(\theta', \theta; x)\chi(x)\,\mathrm{d}^\times x = J_\chi(u, v)$$

for $\chi \in \widehat{k^\times}$.

Now $\frac{1}{2}(1+\mathrm{sgn}_\theta)$ is the characteristic function of $N_\theta(k_\theta^\times)$, so we may rewrite $(*)$:

$$(**) \qquad q^{-(r'+1)/2}\int\!\!\!\!\!\!\!\!\;\;\mathcal{P}_{k^\times} \tfrac{1}{2}(1+\mathrm{sgn}_\theta(x))\cdot j(\theta', \theta; x)\mathcal{H}(\Phi', -sx^{-1})\,\mathrm{d}^\times x.$$

By [Shalika 2004, Lemma 1.3.2] and Lemma 6.2, we have

$$\mathcal{H}(\Phi', b) = |b|^{-1/2}\begin{cases}\mathrm{sgn}_\varpi(b)G_\varpi(\Phi') & r' - \mathrm{ord}(b)\text{ even},\\ 1 & r' - \mathrm{ord}(b)\text{ odd}.\end{cases}$$

We find it useful to describe $\mathcal{H}(\Phi', b)$ without explicit use of cases. As above, $\frac{1}{2}(1+(-1)^n\,\mathrm{sgn}_\epsilon)$ is the characteristic function of $\{b \in k^\times : \mathrm{ord}(b) \equiv n \pmod 2\}$, so we may rewrite

$$\mathcal{H}(\Phi', b) = \tfrac{1}{2}(1+(-1)^{r'}\,\mathrm{sgn}_\epsilon(b))\,\mathrm{sgn}_\varpi(b)G_\varpi(\Phi') + \tfrac{1}{2}(1-(-1)^{r'}\,\mathrm{sgn}_\epsilon(b)).$$

Plugging this into $(**)$ with $b = -st^{-1}$ gives

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2}q^{-(r'+1)/2} & \\ \times \int\!\!\!\!\!\!\!\!\;\;\mathcal{P}_{k^\times} & \tfrac{1}{2}(1+\mathrm{sgn}_\theta(x)) \\ & \times \bigl((1-\gamma_{\mathrm{un}}(s)\,\mathrm{sgn}_\epsilon(x))\gamma_{\mathrm{ram}}(s)\,\mathrm{sgn}_\varpi(x) + (1+\gamma_{\mathrm{un}}(s)\,\mathrm{sgn}_\epsilon(x))\bigr) \\ & \qquad \times |x|^{1/2} j(\theta', \theta; x)\,\mathrm{d}^\times x. \end{aligned}$$

Expanding the product and applying $(\dagger)$ gives the desired formula. □

8B. ***"Deep" Bessel functions.*** By Proposition 8.11, one approach to computing $M^G_{X^*}(Y)$ (hence $\hat{\mu}^G_{X^*}(Y)$, by Proposition 11.2) is to evaluate many Bessel functions, and this is exactly what we do. As Theorem 7.4 makes clear, the behaviour of Bessel functions is more predictable when $m < 2$ than otherwise. We introduce a convenient shorthand for referring to Bessel functions in this range; we will only use it in this section, and Sections 9B and 10C.

Notation 8.12. Define

$$[A;\, B]_{\theta,r'}(\theta') := |\theta|^{1/2} A + q^{-(r'+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} B(\theta').$$

We usually suppress the subscript on $[A;\, B]$ and sometimes write

$$[A;\, B(1),\, B(\epsilon),\, B(\varpi),\, B(\epsilon\varpi)](\theta')$$

for the same quantity.

Proposition 8.13. *With Notations 5.2, 5.8, 8.6, and the notation of Definition 6.5, if $m < 2$, then*

$$|s|^{-1/2} q^{-(r'+1)/2} J_{\nu^{1/2}\chi}(u, v) = \begin{cases} [Q_3(q^{-1/2});\, 1](\theta') & \chi = 1, \\ \gamma_{\mathrm{un}}(s)[\mathrm{sgn}_\epsilon(\theta) Q_3(-q^{-1/2});\, \mathrm{sgn}_\epsilon](\theta') & \chi = \mathrm{sgn}_\epsilon, \\ \gamma_{\mathrm{ram}}(s)^{-1}[\mathrm{sgn}_\varpi(\theta) q^{-1};\, \mathrm{sgn}_\varpi](\theta') & \chi = \mathrm{sgn}_\varpi, \\ -\gamma_{\mathrm{un}}(s)\gamma_{\mathrm{ram}}(s)^{-1}[\mathrm{sgn}_{\epsilon\varpi}(\theta) q^{-1};\, \mathrm{sgn}_{\epsilon\varpi}](\theta') & \chi = \mathrm{sgn}_{\epsilon\varpi}, \end{cases}$$

where

$$Q_3(T) = -T(T^2 + T + 1).$$

The unexpected factor $|s|^{-1/2} q^{-(r'+1)/2}$ on the left-hand side crops up repeatedly in calculations (see, for example, Proposition 8.11), so it simplifies matters to include it in this calculation.

Proof. By Theorem 7.4 and Lemma 5.9,

$$\begin{aligned} J_{\nu^{1/2}\chi}(u, v) &= (\nu^{1/2}\chi)(v)\Gamma(\nu^{-1/2}\chi) + (\nu^{-1/2}\chi)(u)\Gamma(\nu^{1/2}\chi) \\ &= (\nu^{1/2}\chi)(v\theta^{-1}) \\ &\qquad \times \bigl((\nu^{1/2}\chi)(\theta)\Gamma(\nu^{-1/2}\chi) + (\nu^{-1/2}\chi)(uv\theta^{-1})\Gamma(\nu^{1/2}\chi)\bigr) \\ &= |s|^{1/2} q^{(r'+1)/2} \chi(\varpi^{r'+1} s)\bigl[\chi(\theta)\Gamma(\nu^{-1/2}\chi);\, \Gamma(\nu^{1/2}\chi)\cdot\chi\bigr](\theta') \end{aligned}$$

whenever $\chi^2 = 1$.

In particular, using [Sally and Taibleson 1966, Theorem 3.1(i, ii)] to compute the Γ-factors, we see that $|s|^{-1/2} q^{-(r'+1)/2} J_{\nu^{1/2}\chi}(u, v)$ is given by

$$(*)\quad \begin{cases} [Q_3(q^{-1/2});\, 1](\theta') & \chi = 1, \\ \gamma_{\mathrm{un}}(s)[\mathrm{sgn}_\epsilon(\theta) Q_3(-q^{-1/2});\, \mathrm{sgn}_\epsilon](\theta') & \chi = \mathrm{sgn}_\epsilon, \\ \mathrm{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\,\mathrm{sgn}_\varpi)[\mathrm{sgn}_\varpi(\theta) q^{-1};\, \mathrm{sgn}_\varpi](\theta') & \chi = \mathrm{sgn}_\varpi, \\ \gamma_{\mathrm{un}}(s)\,\mathrm{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi})[\mathrm{sgn}_{\epsilon\varpi}(\theta) q^{-1};\, \mathrm{sgn}_{\epsilon\varpi}](\theta') & \chi = \mathrm{sgn}_{\epsilon\varpi}. \end{cases}$$

By [ibid., Theorem 3.1(ii)] again and the fact that $\operatorname{sgn}_{\epsilon\varpi} = \nu^{i\pi/\ln(q)} \operatorname{sgn}_\varpi$, we have $\Gamma(\nu^{1/2} \operatorname{sgn}_{\epsilon\varpi}) = -\Gamma(\nu^{1/2} \operatorname{sgn}_\varpi)$; and, by Lemma 6.3, Definition 6.5, and (6.7),

$$\begin{aligned}\operatorname{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) &= \operatorname{sgn}_\varpi(-1)^{r'+1}\operatorname{sgn}_\varpi(s)\cdot\operatorname{sgn}_\varpi(-1)^{r'+1}G_\varpi(\Phi') \\ &= \operatorname{sgn}_\varpi(s)G_\varpi(\Phi') \\ &= \gamma_{\mathrm{ram}}(s)^{-1}.\end{aligned}$$

This shows that $(*)$ reduces to the table in the statement. ☐

9. Split and unramified orbital integrals

Throughout this section, we have

$$\theta = 1 \text{ or } \theta = \epsilon, \quad \text{so that} \quad r' = r. \tag{9.1}$$

In the split case, $J^1_\chi = J_\chi$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

$$\begin{aligned}M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2}q^{-(r+1)/2} &\\ \times\Big(\big(J_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{un}}(s)J_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u,v)\big)& \\ + \gamma_{\mathrm{ram}}(s)\big(J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u,v) - \gamma_{\mathrm{un}}(s)J_{\nu^{1/2}\operatorname{sgn}_{\epsilon\varpi}}(u,v)\big)\Big)&.\end{aligned} \tag{9.2}$$

In the unramified case, $J^\epsilon_\chi = J^\epsilon_{\chi\operatorname{sgn}_\epsilon}$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

$$\begin{aligned}M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2}q^{-(r+1)/2} &\\ \times\big((1+\gamma_{\mathrm{un}}(s))J^\epsilon_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{ram}}(s)(1-\gamma_{\mathrm{un}}(s))J^\epsilon_{\nu^{1/2}\operatorname{sgn}_\varpi}(u,v)\big)&.\end{aligned} \tag{9.3}$$

By (6.8) and (6.7),

$$\operatorname{sgn}_\varpi(v)G_\varpi(\Phi'_{\varpi^{r+1}}) = \begin{cases}\operatorname{sgn}_\varpi(-1)\gamma_{\mathrm{ram}}(s) = \gamma_{\mathrm{ram}}(s)^{-1} & \theta = 1,\\ \operatorname{sgn}_\varpi(-\epsilon)\gamma_{\mathrm{ram}}(s) = -\gamma_{\mathrm{ram}}(s)^{-1} & \theta = \epsilon.\end{cases} \tag{9.4}$$

9A. ***Far from zero.*** The results of this section are special cases for split and unramified orbital integrals of results of Waldspurger [1995, Proposition VIII.1]. We prove analogues of these results for ramified orbital integrals in Section 10A.

The qualitative behaviour of unramified orbital integrals does not change as we pass from elements of depth less than r to those of depth exactly r; this is unlike the situation for ramified orbital integrals. See Section 10B.

Theorem 9.5. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) \le 0$ *and* X^* *is split or unramified, then* $M^G_{X^*}(Y) = 0$ *unless* X^* *and* Y *lie in* G*-conjugate tori.*

Proof. Recall that $\theta = 1$ if X^* is split, and $\theta = \epsilon$ if X^* is unramified.

By (8.10), $m \ge 2$; in fact, $m > 2$ (indeed, m is odd) unless $\operatorname{ord}(\theta')$ is even.

If $m > 2$, then Proposition 7.5 and (8.8) show that $M^G_{X^*}(Y) = 0$ unless $\theta\theta' \in (k^\times)^2$. By Section 4, it therefore suffices to consider the cases when $\theta = \epsilon$ and $\theta' = \varpi^2\epsilon$, that is, X^* and Y lie in stably, but not rationally, conjugate tori; and when $m = 2$ and $\{\theta, \theta'\} = \{1, \epsilon\}$, that is, one of X^* or Y is split, and the other unramified.

Suppose first that $\theta = \epsilon$ and $\theta' = \varpi^2\epsilon$, so that $\mathrm{ord}(u) = \mathrm{ord}(v) + 2$. By Corollary 7.9, (9.3) becomes $M^G_{X^*}(Y) = 0$.

Now suppose $\{\theta, \theta'\} = \{1, \epsilon\}$ and $m = 2$. By Corollary 7.9, since $\mathrm{ord}(u) = \mathrm{ord}(v)$,

$$J_{\nu^{1/2}}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u, v) \quad \text{and} \quad J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi}}(u, v),$$

so (9.2) agrees with (9.3). We shall work with (9.3), since it is simpler.

By Corollary 7.8 and (8.8), $J_{\nu^\alpha\,\mathrm{sgn}_\varpi}(u, v) = 0$ for all $\alpha \in \mathbb{C}$, in particular, for $\alpha = 1/2$ and $\alpha = 1/2 + i\pi/\ln(q)$. By (8.10), $\mathrm{ord}(s) = r$, so, by Definition 6.5, $\gamma_{\mathrm{un}}(s) = -1$, and (9.3) (hence also (9.2)) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2} J^\epsilon_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = 0. \qquad \square$$

Theorem 9.6. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) \le 0$ *and* X^* *and* Y *lie in a common split or unramified torus* $\boldsymbol{T}$ (*with* $T = \boldsymbol{T}(k)$), *then*

$$M^G_{X^*}(Y) = q^{-(r+1)}|D_{\mathfrak{g}}(Y)|^{-1/2}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G,T)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle),$$

where $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

Proof. The hypothesis implies that $\theta = \theta'$, so $u = v$. By Corollary 7.9,

$$J_{\nu^{1/2}}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u, v) \quad \text{and} \quad J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi}}(u, v),$$

so (9.2) agrees with (9.3). We again work with (9.3), since it is simpler.

By Remark 4.7, $W(G, T_\theta) = \{1, \sigma_\theta\}$, where $\mathrm{Ad}^*(\sigma_\theta)X^* = -X^*$.

We may take the square root w of uv in Proposition 7.5 to be just u. By (8.10),

$$(*) \qquad q^{-m/4} = q^{-(r+1)/2}q^{\mathrm{ord}(s)/2} = q^{-(r+1)/2}|s|^{-1/2}.$$

By Notations 5.2 and 8.6,

$$(**) \qquad \Phi'_{\varpi^{r+1}}(\pm 2w) = \Phi'(\pm 2s\theta) = \Phi(\pm 2\beta s\theta) = \Phi(\pm\langle X^*, Y\rangle)$$

(the last equality following, for example, from (8.5)).

Suppose $\mathrm{ord}(s) \not\equiv r \pmod 2$, so $\gamma_{\mathrm{un}}(s) = 1$ and $\gamma_\Phi(X^*, Y) = 1$. By Corollary 7.9, since $u = v$, (9.3) (hence also (9.2)) becomes

$$(\dagger) \qquad \begin{aligned} M^G_{X^*}(Y) &= \tfrac{1}{2}|s|^{-1/2}q^{-(r+1)/2} \cdot 2 \cdot J^\epsilon_{\nu^{1/2}}(u, v) \\ &= |s|^{-1/2}q^{-(r+1)/2} J_{\nu^{1/2}}(u, v). \end{aligned}$$

Since $m > 2$ and $4 \mid m$ by (8.10), combining Proposition 7.5, $(*)$, and $(**)$ gives

$$(\dagger\dagger) \quad \begin{aligned} J_{\nu^{1/2}}(u,v) &= q^{-(r+1)/2}|s|^{-1/2}\big(\Phi(\langle X^*, Y\rangle) + \Phi(-\langle X^*, Y\rangle)\big) \\ &= q^{-(r+1)/2}|s|^{-1/2} \sum_{\sigma\in W(G,T_\theta)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle) \\ &= q^{-(r+1)/2}|s\theta'|^{-1/2}\gamma_\Phi(X^*, Y) \sum_{\sigma\in W(G,T_\theta)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

The result (in this case) now follows from Lemma 5.9 by combining $(\dagger)$ and $(\dagger\dagger)$.

Suppose now that $\mathrm{ord}(s) \equiv r \pmod 2$, so that $\gamma_{\mathrm{un}}(s) = -1$ and

$$\gamma_\Phi(X^*, Y) = \begin{cases} 1 & \theta = 1, \\ -1 & \theta = \epsilon. \end{cases}$$

Again by Corollary 7.9, since $u = v$, (9.3) (hence also (9.2)) becomes (as in $(\dagger)$)

$$(\dagger') \qquad M^G_{X^*}(Y) = |s|^{-1/2}q^{-(r+1)/2}\gamma_{\mathrm{ram}}(s)J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u,v).$$

Since $4 \nmid m$ by (8.10), if $m > 2$, then combining Proposition 7.5, $(*)$, (9.4), and $(**)$ gives (as in $(\dagger\dagger)$)

$$(\dagger\dagger'_{<r}) \quad \begin{aligned} &J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u,v) \\ &\quad = q^{-(r+1)/2}|s\theta'|^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma\in W(G,T_\theta)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

If $m = 2$, then $|s| = q^{-r}$ and $\mathrm{ord}(u) = -1$ by Lemma 5.9, (8.9), and (8.10). Thus, combining Corollary 7.8, (9.4), and $(**)$ gives

$$(\dagger\dagger'_{=r}) \quad \begin{aligned} &J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u,v) \\ &\quad = q^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma\in W(G,T_\theta)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle) \\ &\quad = q^{-(r+1)/2}|s\theta'|^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma\in W(G,T_\theta)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

The result follows by combining $(\dagger')$ and $(\dagger\dagger'_{<r})$ *or* $(\dagger\dagger'_{=r})$ with Lemma 5.9. □

9B. *Close to zero.*

Theorem 9.7. *If* $\mathrm{d}(X^*)+\mathrm{d}(Y) > 0$, *and* X^* *is split or unramified*, *then let* $\gamma_\Phi(X^*, Y)$ *and* $c_0(X^*)$ *be as in Definitions 6.5 and 6.10*, *respectively. Then*

$$M^G_{X^*}(Y) = c_0(X^*) + \frac{2}{n(X^*)}q^{-(r+1)}|D_{\mathfrak{g}}(Y)|^{-1/2}\gamma_\Phi(X^*, Y),$$

where

$$n(X^*) = \begin{cases} 1 & \textit{for } X^* \textit{ split}, \\ 2 & \textit{for } X^* \textit{ elliptic}. \end{cases}$$

Proof. By (8.10), $m < 2$.

By Proposition 8.13, using Notation 8.12, (9.2) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}\big[Q_3(q^{-1/2}) + Q_3(q^{-1/2}) - q^{-1} - q^{-1};\ 1 + \operatorname{sgn}_\epsilon + \operatorname{sgn}_\varpi + \operatorname{sgn}_{\epsilon\varpi}\big](\theta').$$

Since

$$\textbf{(9.8)}\qquad Q_3(q^{-1/2}) + Q_3(-q^{-1/2}) = -2T^2\big|_{T=q^{-1/2}} = -2q^{-1},$$

this simplifies (by the Plancherel formula on $k^\times/(k^\times)^2$!) to

$$M^G_{X^*}(Y) = [-2q^{-1}; 2, 0, 0, 0].$$

Similarly, (9.3) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}\Big(\tfrac{1}{2}(1+\gamma_{\mathrm{un}}(s))\underbrace{\big[Q_3(q^{-1/2}) + \gamma_{\mathrm{un}}(s)Q_3(-q^{-1/2});\ 1 + \gamma_{\mathrm{un}}(s)\operatorname{sgn}_\epsilon\big]}_{\text{(I)}}$$
$$+ \tfrac{1}{2}(1-\gamma_{\mathrm{un}}(s))\underbrace{\big[-(1-\gamma_{\mathrm{un}}(s))q^{-1};\ (1-\gamma_{\mathrm{un}}(s)\operatorname{sgn}_\epsilon)\operatorname{sgn}_\varpi\big]}_{\text{(II)}}\Big)(\theta').$$

Since $\gamma_{\mathrm{un}}(s) = \pm 1$ (see Definition 6.5), we may replace $\gamma_{\mathrm{un}}(s)$ by 1 in (I) and by -1 in (II), then use (9.8) and check case-by-case to see that the formula simplifies to

$$M^G_{X^*}(Y) = [-q^{-1}; 1, \gamma_{\mathrm{un}}(s), 0, 0](\theta'). \qquad \square$$

10. Ramified orbital integrals

Throughout this section, we have

$$\textbf{(10.1)}\qquad \theta = \varpi, \quad \text{so that} \quad r' = r + \tfrac{1}{2} =: h.$$

Then $J^\varpi_\chi = J^\varpi_{\chi\operatorname{sgn}_\varpi}$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

$$\textbf{(10.2)}\quad M^G_{X^*}(Y)$$
$$= \tfrac{1}{2}|s|^{-1/2}\big((1+\gamma_{\mathrm{ram}}(s))J^\varpi_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))J^\varpi_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u,v)\big).$$

By (6.8),

$$\textbf{(10.3)}\qquad \operatorname{sgn}_\varpi(v)G_\varpi(\Phi'_{\varpi^{h+1}}) = \operatorname{sgn}_\varpi(-\varpi)\gamma_{\mathrm{ram}}(s) = \gamma_{\mathrm{ram}}(s).$$

10A. ***Far from zero.*** As in Section 9A, the results of this section are special cases of [Waldspurger 1995, Proposition VIII.1].

Theorem 10.4. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$ *and* X^* *is ramified, then* $M^G_{X^*}(Y) = 0$ *unless* X^* *and* Y *lie in* G*-conjugate tori.*

Proof. By (8.10), $m > 2$, so Proposition 7.5 and (8.8) show that $M^G_{X^*}(Y) = 0$ unless $\varpi\theta' \in (k^\times)^2$. By Section 4, it therefore suffices to consider the case when $-1 \in (\mathfrak{f}^\times)^2$ (so $\operatorname{sgn}_\varpi(-1) = 1$) and $\theta' = \epsilon^2\varpi$, that is, X^* and Y lie in stably, but not rationally, conjugate tori.

By (8.7), we may take the square root w of uv to be $w = \varpi^{-h}s\epsilon = \epsilon^{-1}u$. Then $u^{-1}w = \epsilon^{-1}$, so Proposition 7.5 shows (whether or not 4 divides m) that, if χ is mildly ramified *and* trivial at -1, then

$$J_{\chi\,\operatorname{sgn}_\varpi}(u, v) = \operatorname{sgn}_\varpi(u^{-1}\varpi)J_\chi(u, v) = -J_\chi(u, v),$$

hence $J^\varpi_\chi(u, v) = 0$. In particular, this equality holds for $\chi = \nu^{1/2}$ and $\chi = \nu^{1/2}\operatorname{sgn}_\epsilon$. It follows from (10.2) that $M^G_{X^*}(Y) = 0$. □

Theorem 10.5. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$, *and* X^* *and* Y *lie in a common ramified torus* $\boldsymbol{T}$ (*with* $T = \boldsymbol{T}(k)$), *then*

$$M^G_{X^*}(Y) = q^{-(h+1)}|D_{\mathfrak{g}}(Y)|^{-1/2}\gamma_\Phi(X^*, Y)\sum_{\sigma\in W(G,T)}\Phi(\langle \operatorname{Ad}^*(\sigma)(X^*), Y\rangle),$$

where $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

Proof. Since we have fixed $\theta = \varpi$, the hypothesis implies that $\theta' = \varpi$. In particular, $u = v$. Write σ_ϖ for the nontrivial element of $W(\boldsymbol{G}, \boldsymbol{T}_\varpi)(k_\varpi)$, so that $\operatorname{Ad}^*(\sigma_\varpi)X^* = -X^*$. It is possible that σ_ϖ is not k-rational. More precisely, by Section 4, we have that

$$W(G, T_\varpi) = \begin{cases} \{1, \sigma_\varpi\} & \operatorname{sgn}_\varpi(-1) = 1, \\ \{1\} & \operatorname{sgn}_\varpi(-1) = -1. \end{cases}$$

By (8.10),

$$(*) \qquad q^{-m/4} = q^{-(h-\operatorname{ord}(s))/2} = q^{-h/2}|s|^{-1/2}.$$

By Corollary 7.9, since $u = v$,

$$J^\varpi_{\nu^{1/2}}(u, v) = J^\varpi_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u, v),$$

so (10.2) becomes

$$(\dagger) \quad M^G_{X^*}(Y) = \tfrac{1}{2}|s|^{-1/2}q^{-(h+1)/2}\big((1+\gamma_{\mathrm{ram}}(s)) + \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)J^\varpi_{\nu^{1/2}}(u, v).$$

It remains to compute $J^\varpi_{\nu^{1/2}}(u, v)$.

We will use Proposition 7.5, but, for simplicity, we want to avoid splitting into cases depending on whether or not $4 \mid m$. By (8.10), the restrictions to $k \setminus \wp^{h-1}$ of $\frac{1}{2}(1 + (-1)^h\operatorname{sgn}_\epsilon) = \frac{1}{2}(1 - \gamma_{\mathrm{un}})$ and $\frac{1}{2}(1 + \gamma_{\mathrm{un}})$ are characteristic functions that indicate whether $4 \mid m$ or $4 \nmid m$, respectively. (We omit $\wp^{h-1}$ because we are concerned with the case where $\mathrm{d}(Y) < r$, so that $\operatorname{ord}(s) < r - \frac{1}{2} = h - 1$.)

Thus, if $\operatorname{sgn}_\varpi(-1) = -1$, then combining Proposition 7.5, $(*)$, and (10.3) gives

$$(*_{\mathrm{ns}}) \qquad \begin{aligned} J_{\nu^\alpha}(u,v) &= q^{-h/2}|s|^{-1/2} \\ &\qquad \times \Big(\tfrac{1}{2}\big((1-\gamma_{\mathrm{un}}(s)) + (1+\gamma_{\mathrm{un}}(s))\gamma_{\mathrm{ram}}(s)\big) \times \Phi'_{\varpi^{h+1}}(2\varpi^{-h}s) \\ &\qquad \overset{(\S)}{+} \tfrac{1}{2}\big((1-\gamma_{\mathrm{un}}(s)) \overset{(\P)}{-} (1+\gamma_{\mathrm{un}}(s))\gamma_{\mathrm{ram}}(s)\big) \\ &\qquad \times \Phi'_{\varpi^{h+1}}(-2\varpi^{-h}s)\Big) \\ &= \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2} \\ &\qquad \times \Big(\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)\Phi(\langle X^*, Y\rangle) \\ &\qquad + \big((1-\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1+\gamma_{\mathrm{ram}}(s))\big)\Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle) \end{aligned}$$

and (changing the sign at (§), but not at (¶)) that

$$\begin{aligned} J_{\nu^\alpha \operatorname{sgn}_\varpi}(u,v) &= \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2} \\ &\qquad \times \Big(\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)\Phi(\langle X^*, Y\rangle) \\ &\qquad - \big((1-\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1+\gamma_{\mathrm{ram}}(s))\big)\Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\Big), \end{aligned}$$

so that

$$(\ddagger_{\mathrm{ns}}) \qquad \begin{aligned} &J^\varpi_{\nu^\alpha}(u,v) \\ &\quad = \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2}\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)\Phi(\langle X^*, Y\rangle). \end{aligned}$$

Similarly, if $\operatorname{sgn}_\varpi(-1) = 1$, then (changing the sign at (¶), but not at (§), in $(*_{\mathrm{ns}})$) we obtain

$$(*_{\mathrm{s}}) \qquad \begin{aligned} J_{\nu^\alpha}(u,v) &= J_{\nu^\alpha \operatorname{sgn}_\varpi}(u,v) \\ &= \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2}\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big) \\ &\qquad \times \big(\Phi(\langle X^*, Y\rangle) + \Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\big), \end{aligned}$$

so that

$$(\ddagger_{\mathrm{s}}) \qquad \begin{aligned} J^\varpi_{\nu^\alpha}(u,v) &= J_{\nu^\alpha}(u,v) \\ &= \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2}\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big) \\ &\qquad \times \big(\Phi(\langle X^*, Y\rangle) + \Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\big). \end{aligned}$$

We may write $(\ddagger_{\mathrm{ns}})$ and $(\ddagger_{\mathrm{s}})$ uniformly as

$$(\ddagger) \qquad \begin{aligned} J^\varpi_{\nu^\alpha}(u,v) &= \tfrac{1}{2}q^{-(h+1)/2}|s\theta'|^{-1/2}\big((1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big) \\ &\qquad \times \sum_{\sigma \in N_G(T_\varpi)/T_\varpi} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

Upon combining (†), (‡), and Lemma 5.9, we obtain the desired formula from

$$\begin{aligned}\big((1+\gamma_{\mathrm{ram}}(s))+\gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)\cdot\big((1+\gamma_{\mathrm{ram}}(s))-\gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)\\ =(1+\gamma_{\mathrm{ram}}(s))^2-\gamma_{\mathrm{un}}(s)^2(1-\gamma_{\mathrm{ram}}(s))^2=4\gamma_{\mathrm{ram}}(s)=4\gamma_\Phi(X^*,Y)\end{aligned}$$

(since $\gamma_{\mathrm{un}}(s)^2=1$). □

10B. ***The bad shell.*** We shall be concerned in this section with the behaviour of $M^G_{X^*}$ (hence $\hat\mu^G_{X^*}$, by Proposition 11.2) at the "bad shell", that is, on those regular, semisimple elements Y such that $\mathrm{d}(Y)=r$. We assume this is the case throughout the section. By (8.10), this implies that $m=2$ and that $\mathrm{ord}(\theta')$ is odd, that is, Y belongs to a ramified torus. By Section 4, we can in fact assume $\mathrm{ord}(\theta')=1$. Then, by Lemma 5.9,

$$\mathrm{ord}(s)=h-1 \quad\Rightarrow\quad \mathrm{sgn}_\epsilon(s)=(-1)^{h-1} \text{ and } |s\theta'|=q^{-h}. \tag{10.6}$$

By Definition 6.5, the formula that holds in the situation of Theorem 10.9 holds also, suitably understood, in the situation of Theorem 10.8. We find it useful to separate them anyway.

Remark 10.7. In this section only, we need to name the specific ramified torus in which we are interested. We therefore assume in Theorems 10.8 and 10.9 that $X^*\in\mathfrak{t}^*_\varpi$. See Remark 6.9 for a discussion of how to handle other ramified tori.

Theorem 10.8. *If* $\mathrm{d}(X^*)+\mathrm{d}(Y)=0$, *and Y lies in a ramified torus that is* not *stably conjugate to $\boldsymbol{T}_\varpi$, then*

$$M^G_{X^*}(Y)=\tfrac12 q^{-(h+1)}\cdot q^{-1/2}|D_{\mathfrak g}(Y)|^{-1/2}\sum_{Z\in(\mathfrak{t}_\varpi)_{r:r+}}\Phi(\langle X^*,Z\rangle)\,\mathrm{sgn}_\varpi(Y^2-Z^2),$$

where we identify the scalar matrices Y^2 and Z^2 with elements of k in the natural way.

Proof. By Section 4, it suffices to consider the case where $\theta'=\epsilon\varpi$.

By Corollary 7.9, since $\mathrm{ord}(u)=\mathrm{ord}(v)$,

$$J^\varpi_{\nu^{1/2}}(u,v)=J^\varpi_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u,v),$$

and, by Corollary 7.8 and (8.8), $J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u,v)=0$, so

$$J^\varpi_{\nu^{1/2}}(u,v)=\tfrac12 J_{\nu^{1/2}}(u,v).$$

Hence, by (10.2) and (10.6),

$$(*)\qquad \begin{aligned} M^G_{X^*}(Y) &= \tfrac14 |s|^{-1/2} q^{-(h+1)/2} \\ &\qquad \times \big((1+\gamma_{\mathrm{ram}}(s)) - (-1)^h \operatorname{sgn}_\epsilon(s)(1-\gamma_{\mathrm{ram}}(s))\big) J_{\nu^{1/2}}(u,v) \\ &= \tfrac14 |s|^{-1/2} q^{-(h+1)/2} \cdot 2 \cdot J_{\nu^{1/2}}(u,v) \\ &= \tfrac12 |s|^{-1/2} J_{\nu^{1/2}}(u,v). \end{aligned}$$

Finally, another application of Corollary 7.8, together with (8.9), gives that

$$J_{\nu^{1/2}}(u,v) = q^{-1} \sum_{c \in \wp^{-1}/R} \Phi'_{\varpi^{h+1}}(2c) \operatorname{sgn}_\varpi(c^2 - (\varpi^{-h} s)^2 \epsilon).$$

Replacing c by $\varpi^{-h}c$ and using (10.6) again allows us to rewrite

$$(**)\qquad J_{\nu^{1/2}}(u,v) = q^{-(h+2)/2} |s\theta'|^{-1/2} \sum_{c \in \wp^{h-1}/\wp^h} \Phi(2\beta\varpi c) \operatorname{sgn}_\varpi(c^2 - s^2\epsilon).$$

By Definition 4.9, the isomorphism $c \mapsto c \cdot \sqrt{\varpi}$ of k with $\mathfrak{t}_\varpi$ identifies $\wp^{h-1}/\wp^h$ with $(\mathfrak{t}_\varpi)_{(h-1/2):(h+1/2)} = (\mathfrak{t}_\varpi)_{r:r+}$. If c is mapped to Z, then (by (8.5), for example) $2\beta\varpi c = \langle X^*, Z\rangle$, and

$$\operatorname{sgn}_\varpi(c^2 - s^2\epsilon) = \operatorname{sgn}_\varpi(s^2\epsilon\varpi - c^2\varpi) = \operatorname{sgn}_\varpi(Y^2 - Z^2).$$

Combining this with $(*)$, $(**)$, and Lemma 5.9 yields the desired formula. □

Theorem 10.9. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) = 0$, *and* $\widetilde{Y}$ *is a stable conjugate of* Y *that lies in a torus with* X^*, *then*

$$\begin{aligned} M^G_{X^*}(Y) &= \tfrac12 q^{-(h+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} \\ &\quad \times \Bigg(\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(\boldsymbol{G}, \boldsymbol{T}_\varpi)} \Phi(\langle \mathrm{Ad}^*(\sigma) X^*, \widetilde{Y}\rangle) \\ &\qquad + q^{-1/2} \sum_{\substack{Z \in (\mathfrak{t}_\varpi)_{r:r+} \\ Z \ne \pm\widetilde{Y}}} \Phi(\langle X^*, Z\rangle) \operatorname{sgn}_\varpi(Y^2 - Z^2) \Bigg), \end{aligned}$$

where $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

Proof. Implicit in the statement is the hypothesis that $\mathfrak{t} = \mathfrak{t}_{\theta'}$ is stably conjugate to $\mathfrak{t}_\varpi$, so that, by Section 4, we have $\theta' = x^2\varpi$ for some $x \in R^\times$. The proof proceeds much as in Theorem 10.8.

By (10.6) and Corollary 7.9, since $\operatorname{ord}(u) = \operatorname{ord}(v)$, (10.2) becomes

$$(*)\qquad M^G_{X^*}(Y) = |s|^{-1/2} q^{-(h+1)/2} J^{\varpi}_{\nu^{1/2}}(u,v).$$

By (8.7), we may take the square root w of uv to be $w = \varpi^{-h} x s$.

Combining Corollary 7.8 with (8.7), (8.9), and (10.6) gives

$$\begin{aligned} J_{\nu^\alpha}(u,v) &= q^{-1} \sum_{\substack{c\in\wp^{-1}/R \\ c\neq\pm\varpi^{-h}xs}} \Phi'_{\varpi^{h+1}}(2c)\,\operatorname{sgn}_\varpi(c^2-(\varpi^{-h}xs)^2) \\ &= q^{-1} \sum_{\substack{c\in\wp^{h-1}/\wp^h \\ c\neq\pm xs}} \Phi(2\beta\varpi c)\,\operatorname{sgn}_\varpi(c^2-x^2s^2) \\ &= q^{-(h+2)/2}|s\theta'|^{-1/2} \sum_{\substack{c\in\wp^{h-1}/\wp^h \\ c\neq\pm xs}} \Phi(2\beta\varpi c)\,\operatorname{sgn}_\varpi(c^2-x^2s^2). \end{aligned}$$

Note that $Y^2 = s^2\theta' = x^2s^2\varpi$, and that

$$\widetilde{Y} := xs\sqrt{\varpi} = \operatorname{Ad}\begin{pmatrix}\sqrt{x} & 0 \\ 0 & \sqrt{x}^{-1}\end{pmatrix} Y$$

is a stable conjugate of Y that lies in $\mathfrak{t}_\varpi$. (Here, $\sqrt{\varpi}$ is an element of $\mathfrak{g}$, but $\sqrt{x}$ is an element of an extension field of k.) As in Theorem 10.8, if $Z = c\cdot\sqrt{\varpi}$, then $\langle X^*, Z\rangle = 2\beta\varpi c$ and $\operatorname{sgn}_\varpi(c^2-x^2s^2) = \operatorname{sgn}_\varpi(Y^2-Z^2)$. That is, upon using again the bijection $\wp^{h-1}/\wp^h \to (\mathfrak{t}_\varpi)_{r:r+}$ given by $c\mapsto c\cdot\sqrt{\varpi}$, we obtain

$$(**_1)\qquad J_{\nu^{1/2}}(u,v) = q^{-(h+2)/2}|s\theta'|^{-1/2} \sum_{\substack{Z\in(\mathfrak{t}_\varpi)_{r:r+} \\ Z\neq 0,\pm\widetilde{Y}}} \Phi(\langle X^*, Z\rangle)\,\operatorname{sgn}_\varpi(Y^2-Z^2).$$

Similarly, combining Corollary 7.8 with (8.9), Lemma 5.9, and (10.3) gives

$$\begin{aligned} (**_\varpi)\quad J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u,v) &= q^{-1/2}\gamma_{\mathrm{ram}}(s)\bigl(\Phi(2\beta\varpi xs)+\Phi(-2\beta\varpi xs)\bigr) \\ &= q^{-(h+1)/2}|s\theta'|^{-1/2}\gamma_{\mathrm{ram}}(s) \sum_{\sigma\in W(\boldsymbol{G},\boldsymbol{T})} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, \widetilde{Y}\rangle). \end{aligned}$$

Combining $(*)$, $(**_1)$, $(**_\varpi)$, and Lemma 5.9 gives the desired formula. □

10C. *Close to zero.*

Theorem 10.10. *If* $\mathrm{d}(X^*)+\mathrm{d}(Y) > 0$, *and* X^* *is ramified, then let* $\gamma_\Phi(X^*, Y)$ *and* $c_0(X^*)$ *be as in Definitions 6.5 and 6.10, respectively. Then*

$$M^G_{X^*}(Y) = c_0(X^*) + q^{-(h+1)}|D_{\mathfrak{g}}(Y)|^{-1/2}\gamma_\Phi(X^*, Y).$$

Proof. By (8.10), $m < 2$.

By Proposition 8.13 and (6.7), using Notation 8.12 changes (10.2) into

$$\begin{aligned} &M^G_{X^*}(Y) \\ &= \tfrac{1}{2}\Bigl(\tfrac{1}{2}(1+\gamma_{\mathrm{ram}}(s))\bigl[Q_3(q^{-1/2})+\gamma_{\mathrm{ram}}(s)q^{-1};\ 1+\gamma_{\mathrm{ram}}(s)^{-1}\operatorname{sgn}_\varpi\bigr] \\ &\quad +\tfrac{1}{2}(1-\gamma_{\mathrm{ram}}(s))\times\bigl[-Q_3(-q^{-1/2})+\gamma_{\mathrm{ram}}(s)q^{-1};\ (1-\gamma_{\mathrm{ram}}(s)^{-1}\operatorname{sgn}_\varpi)\operatorname{sgn}_\epsilon\bigr]\Bigr)(\theta'). \end{aligned}$$

By (9.8) and the fact that

$$Q_3(q^{-1/2}) - Q_3(-q^{-1/2}) = -2T(T^2+1)\big|_{T=q^{-1/2}} = -2q^{-3/2}(q+1),$$

we may check case-by-case to see that this simplifies to

$$M^G_{X^*}(Y) = \big[-\tfrac{1}{2}q^{-3/2}(q+1);\, 1,\, 0,\, \gamma_{\mathrm{ram}}(s),\, 0\big](\theta'). \qquad \square$$

11. An integral formula

Our efforts so far have focused on computing the function $M^G_{X^*}$ of Definition 8.4, whereas we are really interested in the function $\hat{\mu}^G_{X^*}$ of Notation 5.7. We can now show that they are actually equal.

Lemma 11.1. *If $f \in L^1(G)$, then*

$$\int_G f(g)dg = \int_{k_\theta^\times} \int_k f\big(\varphi_\theta^{-1}(\alpha, t)\big)\,\mathrm{d}t\,\mathrm{d}^\times\alpha.$$

In Lemma 11.1, $\mathrm{d}g$, $\mathrm{d}t$, and $\mathrm{d}^\times\alpha$ are Haar measures on the obvious groups. Given any two of them, the third can be chosen so that the identity is satisfied. Since Definition 5.4 requires a measure on $G/C_G(X^*)$, not on G, we do not worry much here about normalisations (although a specific one is used in the proof).

Proof. With respect to the coordinate charts

$$(a, b, c) \mapsto \begin{pmatrix} a & b \\ c & (1+bc)/a \end{pmatrix}$$

(for $a \neq 0$) on G and

$$(a, b, t) \mapsto (a + b\sqrt{\theta}, t)$$

on $k_\theta^\times \times k$, the Jacobian of φ_θ at

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(with $a \neq 0$) is $a^{-1}N_\theta(\alpha)$, where $\varphi_\theta(g) = (\alpha, t)$.

In particular, the Haar measure

$$|a|^{-1}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

on G is carried to the measure

$$|N_\theta(a + b\sqrt{\theta})|^{-1}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}t = |N_\theta(\alpha)|^{-1}\,\mathrm{d}\alpha\,\mathrm{d}t = \mathrm{d}^\times\alpha\,\mathrm{d}t$$

on $k_\theta^\times \times k$, as desired. $\square$

Proposition 11.2. *If $X^* \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$ are regular and semisimple, then*

$$\hat{\mu}^G_{X^*}(Y) = M^G_{X^*}(Y),$$

where $M^G_{X^*}$ is as in Definition 8.4, and the Haar measure $\mathrm{d}\dot{g}$ on $G/C_G(X^*)$ of Notation 5.3 is normalised so that

$$\mathrm{meas}_{\mathrm{d}\dot{g}}(\dot{K}) = \begin{cases} q^{-1}(q+1) & \text{for } X^* \text{ split}, \\ q^{-1}(q-1) & \text{for } X^* \text{ unramified}, \\ \frac{1}{2}q^{-2}(q^2-1) & \text{for } X^* \text{ ramified}, \end{cases}$$

where $\dot{K}$ is the image in $G/C_G(X^*)$ of $\mathrm{SL}_2(R)$.

Proof. We will maintain Notation 5.1. In particular, $X^* \in \mathfrak{t}^*_\theta$.

By the explicit formulae of the previous sections (specifically, Theorems 9.5, 9.6, 9.7, 10.4, 10.5, 10.8, 10.9, and 10.10), $M^G_{X^*} \in C^\infty(\mathfrak{g}^{\mathrm{rss}})$. This result plays the role of [Adler and DeBacker 2004, Corollary A.3.4]; we now imitate the proof of [ibid., Theorem A.1.2].

If $f \in C_c(\mathfrak{g}^{\mathrm{rss}})$, then there is a lattice $\mathscr{L} \subseteq \mathfrak{g}$ such that $f \cdot M^G_{X^*}$ is invariant under translation by $\mathscr{L}$. Then

$$\int_{\mathfrak{g}} f(Y) M^G_{X^*}(Y)\,\mathrm{d}Y = \mathrm{meas}_{\mathrm{d}Y}(\mathscr{L}) \sum_{Y \in \mathfrak{g}/\mathscr{L}} f(Y) \cdot \fint_{k^\times_\theta/C_\theta} \fint_k \Phi(\langle X^*, Y\rangle_{\alpha,t})\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha}.$$

Since the sum is finitely supported, we may bring it inside the integral. By (8.5) and Definition 5.5,

$$(*) \qquad \int_{\mathfrak{g}} f(Y) M^G_{X^*}(Y)\,\mathrm{d}Y$$
$$= \fint_{k^\times_\theta/C_\theta} \fint_k \mathrm{meas}_{\mathrm{d}Y}(\mathscr{L}) \sum_{Y \in \mathfrak{g}/\mathscr{L}} f(Y)\Phi\big(\langle \mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t))X^*, Y\rangle\big)\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha}$$
$$= \fint_{k^\times_\theta/C_\theta} \fint_k \int_{\mathfrak{g}} f(Y)\Phi\big(\langle \mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t))X^*, Y\rangle\big)\,\mathrm{d}Y\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha}$$
$$= \fint_{k^\times_\theta/C_\theta} \fint_k \hat{f}\big(\mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t))X^*\big)\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha},$$

where φ_θ is as in Definition 8.2.

On the other hand, again by Definition 5.5,

$$\hat{\mu}^G_{X^*}(f) := \mu^G_{X^*}(\hat{f}) = \int_{G/T_\theta} \hat{f}\big(\mathrm{Ad}^*(g)X^*\big)\,\mathrm{d}\dot{g} = \int_{\overline{U}\backslash G/T_\theta} \int_{\overline{U}} \hat{f}\big(\mathrm{Ad}^*(\bar{u}g)X^*\big)\,\mathrm{d}\bar{u}\,\mathrm{d}\ddot{g},$$

where

$$\overline{U} = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : b \in k \right\}.$$

By Lemmata 11.1 and 8.3, and $(*)$, if $\mathrm{d}\dot{g}$ is properly normalised, then

$$\hat{\mu}^G_{X^*}(f) = \int_{k_\theta^\times/C_\theta} \int_k \hat{f}\big(\mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha, t))X^*\big)\,\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha} = \int_{\mathfrak{g}} f(Y) M^G_{X^*}(Y)\,\mathrm{d}Y.$$

It remains only to compute the normalisation of $\mathrm{d}\dot{g}$. We do so case-by-case. If X^* is split, so that we may take $\theta = 1$, then the image under φ_1 of

$$(1+\wp_1) \times \wp \subseteq k_1^\times \times k$$

is precisely the kernel K_+ of the (component-wise) reduction map $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathfrak{f})$. Here, we have written $1+\wp_1 = \{(a,b) \in k_1 : a \in 1+\wp, b \in \wp\}$. Thus,

$$(1+\wp_1)C_1/C_1 \times \wp \xrightarrow{\sim} K_+T_1/T_1.$$

Now $N_1 : 1+\wp_1 \to 1+\wp$ is surjective, so by Definitions 2.1 and 8.4, the measure (in $k_1/C_1 \times k$) of the domain is

$$\mathrm{meas}_{\mathrm{d}^\times x}(1+\wp) \cdot \mathrm{meas}_{\mathrm{d}x}(\wp) = q^{-2}.$$

Since $\dot{K} = \mathrm{SL}_2(R)T_1/T_1$ is tiled by

$$\big[\mathrm{SL}_2(R)T_1 \colon K_+T_1\big] = \big[\mathrm{SL}_2(R) \colon K_+(T_1 \cap \mathrm{SL}_2(R))\big] = \big[\mathrm{SL}_2(\mathfrak{f}) \colon \mathsf{T}_1(\mathfrak{f})\big] = q(q+1)$$

copies of K_+T_1/T_1, where

$$\mathsf{T}_1 := \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 - b^2 = 1\right\},$$

we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure $q^{-2} \cdot q(q+1) = q^{-1}(q+1)$.

The remaining cases are easier, since C_θ is contained in the ring R_θ of integers in k_θ, and (for our choices of θ) T_θ is contained in $\mathrm{SL}_2(R)$. If X^* is unramified, so that we may take $\theta = \epsilon$, then the image under φ_ϵ of $R_\epsilon^\times \times R$ is precisely $\mathrm{SL}_2(R)$. Since $N_\epsilon : R_\epsilon^\times \to R^\times$ is surjective, we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure $\mathrm{meas}_{\mathrm{d}^\times x}(R_\epsilon^\times) \cdot \mathrm{meas}_{\mathrm{d}x}(R) = q^{-1}(q-1)$.

If X^* is ramified, so that we may take $\theta = \varpi$, then the image under φ_ϖ of $R_\varpi^\times \times \wp$ is precisely the Iwahori subgroup $\mathcal{I}$, that is, the preimage in $\mathrm{SL}_2(R)$ of

$$\mathsf{B}(\mathfrak{f}) := \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathfrak{f}^\times, b \in \mathfrak{f}\right\}$$

under the reduction map $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathfrak{f})$. Since $N_\varpi : R_\varpi^\times \to R^\times$ has cokernel of order 2, we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure

$$\tfrac{1}{2}\,\mathrm{meas}_{\mathrm{d}^\times x}(R^\times) \cdot \mathrm{meas}_{\mathrm{d}x}(\wp) \cdot \big[\mathrm{SL}_2(\mathfrak{f}) \colon \mathsf{B}\big] = \tfrac{1}{2} q^{-2}(q^2 - 1). \qquad \square$$

Thus, all the results we have proven for $M^G_{X^*}$ are actually results about $\hat{\mu}^G_{X^*}$. We close by summarising some results that can be stated in a fairly uniform fashion (that is, mostly independent of the "type" of X^*, in the sense of Definition 4.4). This theorem does *not* cover everything we have shown about Fourier transforms of semisimple orbital integrals (in particular, it says nothing about the behaviour of ramified orbital integrals on the "bad shell", as in Section 10B); for that, the reader should refer to the detailed results of Sections 9–10.

Theorem 11.3. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$ (*or* $\mathrm{d}(X^*) + \mathrm{d}(Y) \le 0$ *and* X^* *is split or unramified*), *then*

$$\hat{\mu}^G_{X^*}(Y) = q^{-(r'+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} \gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G,T)} \Phi(\langle \mathrm{Ad}^*(\sigma) X^*, Y\rangle)$$

if X^* *and* Y *lie in a common torus* $\boldsymbol{T}$ (*with* $T = \boldsymbol{T}(k)$), *and*

$$\hat{\mu}^G_{X^*}(Y) = 0$$

if X^* *and* Y *do not lie in* G*-conjugate tori. Here*, r' *is as in Notation 5.2*, *and* $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

If $\mathrm{d}(X^*) + \mathrm{d}(Y) > 0$, *then*

$$\hat{\mu}^G_{X^*}(Y) = c_0(X^*) + q^{-(r'+1)} |D_{\mathfrak{g}}(Y)|^{-1/2} \gamma_\Phi(X^*, Y).$$

Here, $\gamma_\Phi(X^*, Y)$ *and* $c_0(X^*)$ *are as in Definitions 6.5 and 6.10*, *respectively.*

Proof. This is an amalgamation of parts of Theorems 9.5, 9.6, 9.7, 10.4, 10.5, and 10.10 and Proposition 11.2. □

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LOREN SPICE
DEPARTMENT OF MATHEMATICS
TEXAS CHRISTIAN UNIVERSITY
TCU BOX 298900
2840 W. BOWIE ST
FORT WORTH, TX 76109
UNITED STATES

l.spice@tcu.edu
http://faculty.tcu.edu/lspice

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