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In this paper, we will study the τ-quasi-Einstein metrics on complete noncompact Riemannian manifolds and get a rigid property. We will also obtain lower and upper estimates for scalar curvatures on these metrics by using the maximum principle.

1. Introduction

For a given smooth potential function f, the τ-Bakry–Émery Ricci curvature tensor

$$\mathrm{Ric}_{f,\tau} = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{\nabla f \otimes \nabla f}{\tau}$$

is always used to replace the Ricci curvature tensor when one tries to study the weighted measure $d\mu = e^{-f}dx$, where $0 < \tau \leq +\infty$ and dx is the Riemann–Lebesgue measure determined by the metric. There has been an active interest in the study of the weighted measure under some conditions about the τ-Bakry–Émery Ricci curvature tensor; see [Li 2005; Wang 2010] and the references therein.

According to [Kim and Kim 2003; Case 2010; Case et al. 2011; Wang 2011], we call a metric g τ-quasi-Einstein with potential function f, if for some constant λ,

$$\mathrm{Ric} + \mathrm{Hess}\, f - \frac{\nabla f \otimes \nabla f}{\tau} = \lambda g, \tag{1-1}$$

where $0 < \tau \leq +\infty$. A τ-quasi-Einstein metric becomes an Einstein metric when the potential function f is constant. We note that an ∞-quasi-Einstein metric indicates a gradient Ricci soliton. As in [Hamilton 1995; Perelman 2002; Cao and Zhu 2006], a gradient Ricci soliton is shrinking, steady or expanding when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

For a positive integer τ, the τ-quasi-Einstein metric is closely relative to the existence of warped product Einstein manifolds [Besse 1987; Case 2010; Case et al. 2011]. Let (M, g) and (N^τ, h) be two Riemannian manifolds. Then, for

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some potential function f on M, the warped product manifold $(M \times N, \tilde{g})$ with product metric

$$\tilde{g} = g \oplus \exp\left(-\frac{2f}{\tau}\right)h$$

is Einstein if and only if (N^τ, h) is Einstein and the Ricci curvature tensor of M satisfies the quasi-Einstein equation (1-1) for some constant λ.

It was proved in [Qian 1997; Wei and Wylie 2007] that a manifold with a τ-quasi-Einstein metric (τ is finite) is automatically compact when $\lambda > 0$. It was also proved in [Ivey 1993] that any expanding or steady gradient Ricci solitons on closed manifolds should be trivial. The same rigid properties for the τ-quasi-Einstein metrics on closed manifolds were proved in [Kim and Kim 2003; Wang 2011]. But for the τ-quasi-Einstein metrics on closed manifolds with $\lambda > 0$, the rigid properties rely on the constant μ which appears in the following identity:

$$R + \frac{\tau - 1}{\tau}|\nabla f|^2 + (\tau - n)\lambda = \mu e^{2f/\tau}, \tag{1-2}$$

where R is the scalar curvature. This identity was proved in [Kim and Kim 2003]. See also [Wang 2011], where the author proved that the quasi-Einstein metrics with $\lambda > 0$ should be trivial when $\mu \le 0$. In fact, the authors of [Lü et al. 2004] constructed nontrivial τ-quasi-Einstein metrics with $\lambda > 0$ and $\tau > 1$, which also satisfy $\mu > 0$.

In this paper, we will study the τ-quasi-Einstein metrics on complete noncompact Riemannian manifolds with $\lambda \le 0$. Our first result is Theorem 1.1, which is about the rigidity.

Theorem 1.1. *Let M be a complete noncompact Riemannian manifold and g a τ-quasi-Einstein metric on M with potential function f and $\lambda \le 0$ a constant. If*

$$R_0^{-2} \int_{B_{2R_0} \setminus B_{R_0}} |\nabla f|^2 \exp\left(-\frac{\tau + 2}{\tau} f\right) dx \to 0 \tag{1-3}$$

as $R_0 \to \infty$, where B_{R_0} denotes the geodesic ball centered at a fixed point $O \in M$ with radius R_0, then e^f is a harmonic function on M, that is, $\Delta e^f = 0$. Moreover, if $\lambda < 0$, then g is trivial in the sense that f is constant.

The following theorem for gradient Ricci solitons was proved in [Zhang 2009]. In fact, part 1 is a consequence of [Chen 2009, Corollary 2.5].

Theorem 1.2. *Let (M^n, g) be a complete noncompact gradient Ricci soliton with potential function f and soliton constant λ.*

(1) *If the gradient Ricci soliton is shrinking or steady, then $R \ge 0$.*

(2) *If the gradient Ricci soliton is expanding, then there exists a positive constant $C(n)$ such that $R \ge C(n)\lambda$.*

Zhang [2011] pointed out that $R \ge n\lambda$ is right in Theorem 1.2(2). The lower bound estimates for scalar curvatures play important roles in the study of geometric properties of gradient Ricci solitons. Based on these estimates, compactness theorems for gradient Ricci solitons were proved in [Zhang 2006] and some results about the volume growth for noncompact gradient Ricci solitons were deduced in [Cao 2009; Cao and Zhou 2010; Munteanu 2009; Zhang 2011].

In [Case et al. 2011], the authors got estimates for R on closed τ-quasi-Einstein metrics. Later, Wang [2011] studied the lower bound estimate for scalar curvature R on complete noncompact τ-quasi-Einstein metrics with $\lambda \le 0$. We state this result as follows.

Theorem 1.3. *Let M be an n-dimensional complete noncompact Riemannian manifold, metric g is τ-quasi-Einstein with potential function f and constant $\lambda \le 0$, where $\tau \ge 1$. If $\mu \le 0$ or $\mu > 0$ and f is bounded from above by a constant C, then*

$$R(y) \ge n\lambda \tag{1-4}$$

for any $y \in M$.

The proof of this theorem in [Wang 2011] relies on a gradient estimate of f, this gradient estimate shows that $|\nabla f|^2$ is bounded from above if $\mu \le 0$ or $\mu > 0$ and f is bounded from above by a constant C. We will give a nontrivial τ-quasi-Einstein metric with $\lambda < 0$, but f is not bounded from above; see Example 2.1. The second main result of this paper is to improve Theorem 1.3. That is to say, we will show that the lower estimate (1-4) is always right for τ-quasi-Einstein metrics with $\lambda \le 0$.

Theorem 1.4. *Let M be an n-dimensional complete noncompact Riemannian manifold, g be a τ-quasi-Einstein metric with potential function f and $\lambda \le 0$ be a constant, where $\tau > 0$. Then (1-4) holds for any $y \in M$.*

Remark 1.5. If $\tau = \infty$, we recover the lower bound estimate for R on a complete noncompact steady or expanding gradient Ricci soliton given in [Zhang 2011].

It remains interesting to find out whether R is bounded from above by a constant for noncompact quasi-Einstein metrics. The following theorem states that the scalar curvature of a quasi-Einstein metric with $\lambda \le 0$ is bounded from above if $\mu \le 0$.

Theorem 1.6. *Let g be a τ-quasi-Einstein metric with $\lambda \le 0$ and $\mu \le 0$. Then*

$$R(y) \le (n - \max\{\tau, 1\})\lambda \tag{1-5}$$

for any $y \in M$.

2. Examples of quasi-Einstein metrics

In this section, we assume that $M = \mathbb{R} \times N^{n-1}$ is a warped product manifold with the product metric given by

$$ds_M^2 = dt^2 + \varphi^2(t)\, ds_N^2,$$

where ds_N^2 is a fixed metric on N and φ is a positive function on $\mathbb{R}$. Consider the orthonormal coframe $\{\theta_\alpha : 2 \le \alpha \le n\}$ on N^{n-1}; then

$$\{\omega_1 = dt,\ \omega_\alpha = \varphi(t)\theta_\alpha : 2 \le \alpha \le n\}$$

is an orthonormal coframe on M^n. We use $R_{M,ijkl}$ and $R_{N,\alpha\beta\gamma\delta}$ to denote the Riemannian curvature tensors of M and N respectively. After the same calculation as in [O'Neill 1983; Wang 2011], we conclude that

$$R_{M,1\alpha ij} = \begin{cases} -(\log\varphi(t))'' - ((\log\varphi(t))')^2 & \text{if } i = 1,\ j = \alpha, \\ (\log\varphi(t))'' + ((\log\varphi(t))')^2 & \text{if } i = \alpha,\ j = 1, \\ 0 & \text{otherwise} \end{cases} \tag{2-1}$$

and

$$R_{M,\alpha\beta ij} = \begin{cases} \varphi^{-2}(t) R_{N,\alpha\beta\gamma\theta} + ((\log\varphi(t))')^2(\delta_{\alpha\theta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\theta}) & \text{if } i = \gamma,\ j = \theta, \\ 0 & \text{otherwise.} \end{cases} \tag{2-2}$$

If we use $R_{N,\alpha\beta}$ to denote the Ricci curvature tensor on N, by (2-1) and (2-2), the Ricci curvature tensor of M can be expressed as

$$R_{M,1i} = -(n-1)\big((\log\varphi(t))'' + ((\log\varphi(t))')^2\big)\delta_{1i}, \tag{2-3}$$

$$R_{M,\alpha\beta} = \varphi^{-2}(t) R_{N,\alpha\beta} - \big((\log\varphi(t))'' + (n-1)((\log\varphi(t))')^2\big)\delta_{\alpha\beta}. \tag{2-4}$$

Example 2.1. For $\tau > 0$, we assume that N is a flat manifold with

$$R_{N,\alpha\beta} = 0.$$

Let

$$f(t,x) = f(t) = \tau t, \quad \varphi(t) = e^{-t}.$$

It is easy to testify that

$$R_{M,ij} + f_{ij} - \frac{f_i f_j}{\tau} = \lambda g_{ij} \tag{2-5}$$

for $\lambda = -(n+\tau-1)$. Hence M is τ-quasi-Einstein with potential function $f = \tau t$ and $\lambda = -(n+\tau-1)$. Moreover, by (2-3) and (2-4), the scalar curvature of M is

$$R_M = -n(n-1),$$

which means that (1-2) follows with $\mu = 0$. It is easy to see that the potential function f is not bounded from above.

Example 2.2. For $\tau > 0$, we assume that N is an Einstein manifold with

$$R_{N,\alpha\beta} = -(n+\tau-2)\delta_{\alpha\beta}.$$

Choose

$$f(t,x) = f(t) = -\tau \log\cosh t, \quad \varphi(t) = \cosh t.$$

It is easy to testify that (2-5) holds for $\lambda = -(n+\tau-1)$. Hence M is τ-quasi-Einstein with potential function $f = -\tau \log\cosh t$ and $\lambda = -(n+\tau-1)$. Moreover, by (2-3) and (2-4), the scalar curvature of M is

$$R_M = -n(n-1) - \frac{(n-1)\tau}{\cosh^2 t},$$

which means that (1-2) follows with

$$\mu = -\tau(\tau+n-2).$$

It is easy to see that $\mu < 0$ and R_M is bounded from above.

3. Basic formulas

In this section, we will first give some basic formulas for quasi-Einstein metrics in Lemma 3.1. These formulas are well-established in [Case et al. 2011; Kim and Kim 2003; Wang 2011].

Lemma 3.1. *If g is a τ-quasi-Einstein metric with potential function f and λ is a constant, then one can get*

$$\tfrac{1}{2}\Delta R - \frac{\tau+2}{2\tau}\nabla f\cdot\nabla R = -\frac{\tau-1}{\tau}\Big|Ric - \frac{1}{n}Rg\Big|^2 - \frac{n+\tau-1}{n\tau}(R-n\lambda)\Big(R - \frac{n(n-1)}{n+\tau-1}\lambda\Big). \tag{3-1}$$

Moreover, there exists a constant μ such that

$$R + \frac{\tau-1}{\tau}|\nabla f|^2 + (\tau-n)\lambda = \mu e^{2f/\tau}. \tag{3-2}$$

And also one can get

$$\nabla\Delta f\cdot\nabla f = \frac{2}{\tau}\Delta f|\nabla f|^2 - 2\mathrm{Ric}(\nabla f,\nabla f) \tag{3-3}$$

$$\Delta f - |\nabla f|^2 - \tau\lambda + \mu e^{2f/\tau} = 0. \tag{3-4}$$

In the following, we will calculate the weighted Laplacian of $\varphi(R+2xe^{2f/\tau})$ by using Lemma 3.1, where $x > 0$ is a constant and φ is a smooth cutoff function.

Lemma 3.2. *Let*

$$Q = \varphi(R + 2xe^{2f/\tau}), \tag{3-5}$$

where $x > 0$ is a constant and φ is a smooth cutoff function. If $\tau > 1$ and $\mu > 0$, then for $\epsilon > 0$,

$$\frac{1}{2}\Delta_f Q \le \frac{\Delta_f\varphi}{2\varphi}Q + \frac{\nabla\varphi\cdot\nabla Q}{\varphi} - \frac{|\nabla\varphi|^2}{\varphi^2}Q + \frac{\varphi^2}{4\epsilon\tau}\left|\frac{\nabla Q}{\varphi} - \frac{Q\nabla\varphi}{\varphi^2}\right|^2 + \frac{4x(n+\tau-1)}{n\tau}Qe^{2f/\tau} - \frac{n+\tau-1}{n\tau\varphi}Q^2 + \left(\frac{2n-2+\tau}{\tau}\lambda - \frac{\epsilon}{(\tau-1)\varphi}\right)Q + \varphi A - \frac{n(n-1)\varphi}{\tau}\lambda^2 + \frac{n-\tau}{\tau-1}\epsilon\lambda \tag{3-6}$$

holds at $y \in M$ with $\varphi(y) \ne 0$, where

$$\Delta_f = \Delta - \nabla f\cdot\nabla \tag{3-7}$$

and A, depending on $x, n, \tau, \mu, \lambda, \epsilon, \varphi$, is defined in (3-14).

Proof. Let

$$G = R + 2xe^{2f/\tau}. \tag{3-8}$$

It is easy to see that

$$\Delta_f e^{2f/\tau} = \Delta e^{2f/\tau} - \nabla e^{2f/\tau}\cdot\nabla f = \left(\frac{4-2\tau}{\tau^2}|\nabla f|^2 + \frac{2}{\tau}\Delta f\right)e^{2f/\tau}, \tag{3-9}$$

which, together with (3-1), shows that, for $\epsilon > 0$,

$$\begin{aligned}\tfrac{1}{2}\Delta_f G &\le \frac{1}{\tau}\nabla R\cdot\nabla f + x\left(\frac{4-2\tau}{\tau^2}|\nabla f|^2 + \frac{2}{\tau}\Delta f\right)e^{2f/\tau} - \frac{n+\tau-1}{n\tau}(R-n\lambda)\left(R - \frac{n(n-1)}{n+\tau-1}\lambda\right)\\ &= \frac{1}{\tau}\nabla G\cdot\nabla f + x\left(-\frac{2}{\tau}|\nabla f|^2 + \frac{2}{\tau}\Delta f\right)e^{2f/\tau} - \frac{n+\tau-1}{n\tau}(R-n\lambda)\left(R - \frac{n(n-1)}{n+\tau-1}\lambda\right)\\ &\le \frac{\varphi}{4\epsilon\tau}|\nabla G|^2 + \frac{\epsilon}{\tau\varphi}|\nabla f|^2 + x\left(-\frac{2}{\tau}|\nabla f|^2 + \frac{2}{\tau}\Delta f\right)e^{2f/\tau}\\ &\qquad - \frac{n+\tau-1}{n\tau}(R-n\lambda)\left(R - \frac{n(n-1)}{n+\tau-1}\lambda\right)\end{aligned}$$

holds at $y \in M$ when $\varphi(y) \ne 0$. By (3-8) and (3-2), we get

$$R = G - 2xe^{2f/\tau} \tag{3-10}$$

and

$$|\nabla f|^2 = -\frac{\tau}{\tau-1}G + \frac{\tau(2x+\mu)}{\tau-1}e^{2f/\tau} + \frac{\tau(n-\tau)}{\tau-1}\lambda. \tag{3-11}$$

Plugging (3-8), (3-10), (3-11) and (3-4) into 3 yields

$$\tag{3-12} \begin{aligned} \tfrac{1}{2}\Delta_f G \le{}& \frac{\varphi}{4\epsilon\tau}|\nabla G|^2 + \frac{4x(n+\tau-1)}{n\tau} G e^{2f/\tau} - \frac{n+\tau-1}{n\tau} G^2 \\ &+ \left(\frac{2n-2+\tau}{\tau}\lambda - \frac{\epsilon}{(\tau-1)\varphi}\right) G - \frac{4x^2(n+\tau-1)+2xn\mu}{n\tau} e^{4f/\tau} \\ &- \left(\frac{4x(n-1)}{\tau}\lambda - \frac{\epsilon(2x+\mu)}{(\tau-1)\varphi}\right) e^{2f/\tau} - \frac{n(n-1)}{\tau}\lambda^2 + \frac{n-\tau}{(\tau-1)\varphi}\epsilon\lambda. \end{aligned}$$

Since for all $a > 0$,

$$-ax^2 + bx \le \frac{b^2}{4a},$$

we conclude that

$$\tag{3-13} -\frac{4x^2(n+\tau-1)+2xn\mu}{n\tau} e^{4f/\tau} - \left(\frac{4x(n-1)}{\tau}\lambda - \frac{\epsilon(2x+\mu)}{(\tau-1)\varphi}\right) e^{2f/\tau} \le A$$

with

$$\tag{3-14} A = \frac{n\tau}{16x^2(n+\tau-1)+8nx\mu}\left(\frac{4x(n-1)}{\tau}\lambda - \frac{\epsilon(2x+\mu)}{(\tau-1)\varphi}\right)^2.$$

It is easy to see that

$$\nabla G = \frac{\nabla Q}{\varphi} - \frac{Q\nabla\varphi}{\varphi^2}$$

and

$$\Delta_f Q = \frac{Q}{\varphi}\Delta_f\varphi + 2\nabla\varphi\cdot\nabla G + \varphi\Delta_f G.$$

Hence

$$\tag{3-15} \Delta_f Q = \frac{\Delta_f\varphi}{\varphi} Q + \frac{2\nabla\varphi\cdot\nabla Q}{\varphi} - \frac{2|\nabla\varphi|^2}{\varphi^2} Q + \varphi\Delta_f G.$$

Plugging (3-5), (3-12) and (3-13) into (3-15) yields (3-6). □

4. A rigid property

In this section, we will prove Theorem 1.1, a rigid property of τ-quasi-Einstein metrics with $\lambda \le 0$ on complete noncompact Riemannian manifolds.

Proof. Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$:

$$\tag{4-1} \theta(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 2, \end{cases}$$

so that

$$\tag{4-2} -10\sqrt{\theta} \le \theta' \le 0.$$

For $R_0 > 0$, let

$$\varphi(x) = \theta\Big(\frac{r(x)}{R_0}\Big)$$

be a cutoff function, where $r(x)$ is the distance function determined by $O \in M$. Then

$$0 \le \varphi \le 1, \quad |\nabla\varphi|(x) \le \frac{C}{R_0}$$

and $\varphi(x) = 1$ on B_{R_0}, $\varphi(x) = 0$ outside of B_{2R_0}. Let

$$\alpha = -\frac{\tau+2}{\tau}.$$

Plugging (1-1) into (3-3) yields

$$-\nabla\Delta f \cdot \nabla f + \nabla|\nabla f|^2 \cdot \nabla f = 2\lambda|\nabla f|^2 + \frac{2}{\tau}|\nabla f|^4 - \frac{2}{\tau}\Delta f|\nabla f|^2. \tag{4-3}$$

Integrating (4-3) on M and using the fact that $\lambda \le 0$, we obtain

$$\begin{aligned} -\int_M \nabla\Delta f \cdot \nabla f \varphi e^{\alpha f}\,dx + \int_M \nabla|\nabla f|^2 \cdot \nabla f \varphi e^{\alpha f}\,dx \\ \le \frac{2}{\tau}\int_M |\nabla f|^4 \varphi e^{\alpha f}\,dx - \frac{2}{\tau}\int_M \Delta f|\nabla f|^2 \varphi e^{\alpha f}\,dx. \end{aligned} \tag{4-4}$$

Integrating by parts yields

$$\int_M \nabla|\nabla f|^2 \cdot \nabla f \varphi e^{\alpha f}\,dx = -\int_M |\nabla f|^2(\Delta f \varphi + \alpha|\nabla f|^2\varphi + \nabla f \cdot \nabla\varphi)e^{\alpha f}\,dx \tag{4-5}$$

and

$$\int_M \nabla\Delta f \cdot \nabla f \varphi e^{\alpha f}\,dx = -\int_M \big((\Delta f)^2\varphi + \alpha\Delta f|\nabla f|^2\varphi + \Delta f\nabla f \cdot \nabla\varphi\big)e^{\alpha f}\,dx. \tag{4-6}$$

Taking (4-5) and (4-6) into (4-4) yields

$$\begin{aligned} &\int_M \big((\Delta f)^2 - 2\Delta f|\nabla f|^2 + |\nabla f|^4\big)\varphi e^{\alpha f}\,dx \\ &\qquad \le -\int_M \big(\Delta f \nabla f \cdot \nabla\varphi - |\nabla f|^2\nabla f \cdot \nabla\varphi\big)e^{\alpha f}\,dx \\ &\qquad \le \Big(\int_M (\Delta f - |\nabla f|^2)^2\varphi e^{\alpha f}\,dx\Big)^{1/2}\Big(\int_{B_{2R_0}\setminus B_{R_0}} \frac{|\nabla f \cdot \nabla\varphi|^2}{\varphi} e^{\alpha f}\,dx\Big)^{1/2}. \end{aligned}$$

Observing that

$$|\nabla f \cdot \nabla\varphi| \le |\nabla f|\,|\nabla\varphi| \le \frac{C}{R_0}|\nabla f|,$$

we get

$$\int_{B_{R_0}} (\Delta f - |\nabla f|^2)^2 e^{\alpha f\ dx} \le \int_M (\Delta f - |\nabla f|^2)^2 \varphi e^{\alpha f} dx \\ \le C R_0^{-2} \int_{B_{2R_0} \setminus B_{R_0}} |\nabla f|^2 e^{\alpha f} dx.$$

Letting $R_0 \to \infty$, by (1-3), we conclude that

$$\int_M (\Delta f - |\nabla f|^2)^2 e^{\alpha f} dx = 0.$$

Hence $\Delta e^f = 0$.

When $\lambda < 0$, we deduce from $\Delta e^f = 0$ that $\Delta f = |\nabla f|^2$. Equation (4-3) is then equivalent to $2\lambda |\nabla f|^2 = 0$, which means that f is constant. □

5. Lower bound of the scalar curvature

In this section, we will prove Theorem 1.4 and Theorem 1.6 by using the weighted Laplacian comparison theorem and the maximum principle. We first introduce the weighted Laplacian comparison theorem, which can be found in [Lott 2003; Wang 2010].

Lemma 5.1. *Let (M, g) be an n-dimensional complete Riemannian manifold, f a real value smooth function on M and $\Delta_f = \Delta - \nabla f \cdot \nabla$ the weighted Laplacian. Assume that the τ-Bakry–Émery Ricci curvature on M is bounded by*

$$\mathrm{Ric}_{f,\tau} \ge \lambda$$

with constant λ and $r(x) = \mathrm{dist}(O, x)$ is the distance function determined by a fixed point O. If a_λ is a solution to the Riccati equation

$$\frac{\partial a_\lambda}{\partial r} = \lambda - \frac{a_\lambda^2}{n+\tau-1}, \qquad \lim_{r \searrow 0} r a_\lambda = n + \tau - 1,$$

then at $y \notin \mathrm{Cut}(O)$,

$$\Delta_f r \le a_\lambda(r).$$

In particular, if $\lambda \le 0$,

$$\Delta_f r \le \frac{n+\tau-1}{r}\left(1 + \sqrt{-\frac{\lambda}{n+\tau-1}}\, r\right).$$

We need the following estimate, which can be proved by using the maximum principle [Pigola et al. 2005; Schoen and Yau 1994; Yau 1975; Cheng and Yau 1975].

Theorem 5.2. *Let M be an n-dimensional complete noncompact Riemannian manifold, g a τ-quasi-Einstein metric with potential function f and $\lambda \le 0$ a constant. We also assume that $\tau > 1$ and $\mu > 0$. Then for $x > 0$,*

$$R(y) + 2xe^{2f/\tau(y)} \ge \frac{n(2n-2+\tau)+n\sqrt{\Delta}}{2(n+\tau-1)}\lambda \tag{5-1}$$

holds for any $y \in M$, where

$$\Delta = \tau^2 + \frac{8(n+\tau-1)(n-1)^2 x}{2x(n+\tau-1)+n\mu}. \tag{5-2}$$

Proof. Consider a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$,

$$\theta(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 2, \end{cases}$$

so that

$$-10\theta^{1/2} \le \theta' \le 0, \theta'' \ge -10. \tag{5-3}$$

For a large enough constant $R_0 > 0$, define the smooth cutoff function $\varphi : M \to \mathbb{R}$ by

$$\varphi(x, t) = \theta\Big(\frac{r(x)}{R_0}\Big).$$

Then

$$\nabla\varphi = \frac{\theta' \nabla r}{R_0}. \tag{5-4}$$

By Lemma 5.1, we have that for $y \in B_{2R_0}$,

$$\begin{aligned} \Delta_f \varphi(y) = \Delta\varphi - \nabla\varphi \cdot \nabla f &= \frac{\theta''}{R_0^2} + \frac{\theta' \Delta_f r}{R_0} \\ &\ge \frac{\theta''}{R_0^2} + \frac{(n+\tau-1)\theta'(1+\sqrt{K}R_0)}{R_0^2}, \end{aligned} \tag{5-5}$$

where

$$K = -\frac{\lambda}{n+\tau-1} \ge 0.$$

Let

$$Q = \varphi G = \varphi(R + 2xe^{2f/\tau}).$$

If for any $R_0 > 0$ the minimal value of G on B_{R_0} is not smaller than zero, then Theorem 5.2 holds. Hence we can assume that for some large enough value of $R_0 > 0$, the minimal value of G on B_{R_0} is negative. If we assume that Q achieves its minimal value at x_0 on B_{2R_0}, then

$$Q(x_0) \le \min_{x \in B_{R_0}} Q(x) = \min_{x \in B_{R_0}} G(x) < 0,$$

which means that x_0 is not on the boundary of B_{2R_0}. Hence $\varphi(x_0) > 0$ and

$$\nabla Q = 0, \tag{5-6}$$

$$\Delta_f Q \geq 0 \tag{5-7}$$

hold at x_0. By (3-6), (5-6) and (5-7), we get that, at x_0,

$$\begin{aligned} 0 \leq \frac{\Delta_f \varphi}{2} Q - \frac{|\nabla\varphi|^2}{\varphi} Q + \frac{|\nabla\varphi|^2}{4\epsilon\tau\varphi} Q^2 + \frac{4x(n+\tau-1)\varphi}{n\tau} Q e^{2f/\tau} - \frac{n+\tau-1}{n\tau} Q^2 \\ + \Big(\frac{2n-2+\tau}{\tau}\varphi\lambda - \frac{\epsilon}{\tau-1}\Big) Q + \varphi^2 A - \frac{n(n-1)\varphi^2}{\tau}\lambda^2 + \frac{n-\tau}{\tau-1}\varphi\epsilon\lambda. \end{aligned} \tag{5-8}$$

Noticing that, at x_0,

$$\frac{4x(n+\tau-1)\varphi}{n\tau} Q e^{2f/\tau} \leq 0. \tag{5-9}$$

Taking (5-4), (5-5) and (5-9) into (5-8), and using (5-3), we get that, at x_0,

$$\begin{aligned} 0 \leq -\frac{105Q}{R_0^2} - \frac{5(n+\tau-1)(1+\sqrt{K}R_0)Q}{R_0^2} + \frac{25}{\epsilon\tau R_0^2} Q^2 - \frac{n+\tau-1}{n\tau} Q^2 \\ + \Big(\frac{2n-2+\tau}{\tau}\varphi\lambda - \frac{\epsilon}{\tau-1}\Big) Q + \varphi^2 A - \frac{n(n-1)\varphi^2}{\tau}\lambda^2 + \frac{n-\tau}{\tau-1}\varphi\epsilon\lambda. \end{aligned} \tag{5-10}$$

By (3-14) and the fact that $\lambda \leq 0$, we have that, at x_0,

$$\varphi^2 A \leq B, \tag{5-11}$$

where

$$B = \frac{n\tau}{16x^2(n+\tau-1)+8nx\mu}\Big(\frac{4x(n-1)}{\tau}\lambda - \frac{\epsilon(2x+\mu)}{\tau-1}\Big)^2. \tag{5-12}$$

For a large enough value of $R_0 > 0$, define

$$\sigma(R_0) = \frac{\inf\{G(x) : x \in B(O, R_0)\}}{\inf\{G(x) : x \in B(O, 2R_0)\}}.$$

It is easy to see that

$$Q(x_0) = \varphi(x_0)G(x_0) \leq \inf\{G(x) : x \in B(O, R_0)\}$$

and

$$Q(x_0) = \varphi(x_0)G(x_0) \geq \varphi(x_0)\inf\{G(x) : x \in B(O, 2R_0)\}.$$

Using the assumption that $\inf\{G(x) : x \in B(O, R_0)\} < 0$, we get that

$$\sigma(R_0) \leq \varphi(x_0) \leq 1. \tag{5-13}$$

From (5-10), (5-11) and (5-13), it follows that, at x_0,

$$0 \le -\frac{105Q}{R_0^2} - \frac{5(n+\tau-1)(1+\sqrt{K}R_0)Q}{R_0^2} + \frac{25}{\epsilon\tau R_0^2}Q^2 - \frac{n+\tau-1}{n\tau}Q^2 + \Big(\frac{2n-2+\tau}{\tau}\sigma(R_0)\lambda - \frac{\epsilon}{\tau-1}\Big)Q + B - \frac{n(n-1)\sigma^2(R_0)}{\tau}\lambda^2 - \frac{|n-\tau|}{\tau-1}\epsilon\lambda. \tag{5-14}$$

Now, assume that R_0 is large enough so that

$$\frac{25}{\epsilon\tau R_0^2} < \frac{n+\tau-1}{n\tau}.$$

Then (5-14) gives us that, at x_0,

$$Q \ge \frac{-E(R_0) - \sqrt{E^2(R_0)+4D(R_0)F(R_0)}}{2D(R_0)}, \tag{5-15}$$

where

$$D(R_0) = \frac{n+\tau-1}{n\tau} - \frac{25}{\epsilon\tau R_0^2},$$

$$E(R_0) = \frac{105+5(n+\tau-1)(1+\sqrt{K}R_0)}{R_0^2} - \frac{(2n-2+\tau)\sigma(R_0)\lambda}{\tau} + \frac{\epsilon}{\tau-1},$$

$$F(R_0) = B - \frac{n(n-1)\sigma^2(R_0)\lambda^2}{\tau} - \frac{|n-\tau|}{\tau-1}\epsilon\lambda.$$

Hence, for all $y \in B_{R_0}$,

$$G(y) = \varphi(y)G(y) \ge Q(x_0) \ge \frac{-E(R_0) - \sqrt{E^2(R_0)+4D(R_0)F(R_0)}}{2D(R_0)}. \tag{5-16}$$

Noting that $0 \le \sigma(R_0) \le 1$ and B is a constant independent of R_0, we deduce from (5-16) that for a large enough value of R_0, G is bounded from below by a constant independent of R_0. Hence

$$\lim_{R_0\to\infty} \sigma(R_0) = 1,$$

which means that

$$D = \lim_{R_0\to\infty} D(R_0) = \frac{n+\tau-1}{n\tau},$$

$$E = \lim_{R_0\to\infty} E(R_0) = -\frac{(2n-2+\tau)\lambda}{\tau} + \frac{\epsilon}{\tau-1},$$

$$F = \lim_{R_0\to\infty} F(R_0) = B - \frac{n(n-1)\lambda^2}{\tau} - \frac{|n-\tau|}{\tau-1}\epsilon\lambda.$$

By (5-16), we obtain that, for all $y \in M$, $x > 0$ and $\epsilon > 0$,

$$G(y) \geq \frac{-E - \sqrt{E^2 + 4DF}}{2D}. \tag{5-17}$$

Note that

$$\lim_{\epsilon \searrow 0} E = -\frac{(2n-2+\tau)\lambda}{\tau}, \quad \lim_{\epsilon \searrow 0} F = \frac{2xn(n-1)^2\lambda^2}{(2x(n+\tau-1)+n\mu)\tau} - \frac{n(n-1)\lambda^2}{\tau}.$$

Letting $\epsilon \searrow 0$ in (5-17) leads to (5-1). □

The following result is useful.

Theorem 5.3. *If* $\lambda < 0$ *and* $\mu < 0$, *then for all* $y \in M$,

$$f(y) \leq \frac{\tau}{2} \ln \frac{\tau\lambda}{\mu}. \tag{5-18}$$

Proof. Let φ be the cutoff function defined in the proof of Theorem 5.2 and

$$H = \varphi e^{2f/\tau}.$$

Noting that

$$\nabla H = \nabla\varphi e^{2f/\tau} + \varphi \nabla e^{2f/\tau},$$

by (3-4) and (3-9), we have that

$$\Delta_f e^{2f/\tau} \geq \frac{2}{\tau} \Delta_f f e^{2f/\tau} = -\frac{2\mu}{\tau} e^{4f/\tau} + 2\lambda e^{2f/\tau}.$$

Hence

$$\begin{aligned} \Delta_f H &= \varphi \Delta_f e^{2f/\tau} + e^{2f/\tau} \Delta_f \varphi + 2\nabla\varphi \cdot \nabla e^{2f/\tau} \\ &\geq -\frac{2\mu\varphi}{\tau} e^{4f/\tau} + 2\lambda\varphi e^{2f/\tau} + \frac{\Delta_f \varphi}{\varphi} H + 2\frac{\nabla\varphi \cdot \nabla H}{\varphi} - 2\frac{|\nabla\varphi|^2}{\varphi^2} H \end{aligned} \tag{5-19}$$

holds at $y \in M$ when $\varphi(y) > 0$. We assume that H achieves its maximum at x_0 on B_{2R_0}. If $\varphi(x_0) = 0$, then for all $x \in B_{R_0}$,

$$e^{2f/\tau(x)} = \varphi(x) e^{2f/\tau(x)} = H(x) \leq H(x_0) = 0,$$

which is impossible, so $\varphi(x_0) > 0$. Noting that

$$\Delta_f H \leq 0, \quad \nabla H = 0$$

hold at x_0, by (5-3), (5-4), (5-5) and (5-19), we get that, at x_0,

$$\begin{aligned}0 &\geq -\frac{2\mu\varphi}{\tau}e^{4f/\tau} + 2\lambda\varphi e^{2f/\tau} + \frac{\Delta_f\varphi}{\varphi}H - \frac{2|\nabla\varphi|^2}{\varphi^2}H \\ &\geq -\frac{2\mu}{\tau\varphi}H^2 + 2\lambda H + \frac{\theta'' + (n+\tau-1)(1+\sqrt{K}R_0)\theta'}{R_0^2\varphi}H - \frac{2|\theta'|^2}{R_0^2\varphi^2}H \\ &\geq -\frac{2\mu}{\tau\varphi}H^2 + 2\lambda H - \frac{210 + 10(n+\tau-1)(1+\sqrt{K}R_0)\sqrt{\varphi}}{R_0^2\varphi}H.\end{aligned}$$

By using the fact that $0 < \varphi(x_0) \leq 1$, we have that, at x_0,

$$\begin{aligned}-\frac{2\mu}{\tau}H^2 &\leq -2\lambda\varphi H + \frac{210 + 10(n+\tau-1)(1+\sqrt{K}R_0)\sqrt{\varphi}}{R_0^2}H \\ &\leq -2\lambda H + \frac{210 + 10(n+\tau-1)(1+\sqrt{K}R_0)}{R_0^2}H\end{aligned} \tag{5-20}$$

or

$$H(x_0) \leq \frac{\tau\lambda}{\mu} - \frac{105\tau + 5\tau(n+\tau-1)(1+\sqrt{K}R_0)}{\mu R_0^2}.$$

Hence, for all $y \in B_{R_0}$,

$$\begin{aligned}e^{2f/\tau(y)} &= \varphi(y)e^{2f/\tau(y)} = H(y) \leq H(x_0)m \\ &\leq \frac{\tau\lambda}{\mu} - \frac{105\tau + 5\tau(n+\tau-1)(1+\sqrt{K}R_0)}{\mu R_0^2}.\end{aligned}$$

Letting $R_0 \to \infty$ yields (5-18). □

Remark 5.4. Equation (1-1) still holds if we shift the function f by a constant. However, from Equation (3-2), the constant μ will change after this shift.

Proof of Theorem 1.4. When $\tau \geq 1$ and $\mu \leq 0$, (1-4) follows from Theorem 1.3. When $\tau > 1$ and $\mu > 0$, (1-4) follows by letting $x \searrow 0$ in (5-1). We only need to consider the case that $0 < \tau \leq 1$. Now if $\mu \geq 0$, (3-2) tells us that

$$R = \mu e^{2f/\tau} + (n-\tau)\lambda + \frac{1-\tau}{\tau}|\nabla f|^2 \geq (n-\tau)\lambda \geq n\lambda.$$

If $\mu < 0$ and $\lambda < 0$, by Theorem 5.3, we have

$$R \geq \mu e^{2f/\tau} + (n-\tau)\lambda \geq n\lambda.$$

Hence (1-4) follows. If $\mu < 0$ and $\lambda = 0$, [Wang 2011, Theorem 3.2] tells us that f is constant, and Theorem 1.4 follows. □

Proof of Theorem 1.6. When $\mu \le 0$, (3-2) tells us that

$$R \le (n-\tau)\lambda + \frac{1-\tau}{\tau}|\nabla f|^2.$$

If $\tau \ge 1$, then $R \le (n-\tau)\lambda$. If $0 < \tau < 1$, by the gradient estimate in [Wang 2011], we have $|\nabla f|^2 \le -\tau\lambda$. Hence $R \le (n-1)\lambda$. □

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