Pacific Journal of Mathematics

HOMOGENEOUS LINKS AND THE SEIFERT MATRIX

PEDRO M. GONZÁLEZ MANCHÓN

Volume 255 No. 2 February 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 255, No. 2, 2012

HOMOGENEOUS LINKS AND THE SEIFERT MATRIX

PEDRO M. GONZÁLEZ MANCHÓN

Homogeneous links were introduced by Peter Cromwell, who proved that the projection surface of these links, given by the Seifert algorithm, has minimal genus. Here we provide a different proof, with a geometric rather than combinatorial flavor. To do this, we first show a direct relation between the Seifert matrix and the decomposition into blocks of the Seifert graph. Precisely, we prove that the Seifert matrix can be arranged in a block triangular form, with small boxes in the diagonal corresponding to the blocks of the Seifert graph. Then we prove that the boxes in the diagonal have nonzero determinant, by looking at an explicit matrix of degrees given by the planar structure of the Seifert graph. The paper also contains a complete classification of homogeneous knots of genus one.

1. Introduction

Throughout this paper, we assume that all links and diagrams are oriented. Let F be a spanning surface for an oriented link L, and let $b : F \times [0, 1] \to \mathbb{R}^3$ be a regular neighborhood. Identify F with $F \times \{0\}$. The associated Seifert matrix $M = (a_{ij})_{1\le i,j\le n}$ of order n is defined by the linking numbers $a_{ij} = \mathrm{lk}(a_i, a_j^+)$, where the a_i are simple closed oriented curves in F whose homology classes form a basis $\mathcal{B}$ of $H_1(F)$, and $a_i^+ = b(a_i \times 1)$ is the lifting of a_i out of F, in $F \times \{1\}$. Then

$$n = \mathrm{rk}\, H_1(F) = 2g(F) + \mu - 1 = 1 - \chi(F),$$

where $g(F)$ and $\chi(F)$ are the genus and Euler characteristic of F, and μ is the number of components of the link. Homology with coefficients in $\mathbb{Z}$ is assumed throughout the paper.

Let $\nabla_L(z)$ and $\Delta_L(x)$ be the Conway and Alexander polynomials of L, in the variables z and x respectively, as defined in [Cromwell 2004]. Upon the substitution $z = x^{-1} - x$, we have $\nabla_L(z) = \Delta_L(x) = \det(xM - x^{-1}M^t)$. Therefore the coefficient c of the highest degree term in $\nabla_L(z)$ is $(-1)^n \det M$ and the degree

Partially supported by Spanish Project MTM2010-19355 and FEDER.
MSC2010: primary 57M25, 57M27; secondary 57M15, 05C50.
Keywords: homogeneous link, projection surface, Seifert graph, Seifert matrix, Conway polynomial, knot genus, blocks of a graph.

of $\nabla_L(z)$ is n, whenever $\det M$ does not vanish. In general $\deg \nabla_L(z) \le n$, which provides the famous lower bound on the genus, $\deg \nabla_L(z) - \mu + 1 \le 2g(F)$, and in particular it allows us to deduce that F is a minimal genus spanning surface for L if $\det M \neq 0$.

Now suppose that the spanning surface F has been constructed by applying the Seifert algorithm to a diagram D of the link L. We briefly summarize the main features of this construction: start with a diagram D in the xy-plane. For each Seifert circle α a Seifert disc a is built in the plane $z = k$, if there are exactly k Seifert circles that contain α; we say that the height of a is k and write $h(a) = k$. This collection of discs lives in the upper half-space $\mathbb{R}^3_+$ and they are stacked in such a way that when viewed from above, the boundary of each disc is visible. To complete the projection surface, insert small twisted rectangles (called bands from now on) at the site of each crossing, choosing the half-twist according to the corresponding crossing. Following [Cromwell 2004], we call F a projection surface.

We can now define a graph G contained in F as follows: take a vertex in each Seifert disc of F and, if two discs are joined by a band, join the corresponding vertices by an edge contained in the band. Label the edge with the sign of the associated crossing in the diagram D. This graph, called the Seifert graph of D, is in fact a planar graph. The rank $\operatorname{rk} G$ of G, as defined in graph theory, is one minus the number of vertices plus the number of edges. Since $\chi(F) = s(D) - c(D)$, where $s(D)$ is the number of Seifert circles and $c(D)$ is the number of crossings of D, it follows that $\operatorname{rk} G = \operatorname{rk} H_1(F)$.

In general, we can consider the decomposition $G = B_1 \cup \cdots \cup B_k$ of the graph G into its blocks, which are the maximal connected subgraphs without cut vertices. The part of the projection surface (bands and Seifert discs) that corresponds to a block B_i is a submanifold of F and will be denoted by F_{B_i}, or simply F_i. The graph G is a deformation retract of the surface F, taking F_i onto B_i; in particular $H_1(F) \cong H_1(G)$ taking $H_1(F_i)$ onto $H_1(B_i)$ and $\operatorname{rk} G = \operatorname{rk} H_1(G)$, an equality sometimes taken as a definition. Now, a basis of $H_1(G)$, hence a basis $\mathcal{B}$ of $H_1(F)$, can be obtained by juxtaposing basis $\mathcal{B}_i$ of $H_1(B_i)$, since the cycles in G are precisely the cycles of its blocks [Diestel 2005, Lemma 3.1.1]. In particular, the rank of G is the sum of the ranks of its blocks.

Let M_i, where $i = 1, \dots, k$, be the Seifert matrix defined by any basis $\mathcal{B}_i$ of $H_1(B_i)$ (hence of $H_1(F_i)$). Our main result is this:

Theorem 6. *Let D be a connected diagram of an oriented link L. Let G be the corresponding Seifert graph and $G = B_1 \cup \cdots \cup B_k$ its decomposition into blocks. Then there is an order in the set of blocks of G for which the Seifert matrix for the projection surface is upper block triangular. More precisely, if M_i is the Seifert matrix that corresponds to any basis $\mathcal{B}_i$ of $H_1(B_i)$, $i = 1, \dots, k$, there exists a* *permutation $\sigma \in S_k$ such that the Seifert matrix takes on the form*

$$\begin{array}{c} \\ \mathcal{B}_{\sigma(1)} \\ \mathcal{B}_{\sigma(2)} \\ \vdots \\ \mathcal{B}_{\sigma(k)} \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{B}^+_{\sigma(1)} & \mathcal{B}^+_{\sigma(2)} & \dots & \mathcal{B}^+_{\sigma(k)} \end{array} \\ \begin{pmatrix} M_{\sigma(1)} & 0 & \dots & 0 \\ * & M_{\sigma(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & M_{\sigma(k)} \end{pmatrix} \end{array}.$$

A link is homogeneous if it has a homogeneous diagram, which is a diagram in which all the edges of each block of its Seifert graph have the same sign. Alternating and positive diagrams (links) are homogeneous diagrams (links). The knot 9_{43} is an example of a homogeneous link that is neither positive nor alternating. Homogeneous links were introduced in [Cromwell 1989]. In knot theory the adjective homogeneous was first applied to a certain class of braids in [Stallings 1978]. Certainly, the closure of a homogeneous braid is a homogeneous diagram, although there are homogeneous links that cannot be presented as the closure of a homogeneous braid, just as there are alternating links that cannot be presented as the closure of alternating braids. In Cromwell proved the following basic result on homogeneous links:

Theorem [Cromwell 1989; 2004]. *Let D be a connected homogeneous diagram of an oriented homogeneous link L and let G be the corresponding Seifert graph. Then the highest degree of $\nabla_L(z)$ is the rank of G. Let $G = B_1 \cup \dots \cup B_k$ be the decomposition of G into blocks and M_i, $i = 1, \dots, k$, the corresponding Seifert matrices. Then* $\det M_i \neq 0$ *for $i = 1, \dots, k$, and the leading coefficient of $\nabla_L(z)$ is*

$$\prod_{i=1}^{k} \epsilon_i^{r_i} |\det M_i|$$

where ϵ_i is the sign of the edges in B_i and $r_i = \operatorname{rk} B_i$.

Corollary. *A projection surface constructed from a connected homogeneous diagram of an oriented link is a minimal-genus spanning surface for the link.*

Cromwell's proof is based on a previous construction of a specific resolving tree for calculating the Conway polynomial [Cromwell 2004, Lemma 7.5.1]. This means that no crossing is switched more than once on any path from the root of the tree to one of its leaves. The skein relation is then considered, at both the level of the diagram and the corresponding Seifert graph, having in mind that to obtain terms involving powers of z when resolving the resolution tree, a crossing must be smoothed in the diagram D, or equivalently, an edge must be deleted from the graph G. A direct proof of the corollary has been recent and independently suggested by M. Hirasawa. The proof, outlined in [Abe 2011] (see also [Ozawa 2011]), is strongly based on a difficult result from [Gabai 1983], which states that the sum of Murasugi of minimal genus surfaces is a minimal-genus surface. Hirasawa applies this result to the portions F_i above defined.

In this paper we give a different proof of Cromwell's theorem, based on the close relation between the Seifert matrix and the decomposition into blocks of the Seifert graph stated in Theorem 6. The key point is the understanding of how the parts of the projection surface corresponding to the blocks are geometrically positioned among them. We remark that Theorem 6 can be useful even when the diagram is not homogeneous. A special case, involving fibered knots of genus two formed by plumbing Hopf bands, was already considered in [Melvin and Morton 1986]. We deal with this topic in Section 2.

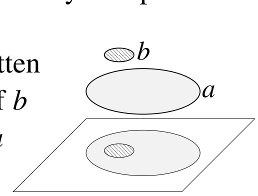
Since a homogeneous block of the Seifert graph corresponds to an alternating diagram, each little box in the diagonal of the Seifert matrix has nonzero determinant, according to the work by K. Murasugi [1958a; 1958b; 1960] and independently Crowell [1959]. Murasugi's proof was accomplished by working on the Alexander matrix of the Dehn presentation of the link, while Crowell worked with the Wirtinger presentation of the fundamental group. In this paper we will prove this result, the second ingredient of our argument, by looking at an explicit matrix of degrees that uses the planar structure of the Seifert graph (Theorem 9). This will be done in Section 3.

Section 4 contains a complete classification of genus-one homogeneous knots.

2. An order for the blocks and the Seifert matrix

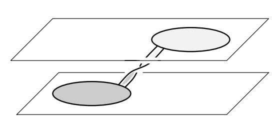
The main achievement of this paper is to prove that there is a certain ordered basis of the first homology group of the projection surface for which the Seifert matrix has a block triangular form. We need first to prove that, in a certain sense, there are only two types of blocks, or more precisely, there are only two possible configurations for the portions F_B associated to a block B.

Let a, b be two Seifert discs. We say that a contains b (written $a \supset b$) if the projection onto the xy-plane of a contains that of b (see figure). Equivalently, the Seifert circle associated to a contains that associated to b, in the xy-plane.



Remark 1. If we project the projection surface onto the xy-plane, the only self-intersections of its boundary are given by the crossings of the original diagram D, and they are produced by the half-twists of the bands.

In particular the arrangement on the right is not possible. As a result we obtain the next lemma.



Lemma 2. *Let a, b be two Seifert discs connected by a band. Then exactly one of the following three statements holds*:

(1) $a \supset b$ *and* $h(b) = h(a) + 1$.

(2) $b \supset a$ *and* $h(a) = h(b) + 1$.

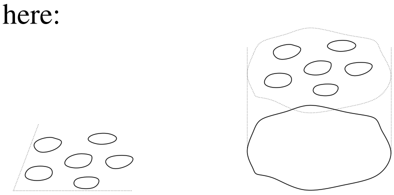
(3) $h(a) = h(b)$.

The proof is easy and left to the reader. Now, we can prove that there are basically two types of blocks. Precisely:

Theorem 3. *Let D be a diagram, F its projection surface and G the corresponding Seifert graph. Then all the Seifert discs associated to a block of G have the same height, except possibly one of them which contains all the other, being its height one less.*

Proof. Suppose that a and b are two Seifert discs with different height connected by a band, both associated to the same block. By Lemma 2 we may assume that one contains the other; say $a \supset b$. It turns out that there is no other Seifert disc associated to the block with height lower than b, since that would make the vertex corresponding to a a cut vertex, according to Remark 1. Analogously, any other disc above b would make (the vertex corresponding to) b a cut vertex. □

Hence we have two possible arrangements for (the Seifert discs that correspond to) a block: type I (*fried eggs* type) and type II (*fried eggs with a pan* type). In a type II block, the pan is the Seifert disc with lowest height. The two types of blocks are illustrated here:



Fried eggs   Fried eggs with a pan

Following Cromwell [1989] or Murasugi [1958a; 1958b] we say that a (Seifert) circle is of type I if it does not contain any other circle; otherwise it is of type II. When a type II circle has other circles outside, it is called a decomposing circle. By definition, a special diagram does not contain any decomposing circle. Note that a type II circle is the boundary of the *pan* of a type II block, assuming that the diagram is connected.

Now, recall from the introduction that the part of the projection surface that corresponds to a block B_i is denoted by F_i, which is a submanifold of F. Recall also that, since the cycles of a graph are the cycles of its blocks, we have that a basis of $H_1(F)$ can be obtained by juxtaposing a basis for each block.

Remark 4. Two different F_i's can have at most one common Seifert disc; hence F is the Murasugi sum of the portions F_i's. The proof by Hirasawa mentioned in the introduction follows from this fact.

In order to prove the main theorem, we need the following result of graph theory:

Lemma 5. *Let G be a connected finite graph with at least one cut vertex. Then there is a block of G which has exactly one cut vertex of G.*

Proof. It can be deduced from Proposition 3.1.2 of [Diestel 2005]. It follows a direct argument: delete any cut vertex v_0 of $C_0 = G$ and consider $C_1 = C_1' \cup \{v_0\}$ where C_1' is any connected component of $C_0 - \{v_0\}$. We remark that, under these assumptions, the cut vertices of C_1 are exactly the cut vertices of G that lie in C_1, except for v_0, and that any block of C_1 is a block of G. If C_1 has no cut vertices, then it is the wanted block. Otherwise we select a cut vertex v_1 of C_1 and consider $C_2 = C_2' \cup \{v_1\}$ where C_2' is a connected component of $C_1 - \{v_1\}$ with $v_0 \notin C_2'$. Repeating this process, we finally get a $k \in \mathbb{N}$ such that C_k has no cut vertices, hence being the wanted block. Otherwise we would obtain an infinite sequence of distinct vertices $\{v_0, v_1, v_2, \ldots\}$ in the finite graph G, a contradiction. □

Theorem 6. *Let D be a connected diagram of an oriented link L. Let G be the corresponding Seifert graph and $G = B_1 \cup \cdots \cup B_k$ its decomposition into blocks. Then there is an order in the set of blocks of G for which the Seifert matrix for the projection surface is upper block triangular. More precisely, if M_i is the Seifert matrix that corresponds to any basis $\mathcal{B}_i$ of $H_1(B_i)$, $i = 1, \ldots, k$, there exists a permutation $\sigma \in S_k$ such that the Seifert matrix takes on the form*

$$\begin{array}{c} \\ \mathcal{B}_{\sigma(1)} \\ \mathcal{B}_{\sigma(2)} \\ \vdots \\ \mathcal{B}_{\sigma(k)} \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{B}^+_{\sigma(1)} & \mathcal{B}^+_{\sigma(2)} & \cdots & \mathcal{B}^+_{\sigma(k)} \end{array} \\ \begin{pmatrix} M_{\sigma(1)} & 0 & \cdots & 0 \\ * & M_{\sigma(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & M_{\sigma(k)} \end{pmatrix} \end{array}$$

Proof. By Lemma 5, there exists a block B which has exactly one cut vertex. Let D be the Seifert disc associated to the unique cut vertex in B. Translated to the surface, this means that the geometric block F_B is separated from the rest of the surface F, with D as the unique intersection.

We may assume, by induction on the number of blocks, that the Seifert matrix for $F \backslash (F_B \backslash D)$ is upper triangular for a suitable order of the rest of blocks B_i's. Suppose now that the positive orientation of the disc D, that looking at $F \times \{1\}$, is upwards. Then, the basis that corresponds to the block B must be added

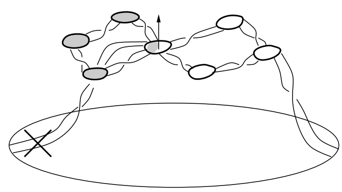
– at the beginning if B is of type I, or D is an egg of the type II block B,

– at the end if D is the pan of the type II block B.

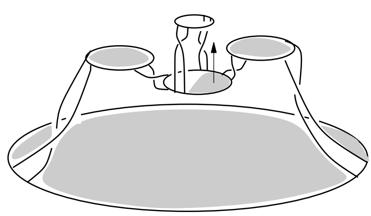
Indeed, if the positive orientation of D were downward, these statements must be interchanged.

In the following displayed figures, the shadowed discs correspond all to the block B; the disc D, partially shadowed, is part of the two considered blocks, B and any other block B_i previously ordered. On D there is an oriented arrow looking upwards, indicating the positive orientation. Suppose now that $g, g_i \in H_1(F)$ correspond to the blocks B and B_i respectively. We have to analyze the three possible cases:

(1) Suppose that B is of type I. We have to see that $\mathrm{lk}(g, g_i^+) = 0$. This can be easily checked if B_i is of type I, or B_i is of type II and the disc D is its pan. And it is also true if B_i is of type II being D an egg of B_i, since in this case the eggs would be on different half parts of the pan. To see this, project both blocks B and B_i onto the plane $z = h(D) - 1$, hence the Seifert discs at height $h(D)$ are now nested inside the pan of the block B_i (*all the eggs in the same pan*). By Remark 1 there is no intersections other than those given by the half-twists of the bands, which means that the two blocks are basically in separated half parts of the pan of B_i. In particular, a band crossed like this is not possible:



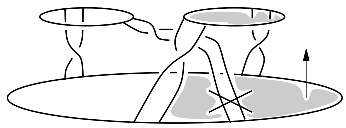
(2) Suppose that B is of type II, and the disc D is an egg of B. As in the previous case, we have to see that $\mathrm{lk}(g, g_i^+) = 0$. This can be easily checked if any other block B_i is of type I, or (see figure below) B_i is of type II and the disc D is the pan of B_i.



Note that D cannot be an egg of another type II block B_i. Indeed, if it were, again by Remark 1, the pan would be the same for B and B_i, hence the blocks B and B_i would share at least two vertices. But, by their maximality, different blocks of G overlap in at most one vertex.

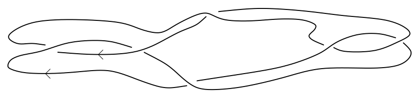
(3) Suppose that B is of type II, and the disc D is its pan. In this case we have to see that $\mathrm{lk}(g_i, g^+) = 0$. This can be easily checked if the block B_i is of type I, or

the disc D is an egg of a type II block B_i. And it is also true if the disc D is the pan of another type II block B_i, since in this case, by a similar argument to that used in the first case, the eggs would be on different half parts of the pan, the crossed band shown here not being possible:

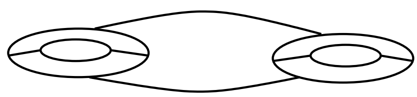


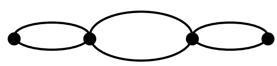
□

Example 7. Suppose that we wish to find the block triangular form for the Seifert matrix of the link shown here:

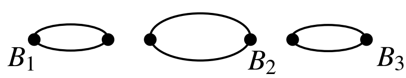


We draw the corresponding Seifert circles and Seifert graph:

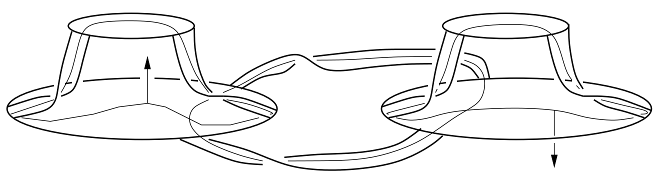




We decompose the Seifert graph into blocks B_1, B_2 and B_3, from left to right:



Here is projection surface:



We can consider $B = B_1$ as the block with only one cut vertex. Then, if the positive orientation of D is upwards, for the other two blocks the suitable basis is given by the order of blocks $\{B_3, B_2\}$, which gives the matrix

$$\begin{array}{ccc} & \mathcal{B}_3^+ & \mathcal{B}_2^+ \\ \mathcal{B}_3 & * & 0 \\ \mathcal{B}_2 & * & * \end{array}$$

Since $B = B_1$ is a block of type II and the disc D that corresponds to the cut vertex is a pan of B, according to the proof of Theorem 6 we must add the basis for B at the end, obtaining the order $\{B_3, B_2, B_1\}$ and the matrix

$$\begin{array}{cccc} & \mathcal{B}_3^+ & \mathcal{B}_2^+ & \mathcal{B}_1^+ \\ \mathcal{B}_3 & * & 0 & 0 \\ \mathcal{B}_2 & * & * & 0 \\ \mathcal{B}_1 & * & * & * \end{array}$$

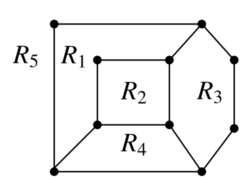
3. The box matrix associated to a block

Recall from the introduction that the coefficient c of the highest degree term in $\nabla_L(z)$ is equal to $(-1)^n \det M$ and the degree of $\nabla_L(z)$ is $n = \operatorname{rk} H_1(F)$, whenever $\det M$ does not vanish. By Theorem 6, $\det M = \prod_{i=1}^k \det M_i$ where M_i is the Seifert matrix that corresponds to the surface F_i associated to the block B_i of G. Then, in order to prove the theorem stated in the introduction, it is enough to show that, if B_i is a block with rank r_i and all its edges have sign ϵ_i, then the determinant of its Seifert matrix does not vanish and has sign $(-\epsilon_i)^{r_i}$. Indeed, since $n = \operatorname{rk} G$ is the sum of the ranks r_i of its blocks, we would have

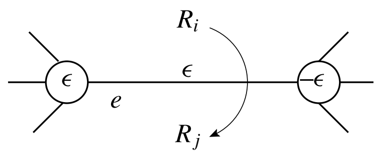
$$\begin{aligned} c = (-1)^n \det M &= (-1)^n \prod_{i=1}^k \det M_i \\ &= (-1)^n \prod_{i=1}^k (-\epsilon_i)^{r_i} |\det M_i| = \prod_{i=1}^k \epsilon_i^{r_i} |\det M_i| \neq 0. \end{aligned}$$

Now, the part of the diagram that corresponds to a homogeneous block is alternating (in fact, it is a special alternating diagram), and the result for these links follows from [Murasugi 1960] and [Crowell 1959]. Murasugi's proof was accomplished by working on the Alexander matrix of the Dehn presentation, while Crowell worked with the Wirtinger presentation of the fundamental group of the link. In fact, Crowell's paper rests on a striking application of a graph theoretical result, the Bott–Mayberry matrix tree theorem, an approach also explained in [Burde and Zieschang 2003, Proposition 13.24]. In this section we will prove it (Theorems 9 and 10) by looking at an explicit matrix of degrees defined using the planar structure of the Seifert graph.

Let D be an oriented diagram, F its projection surface and G the corresponding Seifert graph. Let B be a block of G. A basis $\{g_1, \ldots, g_r\}$ of $H_1(B)$ (hence of $H_1(F_B)$) can be obtained collecting the counterclockwise oriented cycles defined by the boundaries of the bounded regions R_i defined by B. Let R_{r+1} be the unbounded region defined by this planar graph B (like R_5 in the figure on the right).



The Seifert graph is a bipartite graph because the projection surface is orientable, hence every circuit in the graph must have an even length. In particular, we can choose a sign for an arbitrary vertex, and extend this labelling to the other vertices in an alternating fashion, when moving along the edges. We also have, for each edge e in B, its corresponding sign $\epsilon(e)$ (if the original diagram is homogeneous, this sign is constant in the block). We define E_{ij} as the set of edges in $\partial R_i \cap \partial R_j$ with the sign arrangement exemplified by the figure. (The edge e belongs to E_{ij} with this arrangement of signs.)



It turns out that $\mathrm{lk}(g_i, g_i^+) = \frac{1}{2}\sum_{e\in\partial R_i} -\epsilon(e)$ and $\mathrm{lk}(g_i, g_j^+) = \sum_{e\in E_{ij}} \epsilon(e)$.

In particular, if the block is homogeneous, let say with sign ϵ, then

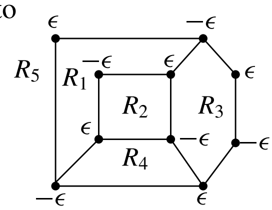
$$\mathrm{lk}(g_i, g_i^+) = -\epsilon k_i,$$

where $2k_i$ is the number of edges in ∂R_i, and

$$\mathrm{lk}(g_i, g_j^+) = \epsilon |E_{ij}|.$$

In other words, $\mathrm{lk}(g_i, g_j^+)$ is the number (with sign ϵ) of the edges e in the frontier of the regions R_i and R_j, such that one leaves the $-\epsilon$ signed vertex on the left when going from R_i to R_j through the edge e (see figure above).

As an example, we display the Seifert matrix associated to the graph of the previous page, assuming that the top left vertex is labelled with sign ϵ; the figure on the right shows the other vertex labels (note that this constitutes a homogeneous block; all the edges have sign ϵ):

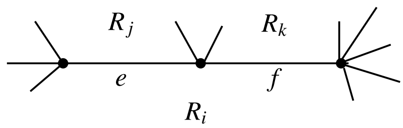


$$\begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array} \begin{array}{c} \begin{array}{cccc} g_1^+ & g_2^+ & g_3^+ & g_4^+ \end{array} \\ \begin{pmatrix} -3\epsilon & \epsilon & \epsilon & 0 \\ \epsilon & -2\epsilon & 0 & \epsilon \\ 0 & \epsilon & -3\epsilon & 0 \\ \epsilon & 0 & \epsilon & -2\epsilon \end{pmatrix} \end{array}$$

The sets E_{ij} satisfy two properties, which will play later a central role, especially in Theorem 9:

(1) If $e \in \partial R_i \cap \partial R_j$, then $e \in E_{ij} \Longleftrightarrow e \notin E_{ji}$, and in particular $|E_{ij}| + |E_{ji}|$ is the cardinal of the edges in $\partial R_i \cap \partial R_j$.

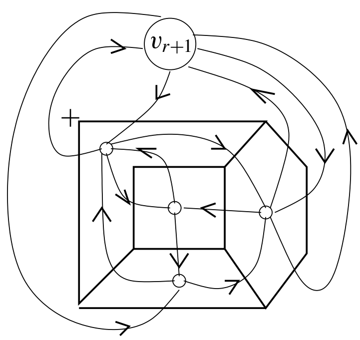
(2) Consider two consecutive edges e and f in the boundary of a certain region R_i, which separate R_i from R_j and R_k respectively (see figure), with possibly $j = k$. Suppose that both edges have the same sign, which is the case if we have a homogeneous graph. Then $e \in E_{ij} \Longleftrightarrow f \in E_{ki}$.



Remark 8. For a homogeneous block with sign ϵ, the sum of two transposed elements in the corresponding Seifert matrix gives

$$\mathrm{lk}(g_i, g_j^+) + \mathrm{lk}(g_j, g_i^+) = \epsilon|E_{ij}| + \epsilon|E_{ji}| = \epsilon\ |\partial R_i \cap \partial R_j|.$$

The directed dual graph. A description of the Seifert matrix corresponding to a homogeneous block can be better understood as a certain matrix of degrees for the oriented dual graph. To construct the directed dual graph we draw a vertex v_i in the region R_i, including a vertex v_{r+1} for the unbounded region R_{r+1}, and for each edge e in $\partial R_i \cap \partial R_j$ we draw an edge $\bar{e}$ joining v_i and v_j, the edge $\bar{e}$ intersecting the original graph only in e. Moreover, the edge $\bar{e}$ is oriented from v_i to v_j if (and only if) $e \in E_{ij}$. The directed dual graph in the case of our running example is exhibited in the figure, assuming the sign $\epsilon = +1$ for all the edges and for the top left vertex.



Note that the edges incident at any vertex have alternative orientations, which is equivalent to the second property of the sets E_{ij}'s. In particular the degrees of the vertices are even numbers.

We now define $m_{ii} = -\epsilon\ \deg_i$ and $m_{ij} = \epsilon \deg_{ij}$, where $\deg_i$ is the number of edges leaving (or going to) v_i and $\deg_{ij}$ is the number of edges from v_i to v_j. It turns out that the matrix $(m_{ij})_{1 \le i, j \le r+1}$ has determinant zero, and we obtain the Seifert matrix of the block by just deleting its last row and column. In our running example, for $\epsilon = +1$, we would have

$$\begin{pmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 2 \\ 1 & 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & -3 \end{pmatrix}.$$

One should note that this is essentially what Proposition 13.21 in [Burde and Zieschang 2003] states, where the adjective special is applied to a diagram if the union of the black regions (assuming a chessboard coloring in which the unbounded region is white) is the image of a Seifert surface under the projection that defines the diagram.

Properties of the matrix for a homogeneous block. Sard matrices. Let ϵ be a sign, $+1$ or -1. A square matrix A is said to be ϵ-signed if its diagonal elements have sign $-\epsilon$ (in particular they do not vanish) and the elements out of the diagonal are zero or have sign ϵ. The matrix A is said to be row-dominant if for any row i we have $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$. The matrix A is said to be strictly ascending row-dominant (abbreviated, *sard*) if A is row-dominant and, in addition, there is an order of its rows $i_1 < \cdots < i_r$ such that $|a_{i_r i_r}| > 0$ and for any $k \in \{1, \dots, r-1\}$ we have that $|a_{i_k i_k}| > \sum_{j\neq i_1, i_2, \dots, i_k} |a_{i_k j}|$.

The following matrix can be seen to be $(+)$-signed and sard choosing the order 3, 1, 2 for its rows (note that the condition $|a_{i_r i_r}| > 0$ is for sure if A is ϵ-signed):

$$\begin{pmatrix} -3 & 0 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

Theorem 9. *Let B be a homogeneous block with sign ϵ. Then there exists a basis of $H_1(B)$ such that the associated Seifert matrix M is ϵ-signed and sard.*

Proof. Consider the basis of $H_1(B)$ given by the counterclockwise oriented cycles $\{g_1, \dots, g_r\}$, boundaries of the bounded regions R_i of B. Then the Seifert matrix $M = (a_{ij})_{1\leq i,j\leq r}$ is obviously ϵ-signed since $a_{ii} = \mathrm{lk}(g_i, g_i^+) = -\epsilon k_i$ where k_i is half the number of edges in the boundary of R_i, and $a_{ij} = \mathrm{lk}(g_i, g_j^+) = \epsilon |E_{ij}|$ if $i \neq j$. To see that A is row-dominant note that $|a_{ii}| = k_i$, and on the other hand $\sum_{j\neq i} |a_{ij}| = \sum_{j\neq i} |E_{ij}| \leq k_i$, the inequality by the second property of the sets E_{ij}.

We finally check that the matrix M is sard, by finding an order $i_1 < \cdots < i_r$ for its rows such that $|a_{i_k i_k}| > \sum_{j\neq i_1, i_2, \dots, i_k} |a_{i_k j}|$ for any $k \in \{1, \dots, r-1\}$. By the second property of the sets E_{ij} there is always a bounded region R_i such that $E_{i,r+1} \neq \varnothing$. The corresponding row is chosen to be the first one in this order, that is, $i_1 = i$. Note that, since $|a_{ii}| \geq \sum_{1\leq j\leq r+1, j\neq i} |E_{ij}|$ and $E_{i,r+1} \neq \varnothing$, it follows that $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$. Now, when we delete the i-th row and column, the remaining matrix corresponds to the graph that remains after deleting the region R_i (precisely, deleting the intersection between R_i and R_{r+1}). The region can be also taken in such a way that the remaining graph is still a homogeneous block, hence the repetition of this process provides the wanted order for the rows of M. □

The determinant for a homogeneous block. In this section we will prove that, given a block with sign ϵ and rank r, the determinant of the corresponding submatrix is nonzero, and its sign is equal to $(-\epsilon)^r$. To see this we just need a purely algebraic result due to Murasugi. For the convenience of the reader, we reproduce here its proof in a slightly different way:

Theorem 10 [Murasugi 1960, Section 2]. *Let A be a square matrix of order r, ϵ-signed and sard. Then* $\det A < 0$ *if* $\epsilon = +1$ *and r is odd*, *and* $\det A > 0$ *otherwise. In other words*, $\det A$ *does not vanish and has sign* $(-\epsilon)^r$.

Proof. By induction on r. The case $r = 1$ (odd) is trivial; for $A = (a)$ we have $\det A = a$, and the result follows from the fact that A is ϵ-signed.

Now assume the statement for cases 1 to $r-1$, and consider the case r. Since A is sard, there is an order $i_1 < \cdots < i_r$ of the rows such that for any $k \in \{1, \dots, r-1\}$ we have $|a_{i_k i_k}| > \sum_{j\neq i_1,i_2,\dots,i_k} |a_{i_k j}|$. In particular, we have

$$a_{i_1 i_1} = -\sum_{j\neq i_1} a_{i_1 j} - \lambda$$

with $\lambda \neq 0$ and sign ϵ. We now develop the determinant by the i_1-row, obtaining

$$\det A = \det \begin{pmatrix} & \dots & \\ a_{i_1 1} \ \dots \ a_{i_1 i_1} \ \dots \ a_{i_1 r} \\ & \dots & \end{pmatrix} = x - \lambda y,$$

where

$$x = \det \begin{pmatrix} & \dots & \\ a_{i_1 1} \ \dots \ -\sum_{j\neq i_1} a_{i_1 j} \ \dots \ a_{i_1 r} \\ & \dots & \end{pmatrix}$$

and y is the determinant of the square matrix of order $r-1$, obtained by deleting the i_1-th row and column. Since this matrix is also ϵ-signed and sard, by induction $y = (-\epsilon)^{r-1}|y| \neq 0$. Moreover, if each $a_{i_1 j} = 0$ for $j \neq i_1$ then $x = 0$ obviously; otherwise it is a square matrix of order r, ϵ-signed and row-dominant, and by Lemma 11, either $x = 0$ or x has sign $(-\epsilon)^r$. Then

$$\det A = x - \lambda y = (-\epsilon)^r|x| - \epsilon|\lambda|(-\epsilon)^{r-1}|y| = (-\epsilon)^r(|x| + |\lambda||y|)$$

and the result follows since $|\lambda| > 0$, $|y| > 0$ and $|x| \geq 0$. □

Lemma 11. *Let A be a square matrix of order r, ϵ-signed and row-dominant. Then* $\det A \le 0$ *if $\epsilon = +1$ and r is odd, and* $\det A \ge 0$ *otherwise.*

Proof. For technical reasons in the induction argument, we will prove this result for a slightly wider category of matrices, the weak ϵ-signed and row-dominant matrices. For this matrices the condition of being ϵ-signed is relaxed for allowing zeros in the diagonal.

We proceed by induction on r. The case $r = 1$ is trivial. Assume now the statement for cases 1 to $r-1$, and consider the case r. Since A is weak ϵ-signed and row-dominant, each diagonal element of A can be written as $a_{ii} = -\sum_{j\neq i} a_{ij} - \lambda_i$ with $\lambda_i = 0$ or with sign ϵ.

Let A_i be the same matrix as A except for possibly the first i elements of its diagonal, where a_{ii} is replaced by $a_{ii} + \lambda_i$. Let $A_0 = A$. It turns out that

$$\det A = \det A_r - \sum_{i=1}^{r} \lambda_i \ \det\big((A_{i-1})_i^i\big),$$

where the notation B_i^i is used to denote the matrix obtained from B by deleting its i-th row and i-th column. This follows from the fact that, for $k = 1, \dots, r$,

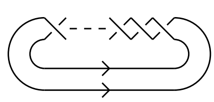
$$\det A_{k-1} = \det A_k - \lambda_k \det\big((A_{k-1})_k^k\big).$$

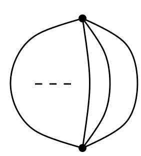
Note that the determinant of A_r is equal to zero, since the sum of all the elements of each row is zero. Moreover, each matrix $(A_{i-1})_i^i$ is also weak ϵ-signed and row-dominant, and has order $r-1$. By induction, its determinant is zero or has sign $(-\epsilon)^{r-1}$. Since each λ_i is zero or has sign $-\epsilon$, the result follows. □

Here is an application of the argument developed in this section:

Claim. *Let L be an oriented link which has a special alternating diagram. Then the leading coefficient of $\nabla_L(z)$ is ± 1 if and only if L is the connected sum of $(2, q)$-torus links.*

Proof. Assume that L is the connected sum of $(2, q)$-torus links. Since $\nabla_{L\sharp L'}(z) = \nabla_L(z)\nabla_{L'}(z)$, it is enough to show that the leading coefficient of $\nabla_L(z)$ is ± 1 if L is a $(2, q)$-torus link. The diagram of L is then of this form, or its mirror image:



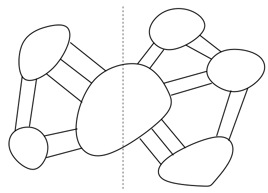


It has q crossings, all with the same sign ϵ. The corresponding Seifert graph, shown on the right, is a homogeneous block B with two vertices and q edges, all of them with sign ϵ.

Following the process explained at the beginning of this section, we obtain the Seifert matrix $M = (m_{i,j})_{i,j=1,\dots,q-1}$ where $m_{i,i} = -\epsilon$ for $i = 1, \dots, q-1$,

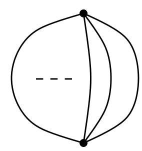
$m_{i,i+1} = \epsilon$ for $i = 1, \ldots, q-2$, and $m_{i,j} = 0$ otherwise. Then the leading coefficient of $\nabla_L(z)$ is $\epsilon^{\operatorname{rk} B} |\det M| = \epsilon^{q-1}$ since $\operatorname{rk} B = 1 - v + e = 1 - 2 + q = q - 1$.

Suppose now that the leading coefficient of $\nabla_L(z)$ is ± 1, and L has a special alternating diagram D. Then D is the connected sum of diagrams $D_1, \ldots, D_r$, where each D_i is a diagram (of a link L_i) such that its Seifert graph has only one (homogeneous) block:



Clearly, $L = \sharp_{i=1}^r L_i$. Since $\nabla_L(z) = \prod_{i=1}^r \nabla_{L_i}(z)$ and $\nabla_L(z) \in Z[z^{\pm 1}]$, the leading coefficient of each $\nabla_{L_i}(z)$ is ± 1. Hence it is enough to prove that L is a $(2, q)$-torus link assuming that the leading coefficient of $\nabla_L(z)$ is ± 1, and L has a diagram D whose associated Seifert graph is a homogeneous block B, let's say with sign ϵ.

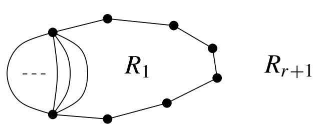
We will prove that B has the desired form (reproduced on the right for convenience). We do this by induction on the number of edges of B. With this aim, we order the r bounded regions of B as in the proof of Theorem 9. The corresponding Seifert matrix A is then ϵ-signed and sard, and by the proof of Theorem 10, we have



$$\det A = (-\epsilon)^r (|x| + |\lambda|\, |y|),$$

where $y = \det A_1^1 \neq 0$. Since the leading coefficient of $\nabla_L(z)$ is ± 1, we have $\det A = \pm 1$; since λ and y are nonzero integers, we have $y = \det A_1^1 = \pm 1$.

Now, according to the proof of Theorem 9, A_1^1 is the Seifert matrix associated to the diagram D' whose Seifert graph is $B' = B \setminus (R_1 \cap R_{r+1})$, where R_{r+1} is the unbounded region of B. Since B' is still a homogeneous block, by induction we have that B' has the form shown above and to the right, and B adds a path connecting the two vertices of B' in the unbounded region of B':



Let $2k$ be the number of edges bounding R_1 in B. Then the original Seifert matrix is

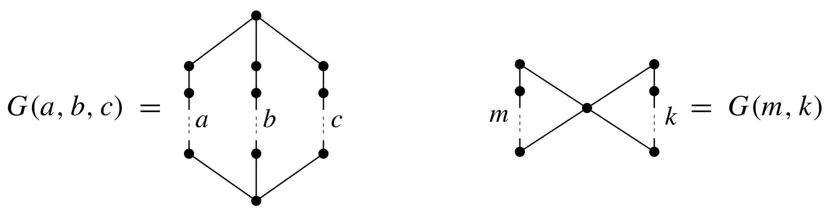
$$A = \left(\begin{array}{c|c} k & \pm 1\ 0\ \cdots\ 0 \\ \hline 0 & \\ \vdots & A_1^1 \\ 0 & \end{array}\right)$$

or its transpose, in any case with determinant $\pm k$. Hence $k = 1$ and the result follows. □

Corollary 12. *Let L be an oriented homogeneous link. Then the leading coefficient of* $\nabla_L(z)$ *equals* ± 1 *if and only if L is the Murasugi sum of connected sums of* $(2, q)$*-torus links.*

4. Homogeneous knots of genus one

We finish the paper with a complete classification of the family of homogeneous knots of genus one. Let D be a homogeneous diagram of a homogeneous knot K of genus one. Let F and G be respectively the projection surface and the Seifert graph associated to the diagram D. We already know that the genus of F is exactly the genus of the knot. Since $2g(F)+\mu-1=\operatorname{rk} G$ and K is a link with one component, we deduce that G has rank two. Here are the two types of graphs:



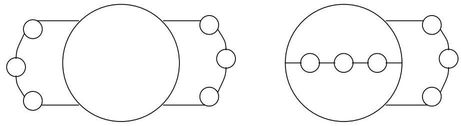
Homogeneous graphs with rank two: one and two blocks

As indicated, we name them $G(a, b, c)$ and $G(m, k)$, respectively; the absolute values of the integers a, b, c, m, k are the numbers of corresponding edges, and their signs are the signs of these edges.

Note that these graphs could have some tails, but this would not affect to the knot type. Since $G(a, b, c)$ is homogeneous and has only one block, a, b, c must have all the same sign; since F is orientable, they have also the same parity. On the contrary, $G(m, k)$ has two blocks, hence m and k can have different signs, but both must be even because of the orientability. Note also that the second graph can be considered a degenerated form of the first one, with $b = 0$.

In general, the Seifert graph does not determine the link where it comes from, although in the first case it does. In $G(a, b, c)$ there are exactly two trivalent vertices; the corresponding Seifert circles can be one inside the other, or separated. When viewed this in the sphere S^2 there is no difference, and the corresponding knot is the pretzel knot with diagram $P(a, b, c)$. Moreover, since $P(a, b, c)$ must be a knot, the numbers a, b, c should be all odd, or exactly one of them should be even. It follows that all of them are odd.

Now consider the graph with two blocks, $G(m, k)$. There is only one vertex with valence four, given the two possible configurations for the Seifert circles shown in the figure (which illustrates the case $|m| = |k| = 4$):



The first configuration corresponds to a link with three components, and the second corresponds to a knot K (see figure below). Moreover, the knot K obtained is also a pretzel knot, given by the pretzel diagram $D(m, k) = P(m, \epsilon, \overset{|k|}{\dots}, \epsilon)$, where m and k are even integers and ϵ is the sign of k. For example, $D(4, -2) = P(4, -1, -1)$ is the example in the right diagram:

What we have done is to prove the following result:

Theorem 13. *A genus-one knot is homogeneous if and only if it belongs to one of the two following classes of knots*:

(1) *Pretzel knots with diagram $P(a, b, c)$, where a, b, c are odd integers with the same sign.*

(2) *Pretzel knots with diagram $D(m, k) = P(m, \epsilon, \overset{|k|}{\dots}, \epsilon)$, where m and k are nonzero even integers and $\epsilon = k/|k|$ is the sign of k.*

This classification and some partial information from the Jones polynomial allow us to give another proof of the following result:

Corollary 14 [Cromwell 1989]. *Pretzel knots $P(p, -q, -r)$ with $3 \leq p \leq q \leq r$, all of them odd, are not homogeneous.*

In the original proof, Cromwell calculated the Homfly polynomial $P(v, z) = \sum_{i=0}^{r} \alpha_i(v) z^i$ and checked that $\alpha_r(v)$ contains terms of both signs [Cromwell 1989, Theorem 10]. But, for homogeneous links, these coefficients are all nonnegative or all nonpositive, according to a result [ibid, Corollary 4.3] due to Traczyk.

Proof. We want to prove that the knot K defined by a pretzel diagram $P(p, -q, -r)$ is not homogeneous. First note that K has genus one, since the projection surface defined by the diagram $P(p, -q, -r)$ has Euler characteristic -1, hence genus one, and K is not the trivial knot; for example, according to [Manchón 2003, Theorem 2, case (iv)(a)], the span of its Jones polynomial (with normalization

$-t^{-1/2} - t^{1/2})$ is $p + q + r - \min\{p, q - 1\}$, which is different from one since $3 \le p \le q, r$.

Now, the lowest degree and the coefficient of the highest degree term of the Jones polynomial tell us that K does not belong to any of the two classes of homogeneous knots of genus one given by Theorem 13, as the following table shows:

Knot diagram		Lowest degree	Coefficient of the highest degree term
$P(p, -q, -r)$	$3 \le p < q \le r$	$1/2$	-1
	$3 \le p = q \le r$	$-1/2$	
$P(a, b, c)$	$0 \le a, b, c$	$-3/2 - a - b - c$	
	$a, b, c \le 0$	$1/2$	1
$D(m, k)$	$m, k > 0$	$-m - 1/2$	
	$m < 0, k > 0$	$1/2$	1
	$m > 0, k < 0$	$k - m - 1/2$	
	$m, k < 0$	$k - 1/2$	

□

Note that the Conway polynomial together with the span of the Jones polynomial are not enough in order to prove Corollary 14. According to the values displayed in the following table, we have for example that the knots defined by the diagrams $P(3, -45, -91)$ and $P(11, 23, 101)$ share Conway polynomial and the span of their Jones polynomials, and the same happens to the pair of knots defined by the diagrams $P(11, -15, -15)$ and $D(-4, 26)$.

Knot diagram		Conway polynomial $1 + \lambda z^2$, where λ is	Jones polynomial span
$P(p, -q, -r)$	$3 \le p < q \le r$	$(qr - pq - pr + 1)/4$	$q + r$
	$3 \le p = q \le r$		$q + r + 1$
$P(a, b, c)$	$0 \le a, b, c$	$(ab + ac + bc + 1)/4$	$1 + a + b + c$
	$a, b, c \le 0$		$1 - a - b - c$
$D(m, k)$	$m, k > 0$	$mk/4$	$1 + m + k$
	$m < 0, k > 0$		$k - m$
	$m > 0, k < 0$		$m - k$
	$m, k < 0$		$1 - m - k$

We also have the following result (as above, the Jones polynomial of the pretzel links and their spans have been calculated following [Manchón 2003]):

Corollary 15. *At least one of the extreme coefficients of the Jones polynomial of a homogeneous knot of genus one is* -1.

Finally, we remark that Stoimenow [2008] has showed that a genus-two homogeneous knot is alternating or positive. Jong and Kishimoto [2009] have studied genus-two positive knots extensively.

Acknowledgments

I am grateful to Prof. Hugh R. Morton for several helpful comments on a previous version of this paper. I am also grateful to the referee for a number of helpful suggestions for improving the article. An overview of this work was presented at the Conference *Knots in Poland III*; I am grateful to its organizers for their hospitality.

References

[Abe 2011] T. Abe, "The Rasmussen invariant of a homogeneous knot", *Proc. Amer. Math. Soc.* **139**:7 (2011), 2647–2656. MR 2012c:57010 Zbl 1226.57004

[Burde and Zieschang 2003] G. Burde and H. Zieschang, *Knots*, 2nd ed., De Gruyter Studies in Mathematics **5**, De Gruyter, Berlin, 2003. MR 2003m:57005 Zbl 1009.57003

[Cromwell 1989] P. R. Cromwell, "Homogeneous links", *J. London Math. Soc.* (2) **39**:3 (1989), 535–552. MR 90f:57001 Zbl 0685.57004

[Cromwell 2004] P. R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004. MR 2005k:57011 Zbl 1066.57007

[Crowell 1959] R. H. Crowell, "Genus of alternating link types", *Ann. of Math.* (2) **69**:2 (1959), 258–275. MR 20 #6103b Zbl 0111.35803

[Diestel 2005] R. Diestel, *Graph theory*, 3rd ed., Grad. Texts in Math. **173**, Springer, Berlin, 2005. MR 2006e:05001 Zbl 1093.05001

[Gabai 1983] D. Gabai, "The Murasugi sum is a natural geometric operation", pp. 131–143 in *Low dimensional topology* (San Francisco, 1981), edited by S. J. Lomonaco, Jr., Contemp. Math. **20**, Amer. Math. Soc., Providence, RI, 1983. MR 85d:57003 Zbl 0524.57004

[Jong and Kishimoto 2009] I. D. Jong and K. Kishimoto, "On positive knots of genus two", preprint, Osaka City University, 2009, available at http://jong.web.fc2.com/posi.pdf.

[Manchón 2003] P. M. G. Manchón, "On the Kauffman bracket of pretzel links", *Marie Curie Fellowships Annals* **2** (2003), 118–122.

[Melvin and Morton 1986] P. M. Melvin and H. R. Morton, "Fibred knots of genus 2 formed by plumbing Hopf bands", *J. London Math. Soc.* (2) **34**:1 (1986), 159–168. MR 87m:57006 Zbl 0576.57008

[Murasugi 1958a] K. Murasugi, "On the genus of the alternating knot, I", *J. Math. Soc. Japan* **10**:1 (1958), 94–105. MR 20 #6103a Zbl 0084.19301

[Murasugi 1958b] K. Murasugi, "On the genus of the alternating knot, II", *J. Math. Soc. Japan* **10**:3 (1958), 235–248. MR 20 #6103a Zbl 0106.16701

[Murasugi 1960] K. Murasugi, "On alternating knots", *Osaka Math. J.* **12**:2 (1960), 277–303. MR 25 #563 Zbl 0113.38603

[Ozawa 2011] M. Ozawa, "Essential state surfaces for knots and links", *J. Austral. Math. Soc.* **391** (2011), 365–390.

[Stallings 1978] J. R. Stallings, "Constructions of fibred knots and links", pp. 55–60 in *Algebraic and geometric topology, II* (Stanford, CA, 1976), edited by R. J. Milgram, Proc. Sympos. Pure Math. **32**, Amer. Math. Soc., Providence, RI, 1978. MR 80e:57004 Zbl 0394.57007

[Stoimenow 2008] A. Stoimenow, "Knots of (canonical) genus two", *Fund. Math.* **200**:1 (2008), 1–67. MR 2009m:57015 Zbl 1190.57008

Received February 16, 2011. Revised August 29, 2011.

PEDRO M. GONZÁLEZ MANCHÓN
DEPARTMENT OF APPLIED MATHEMATICS
EUITI – UNIVERSIDAD POLITÉCNICA DE MADRID
RONDA DE VALENCIA 3
28012 MADRID
SPAIN

pedro.gmanchon@upm.es
http://gestion.euiti.upm.es/index/departamentos/matematicas/manchon/index.htm

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 255 No. 2 February 2012

0030-8730(201202)255:2;1-B