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TOPOLOGICAL CLASSIFICATION OF QUASITORIC MANIFOLDS WITH SECOND BETTI NUMBER 2

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In memory of the late Professor Hyo Chul Myung.

A quasitoric manifold is a 2n-dimensional compact smooth manifold with a locally standard action of an n-dimensional torus whose orbit space is a simple polytope. We classify quasitoric manifolds with second Betti number $\beta_2 = 2$ topologically. Interestingly, they are distinguished by their cohomology rings up to homeomorphism.

1. Introduction

The notion of a quasitoric manifold was introduced by Davis and Januszkiewicz [1991]. A *quasitoric manifold M* is a 2n-dimensional compact smooth manifold with a locally standard action of an n-dimensional torus $T^n = (S^1)^n$, whose orbit space can be identified with an n-dimensional simple polytope P. Here, the orbit map $\pi: M \to P$ maps every k-dimensional orbit to a point in the interior of a codimension-k face of P for $k = 0, \ldots, n$. A typical example of a quasitoric manifold is a complex projective space $\mathbb{C}P^n$ of complex dimension n with the standard T^n-action whose orbit space is the n-simplex Δ^n.

A quasitoric manifold is a topological analogue of a nonsingular projective toric variety. A *toric variety X* of complex dimension n is a normal algebraic variety which admits an action of an algebraic torus $(\mathbb{C}^*)^n$ having a dense orbit. We call a nonsingular compact toric variety a *toric manifold*. Note that we have the restricted action of $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$ on a toric manifold X. One can easily show that this action is locally standard, and if X is projective, then there is a moment map

whose image is a simple convex polytope. Hence, all projective toric manifolds are quasitoric manifolds. However, the converse is not always true. For instance, $\mathbb{C}P^2 \# \mathbb{C}P^2$ with an appropriate T^2-action is a quasitoric manifold over $\Delta^1 \times \Delta^1$ but not a toric manifold, because there is no almost complex structure on $\mathbb{C}P^2 \# \mathbb{C}P^2$. Therefore, the notion of a quasitoric manifold can be regarded as a topological generalization of that of a projective toric manifold in algebraic geometry.

We shall investigate quasitoric manifolds M with second Betti number $\beta_2 = 2$. As will be remarked in Section 3, the orbit space of M can be identified with a product of two simplices. The classification of projective toric manifolds with $\beta_2 = 2$ as varieties was completed by Kleinschmidt [1988]. More generally, toric manifolds over a product of simplices were studied by Dobrinskaya [2001] and Choi et al. [2010a]. These toric manifolds are known as generalized Bott manifolds. In particular, toric manifolds with $\beta_2 = 2$ are two-stage generalized Bott manifolds, which will be explained in Section 3. It is shown in [Choi et al. 2010b] that all two-stage generalized Bott manifolds are classified by their cohomology rings, which gives the smooth classification of toric manifolds with $\beta_2 = 2$.

The purpose of this paper is to classify quasitoric manifolds with $\beta_2 = 2$ up to homeomorphism. For this, we show that if the cohomology ring of a quasitoric manifold is isomorphic to that of a two-stage generalized Bott manifold, then the quasitoric manifold is homeomorphic to a two-stage generalized Bott manifold. We also show that for a polytope which is the product of two simplices there are only finitely many quasitoric manifolds over the polytope, which are not homeomorphic to generalized Bott manifolds. As we will see in the paragraph after (3-1) on page 25, any quasitoric manifold with $\beta_2 = 2$ can be written as $M_{a,b}$ for some $a \in \mathbb{Z}^m$ and $b \in \mathbb{Z}^n$, where the orbit space of $M_{a,b}$ is $\Delta^n \times \Delta^m$. Then we have the following topological classification.

Theorem 1.1. *Any quasitoric manifold with second Betti number $\beta_2 = 2$ is homeomorphic to either a two-stage generalized Bott manifold or to $M_{s,r}$, where $s :=$ $(2, \ldots, 2, 0, \ldots, 0) \in \mathbb{Z}^m$ and $r := (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ have at most $\lfloor \frac{m+1}{2} \rfloor$ and $\lfloor \frac{n+1}{2} \rfloor$ nonzero components, respectively. Moreover, if n or m is 1, then $M_{s,r}$ is a two-stage generalized Bott manifold, or $\mathbb{C}P^{m+n} \# \mathbb{C}P^{m+n}$, or $M_{2,(1,0,\ldots,0)}$.*

More precise classification results are summarized in Section 8. Note that there is an interesting quasitoric manifold over $\Delta^n \times \Delta^1$ which is homeomorphic to a generalized Bott manifold, but has no T^{n+1}-invariant almost complex structure; namely, $M_{2,(1,0)}$ is such a quasitoric manifold that is homeomorphic to a generalized Bott manifold $M_{2,(0,0)}$, as we will see in Lemma 5.4.

Furthermore, we can show that $M_{a,b}$ and $M_{a',b'}$ with $M_{a,b}/T$ and $M_{a',b'}/T$ combinatorially equivalent to $\Delta^n \times \Delta^m$ are homeomorphic if and only if their cohomology rings are isomorphic as graded rings. In addition, the combinatorial

types of certain polytopes are completely determined by the cohomology rings of quasitoric manifolds over those polytopes, see [Choi et al. 2010c]. Products of simplices belong to the class of polytopes that have this property. That is, for a quasitoric manifold M, if the cohomology ring of M is isomorphic to that of $M_{a,b}$, then the orbit space of M is combinatorially equivalent to the orbit space of $M_{a,b}$.

As a consequence, we have the following main theorem of this paper, which does not include any assumption on the type of the base polytope:

Theorem 1.2. *Two quasitoric manifolds with $\beta_2 = 2$ are homeomorphic if and only if their cohomology rings are isomorphic as graded rings.*

This research is motivated by the *cohomological rigidity problem* for quasitoric manifolds which asks whether the homeomorphism types of quasitoric manifolds are distinguished by their cohomology rings or not, see [Masuda and Suh 2008] for the problem and other related problems. In general, the cohomological rigidity problem is rather bold because the cohomology ring as an invariant is not sufficient to determine topological types of manifolds. Indeed, many classical results such as [Hsiang 1966] provide many examples of pairs of manifolds which are homotopic but not homeomorphic. However, many $2n$-dimensional manifolds do not have T^n-symmetry, and, so far, there is no counterexample for the cohomological rigidity problem. On the contrary, some affirmative partial evidence is given by recent papers such as [Masuda and Panov 2008; Choi et al. 2010b; Choi and Suh 2011; Choi and Masuda 2009] and others. Theorem 1.2 also gives another affirmative partial answer to the rigidity problem. For more information about rigidity problem, we refer the reader to the survey paper [Choi et al. 2011].

This paper is organized as follows. In Section 2, we recall general facts on quasitoric manifolds and moment angle manifolds. In Section 3, we introduce generalized Bott manifolds, and deal with the cohomology rings of quasitoric manifolds with $\beta_2 = 2$. We find a necessary and sufficient condition for a quasitoric manifold to be equivalent to a generalized Bott manifold in some specific cases in Section 4. In Sections 5 and 6, we prove Theorem 1.1, and prepare to prove Theorem 1.2 by classifying quasitoric manifolds $M_{a,b}$ and $M_{s,r}$ up to homeomorphism. In Section 7, we give a full proof of Theorem 1.2. In the final section, we give the complete topological classification of quasitoric manifolds with $\beta_2 = 2$.

2. Preliminaries

An n-dimensional (combinatorial) polytope is called *simple* if exactly n facets (codimension-one face) meet at each vertex. Let P be a simple polytope of dimension n with d facets, and let $\mathcal{F}(P) = \{F_1, \ldots, F_d\}$ be the set of facets of P. Now consider a map

$$\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^n$$

which satisfies the following *nonsingularity condition*:

(2-1) $\lambda(F_{i_1}), \ldots, \lambda(F_{i_\alpha})$ form a part of an integral basis of $\mathbb{Z}^n$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_\alpha}$ is nonempty.

Such λ is called a *characteristic function*, and $\lambda(F_i)$ is called a *facet vector* of F_i. For a characteristic function $\lambda \colon \mathcal{F}(P) \to \mathbb{Z}^n$ and a face F of P, we denote by $T(F)$ the subgroup of T^n corresponding to the unimodular subspace of $\mathbb{Z}^n$ spanned by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_\alpha})$, where $F = F_{i_1} \cap \cdots \cap F_{i_\alpha}$.

Given a characteristic function λ on P, we construct a manifold

(2-2) $$M(\lambda) := T^n \times P / \sim,$$

where $(t, p) \sim (s, q)$ if and only if $p = q$ and $t^{-1}s \in T(F(p))$, where $F(p)$ is the face of P which contains $p \in P$ in its relative interior. The standard T^n-action on T^n induces a free action of T^n on $T^n \times P$, which descends to an effective action on $M(\lambda)$ whose orbit space is P. Since this action is locally standard, $M(\lambda)$ is indeed a quasitoric manifold over P.

Two quasitoric manifolds M_1 and M_2 over P are said to be *equivalent* if there is a θ-equivariant homeomorphism $f \colon M_1 \to M_2$, that is, $f(gm) = \theta(g) \cdot f(m)$ for $g \in T^n$ and $m \in M_1$, which covers the identity map on P for some automorphism θ of T^n. It is obvious from the definition of the equivalence that $M(\lambda_1)$ and $M(\lambda_2)$ are equivalent if there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(\mathbb{Z}, n)$ such that $\lambda_1 = \sigma \circ \lambda_2$. By Davis and Januszkiewicz [1991], every quasitoric manifold is represented by a pair of P and λ up to equivalence.

Note that one may assign an $n \times d$ matrix Λ to a characteristic function λ by

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_d)) = (A | B),$$

where A is an $n \times n$ matrix and B is an $n \times (d - n)$ matrix. We call Λ a *characteristic matrix*. By additionally setting $F_1 \cap \cdots \cap F_n \neq \varnothing$, we may assume that the matrix $A = (\lambda(F_1), \ldots, \lambda(F_n))$ is invertible from the nonsingularity condition (2-1). Moreover, the inverse A^{-1} belongs to $\mathrm{GL}(\mathbb{Z}, n)$. Thus, up to equivalence, the corresponding matrix Λ can be represented by $(E_n | A^{-1}B)$, where E_n is the identity matrix of size n.

Remark 2.1. Let Λ be the above characteristic matrix corresponding to a quasitoric manifold M. If we let

$$D_{k,n} := \mathrm{diag}(1, \ldots, 1, -1, 1, \ldots, 1)$$

be the diagonal $n \times n$ matrix whose k-th diagonal entry is -1 and the others are 1, then the matrix $D_{k,n} \Lambda D_{\ell,d}$ is the matrix obtained from Λ by changing the signs of k-th row and ℓ-th column, where $1 \le k \le n$ and $1 \le \ell \le d$. Since two vectors $\lambda(F_i)$ and $-\lambda(F_i)$ determine the same circle subgroup of T^n, the sign of a facet vector

does not affect the corresponding quasitoric manifold from the construction (2-2). Thus $\Lambda D_{\ell,d}$ is still a characteristic matrix corresponding to M. Hence $D_{k,n}\Lambda D_{\ell,d}$ can also be a characteristic matrix corresponding to M, up to equivalence, because $D_{k,n} \in \mathrm{GL}(\mathbb{Z}, n)$.

Let $\mathbb{Z}[v_1, \ldots, v_d]$ denote the polynomial ring in d variables v_i over $\mathbb{Z}$ with $\deg v_i = 2$. We identify each $F_i \in \mathscr{F}(P)$ with the indeterminate v_i. The *face ring* (or *Stanley–Reisner ring*) $\mathbb{Z}(P)$ of P is the quotient ring

$$\mathbb{Z}(P) = \mathbb{Z}[v_1, \ldots, v_d]/I_P,$$

where I_P is the ideal generated by the monomials $v_{i_1} \cdots v_{i_\ell}$ with $F_{i_1} \cap \cdots \cap F_{i_\ell} = \varnothing$.

Let M be a quasitoric manifold over P with projection $\pi : M \to P$ and the characteristic function λ. Then one can find an isomorphism between $\mathbb{Z}(P)$ and the equivariant cohomology ring of M with $\mathbb{Z}$ coefficients,

$$H_T^*(M) \cong \mathbb{Z}[v_1, \ldots, v_d]/I_P = \mathbb{Z}(P),$$

where v_j is the equivariant Poincaré dual of the codimension two invariant submanifold $M_j = \pi^{-1}(F_j)$ in M. Note that $H_T^*(M)$ is not only a ring but also a $H^*(BT) = \mathbb{Z}[t_1, \ldots, t_n]$-module via the map p^*, where $p : ET \times_T M \to BT$ is the natural projection, and p^* takes t_i to $\theta_i := \lambda_{i1}v_1 + \cdots + \lambda_{id}v_d \in \mathbb{Z}(P)$, where $\lambda(F_i) = (\lambda_{1i}, \ldots, \lambda_{ni})^T \in \mathbb{Z}^n$ for $i = 1, \ldots, n$. Since everything has vanishing odd degrees, $H_T^*(M)$ is a free $H^*(BT)$-module. Hence the kernel of the morphism $\mathbb{Z}(P) = H_T^*(M) \to H^*(M)$ is the ideal J_λ of $\mathbb{Z}(P)$ generated by $\theta_1, \ldots, \theta_n$. Therefore, we have

$$(2\text{-}3) \qquad H^*(M) = \mathbb{Z}[v_1, \ldots, v_d]/(I_P + J_\lambda).$$

See [Davis and Januszkiewicz 1991] for more details of the previous argument.

Now let P be an n-dimensional simple polytope with d facets. Davis and Januszkiewicz [1991] constructed a T^d-manifold $\mathscr{Z}_P$ that is now called the *moment angle manifold* of P. Let $\mathscr{F}(P) = \{F_1, \ldots, F_d\}$ be the set of facets of P. For each facet F_i let T_{F_i} denote the one-dimensional coordinate subgroup of $T^{\mathscr{F}(P)} \cong T^d$ corresponding to F_i. We assign to every face $F = F_{i_1} \cap \cdots \cap F_{i_\ell}$ the coordinate subtorus

$$T_F = \prod_{j=1}^{\ell} T_{F_{i_j}} \subset T^d.$$

Then the moment angle manifold of P can be constructed as

$$\mathscr{Z}_P = T^d \times P^n/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_{F(p)}$. From the definition of $\mathscr{Z}_P$, we can see easily that $\mathscr{Z}_{P_1 \times P_2} = \mathscr{Z}_{P_1} \times \mathscr{Z}_{P_2}$ for any simple polytopes P_1 and P_2.

Example 2.2. It is not so hard to see that the moment angle manifold $\mathscr{Z}_{\Delta^n}$ of an n-simplex is homeomorphic to a sphere S^{2n+1}, hence $\mathscr{Z}_{\Delta^n \times \Delta^m} = S^{2n+1} \times S^{2m+1}$.

Let us fix a characteristic function λ on P, and let $M(\lambda)$ be the quasitoric manifold constructed in (2-2). There is a natural identification $\psi_k : \mathbb{Z}^k \to \mathrm{Hom}(S^1, T^k)$ given by $(a_1, \ldots, a_k) \mapsto (t \mapsto (t^{a_1}, \ldots, t^{a_k}))$ for any positive integer k. Hence the characteristic matrix Λ corresponding to λ induces a surjective homomorphism $\overline{\lambda} : T^d \to T^n$ by $\overline{\lambda}(\psi_d(e_i)(t)) = \psi_n(\lambda(F_i))(t)$ for $t \in S^1$, where e_i is the standard i-th basis vector of $\mathbb{Z}^d$ for $i = 1, \ldots, d$. Then $\ker \overline{\lambda}$ is a $(d-n)$-dimensional subtorus of T^d. From the nonsingularity condition (2-1), $\ker \overline{\lambda}$ meets every isotropy subgroup at the unit. Thus $\ker \overline{\lambda}$ acts freely on $\mathscr{Z}_P$, and the map

$$(\overline{\lambda}, \mathrm{id}) : T^d \times P^n \to T^n \times P^n$$

induces a principal T^{d-n}-bundle $\mathscr{Z}_P$ over $M(\lambda)$. We thus have:

Proposition 2.3 [Buchstaber and Panov 2002, Proposition 6.5]. *The subtorus* $\ker \overline{\lambda}$ *acts freely on* $\mathscr{Z}_P$, *thereby defining a principal* T^{d-n}-*bundle* $\mathscr{Z}_P \to M(\lambda)$.

Let $M(\lambda_1)$ and $M(\lambda_2)$ be two quasitoric manifolds over a simple polytope P. If a self map φ of the moment angle manifold $\mathscr{Z}_P$ is θ-equivariant, that is, there exists an isomorphism $\theta : \ker \overline{\lambda}_1 \to \ker \overline{\lambda}_2$ such that $\varphi(t \cdot x) = \theta(t) \cdot \varphi(x)$ for all $t \in \ker \overline{\lambda}_1$ and $x \in \mathscr{Z}_P$, then there is a natural induced map $\overline{\varphi}$ from $M(\lambda_1)$ to $M(\lambda_2)$:

$$
\begin{array}{ccc}
\mathscr{Z}_P & \xrightarrow{\ \varphi\ } & \mathscr{Z}_P \\
{\scriptstyle /\ker \overline{\lambda}_1} \downarrow & & \downarrow {\scriptstyle /\ker \overline{\lambda}_2} \\
M(\lambda_1) & \xrightarrow[\ \overline{\varphi}\ ]{} & M(\lambda_2)
\end{array}
$$

Thus if we construct a θ-equivariant homeomorphism φ from the moment angle manifold $\mathscr{Z}_P$ to itself, then the induced map $\overline{\varphi}$ is a homeomorphism from $M(\lambda_1)$ to $M(\lambda_2)$.

3. Quasitoric manifolds with $\beta_2 = 2$

The main interest of the present paper is focused on quasitoric manifolds with second Betti number $\beta_2 = 2$. Let P be an ℓ-dimensional simple polytope with d facets, and let M be a quasitoric manifold over P with the characteristic function λ. Since J_λ consists of ℓ linear combinations of $v_1, \ldots, v_d$ and I_P does not contain a linear combination in (2-3), we can see that the second Betti number of M is $d - \ell$. Thus if P supports a quasitoric manifold with $\beta_2 = 2$, then it has exactly $\ell + 2$

facets, and hence P is combinatorially equivalent to a product of two simplices as is well-known, see chapter 6 in [Grünbaum 2003]. Therefore we may assume that $P = \Delta^n \times \Delta^m$.

Now consider a quasitoric manifold M of dimension $2(n+m)$ over $\Delta^n \times \Delta^m$. Consider the facets of $\Delta^n \times \Delta^m$ in the following order:

$$F_1 \times \Delta^m, \ldots, F_n \times \Delta^m, \Delta^n \times G_1, \ldots, \Delta^n \times G_m, F_{n+1} \times \Delta^m, \Delta^n \times G_{m+1},$$

where F_i's are the facets of Δ^n and G_j's are the facets of Δ^m. Then the first $(n+m)$ facets meet at a vertex. Thus, by Remark 2.1, the characteristic matrix Λ corresponding to M is of the form

(3-1)
$$\Lambda = (E_{n+m} | \Lambda_*) = \begin{pmatrix} 1 & & & & & -1 & -b_1 \\ & \ddots & & & 0 & \vdots & \vdots \\ & & 1 & & & -1 & -b_n \\ & & & 1 & & -a_1 & -1 \\ & 0 & & & \ddots & \vdots & \vdots \\ & & & & 1 & -a_m & -1 \end{pmatrix}$$

up to equivalence, where $1 - a_j b_i = \pm 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ because of the nonsingularity condition (2-1) of the characteristic function. From now on we denote such M by $M_{a,b}$ for $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$. Hence, from (2-3), the cohomology ring of $M_{a,b}$ with $\mathbb{Z}$ coefficients is

(3-2)
$$H^*(M_{a,b}) = \mathbb{Z}[x_1, x_2] \Big/ \Big\langle x_1 \prod_{i=1}^{n} (x_1 + b_i x_2), \ x_2 \prod_{j=1}^{m} (a_j x_1 + x_2) \Big\rangle.$$

A (complex) *generalized Bott tower* of height h, or an *h-stage generalized Bott tower*, is a sequence

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}$$

of manifolds $B_i = P\big(\mathbb{C} \oplus \bigoplus_{j=1}^{\ell_i} \xi_{i,j}\big)$, where $\xi_{i,j}$ is a complex line bundle, $\mathbb{C}$ is the trivial complex line bundle over B_{i-1} for each $i = 1, \ldots, h$, and $P(\cdot)$ stands for the projectivization. We call B_i *the i-stage generalized Bott manifold*.

The Whitney sum of ℓ complex line bundles admits a canonical T^ℓ-action. Assume B_{i-1} admits an effective $T^{\sum_{k=1}^{i-1} \ell_k}$-action. Since $H^1(B_{i-1}) = 0$, it lifts to an action on $\xi_i = \bigoplus_{j=1}^{\ell_i} \xi_{i,j}$; see [Hattori and Yoshida 1976]. Moreover, it commutes with the canonical T^{ℓ_i}-action on ξ_i, and hence, it induces an effective $T^{\sum_{k=1}^{i} \ell_k}$-action on B_j. Thus, we can define an effective half-dimensional torus action on B_h inductively. One can show that this action is locally standard and its orbit space is a product of h simplices $\prod_{i=1}^{h} \Delta^{\ell_i}$. Thus a two-stage generalized Bott manifold is a quasitoric manifold over $P = \Delta^{\ell_1} \times \Delta^{\ell_2}$ and has $\beta_2 = 2$.

Remark 3.1. In fact, a generalized Bott manifold is not only a quasitoric manifold but also a (projective) toric manifold. All toric manifolds admit T^n-invariant complex structures. Hence, by Choi et al. [2010a, Theorem 6.4], all toric manifolds over a product of simplices are generalized Bott manifolds.

We already know a necessary and sufficient condition for a quasitoric manifold M to be equivalent to a generalized Bott manifold by the following proposition.

Proposition 3.2 [Choi et al. 2010a]. *Let M be a quasitoric manifold over the polytope $P = \prod_{i=1}^{h} \Delta^{\ell_i}$, and let Λ_* be an $h \times h$ vector matrix associated with M.[1] Then M is equivalent to a generalized Bott manifold if and only if Λ_* is conjugate to an $h \times h$ lower triangular vector matrix.*

Moreover, the following theorem gives a smooth classification of two-stage generalized Bott manifolds.

Theorem 3.3 [Choi et al. 2010b]. *Let $B_2 = P(\oplus_{i=0}^{m} \gamma^{u_i})$ and $B_2' = P(\oplus_{i=0}^{m} \gamma^{u_i'})$, where $u_0 = u_0' = 0$ and γ^{u_i} denotes the complex line bundle over $B_1 = \mathbb{C}P^n$ whose first Chern class is $u_i \in H^2(B_1)$. Then the following are equivalent*:

(1) *There exists $\epsilon = \pm 1$ and $w \in H^2(B_1)$ such that*

$$\prod_{i=0}^{m}(1 + \epsilon(u_i' + w)) = \prod_{i=0}^{m}(1 + u_i) \quad in \ H^*(B_1).$$

(2) *B_2 and B_2' are diffeomorphic.*

(3) *$H^*(B_2)$ and $H^*(B_2')$ are isomorphic as graded rings.*

When a quasitoric manifold M is equivalent to a two-stage generalized Bott manifold, we may assume that $M = M_{\boldsymbol{a},\boldsymbol{0}}$. In this case, M is a $\mathbb{C}P^m$-bundle over $\mathbb{C}P^n$, and $H^*(M_{\boldsymbol{a},\boldsymbol{0}})$ is of the form

$$(3\text{-}3) \qquad H^*(M_{\boldsymbol{a},\boldsymbol{0}}) = \mathbb{Z}[x_1, x_2] \Big/ \Big\langle x_1^{n+1}, \ x_2 \prod_{j=1}^{m}(a_j x_1 + x_2) \Big\rangle.$$

If a quasitoric manifold M with $\beta_2 = 2$ is not equivalent to a generalized Bott manifold, then we may assume that $M = M_{\boldsymbol{a},\boldsymbol{b}}$ for some nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by Proposition 3.2. Then $a_j b_i = 2$ for some i and j. Without loss of generality, we may assume that a_j is 0 or ± 2 and b_i is 0 or ± 1. Note that the signs of nonzero a_j's and b_i's are the same and, by Remark 2.1, $M_{\boldsymbol{a},\boldsymbol{b}}$ is equivalent to $M_{-\boldsymbol{a},-\boldsymbol{b}}$.[2] Hence,

---

[1] In fact, Λ_* is a $(\sum_{i=1}^{h} \ell_i) \times h$ matrix. Then Λ_* can be viewed as an $h \times h$ vector matrix whose entries in the i-th row are vectors in $\mathbb{Z}^{\ell_i}$. A more precise description of (a transpose version of) Λ_* is explained on page 114 in [Choi et al. 2010a].

[2] We can see this easily by the following steps; 1) change the signs of the first n row vectors of the characteristic matrix (3-1), 2) change the signs of the first n column vectors and the $(n + m + 2)$-nd of the resulting matrix. Then we can obtain the characteristic matrix corresponding to $M_{-\boldsymbol{a},-\boldsymbol{b}}$.

we may assume that the nonzero a_j is 2, and the nonzero b_i is 1. Now let s be the number of $a_j = 2$ for $j = 1, \ldots, m$ and r the number of $b_i = 1$ for $i = 1, \ldots, n$. Then, the cohomology ring of $M_{a,b}$ is isomorphic to

$$(3\text{-}4) \qquad \mathbb{Z}[x_1, x_2] / \langle x_1^{n+1-r}(x_1 + x_2)^r, \ x_2^{m+1-s}(2x_1 + x_2)^s \rangle$$

for some $1 \leq r \leq n$ and $1 \leq s \leq m$.

We close this section by giving another construction of quasitoric manifolds over $\Delta^n \times \Delta^m$ from the moment angle manifold $\mathcal{Z}_{\Delta^n \times \Delta^m}$.

Remark 3.4. Note that the moment angle manifold $\mathcal{Z}_{\Delta^n \times \Delta^m}$ is

$$S^{2n+1} \times S^{2m+1} = \{ (\boldsymbol{w}, \boldsymbol{z}) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} : |\boldsymbol{w}| = 1, \ |\boldsymbol{z}| = 1 \},$$

which has the standard T^{n+m+2}-action of the componentwise complex multiplication. Let λ be a characteristic function corresponding to $M_{a,b}$, and let $K_{a,b}$ be the image of the homomorphism $\mu : T^2 \to T^{n+m+2}$ defined by

$$(3\text{-}5) \qquad \begin{pmatrix} 1 & b_1 \\ \vdots & \vdots \\ 1 & b_n \\ 1 & 0 \\ a_1 & 1 \\ \vdots & \vdots \\ a_m & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the action of the two-torus $K_{a,b}$ on $S^{2n+1} \times S^{2m+1}$ defined by

$$\mu(t_1, t_2) \cdot \big((w_1, \ldots, w_{n+1}), (z_1, \ldots, z_{m+1}) \big)$$
$$= \big((t_1 t_2^{b_1} w_1, \ldots, t_1 t_2^{b_n} w_n, t_1 w_{n+1}), (t_1^{a_1} t_2 z_1, \ldots, t_1^{a_m} t_2 z_m, t_2 z_{m+1}) \big)$$

is free because of the nonsingularity condition (2-1) of λ. Moreover, this action is exactly equal to the $(\ker \bar{\lambda})$-action on $\mathcal{Z}_{\Delta^n \times \Delta^m}$, where a homomorphism $\bar{\lambda}$ is defined on page 24, and the quasitoric manifold $M_{a,b}$ is the orbit space $\mathcal{Z}_{\Delta^n \times \Delta^m} / K_{a,b}$ with the action of T^{n+m} defined by

$$(t_1, \ldots, t_{n+m}) \cdot \big[(w_1, \ldots, w_{n+1}), (z_1, \ldots, z_{m+1}) \big]$$
$$= \big[(t_1 w_1, \ldots, t_n w_n, w_{n+1}), (t_{n+1} z_1, \ldots, t_{n+m} z_m, z_{m+1}) \big].$$

See [Choi et al. 2010a] for more details.

In other words, the subtorus $K_{a,b} \subset T^{n+m+2}$ is represented by the unimodular subgroup of $\mathbb{Z}^{n+m+2}$ spanned by the two vectors $(1, \ldots, 1, a_1, \ldots, a_m, 0)$ and $(b_1, \ldots, b_n, 0, 1, \ldots, 1)$. Note that these two vectors generate the null space of the matrix

(3-6)
$$\begin{pmatrix} 1 & & -1 & & & -b_1 \\ & \ddots & & \vdots & 0 & \vdots \\ & & 1 & -1 & & -b_n \\ & & -a_1 & 1 & & -1 \\ 0 & & \vdots & & \ddots & \vdots \\ & & -a_m & & 1 & -1 \end{pmatrix}$$

which is obtained from the matrix Λ in (3-1) by changing the order of the facets of $\Delta^n \times \Delta^m$ to

$$F_1 \times \Delta^m, \ldots, F_n \times \Delta^m, F_{n+1} \times \Delta^m, \Delta^n \times G_1, \ldots, \Delta^n \times G_m, \Delta^n \times G_{m+1}.$$

4. Quasitoric manifolds equivalent to a generalized Bott manifold

Recall that the cohomological rigidity problem asks whether two quasitoric manifolds are homeomorphic if their cohomology rings are isomorphic. As an intermediate step toward the answer to the question for quasitoric manifolds homeomorphic to generalized Bott manifolds, we can ask the following question: *is a quasitoric manifold over a product of simplices equivalent (or homeomorphic) to a generalized Bott manifold if its cohomology ring is isomorphic to that of a generalized Bott manifold?* When the orbit space is $\Delta^1 \times \Delta^1$, then the answer is affirmative by Choi and Suh [2011]. Assume that a two-stage generalized Bott manifold is a $\mathbb{C}P^m$-bundle over $\mathbb{C}P^n$. In this section we answer to this question for $m > 1$ case. For the case of $m = 1$, we will show in the next section that if a quasitoric manifold M has the cohomology ring of the type (3-3), then M is homeomorphic (but not necessarily equivalent) to a generalized Bott manifold.

Proposition 4.1. *Let M be a quasitoric manifold over $\Delta^n \times \Delta^m$ with $m > 1$. If there is a generalized Bott tower $B_2 \to \mathbb{C}P^n \to B_0$ such that the fiber of $B_2 \to \mathbb{C}P^n$ has complex dimension m and $H^*(B_2) \cong H^*(M)$, then M is equivalent to a generalized Bott manifold.*

Proof. From (3-3), the cohomology ring of B_2 can be given by

$$H^*(B_2) = \mathbb{Z}[x_1, x_2] \Big/ \Big\langle x_1^{n+1}, \; x_2 \prod_{j=1}^{m} (a_j x_1 + x_2) \Big\rangle,$$

and from (3-2), the cohomology ring of M can be given by

$$H^*(M) = \mathbb{Z}[y_1, y_2] \Big/ \Big\langle y_1 \prod_{i=1}^{n} (y_1 + d_i y_2), \ y_2 \prod_{j=1}^{m} (c_j y_1 + y_2) \Big\rangle.$$

For simplicity, let $\mathscr{I} \subset \mathbb{Z}[x_1, x_2]$ be the ideal generated by the homogeneous poly-
nomials x_1^{n+1} and $x_2 \prod_{j=1}^{m} (a_j x_1 + x_2)$ and let $\mathscr{J} \subset \mathbb{Z}[y_1, y_2]$ be also the ideal
generated by the homogeneous polynomials

$$y_1 \prod_{i=1}^{n} (y_1 + d_i y_2) \quad \text{and} \quad y_2 \prod_{j=1}^{m} (c_j y_1 + y_2).$$

Then we have $H^*(B_2) = \mathbb{Z}[x_1, x_2]/\mathscr{I}$ and $H^*(M) = \mathbb{Z}[y_1, y_2]/\mathscr{J}$.

From the hypothesis, there is a ring isomorphism $\phi : H^*(B_2) \to H^*(M)$ which
preserves the grading. Then the map ϕ lifts to a grading preserving isomorphism
$\bar{\phi} : \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[y_1, y_2]$ with $\bar{\phi}(\mathscr{I}) = \mathscr{J}$. Note that if we let $\bar{\phi}(x_i) = g_{i1} y_1 + g_{i2} y_2$,
$i = 1, 2$, and denote the matrix of $\bar{\phi}$ by G, then

$$\det G = \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \pm 1.$$

We prove the proposition by showing that either

$$c_1 = \cdots = c_m = 0 \quad \text{or} \quad d_1 = \cdots = d_n = 0.$$

Then, by Proposition 3.2, M is equivalent to a generalized Bott manifold. We
consider the three cases (1) $n < m$, (2) $n = m$, and (3) $1 < m < n$ separately.

Case 1: $n < m$. Since $\bar{\phi}(\mathscr{I}) = \mathscr{J}$ and $m > n$, we have

(4-1) $$\bar{\phi}(x_1^{n+1}) = \alpha y_1 \prod_{i=1}^{n} (y_1 + d_i y_2),$$

where α is an integer. Then the set of prime divisors of x_1^{n+1} is mapped by $\bar{\phi}$ to the
set of prime divisors of $\alpha y_1 \prod_{i=1}^{n} (y_1 + d_i y_2)$. Since x_1 is the only prime divisor
of x_1^{n+1}, we must have $\alpha \neq 0$ and $d_i = 0$ for all i, which proves the proposition in
this case.

Case 2: $n = m$. Since $\bar{\phi}(\mathscr{I}) = \mathscr{J}$ and $n = m$, we have

(4-2) $$\bar{\phi}(x_1^{n+1}) = \alpha y_1 \prod_{i=1}^{n} (y_1 + d_i y_2) + \alpha' y_2 \prod_{j=1}^{n} (c_j y_1 + y_2),$$

where α and α' are integers.

(i) If α is zero, then by an argument similar to the one in Case 1, we have $\alpha' \neq 0$
and $c_j = 0$ for all j, which proves the proposition.

(ii) If α' is zero, then an argument similar to the one in Case 1 shows that $\alpha \neq 0$ and $d_i = 0$ for all i, which proves the proposition.

(iii) Now assume that neither α nor α' is zero. Plugging $\bar{\phi}(x_1) = g_{11}y_1 + g_{12}y_2$ into (4-2), we have

$$(4\text{-}3) \qquad (g_{11}y_1 + g_{12}y_2)^{n+1} = \alpha y_1 \prod_{i=1}^{n}(y_1 + d_i y_2) + \alpha' y_2 \prod_{j=1}^{n}(c_j y_1 + y_2).$$

Then we can see that $\alpha = g_{11}^{n+1}$ and $\alpha' = g_{12}^{n+1}$ by comparing the coefficients of y_1^{n+1} and y_2^{n+1} on both sides of (4-3). Hence we have

$$(4\text{-}4) \qquad (g_{11}y_1 + g_{12}y_2)^{n+1} = g_{11}^{n+1} y_1 \prod_{i=1}^{n}(y_1 + d_i y_2) + g_{12}^{n+1} y_2 \prod_{j=1}^{n}(c_j y_1 + y_2)$$

as polynomials. Plug $y_1 = y_2 = 1$ and $y_1 = 1$, $y_2 = -1$ into (4-4) to get the system of equations

$$(4\text{-}5) \qquad \begin{aligned} (g_{11} + g_{12})^{n+1} &= g_{11}^{n+1} \prod_{i=1}^{n}(1 + d_i) + g_{12}^{n+1} \prod_{j=1}^{n}(c_j + 1) \\ (g_{11} - g_{12})^{n+1} &= g_{11}^{n+1} \prod_{i=1}^{n}(1 - d_i) - g_{12}^{n+1} \prod_{j=1}^{n}(c_j - 1). \end{aligned}$$

Note that $1 - d_i c_j = \pm 1$ for all $1 \leq i, j \leq n$ from the nonsingularity condition (2-1). If we show that $d_i c_j = 0$ for all $1 \leq i, j \leq n$, then we are done. Indeed, if $c_{j_0} \neq 0$ for some $1 \leq j_0 \leq n$, then $d_i c_{j_0} = 0$ for all $1 \leq i \leq n$ implies that $d_i = 0$ for all $1 \leq i \leq n$. Otherwise $c_j = 0$ for all $1 \leq j \leq n$, which proves the proposition.

We will now show that $d_i c_j = 0$ for all $1 \leq i, j \leq n$. Suppose not, that is, $d_{i_0} c_{j_0} \neq 0$ for some $1 \leq i_0, j_0 \leq n$. Then from the nonsingularity condition we have $d_{i_0} c_{j_0} = 2$. For simplicity we may assume that $d_1 c_1 = 2$, so either $d_1 = 1$ and $c_1 = 2$, or $d_1 = 2$ and $c_1 = 1$ (up to equivalence). But these two cases are symmetric because $n = m$. Thus it is enough to consider the case $d_1 = 1$ and $c_1 = 2$.

Since $1 - d_i c_j = \pm 1$ for all $1 \leq i, j \leq n$, we have that $d_i = 1$ or 0 for $i = 2, \ldots, n$ and $c_j = 2$ or 0 for $j = 2, \ldots, n$. Plug these into (4-5) to get

$$(4\text{-}6) \qquad (g_{11} + g_{12})^{n+1} = 2^r g_{11}^{n+1} + 3^s g_{12}^{n+1},$$

$$(4\text{-}7) \qquad (g_{11} - g_{12})^{n+1} = -g_{12}^{n+1}(-1)^{n+1-s}$$

for some $1 \leq r, s \leq n$. From (4-7), we have $g_{11} = 0$ or $g_{11} = 2g_{12}$. If we have $g_{11} = 0$, then (4-6) implies $g_{12}^{n+1} = 3^s g_{12}^{n+1}$. Therefore $g_{12} = 0$, which contradicts $\det(G) \neq 0$. Otherwise, that is if $g_{11} = 2g_{12}$, then by plugging $g_{11} = 2g_{12}$ into (4-6) we have

$$3^{n+1} g_{12}^{n+1} = 2^{r+n+1} g_{12}^{n+1} + 3^s g_{12}^{n+1}.$$

Therefore, g_{12} is zero and so is g_{11}, which also contradicts $\det(G) \neq 0$. These contradictions arise because we assumed that $d_{i_0} c_{j_0} = 2$ for some $1 \leq i_0, j_0 \leq n$. This shows that $d_i c_j = 0$ for all $1 \leq i, j \leq n$.

Case 3: $1 < m < n$. Since $n > m$, we have

$$(4\text{-}8) \qquad \bar{\phi}(x_2 \prod_{j=1}^{m} (a_j x_1 + x_2)) = \alpha y_2 \prod_{j=1}^{m} (c_j y_1 + y_2)$$

for some nonzero integer α. Plugging $\bar{\phi}(x_i) = g_{i1} y_1 + g_{i2} y_2$ into (4-8), we have

$$(4\text{-}9) \quad (g_{21} y_1 + g_{22} y_2) \prod_{j=1}^{m} ((a_j g_{11} + g_{21}) y_1 + (a_j g_{12} + g_{22}) y_2) = \alpha y_2 \prod_{j=1}^{m} (c_j y_1 + y_2).$$

Comparing the coefficients of y_2^{n+1} on both sides of (4-9), we see that

$$\alpha = g_{22} \prod_{j=1}^{m} (a_j g_{12} + g_{22})$$

and we have

$$(4\text{-}10) \quad (g_{21} y_1 + g_{22} y_2) \prod_{j=1}^{m} ((a_j g_{11} + g_{21}) y_1 + (a_j g_{12} + g_{22}) y_2)$$

$$= g_{22} \prod_{j=1}^{m} (a_j g_{12} + g_{22}) y_2 \prod_{j=1}^{m} (c_j y_1 + y_2).$$

By comparing the coefficients of y_1^{m+1} on both sides, we get

$$g_{21} \prod_{j=1}^{m} (a_j g_{11} + g_{21}) = 0.$$

If $g_{21} = 0$, then $\det(G) = g_{11} g_{22} = \pm 1$, and hence $g_{11} = \pm 1$. If $a_j g_{11} + g_{21} = 0$ for some $1 \leq j \leq m$, then

$$\det(G) = g_{11} g_{22} - g_{12} g_{21} = g_{11}(g_{22} + a_j g_{12}) = \pm 1.$$

Hence $g_{11} = \pm 1$ in this case too.

As in Case 2, it is enough to show that $d_i c_j = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Suppose otherwise, that is, $d_{i_0} c_{j_0} = 2$ as before.

(i) Suppose $c_{j_0} = 2$. Then $d_{i_0} = 1$, and $c_j = 0$ or 2 for all $1 \leq j \leq m$ and $d_i = 0$ or 1 for all $1 \leq i \leq n$. Let s be the number of c_j's equal to 2.

(i-1) First consider the case $0 < s < m$. In this case we may assume $c_1 = 2$ and $c_m = 0$ for simplicity. Since $\bar{\phi}(x_1^{n+1}) \in \mathcal{J}$, we have

$$(4\text{-}11) \qquad \bar{\phi}(x_1^{n+1}) = \alpha y_1 \prod_{i=1}^{n} (y_1 + d_i y_2) + f(y_1, y_2) y_2 \prod_{j=1}^{m} (c_j y_1 + y_2),$$

where α is an integer and $f(y_1, y_2)$ is a homogeneous polynomial of degree $n-m$. Plugging $\overline{\phi}(x_1) = g_{11}y_1 + g_{12}y_2$ into (4-11), we have

$$(4\text{-}12) \quad (g_{11}y_1 + g_{12}y_2)^{n+1} = \alpha y_1 \prod_{i=1}^{n}(y_1 + d_i y_2) + f(y_1, y_2)y_2 \prod_{j=1}^{m}(c_j y_1 + y_2).$$

If $\alpha = 0$, then $g_{11} = 0$, so $c_j = 0$ for all $j = 1, \ldots, m$. This is a contradiction to the assumption $c_1 = 1$. Hence $\alpha \neq 0$. Comparing the coefficients of y_1^{n+1} on both sides of (4-12), we can see that $\alpha = g_{11}^{n+1}$ and we have

$$(4\text{-}13) \quad (g_{11}y_1 + g_{12}y_2)^{n+1} = g_{11}^{n+1}y_1 \prod_{i=1}^{n}(y_1 + d_i y_2) + f(y_1, y_2)y_2 \prod_{j=1}^{m}(c_j y_1 + y_2),$$

as polynomials in y_1 and y_2. Since $c_m = 0$, comparing the coefficients of $y_1^n y_2$ on both sides of (4-13), we get the equation

$$(4\text{-}14) \qquad\qquad (n+1)g_{11}^n g_{12} = g_{11}^{n+1}(d_1 + \cdots + d_n).$$

Since $g_{11} = \pm 1$ and $d_i = 0$ or 1 with $d_1 + \cdots + d_n \leq n$, the last equation gives a contradiction. So $s < m$ cannot happen.

(i-2) Now suppose $s = m$, that is, $c_1 = \cdots = c_m = 2$. In this case there is a ring isomorphism ψ from the cohomology ring

$$H^*(M) = \mathbb{Z}[y_1, y_2] / \langle y_1^{n+1-r}(y_1 + y_2)^r, \, y_2(2y_1 + y_2)^m \rangle$$

to the ring

$$\mathbb{Z}[Y_1, Y_2] / \langle Y_1^{n+1-r}(Y_1 + Y_2)^r, \, Y_2^m(2Y_1 + Y_2) \rangle$$

given by $\psi(y_1) = -Y_1$, $\psi(y_2) = 2Y_1 + Y_2$. In other words, if $s = m$, then $H^*(M)$ is isomorphic to a ring

$$\mathbb{Z}[y_1, y_2] / \left\langle y_1 \prod_{i=1}^{n}(y_1 + d_i y_2), \, y_2 \prod_{j=1}^{m}(c_j y_1 + y_2) \right\rangle$$

with $c_1 = 2$, $c_2 = \cdots = c_m = 0$. We are thus in the $s = 1$ case. But by the previous argument this induces a contradiction.

(ii) Suppose $c_{j_0} = 1$. Then $d_{i_0} = 2$. As before let r be the number of c_j equal to 1.

(ii-1) First consider the case when $0 < r < m$. In this case we may assume that $c_1 = 1$ and $c_m = 0$. By the same argument as above, (4-13) and (4-14) also hold. Since $g_{11} = \pm 1$, we have $(n+1)g_{12} = g_{11}(d_1 + \cdots + d_n) = 2g_{11}s$, where s is the number of d_i's equal to 2, and $0 < s \leq n$. This equality holds if and only if $g_{11} = g_{12}$, $s = (n+1)/2$, and n is odd. By plugging $y_1 = 1$ and $y_2 = -1$ into (4-13), we have $0 = g_{11}^{n+1}\prod_{i=1}^{n}(1 - d_i)$ which is a contradiction. This shows that $0 < r < m$ is impossible.

(ii-2) Now suppose $r = m$, that is, $c_1 = \cdots = c_m = 1$. Then by the ring isomorphism given by $\psi(y_1) = -Y_1$ and $\psi(y_2) = Y_1 + Y_2$, $H^*(M)$ is isomorphic to the ring

$$\mathbb{Z}[Y_1, Y_2] \big/ \big\langle Y_1^{n+1-s}(Y_1 + 2Y_2)^s, Y_2^m(Y_1 + Y_2) \big\rangle,$$

which brings us to the case when $r = 1$. By the previous argument, this case also induces a contradiction.

We thus have proved that $d_i c_j = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$, which proves the proposition. $\square$

5. Quasitoric manifolds over $\Delta^n \times \Delta^1$

In this section, we restrict our attention to the case where the orbit space is $\Delta^n \times \Delta^1$.

Example 5.1 [Davis and Januszkiewicz 1991]. Projective toric manifolds over $\Delta^1 \times \Delta^1$ are *Hirzebruch surfaces* $\Sigma_a = P(\mathbb{C} \oplus \gamma^{\otimes a})$ for $a \in \mathbb{Z}$, where γ is the tautological line bundle over $\mathbb{C}P^1$. By Hirzebruch [1951], Σ_a is diffeomorphic to Σ_b if and only if a is congruent to b modulo 2. Hence a projective toric manifold over $\Delta^1 \times \Delta^1$ is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. On the other hand, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is the unique quasitoric manifold over $\Delta^1 \times \Delta^1$ which is not a projective toric manifold. Hence there are only three topological types of quasitoric manifolds over $\Delta^1 \times \Delta^1$: $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and $\mathbb{C}P^2 \# \mathbb{C}P^2$.

Let M be a quasitoric manifold over $\Delta^n \times \Delta^1$. As in Section 3, we order the facets of $\Delta^n \times \Delta^1$ as follows:

(5-1) $F_1 \times \Delta^1, \ldots, F_n \times \Delta^1, \Delta^n \times G_1, F_{n+1} \times \Delta^1, \Delta^n \times G_2,$

where F_i's are facets of Δ^n and G_i are facets of Δ^1. Up to equivalence of quasitoric manifolds we may assume that the characteristic function λ on the ordered facets gives the following $(n+1) \times (n+3)$ matrix

(5-2) $$\Lambda = \begin{pmatrix} 1 & \cdots & 0 & 0 & -1 & -b_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -1 & -b_n \\ 0 & \cdots & 0 & 1 & -a & -1 \end{pmatrix},$$

namely, $\lambda(F_i \times \Delta^1) = e_i$ for $0 \leq i \leq n$, $\lambda(\Delta^n \times G_1) = e_{n+1}$,

$\lambda(F_{n+1} \times \Delta^1) = (-1, \ldots, -1, -a)^T$ and $\lambda(\Delta^n \times G_2) = (-b_1, \ldots, -b_n, -1)^T$.

We denote such M by $M_{a,b}$ for $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$. Moreover, by the nonsingularity condition (2-1), we have $ab_i = 0$ or 2 for $i = 1, \ldots, n$.

We first consider the case $ab_i = 0$ for all $i = 1, \ldots, n$. Then either $a = 0$ or $(b_1, \ldots, b_n)$ is a zero vector. Then $M_{a,b}$ is equivalent to a generalized Bott

manifold by Proposition 3.2. More precisely, $M_{a,\mathbf{0}} = P(\mathbb{C} \oplus \gamma^{\otimes a}) \to \mathbb{C}P^n$, and $M_{\mathbf{0},\mathbf{b}} = P(\mathbb{C} \oplus (\bigoplus_{j=1}^n \gamma^{\otimes b_j})) \to \mathbb{C}P^1$. In this case, $M_{a,\mathbf{b}}$ is a projective toric manifold. Here we classify all projective toric manifolds over $\Delta^n \times \Delta^1$ smoothly.

Proposition 5.2. *Let n be a positive integer greater than 1.*

(1) Let $M_{a,\mathbf{0}}$ denote the two-stage generalized Bott manifold

$$M_{a,\mathbf{0}} = B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{a \text{ point}\},$$

where $B_1 = \mathbb{C}P^n$, $B_2 = P(\mathbb{C} \oplus \gamma^{\otimes a})$, and γ is the tautological line bundle over $\mathbb{C}P^n$. Then $M_{a,\mathbf{0}}$ is diffeomorphic to $M_{a',\mathbf{0}}$ if and only if $|a| = |a'|$.

(2) Let $M_{\mathbf{0},\mathbf{b}}$ denote the two-stage generalized Bott manifold

$$M_{\mathbf{0},\mathbf{b}} = B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{a \text{ point}\},$$

where $B_1 = \mathbb{C}P^1$, $B_2 = P(\mathbb{C} \oplus (\bigoplus_{i=1}^n \gamma^{\otimes b_i}))$ for $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, and γ is the tautological line bundle over $\mathbb{C}P^1$. Then $M_{\mathbf{0},\mathbf{b}}$ is diffeomorphic to $M_{\mathbf{0},\mathbf{b}'}$ if and only if there is $\epsilon = \pm 1$ such that $\epsilon \sum_{i=1}^n b_i \equiv \sum_{i=1}^n b_i' \pmod{n+1}$.

Proof. (1) Note that

$$H^*(M_{a,\mathbf{0}}) = \mathbb{Z}[x_1, x_2] / \langle x_1^{n+1}, x_2(ax_1 + x_2) \rangle,$$

and $\pi_2^*(H^*(B_1)) \cong \mathbb{Z}[x_1]/x_1^{n+1} \subset H^*(M_{a,\mathbf{0}})$. By Theorem 3.3, $M_{a,\mathbf{0}}$ and $M_{a',\mathbf{0}}$ are diffeomorphic if and only if there exist $\epsilon = \pm 1$ and $w \in \mathbb{Z}$ such that

$$(1 + \epsilon w x_1)(1 + \epsilon(a+w)x_1) = (1 + a'x_1)$$

in $\mathbb{Z}[x_1]/x_1^{n+1}$. Hence, we have $\epsilon(a + 2w) = a'$ and $w(a+w)x_1^2 = 0$. Since $n > 1$, $x_1^2 \neq 0$ in $\mathbb{Z}[x_1]/x_1^{n+1}$. Therefore $w(a+w) = 0$, hence w is either 0 or $-a$. In any case, we obtain $a' = \pm a$.

(2) Note that

$$H^*(M_{\mathbf{0},\mathbf{b}}) = \mathbb{Z}[x_1, x_2] \Big/ \Big\langle x_1 \prod_{i=1}^n (x_1 + b_i x_2), x_2^2 \Big\rangle$$

and $\pi_2^*(H^*(B_1)) \cong \mathbb{Z}[x_2]/x_2^2 \subset H^*(M_{\mathbf{0},\mathbf{b}})$. By Theorem 3.3, $M_{\mathbf{0},\mathbf{b}}$ and $M_{\mathbf{0},\mathbf{b}'}$ are diffeomorphic if and only if there exist $\epsilon = \pm 1$ and $w \in \mathbb{Z}$ such that

$$\prod_{i=0}^n (1 + \epsilon(b_i + w)x_2) = \prod_{i=0}^n (1 + b_i' x_2)$$

in $\mathbb{Z}[x_2]/x_2^2$, where $b_0 = b_0' = 0$. Since $x_2^2 = 0$ we only have the condition $\epsilon \sum_{i=1}^n b_i + (n+1)w = \sum_{i=1}^n b_i'$. $\square$

Now, we consider the case $ab_i = 2$ for some i. In this case, $M_{a,b}$ cannot be equivalent to a generalized Bott manifold. However, as we will see later, they can be homeomorphic to generalized Bott manifolds. Note that, by Remark 2.1, we may assume that a and the nonzero b_i's have the positive sign. If $ab_i = 2$ for some $i = 1, \ldots, n$, then a must be either 1 or 2.

Let s be the number of the nonzero b_i's. Then, by (3-2), we have

$$H^*(M_{a,b}) = \mathbb{Z}[x_1, x_2]/\langle x_1^{n+1-s}(x_1 + bx_2)^s, x_2(ax_1 + x_2)\rangle,$$

where $ab = 2$.

Here, we classify all quasitoric manifold which are not equivalent to projective toric manifolds over $\Delta^n \times \Delta^1$ topologically. To do this, we prepare two lemmas.

Lemma 5.3. *For any $b \in \mathbb{Z}^n$, $M_{1,b}$ is homeomorphic to either $\mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$ or $\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$.*

Proof. Let N be a quasitoric manifold over an $(n+1)$-dimensional polytope P with the characteristic function λ. Let $F_1, \ldots, F_{n+1}$ be the facets of P meeting at a vertex q of P. Then from the nonsingularity condition (2-1) we have

$$\det(\lambda(F_1), \ldots, \lambda(F_{n+1})) = \pm 1.$$

Let $\mathrm{vc}(P)$ be the vertex cut of P about the vertex q of P, and let G be the new facet of $\mathrm{vc}(P)$ obtained from the vertex cut. Let $F_1, \ldots, F_{n+1}$ still denote the facets surrounding the facet G as in Figure 1. If we extend the characteristic function λ to the facets of $\mathrm{vc}(P)$, then the corresponding quasitoric manifold over $\mathrm{vc}(P)$ is a connected sum of N with $\mathbb{C}P^{n+1}$ or $\overline{\mathbb{C}P^{n+1}}$.
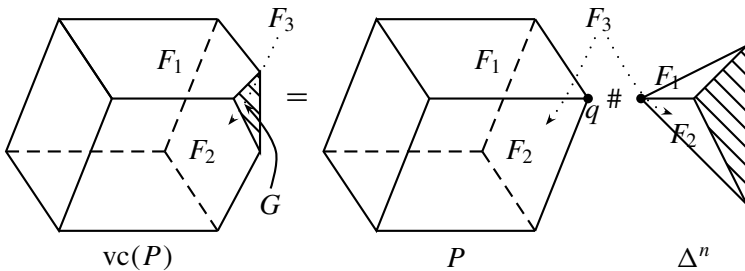
Figure 1. The vertex cut of a polytope P.

Recall the ordering (5-1) of the facets of $\Delta^n \times \Delta^1$. Since $\Delta^n \times \Delta^1$ can be viewed as a vertex cut of Δ^{n+1}, the condition

$$\det(\lambda(F_1 \times \Delta^1), \ldots, \lambda(F_n \times \Delta^1), \lambda(F_{n+1} \times \Delta^1)) = -a = -1$$

implies that the characteristic function λ can be considered as the one extended from a characteristic function on Δ^{n+1}. Therefore $M_{1,b}$ is homeomorphic to either $\mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$ or $\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$. $\qquad\square$

As we have seen in Remark 3.4, the moment angle manifold $\mathcal{Z}_{\Delta^n \times \Delta^1}$ is

$$S^{2n+1} \times S^3 = \left\{(\boldsymbol{w}, \boldsymbol{z}) \in \mathbb{C}^{n+1} \times \mathbb{C}^2 : |\boldsymbol{w}| = 1, \ |\boldsymbol{z}| = 1\right\}$$

and the subtorus $\ker \overline{\lambda} \subset T^{n+3}$ is represented by the unimodular subgroup of $\mathbb{Z}^{n+3}$ spanned by $(1, \ldots, 1, a, 0)$ and $(b_1, \ldots, b_n, 0, 1, 1)$. In this section, we denote the subtorus $\ker \overline{\lambda}$ by $K_{a,b}$.

Assume that we have two quasitoric manifolds $M_{a,b}$ and $M_{a',b'}$. If there is a θ-equivariant homeomorphism φ from $\mathcal{Z}_{\Delta^n \times \Delta^1}$ with the action of the subgroup $K_{a,b} \subset T^{n+3}$ to $\mathcal{Z}_{\Delta^n \times \Delta^1}$ with the action of the subgroup $K_{a',b'} \subset T^{n+3}$, where θ is an isomorphism from $K_{a,b}$ to $K_{a',b'}$, then φ induces a homeomorphism

$$\overline{\varphi} : M_{a,b} = \mathcal{Z}_{\Delta^n \times \Delta^1}/K_{a,b} \to M_{a',b'} = \mathcal{Z}_{\Delta^n \times \Delta^1}/K_{a',b'}.$$

Lemma 5.4. *Let $n > 1$, $\boldsymbol{b} = (b, \ldots, b, 0, \ldots, 0) \in \mathbb{Z}^n$, and $ab = 2$. Then we have*

(1) *$M_{a,(b,0,\ldots,0)}$ is homeomorphic to $M_{a,(b,\ldots,b)}$, and*

(2) *$M_{a,b}$ is either homeomorphic to $M_{a,\boldsymbol{0}}$ if s is even, or $M_{a,(b,0,\ldots,0)}$ if s is odd, where s is the number of b's in $\boldsymbol{b}$.*

In particular, if n is even, then $M_{a,b}$ is homeomorphic to $M_{a,\boldsymbol{0}}$.

Proof. (1) Let $\boldsymbol{b} = (b, 0, \ldots, 0)$ and $\boldsymbol{b}' = (b, \ldots, b)$. Then, by (3-5), there are isomorphisms $\mu \colon T^2 \to K_{a,b} \subset T^{n+3}$ and $\mu' \colon T^2 \to K_{a,b'} \subset T^{n+3}$ defined by

$$\begin{pmatrix} 1 & b \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ a & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & b \\ \vdots & \vdots \\ 1 & b \\ 1 & 0 \\ a & 1 \\ 0 & 1 \end{pmatrix},$$

respectively. We set $(\boldsymbol{w}, \boldsymbol{z}) = (w_1, \ldots, w_{n+1}, z_1, z_2) \in S^{2n+1} \times S^3 \subset \mathbb{C}^{n+1} \times \mathbb{C}^2$. We define an isomorphism

$$\theta \colon K_{a,b} \to K_{a,b'}, \quad \mu(t_1, t_2) \mapsto \mu'(t_1 t_2^b, t_2^{-1})$$

and a map $\varphi \colon S^{2n+1} \times S^3 \to S^{2n+1} \times S^3$ by

$$\varphi(w_1, \ldots, w_{n+1}, z_1, z_2) = (w_{n+1}, w_2, \ldots, w_n, w_1, z_1, \overline{z}_2).$$

Let us check that φ is θ-equivariant:

$$\varphi(\mu(t_1, t_2) \cdot (\boldsymbol{w}, \boldsymbol{z}))$$
$$= \varphi(t_1 t_2^b w_1, t_1 w_2, \ldots, t_1 w_{n+1}, t_1^a t_2 z_1, t_2 z_2)$$
$$= (t_1 w_{n+1}, t_1 w_2, \ldots, t_1 w_n, t_1 t_2^b w_1, t_1^a t_2 z_1, t_2^{-1} \bar{z}_2)$$
$$= (t_1 t_2^b (t_2^{-1})^b w_{n+1}, \ldots, t_1 t_2^b (t_2^{-1})^b w_n, t_1 t_2^b w_1, (t_1 t_2^b)^a t_2^{-1} z_1, t_2^{-1} \bar{z}_2)$$
$$= \mu'(t_1 t_2^b, t_2^{-1}) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z})$$
$$= \theta(\mu(t_1, t_2)) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z})$$

because $ab = 2$. Hence φ is a θ-equivariant homeomorphism which induces a homeomorphism $\overline{\varphi} : M_{a,b} \to M_{a,b'}$.

(2) By Lemma 5.3, $M_{1,b}$ is homeomorphic to $\mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$ or $\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$. Note that $M_{1,0} = \mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$. If n is even, $\mathbb{C}P^{n+1}$ has an orientation-reversing self-homeomorphism. Thus $\mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$ is homeomorphic to $\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$. So each $M_{1,b}$ is homeomorphic to $M_{1,0}$. If n is odd, then we have

$$H^*(M_{1,b}) = \begin{cases} H^*(\mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}) & \text{if } s \text{ is even,} \\ H^*(\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}) & \text{if } s \text{ is odd.} \end{cases}$$

We note that $H^*(M_{1,0})$ and $H^*(M_{1,(2,0,\ldots,0)})$ are not isomorphic as graded rings. (We refer the reader to the proof of Theorem 5.5 below.) Therefore, the manifold $M_{1,b}$ is either homeomorphic to $M_{1,0} = \mathbb{C}P^{n+1} \# \overline{\mathbb{C}P^{n+1}}$ if s is even, or $M_{1,(2,0,\ldots,0)} = \mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$ if s is odd.

Now, consider the case $a = 2$. Let $\boldsymbol{b} = (1, \ldots, 1, 0, \ldots, 0)$, where the first s elements are 1, $\boldsymbol{b}' = \boldsymbol{0}$, and $\boldsymbol{b}'' = (1, 0, \ldots, 0)$. Then, by (3-5), there are isomorphisms $\mu : T^2 \to K_{2,b} \subset T^{n+3}$, $\mu' : T^2 \to K_{2,b'} \subset T^{n+3}$, and $\mu'' : T^2 \to K_{2,b''} \subset T^{n+3}$ defined by

$$\begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix},$$

respectively.

If s is even, we define an isomorphism $\theta : K_{2,b} \to K_{2,b'}$ by

$$\theta(\mu(t_1, t_2)) = \mu'(t_1^{-1}, t_2^{-1})$$

and a map $\varphi \colon S^{2n+1} \times S^3 \to S^{2n+1} \times S^3$ by

$$(\boldsymbol{w}, \boldsymbol{z}) \mapsto (\overline{z}_1 w_1 + z_2 \overline{w}_2, -z_2 \overline{w}_1 + \overline{z}_1 w_2, \ldots,$$
$$- z_2 \overline{w}_{s-1} + \overline{z}_1 w_s, \overline{w}_{s+1}, \ldots, \overline{w}_{n+1}, \overline{z}_1, \overline{z}_2).$$

This map is well-defined because $(\overline{z}_1 w_{k-1} + z_2 \overline{w}_k, -z_2 \overline{w}_{k-1} + \overline{z}_1 w_k)$ comes from the multiplication of quaternion numbers $\overline{z_1 + z_2 \, \boldsymbol{j}}$ and $w_{k-1} + w_k \, \boldsymbol{j}$ for even k with $2 \le k \le s$. Then this map φ is θ-equivariant because

$$\varphi(\mu(t_1, t_2) \cdot (\boldsymbol{w}, \boldsymbol{z}))$$
$$= \varphi(t_1 t_2 w_1, \ldots, t_1 t_2 w_s, t_1 w_{s+1}, \ldots, t_1 w_{n+1}, t_1^2 t_2 z_1, t_2 z_2)$$
$$= \left(t_1^{-1} (\overline{z}_1 w_1 + z_2 \overline{w}_2), t_1^{-1} (-z_2 \overline{w}_1 + \overline{z}_1 w_2), \ldots, \right.$$
$$\left. t_1^{-1} (-z_2 \overline{w}_{s-1} + \overline{z}_1 w_s), t_1^{-1} \overline{w}_{s+1}, \ldots, t_1^{-1} \overline{w}_{n+1}, t_1^{-2} t_2^{-1} \overline{z}_1, t_2^{-1} \overline{z}_2 \right)$$
$$= \mu'\!\left(t_1^{-1}, t_2^{-1} \right) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z})$$
$$= \theta(\mu(t_1, t_2)) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z}).$$

Hence φ induces a homeomorphism $\overline{\varphi} \colon M_{2,\boldsymbol{b}} \to M_{2,\boldsymbol{b}'}$.

If s is odd, we define an isomorphism $\theta \colon K_{2,\boldsymbol{b}} \to K_{2,\boldsymbol{b}''}$ by

$$\theta(\mu(t_1, t_2)) = \mu''(t_1^{-1}, t_2^{-1})$$

and a map $\varphi \colon S^{2n+1} \times S^3 \to S^{2n+1} \times S^3$ by

$$(\boldsymbol{w}, \boldsymbol{z}) \mapsto (\overline{w}_1, \overline{z}_1 w_2 + z_2 \overline{w}_3, -z_2 \overline{w}_2 + \overline{z}_1 w_3, \ldots,$$
$$- z_2 \overline{w}_{s-1} + \overline{z}_1 w_s, \overline{w}_{s+1}, \ldots, \overline{w}_{n+1}, \overline{z}_1, \overline{z}_2).$$

Then this map φ is also θ-equivariant because

$$\varphi(\mu(t_1, t_2) \cdot (\boldsymbol{w}, \boldsymbol{z}))$$
$$= \varphi(t_1 t_2 w_1, \ldots, t_1 t_2 w_s, t_1 w_{s+1}, \ldots, t_1 w_{n+1}, t_1^2 t_2 z_1, t_2 z_2)$$
$$= \left(t_1^{-1} t_2^{-1} \overline{w}_1, t_1^{-1} (\overline{z}_1 w_2 + z_2 \overline{w}_3), t_1^{-1} (-z_2 \overline{w}_2 + \overline{z}_1 w_3), \ldots, \right.$$
$$\left. t_1^{-1} (-z_2 \overline{w}_{s-1} + \overline{z}_1 w_s), t_1^{-1} \overline{w}_{s+1}, \ldots, t_1^{-1} \overline{w}_{n+1}, t_1^{-2} t_2^{-1} \overline{z}_1, t_2^{-1} \overline{z}_2 \right)$$
$$= \mu''\!\left(t_1^{-1}, t_2^{-1} \right) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z})$$
$$= \theta(\mu(t_1, t_2)) \cdot \varphi(\boldsymbol{w}, \boldsymbol{z}).$$

Hence φ induces a homeomorphism $\overline{\varphi} \colon M_{2,\boldsymbol{b}} \to M_{2,\boldsymbol{b}''}$. $\hfill\square$

Now, we are ready to prove the following topological classification of quasitoric manifolds over $\Delta^n \times \Delta^1$ which are not projective toric manifolds.

Theorem 5.5. *Let $n > 1$, $\boldsymbol{b} = (b, \ldots, b, 0, \ldots, 0) \in \mathbb{Z}^n$, and $ab = 2$. Then the homeomorphism classes of quasitoric manifolds $M_{a,\boldsymbol{b}}$ are represented by*

(1) $M_{1,\mathbf{0}}$ and $M_{2,\mathbf{0}}$, if n is even, or

(2) $M_{1,\mathbf{0}}$, $M_{2,\mathbf{0}}$, $M_{1,(2,0,\dots,0)}$ and $M_{2,(1,0,\dots,0)}$, if n is odd.

Furthermore, the cohomology rings of the different classes are distinct.

Proof. By Lemma 5.4, each quasitoric manifold over $\Delta^n \times \Delta^1$ is homeomorphic to one of the given manifolds. Hence, it is enough to show the last statement.

We note that, by Proposition 5.2, the cohomology rings of $M_{1,\mathbf{0}}$ and $M_{2,\mathbf{0}}$ are distinct. Thus, it suffices to show that if n is odd and $a'b' = 2$, then we have $H^*(M_{a,\mathbf{0}}) \not\cong H^*(M_{a',(b',0,\dots,0)})$ and $H^*(M_{1,(2,0,\dots,0)}) \not\cong H^*(M_{2,(1,0,\dots,0)})$.

We denote $M = M_{1,(2,0,\dots,0)}$ and $N = M_{2,(1,0,\dots,0)}$. Then

$$H^*(M) = \mathbb{Z}[x_1, x_2]/\langle x_1^n(x_1 + 2x_2), x_2(x_1 + x_2)\rangle,$$
$$H^*(N) = \mathbb{Z}[y_1, y_2]/\langle y_1^n(y_1 + y_2), y_2(2y_1 + y_2)\rangle.$$

We first claim that $H^*(M_{a,\mathbf{0}})$ is neither isomorphic to $H^*(M)$ nor $H^*(N)$ if n is odd and greater than 1. Since $x_1 x_2 = -x_2^2$ and $x_1^{n+1} = -2x_2 x_1^n$ in $H^*(M)$, for any linear element $cx_1 + dx_2 \in H^*(M)$, we have

$$
\begin{aligned}
(cx_1 + dx_2)^{n+1} &= \sum_{i=0}^{n+1} \binom{n+1}{i}(cx_1)^i (dx_2)^{n+1-i} \\
&= (cx_1)^{n+1} + \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}c^i d^{n+1-i} x_2^{n+1} \\
&= 2c^{n+1}x_2^{n+1} + \sum_{i=0}^{n}\binom{n+1}{i}(-c)^i d^{n+1-i} x_2^{n+1} \\
&= (c^{n+1} + (-c+d)^{n+1})x_2^{n+1}
\end{aligned}
$$

in $H^*(M)$. Since x_2^{n+1} does not vanish in $H^*(M)$, $(cx_1 + dx_2)^{n+1}$ cannot be zero in $H^*(M)$ for odd $n > 1$. Similarly, we can see that

$$(cy_1 + dy_2)^{n+1} = \tfrac{1}{2}\big(c^{n+1} + (c - 2d)^{n+1}\big)y_1^{n+1}$$

cannot be zero in $H^*(N)$ for odd $n > 1$. Since there is a linear element in $H^*(M_{a,\mathbf{0}})$ whose $(n+1)$-st power vanishes, $H^*(M_{a,\mathbf{0}})$ can neither be isomorphic to $H^*(M)$ nor $H^*(N)$ for odd $n > 1$.

We finally claim that $H^*(M)$ is not isomorphic to $H^*(N)$. Suppose that there is a grading preserving isomorphism

$$\phi : H^*(M) = \mathbb{Z}[x_1, x_2]/\mathcal{I}_M \to H^*(N) = \mathbb{Z}[y_1, y_2]/\mathcal{I}_N$$

which lifts to a grading preserving isomorphism $\overline{\phi} : \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[y_1, y_2]$ with $\overline{\phi}(\mathscr{I}_M) = \mathscr{I}_N$. Since $\overline{\phi}(\mathscr{I}_M) = \mathscr{I}_N$ and $n > 1$, we have

$$(5\text{-}3) \qquad\qquad \overline{\phi}(x_2(x_1 + x_2)) = \alpha y_2(2y_1 + y_2),$$

where α is a nonzero integer. The prime divisors of the left hand side of (5-3) generate $\mathbb{Z}[x_1, x_2]$ as a $\mathbb{Z}$-algebra, whereas the prime divisors of the right hand side of (5-3) do not generate $\mathbb{Z}[y_1, y_2]$. Therefore, $H^*(M)$ and $H^*(N)$ cannot be isomorphic. $\qquad\square$

Corollary 5.6. *Two quasitoric manifolds over $\Delta^n \times \Delta^1$ are homeomorphic if their cohomology rings are isomorphic as graded rings. In particular*,

(1) *if n is even, then M is homeomorphic to a generalized Bott manifold $M_{a,\mathbf{0}}$ or to $M_{0,\mathbf{b}}$*,

(2) *if n is odd, then M is homeomorphic to a generalized Bott manifold, or to $M_{1,(2,0,...,0)} \cong \mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$, or to $M_{2,(1,0,...,0)}$*.

Proof. Let M and N be quasitoric manifolds over $\Delta^n \times \Delta^1$. Assume that we have $H^*(M) \cong H^*(N)$. When $n = 1$, M is homeomorphic to N by Example 5.1.

Now consider the case when $n > 1$. If M is equivalent to a generalized Bott manifold $M_{0,\mathbf{b}}$, then so is N by Proposition 4.1, so M and N are homeomorphic by Theorem 3.3.

If M is equivalent to a generalized Bott manifold $M_{a,\mathbf{0}}$, then $N := M_{a',b'}$ must be homeomorphic to a generalized Bott manifold $M_{a',\mathbf{0}}$ because $H^*(M_{a,\mathbf{0}})$ cannot be isomorphic to $H^*(M_{a',(b',0,...,0)})$ as in the proof of Theorem 5.5. Therefore M and N are homeomorphic by Theorem 3.3.

If neither M nor N is equivalent to a generalized Bott manifold, then the assertion is true by Theorem 5.5.

Hence, for any case, M is homeomorphic to N. The latter statement of the corollary immediately follows Theorem 5.5. $\qquad\square$

The above corollary proves a part of Theorem 1.1.

Example 5.7. There are quasitoric manifolds homeomorphic but not equivalent to generalized Bott manifolds. For example, $M_{2,(1,1,0,...,0)}$ is homeomorphic to a generalized Bott manifold $M_{2,(0,...,0)}$. But $M_{2,(1,1,0,...,0)}$ is not equivalent to a generalized Bott manifold by Proposition 3.2.

6. Quasitoric manifolds over $\Delta^n \times \Delta^m$ with $n, m > 1$

Let $M_{a,b}$ be a quasitoric manifold over $\Delta^n \times \Delta^m$ with $n, m > 1$, as defined in Section 3, whose characteristic matrix is of the form (3-6). Define two vectors $\boldsymbol{s}$

and $\boldsymbol{r}$ by

(6-1) $\qquad \boldsymbol{s} := (\underbrace{2, \ldots, 2}_{s}, 0, \ldots, 0) \in \mathbb{Z}^m \text{ and } \boldsymbol{r} := (\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0) \in \mathbb{Z}^n,$

where $1 \le s \le m$ and $1 \le r \le n$. If a quasitoric manifold M with $\beta_2 = 2$ is not equivalent to a generalized Bott manifold, then M is equivalent to $M_{\boldsymbol{s},\boldsymbol{r}}$ for some $\boldsymbol{s}$ and $\boldsymbol{r}$.

In this section we prove Theorem 1.1 and Theorem 1.2 when $n, m > 1$. In doing so, we follow the same strategy as in Section 5. Assume that we have two quasitoric manifolds $M_{\boldsymbol{a},\boldsymbol{b}}$ and $M_{\boldsymbol{a}',\boldsymbol{b}'}$. If there is a θ-equivariant homeomorphism φ from $\mathcal{Z}_{\Delta^n \times \Delta^m}$ with the subtorus $K_{\boldsymbol{a},\boldsymbol{b}} \subset T^{n+m+2}$-action to $\mathcal{Z}_{\Delta^n \times \Delta^m}$ with the subtorus $K_{\boldsymbol{a}',\boldsymbol{b}'} \subset T^{n+m+2}$-action, where θ is an isomorphism from $K_{\boldsymbol{a},\boldsymbol{b}}$ to $K_{\boldsymbol{a}',\boldsymbol{b}'}$, then φ induces a homeomorphism

$$\overline{\varphi} : M_{\boldsymbol{a},\boldsymbol{b}'} = \mathcal{Z}_{\Delta^n \times \Delta^m} / K_{\boldsymbol{a},\boldsymbol{b}} \to M_{\boldsymbol{a}',\boldsymbol{b}'} = \mathcal{Z}_{\Delta^n \times \Delta^m} / K_{\boldsymbol{a}',\boldsymbol{b}'}.$$

Lemma 6.1. *Two quasitoric manifolds $M_{\boldsymbol{s},\boldsymbol{r}}$ and $M_{\boldsymbol{s}',\boldsymbol{r}'}$ are homeomorphic if the two pairs $(\boldsymbol{s}, \boldsymbol{r})$ and $(\boldsymbol{s}', \boldsymbol{r}')$ satisfy*

$$s = s' \text{ or } s + s' = m + 1 \quad and \quad r = r' \text{ or } r + r' = n + 1$$

where $\boldsymbol{s}, \boldsymbol{s}' \in \mathbb{Z}^m$ and $\boldsymbol{r}, \boldsymbol{r}' \in \mathbb{Z}^n$ are vectors as in (6-1).

Proof. As we have seen in Remark 3.4, the moment angle manifold $\mathcal{Z}_{\Delta^n \times \Delta^m}$ is

$$S^{2n+1} \times S^{2m+1} = \left\{ (\boldsymbol{w}, \boldsymbol{z}) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} : |\boldsymbol{w}| = 1, \ |\boldsymbol{z}| = 1 \right\},$$

and the subtorus $K_{\boldsymbol{s},\boldsymbol{r}}$ in T^{n+m+2} is represented by the unimodular subgroup of $\mathbb{Z}^{n+m+2}$ spanned by

$$\boldsymbol{u}_s := (\underbrace{1, \ldots, 1}_{n+1}, \underbrace{2, \ldots, 2}_{s}, 0, \ldots, 0) \quad and \quad \boldsymbol{v}_r := (\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0, \underbrace{1, \ldots, 1}_{m+1}).$$

That is, there is an isomorphism $\mu : T^2 \to K_{\boldsymbol{s},\boldsymbol{r}}$ defined by the matrix $\left(\begin{smallmatrix} \boldsymbol{u}_s^T & \boldsymbol{v}_r^T \end{smallmatrix} \right)$.

First consider the case when $\boldsymbol{s} = \boldsymbol{s}'$, $r \le \lfloor \frac{n+1}{2} \rfloor$, and $r' = n + 1 - r$. Then we have an isomorphism $\mu' : T^2 \to K_{\boldsymbol{s}',\boldsymbol{r}'}$ defined by the matrix $\left(\begin{smallmatrix} \boldsymbol{u}_s^T & \boldsymbol{v}_{n+1-r}^T \end{smallmatrix} \right)$.

We set $(\boldsymbol{w}, \boldsymbol{z}) = (w_1, \ldots, w_{n+1}, z_1, \ldots, z_{m+1}) \in S^{2n+1} \times S^{2m+1} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$. Now we define an isomorphism $\theta : K_{\boldsymbol{s},\boldsymbol{r}} \to K_{\boldsymbol{s}',\boldsymbol{r}'}$ by $\mu(t_1, t_2) \mapsto \mu'(t_1 t_2, t_2^{-1})$ and a map $\varphi : S^{2n+1} \times S^{2m+1} \to S^{2n+1} \times S^{2m+1}$ by

$$\varphi(w_1, \ldots, w_{n+1}, z_1, \ldots, z_{m+1})$$
$$= (w_{r+1}, \ldots, w_{n+1}, w_1, \ldots, w_r, z_1, \ldots, z_s, \overline{z}_{s+1}, \ldots, \overline{z}_{m+1}).$$

Let us check that φ is θ-equivariant:

$$\varphi(\mu(t_1, t_2) \cdot (\mathbf{w}, \mathbf{z})) = \varphi(t_1 t_2 w_1, \ldots, t_1 t_2 w_r, t_1 w_{r+1}, \ldots, t_1 w_{n+1},$$
$$t_1^2 t_2 z_1, \ldots, t_1^2 t_2 z_s, t_2 z_{s+1}, \ldots, t_2 z_{m+1})$$
$$= (t_1 w_{r+1}, \ldots, t_1 w_{n+1}, t_1 t_2 w_1, \ldots, t_1 t_2 w_r,$$
$$t_1^2 t_2 z_1, \ldots, t_1^2 t_2 z_s, t_2^{-1} \overline{z}_{s+1}, \ldots, t_2^{-1} \overline{z}_{m+1})$$
$$= \mu'(t_1 t_2, t_2^{-1}) \cdot \varphi(\mathbf{w}, \mathbf{z}) = \theta(\mu(t_1, t_2)) \cdot \varphi(\mathbf{w}, \mathbf{z}).$$

Hence φ induces a homeomorphism $\overline{\varphi}$ from $M_{s,r}$ to $M_{s',r'}$.

We now consider the case when $s \le \lfloor \frac{m+1}{2} \rfloor$, $s' = m + 1 - s$, and $\mathbf{r} = \mathbf{r}'$. Then we have an isomorphism $\mu'' : T^2 \to K_{s',r'}$ defined by the matrix $\left(\mathbf{u}_{m+1-2}^T \ \mathbf{v}_r^T \right)$.

We define an isomorphism $\theta : K_{s,r} \to \theta(K_{s',r'})$ by $\theta(\mu(t_1, t_2)) \mapsto \mu''(t_1^{-1}, t_1^2 t_2)$, and a map ϕ by

$$\varphi(w_1, \ldots, w_{n+1}, z_1, \ldots, z_{m+1})$$
$$= (w_1, \ldots, w_r, \overline{w}_{r+1}, \ldots, \overline{w}_{n+1}, z_{s+1}, \ldots, z_{m+1}, z_1, \ldots, z_s).$$

Then,

$$\varphi(\mu(t_1, t_2) \cdot (\mathbf{w}, \mathbf{z})) = (t_1 t_2 w_1, \ldots, t_1 t_2 w_r, t_1^{-1} \overline{w}_{r+1}, \ldots, t_1^{-1} \overline{w}_{n+1},$$
$$t_2 z_{s+1}, \ldots, t_2 z_{m+1}, t_1^2 t_2 z_1, \ldots, t_1^2 t_2 z_s)$$
$$= \mu''(t_1^{-1}, t_1^2 t_2) \cdot \varphi(\mathbf{w}, \mathbf{z}) = \theta(\mu(t_1, t_2)) \cdot \varphi(\mathbf{w}, \mathbf{z}).$$

Thus φ is a θ-equivariant homeomorphism which induces a homeomorphism $\overline{\varphi}$ from $M_{s,r}$ to $M_{s',r'}$.

Finally, we note that the case when $r = n + 1 - r'$ and $s = m + 1 - s'$ immediately follows from the composition of the above two cases. $\square$

Theorem 6.2. *Let $M_{s,r}$ and $M_{s',r'}$ be quasitoric manifolds as defined above. Then the following are equivalent*:

(1) $s = s'$ *or* $s + s' = m + 1$, *and* $r = r'$ *or* $r + r' = n + 1$.

(2) $H^*(M_{s,r})$ *and* $H^*(M_{s',r'})$ *are isomorphic.*

(3) $M_{s,r}$ *and* $M_{s',r'}$ *are homeomorphic.*

Proof. By Lemma 6.1, it suffices to show that (2) implies (1). Let $\mathscr{I} \subset \mathbb{Z}[x_1, x_2]$ and $\mathscr{J} \subset \mathbb{Z}[y_1, y_2]$ be the homogeneous ideals

$$\mathscr{I} = \left\langle x_1^{n+1-r}(x_1 + x_2)^r, x_2^{m+1-s}(2x_1 + x_2)^s \right\rangle,$$
$$\mathscr{J} = \left\langle y_1^{n+1-r'}(y_1 + y_2)^{r'}, y_2^{m+1-s'}(2y_1 + y_2)^{s'} \right\rangle.$$

Then we have

$$H^*(M_{s,r}) = \mathbb{Z}[x_1, x_2]/\mathscr{I} \quad \text{and} \quad H^*(M_{s',r'}) = \mathbb{Z}[y_1, y_2]/\mathscr{J}.$$

Then the cohomology ring isomorphism $\phi : H^*(M_{s,r}) \to H^*(M_{s',r'})$ lifts to a grading preserving isomorphism $\bar{\phi} : \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[y_1, y_2]$ with $\bar{\phi}(\mathscr{I}) = \mathscr{J}$. We divide the proof into three cases: (1) $n > m$, (2) $n < m$, and (3) $n = m$.

Case 1: $n > m$. Since $\bar{\phi}(x_2^{m+1-s}(2x_1 + x_2)^s) \in \mathscr{J}$ and $n > m$, we have

$$(6\text{-}2) \qquad \bar{\phi}(x_2^{m+1-s}(2x_1 + x_2)^s) = \alpha y_2^{m+1-s'}(2y_1 + y_2)^{s'}$$

for some nonzero integer α. Comparing the multiplicities of the prime divisors of both sides of (6-2), we can easily see that $s = s'$ or $s = m + 1 - s'$. Thus $\bar{\phi}(x_2)$ is either $\pm y_2$ or $\pm(2y_1 + y_2)$. Then we obtain the following four cases: when $s = s'$,

$$\begin{cases} \bar{\phi}(x_1) = \mp(y_1 + y_2) & \text{and} \quad \bar{\phi}(x_2) = \pm y_2, \text{ (i)} \\ \bar{\phi}(x_1) = \pm y_1 & \text{and} \quad \bar{\phi}(x_2) = \pm y_2, \text{ (ii)} \end{cases}$$

and when $s + s' = m + 1$,

$$\begin{cases} \bar{\phi}(x_1) = \mp(y_1 + y_2) & \text{and} \quad \bar{\phi}(x_2) = \pm(2y_1 + y_2), \text{ (iii)} \\ \bar{\phi}(x_1) = \mp y_1 & \text{and} \quad \bar{\phi}(x_2) = \pm(2y_1 + y_2). \text{ (iv)} \end{cases}$$

One can check that cases (i) and (iii) imply that $r + r' = n + 1$ and cases (ii) and (iv) imply that $r = r'$, which proves the implication (2) $\Rightarrow$ (1) in this case.

Case 2: $n < m$. This case is quite analogous to Case 1, so we skip the proof.

Case 3: $n = m$. Since $\bar{\phi}(\mathscr{I}) = \mathscr{J}$, we have

$$(6\text{-}3) \qquad \begin{aligned} \bar{\phi}(x_1^{n+1-r}(x_1 + x_2)^r) &= \alpha y_1^{n+1-r'}(y_1 + y_2)^{r'} + \alpha' y_2^{n+1-s'}(2y_1 + y_2)^{s'}, \\ \bar{\phi}(x_2^{n+1-s}(2x_1 + x_2)^s) &= \beta y_1^{n+1-r'}(y_1 + y_2)^{r'} + \beta' y_2^{n+1-s'}(2y_1 + y_2)^{s'}, \end{aligned}$$

where α, α', β, and β' are integers. Note that either α or α' is nonzero, and either β or β' is nonzero. We first show that α' and β are zero and then prove the theorem in this case.

Let G be the matrix of $\bar{\phi}$. Plugging $\bar{\phi}(x_i) = g_{i1}y_1 + g_{i2}y_2$, $i = 1, 2$, into (6-3), we have

$$(6\text{-}4) \quad (g_{11}y_1 + g_{12}y_2)^{n+1-r}((g_{11} + g_{21})y_1 + (g_{12} + g_{22})y_2)^r = \\ \alpha y_1^{n+1-r'}(y_1 + y_2)^{r'} + \alpha' y_2^{n+1-s'}(2y_1 + y_2)^{s'}$$

and

$$(6\text{-}5) \quad (g_{21}y_1 + g_{22}y_2)^{n+1-s}((2g_{11} + g_{21})y_1 + (2g_{12} + g_{22})y_2)^s = \\ \beta y_1^{n+1-r'}(y_1 + y_2)^{r'} + \beta' y_2^{n+1-s'}(2y_1 + y_2)^{s'},$$

where the determinant of G is ± 1.

Suppose that none of α, α', β, and β' are zero. Then by comparing the coefficients of y_1^{n+1} and y_2^{n+1} on both sides of (6-4), we have $\alpha = g_{11}^{n+1-r}(g_{11} + g_{21})^r$ and

$\alpha' = g_{12}^{n+1-r}(g_{12} + g_{22})^r$. By comparing the coefficients of y_1^{n+1} and y_2^{n+1} on both sides of (6-5), we have $\beta = g_{21}^{n+1-s}(2g_{11} + g_{21})^s$ and $\beta' = g_{22}^{n+1-s}(2g_{12} + g_{22})^s$. Hence we have a system of polynomial equations:

$$
\begin{aligned}
(6\text{-}6) \quad (g_{11}y_1 + g_{12}y_2)^{n+1-r}&((g_{11} + g_{21})y_1 + (g_{12} + g_{22})y_2)^r \\
&= g_{11}^{n+1-r}(g_{11} + g_{21})^r y_1^{n+1-r'}(y_1 + y_2)^{r'} \\
&\quad + g_{12}^{n+1-r}(g_{12} + g_{22})^r y_2^{n+1-s'}(2y_1 + y_2)^{s'},
\end{aligned}
$$

$$
\begin{aligned}
(6\text{-}7) \quad (g_{21}y_1 + g_{22}y_2)^{n+1-s}&((2g_{11} + g_{21})y_1 + (2g_{12} + g_{22})y_2)^s \\
&= g_{21}^{n+1-s}(2g_{11} + g_{21})^s y_1^{n+1-r'}(y_1 + y_2)^{r'} \\
&\quad + g_{22}^{n+1-s}(2g_{12} + g_{22})^s y_2^{n+1-s'}(2y_1 + y_2)^{s'}.
\end{aligned}
$$

We first show that $\alpha' = 0$. Plug $y_1 = 1$ and $y_2 = -1$ into (6-6) to get the equation

$$
(6\text{-}8) \quad (g_{11}-g_{12})^{n+1-r}((g_{11}+g_{21})-(g_{12}+g_{22}))^r = g_{12}^{n+1-r}(g_{12}+g_{22})^r(-1)^{n+1-s'}.
$$

Since we assume that α' is not zero, $g_{12}(g_{12} + g_{22}) \neq 0$. Then, by (6-8), we have

$$
\left(\frac{g_{11}}{g_{12}} - 1\right)^{n+1-r} \left(\frac{g_{11}+g_{21}}{g_{12}+g_{22}} - 1\right)^r = (-1)^{n+1-s'}.
$$

Thus $g_{11}/g_{12} = 2$ or 0, and $(g_{11} + g_{21})/(g_{12} + g_{22}) = 2$ or 0. In these cases, both g_{11} and g_{21} are even, which contradicts $\det(G) = \pm 1$. Hence, α' is zero.

We next show that $\beta = 0$. Plug $y_1 = 1$ and $y_2 = -2$ into (6-7) to get the equation

$$
(6\text{-}9) \quad (g_{21}-2g_{22})^{n+1-s}((2g_{11}+g_{21})-2(2g_{12}+g_{22}))^s = g_{21}^{n+1-s}(2g_{11}+g_{21})^s(-1)^{r'}.
$$

Since we assume that β is not zero, $g_{21}(2g_{11} + g_{21}) \neq 0$. Then, by (6-9), we have

$$
\left(1 - \frac{2g_{22}}{g_{21}}\right)^{n+1-s} \left(1 - \frac{2(2g_{12} + g_{22})}{2g_{11} + g_{21}}\right)^s = (-1)^{r'}.
$$

Thus $g_{22}/g_{21} = 0$ or 1, and $(2g_{12} + g_{22})/(2g_{11} + g_{21}) = 0$ or 1. In these cases, $\det G \neq \pm 1$ which is a contradiction. Hence, β is zero.

Now we will show that $s = s'$ or $s + s' = m + 1$, and $r = r'$ or $r + r' = n + 1$. Since both α' and β are zero, we have

$$
\begin{aligned}
\overline{\phi}(x_1^{n+1-r}(x_1 + x_2)^r) &= \alpha y_1^{n+1-r'}(y_1 + y_2)^{r'}, \\
\overline{\phi}(x_2^{n+1-s}(2x_1 + x_2)^s) &= \beta' y_2^{n+1-s'}(2y_1 + y_2)^{s'}.
\end{aligned}
$$

Hence, by using the same argument as in Case 1, we can show that $s = s'$ or $s + s' = m + 1$, and $r = r'$ or $r + r' = n + 1$. $\qquad \square$

Lemma 6.3. *If $n \neq m$, then two quasitoric manifolds $M_{s,r}$ and $M_{r',s'}$ are not homeomorphic for any chosen vectors s, $r' \in \mathbb{Z}^m$ and r, $s' \in \mathbb{Z}^n$ as in* (6-1). *That is,*

$$s := (\underbrace{2, \ldots, 2}_{s}, 0, \ldots, 0), \quad r' := (\underbrace{1, \ldots, 1}_{r'}, 0, \ldots, 0) \in \mathbb{Z}^m,$$

$$r := (\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0), \quad s' := (\underbrace{2, \ldots, 2}_{s'}, 0, \ldots, 0) \in \mathbb{Z}^n.$$

Proof. It is enough to show the case when $n < m$. First let $\mathscr{I} \subset \mathbb{Z}[x_1, x_2]$ and $\mathscr{J} \subset \mathbb{Z}[y_1, y_2]$ be the homogeneous ideals

$$\mathscr{I} = \left\langle x_1^{n+1-r}(x_1 + x_2)^r, x_2^{m+1-s}(2x_1 + x_2)^s \right\rangle,$$
$$\mathscr{J} = \left\langle y_1^{n+1-s'}(y_1 + 2y_2)^{s'}, y_2^{m+1-r'}(y_1 + y_2)^{r'} \right\rangle.$$

Then we have $H^*(M_{s,r}) = \mathbb{Z}[x_1, x_2]/\mathscr{I}$ and $H^*(M_{r',s'}) = \mathbb{Z}[y_1, y_2]/\mathscr{J}$.

Suppose that $M_{s,r}$ and $M_{r',s'}$ are homeomorphic for some vectors s, $r' \in \mathbb{Z}^m$ and r, $s' \in \mathbb{Z}^n$. Then the ring isomorphism $\phi : H^*(M_{s,r}) \to H^*(M_{r',s'})$ lifts to a grading preserving isomorphism $\overline{\phi} : \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[y_1, y_2]$ with $\overline{\phi}(\mathscr{I}) = \mathscr{J}$. Then

$$\overline{\phi}(x_1^{n+1-r}(x_1 + x_2)^r) = \alpha y_1^{n+1-s'}(y_1 + 2y_2)^{s'}$$

for some nonzero integer α. But this is a contradiction because the prime divisors of the left hand side generate $\mathbb{Z}[x_1, x_2]$ as a $\mathbb{Z}$-algebra, whereas the prime divisors of the right hand side do not generate $\mathbb{Z}[y_1, y_2]$.

Therefore, there is no isomorphism between $H^*(M_{s,r})$ and $H^*(M_{r',s'})$, so $M_{s,r}$ and $M_{r',s'}$ are not homeomorphic. $\square$

Theorem 6.4. *Two quasitoric manifolds over $\Delta^n \times \Delta^m$ with $n, m > 1$ are homeomorphic if and only if their cohomology rings are isomorphic as graded rings.*

Proof. Let M and N be quasitoric manifolds over $\Delta^n \times \Delta^m$, with $H^*(M) \cong H^*(N)$.

If M is equivalent to a generalized Bott manifold, then N is also equivalent to a generalized Bott manifold by Proposition 4.1, so M and N are homeomorphic by Theorem 3.3.

If M is equivalent to $M_{s,r}$, then N is equivalent to either $M_{s',r'}$ or $M_{r',s'}$ by Proposition 4.1. But by Lemma 6.3, N must be equivalent to $M_{s',r'}$. Thus M and N are homeomorphic by Theorem 6.2.

Hence, in any case, M is homeomorphic to N. $\square$

Corollary 6.5. *Let $N(n, m)$ be the number of quasitoric manifolds over $\Delta^n \times \Delta^m$ which are not homeomorphic to generalized Bott manifolds.*

(1) *When $n = m$, $N(n, n) = \left\lfloor \frac{n+1}{2} \right\rfloor \times \left\lfloor \frac{n+1}{2} \right\rfloor$.*

(2) *When $n \neq m$ and $n, m > 1$, $N(n, m) = 2\left\lfloor \frac{n+1}{2} \right\rfloor \times \left\lfloor \frac{m+1}{2} \right\rfloor$.*

(3) *$N(n, 1) = 0$ for even n and $N(n, 1) = 2$ for odd $n \geq 3$.*

Proof. This follows from Corollary 5.6, Theorem 6.2, and Lemma 6.3.   □

7. Proof of Theorem 1.2

A simple polytope P is said to be *cohomologically rigid* if there exists a quasitoric manifold M over P, and whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M) \cong H^*(N)$ there is a combinatorial equivalence $P \approx Q$. By Choi et al. [2010c], a product of simplices is cohomologically rigid.

Let M and M' be quasitoric manifolds with $\beta_2 = 2$. Then they are supported by the polytopes combinatorially equivalent to products of two simplices, say $\Delta^n \times \Delta^m$ and $\Delta^{n'} \times \Delta^{m'}$, respectively. Since products of simplices are cohomologically rigid, if $H^*(M) = H^*(M')$, then $\{n, m\} = \{n', m'\}$. In other words, two quasitoric manifolds over distinct products of simplices cannot have the same cohomology rings.

By Corollary 5.6 and Theorem 6.4, all quasitoric manifolds over a certain product of two simplices are classified by their cohomology rings. Hence, all quasitoric manifolds with $\beta_2 = 2$ are classified by their cohomology rings as graded rings.

8. Classification of quasitoric manifolds with $\beta_2 = 2$

Let $\boldsymbol{u} = (u_1, \ldots, u_k), \boldsymbol{u}' = (u'_1, \ldots, u'_k) \in \mathbb{Z}^k$ and let ℓ be a positive integer. We define $\boldsymbol{u}$ is *equivalent* to $\boldsymbol{u}'$ with respect to ℓ, denote it by $\boldsymbol{u} \sim_\ell \boldsymbol{u}'$, if there is $\epsilon = \pm 1$ and $w \in \mathbb{Z}$ such that

$$\prod_{i=1}^{k}(1 + u_i x) = (1 + \epsilon w x)\prod_{i=1}^{k}(1 + \epsilon(u'_i + w)x) \quad \text{in } \mathbb{Z}[x]/x^{\ell+1}.$$

Then from Theorem 3.3, Example 5.1, Corollary 5.6, and Theorems 6.2 and 6.4, we have the following topological classification.

Theorem 8.1. (1) *The homeomorphism classes of quasitoric manifolds over the polytope $\Delta^n \times \Delta^m$ with $n \neq m$ ($n, m > 1$) are represented by*:

- $M_{\boldsymbol{0},\boldsymbol{0}} = \mathbb{C}P^n \times \mathbb{C}P^m$, *a trivial generalized Bott manifold.*

- $M_{\boldsymbol{a},\boldsymbol{0}}$ *for $\boldsymbol{a} \in (\mathbb{Z}^m - \boldsymbol{0})/\sim_n$, nontrivial generalized Bott manifolds.*

- $M_{\boldsymbol{0},\boldsymbol{b}}$ *for $\boldsymbol{b} \in (\mathbb{Z}^n - \boldsymbol{0})/\sim_m$, nontrivial generalized Bott manifolds.*

- $M_{\boldsymbol{s},\boldsymbol{r}}$ *for $\boldsymbol{s} := (2, \ldots, 2, 0, \ldots, 0) \in \mathbb{Z}^m$ and $\boldsymbol{r} := (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$,*

- $M_{\boldsymbol{s},\boldsymbol{r}}$ *for $\boldsymbol{s} := (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^m$ and $\boldsymbol{r} := (2, \ldots, 2, 0, \ldots, 0) \in \mathbb{Z}^n$,*

where the number of nonzero components in $\boldsymbol{s}$, respectively $\boldsymbol{r}$, is positive and less than or equal to $\lfloor \frac{m+1}{2} \rfloor$, respectively $\lfloor \frac{n+1}{2} \rfloor$.

(2) *The homeomorphism classes of quasitoric manifold over $\Delta^n \times \Delta^n$ ($n > 1$) are represented by*:

- $M_{\mathbf{0},\mathbf{0}} = \mathbb{C}P^n \times \mathbb{C}P^n$.
- $M_{\mathbf{a},\mathbf{0}}$ *for* $\mathbf{a} \in (\mathbb{Z}^n - \mathbf{0})/\sim_n$.
- $M_{\mathbf{s},\mathbf{r}}$ *for* $\mathbf{s} := (2, \ldots, 2, 0, \ldots, 0) \in \mathbb{Z}^n$ *and* $\mathbf{r} := (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$,

where the number of nonzero components in $\mathbf{s}$ and $\mathbf{r}$ are positive and less than or equal to $\left\lfloor \frac{n+1}{2} \right\rfloor$.

(3) *The homeomorphism classes of quasitoric manifolds over $\Delta^1 \times \Delta^n$ ($n > 1$ is odd) are represented by*:

- $M_{\mathbf{0},\mathbf{0}} = \mathbb{C}P^1 \times \mathbb{C}P^n$.
- $M_{a,\mathbf{0}}$ *for* $a \in \mathbb{N}$.
- $M_{\mathbf{0},\mathbf{b}}$ *for* $\mathbf{b} \in (\mathbb{Z}^n - \mathbf{0})/\sim_1$ (*see Proposition 5.2*).
- $\mathbb{C}P^{n+1} \# \mathbb{C}P^{n+1}$.
- $M_{2,(1,0,\ldots,0)}$.

(4) *The homeomorphism classes of quasitoric manifolds over $\Delta^1 \times \Delta^n$ (n is even) are represented by*:

- $M_{\mathbf{0},\mathbf{0}} = \mathbb{C}P^1 \times \mathbb{C}P^n$.
- $M_{a,\mathbf{0}}$ *for* $a \in \mathbb{N}$.
- $M_{\mathbf{0},\mathbf{b}}$ *for* $\mathbf{b} \in (\mathbb{Z}^n - \mathbf{0})/\sim_1$ (*see Proposition 5.2*).

(5) *The homeomorphism classes of quasitoric manifolds over $\Delta^1 \times \Delta^1$ are represented by*:

- $M_{0,0} = \mathbb{C}P^1 \times \mathbb{C}P^1$.
- $M_{0,1} = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.
- $M_{2,1} = \mathbb{C}P^2 \# \mathbb{C}P^2$.

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SUYOUNG CHOI
DEPARTMENT OF MATHEMATICS
AJOU UNIVERSITY
SAN 5, WONCHEON-DONG, YEONGTONG-GU
SUWON 443-749
SOUTH KOREA
schoi@ajou.ac.kr

SEONJEONG PARK
DEPARTMENT OF MATHEMATICAL SCIENCES
KAIST
291 DAEHAK-RO, YUSEONG-GU
DAEJEON 305-701
SOUTH KOREA

psjeong@kaist.ac.kr

DONG YOUP SUH
DEPARTMENT OF MATHEMATICAL SCIENCES
KAIST
291 DAEHAK-RO, YUSEONG-GU
DAEJEON 305-701
SOUTH KOREA

dysuh@math.kaist.ac.kr

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