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DEFORMATION RETRACTS TO THE FAT DIAGONAL AND APPLICATIONS TO THE EXISTENCE OF PEAK SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

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We consider the equation $-\varepsilon^2\Delta u = u^p - u^q$ in a bounded, smooth domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions when either

$$q = 1 < p < \frac{N+2}{N-2} \quad \text{or} \quad \frac{N}{N-2} < q < p < \frac{N+2}{N-2} \text{ and } N \geq 3.$$

We prove the existence of multiple positive solutions in the case of small diffusion provided the domain Ω is not contractible.

1. Introduction

We consider the equation

$$-\epsilon^2\Delta u = f(u) \text{ in } \Omega, \quad \text{where } u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, the nonlinearity f is either

$$f(y) = y^p - y \qquad \text{with } 1 < p < \frac{N+2}{N-2}, \text{ or} \tag{2}$$

$$f(y) = y^p - y^q \qquad \text{with } \frac{N}{N-2} < q < p < \frac{N+2}{N-2} \text{ and } N \geq 3. \tag{3}$$

Problem (1) with (2) arises in various mathematical models in biological population theory, chemical reactor theory, and so on. Many results show that solutions of (1) may exhibit sharp peaks near a certain number of points. In particular, many papers have sought to prove the existence of single and multiple peak solutions and to find the peaks as well as the profile of the spikes as $\epsilon \to 0^+$. Ni and Wei [1995] showed that for $\epsilon > 0$ sufficiently small, problem (1) has a positive least energy solution that concentrates at the *most centered* part of the domain, that is, the maximum point of the distance function from the boundary $\partial\Omega$. Since then, many authors have looked for higher energy solutions. For papers that study the

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effect of the geometry of the domain on the existence of positive solutions with single and multiple peaks, see [Benci and Cerami 1987; Cao et al. 1996; Dancer 1988; 1990; Dancer and Yan 1999a; 1999b; Dancer and Wei 1998; del Pino et al. 2000a; 2000b; Grossi and Pistoia 2000; Li and Nirenberg 1998; Wei 1998] and references therein.

Problem (1) with (3) was first studied in [Dancer and Santra 2010] where the authors showed that for $\epsilon > 0$ small the problem has a positive least energy solution that concentrates at an *harmonic center* of Ω, that is, a critical point of the Robin's function $H(x, x)$, where $H(x, y)$ is the regular part of the Green function $G(x, y)$ of $-\Delta$ on Ω with Dirichlet boundary conditions. Namely

$$G(x, y) := \frac{\gamma_N}{|x-y|^{N-2}} - H(x, y) \quad \text{and} \quad \gamma_N = \frac{1}{(N-2)|\partial B|}, \tag{4}$$

where $|\partial B|$ denotes the surface area of the unit sphere in $\mathbb{R}^N$.

The following result concerns the existence of single and multiple peak solutions and is quite standard; see Section 5.

Theorem 1.1. *Assume (2) and k is a positive integer. Let*

$$Z(\boldsymbol{x}) := \sum_{i=1}^{k} H(x_i, x_i) - \sum_{\substack{i,j=1 \\ i\neq j}}^{k} G(x_i, x_j) \quad \textit{where } \boldsymbol{x} := (x_1, \ldots, x_k) \in \Omega^k. \tag{5}$$

Assume ξ_0 is a C^1-stable critical point of the function Z.

Then, for ϵ small enough, there is a solution of (1) with exactly k sharp peaks $\boldsymbol{x}_{1\epsilon}, \ldots, \boldsymbol{x}_{k\epsilon}$ such that $\boldsymbol{x}_\epsilon := (\boldsymbol{x}_{1\epsilon}, \ldots, \boldsymbol{x}_{k\epsilon}) \to \boldsymbol{x}_0$ as $\epsilon \to 0$.

The function Z plays the same role in constructing single and multiple peak solutions for problems defined on two-dimensional domains; see [Esposito et al. 2005; del Pino et al. 2005; Esposito et al. 2006].

In this paper, we prove the existence of multiple positive solutions in the case of small diffusion provided the domain Ω is not contractible for problem (1) with (2) or (3). Our main contribution is a new topological result on fat diagonals, which seems to be of independent interest. We then use it to obtain results under weaker topological assumptions than before. In fact, our topological assumption on the domain is the natural one.

Theorem 1.2. *Assume (2), k is a positive integer and Ω is not contractible. Then, for ϵ small enough, there is a solution of (1) with exactly k sharp peaks.*

Theorem 1.3. *Assume (3), k is a positive integer and Ω is not contractible. Then, for ϵ small enough, there is a solution of (1) with exactly k sharp peaks.*

The proof of the two theorems are similar but there are noticeable differences because the corresponding reduced energies are quite different. Even the location of the peak solutions is different.

If Ω is a ball, or more generally a GNN domain in the sense of Gidas, Ni and Nirenberg [Gidas et al. 1979], then their theorem implies that a positive solution has only one sharp peak and is unique. Thus the behavior quite different. In addition if $p \geq (N+2)/(N-2)$ and $1 < q < (N+2)/(N-2)$ and Ω is star shaped, the Pohozaev identity implies that there is no positive solution at all, while if $p = 1$ and $1 < q < (N+2)/(N-2)$ the positive solution is unique but does not have a sharp peak.

Theorem 1.2 was proved in [Dancer and Yan 1999a] under the stronger assumption that Ω has nontrivial reduced homology. We feel our techniques have other uses. In particular, we suspect Theorem 1.3 holds for $1 < q < p < (N+2)/(N-2)$. The main difficulty is probably to prove an analogue of Lemma 4.1. We would expect the multipeak locations to be different.

The homology version of Theorem 1.3 could also be proved by using the techniques in [Clapp et al. 2008], but they did not seem sufficient to prove the full Theorem 1.3.

Finally, if one is willing to assume the topological result in Section 1, one can read the rest of the paper without needing much topology.

2. A topological result

The result proved here may be known, but we could not find a reference. It does seem likely to have other uses.

If Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary and k is an integer with $k \geq 2$, define the *fat diagonal* $\mathfrak{F}_k$ to be

$$\mathfrak{F}_k := \{(x_1, \ldots, x_k) \in \Omega^k : x_i = x_j \text{ for some } i \neq j\}.$$

Theorem 2.1. *If Ω is not contractible, then $\mathfrak{F}_k$ is not a strong deformation retract of Ω^k.*

Lemma 2.2 (Van Kampen's theorem [Whitehead 1978, p. 94]). *Assume that X_1 and X_2 are topological spaces and let $X_0 := X_1 \cap X_2$. Suppose that*

(i) *(X_i, X_0) for $i = 1, 2$ are neighborhood deformation retract (NDR) pairs in the sense of* [Spanier 1966] *and*

(ii) *X_1, X_2 and X_0 are connected and the natural induced inclusions of $\Pi_1(X_0)$ into $\Pi_1(X_i)$ for $i = 1, 2$ are one-to-one and not onto.*

Then

$$\Pi_1(X_1 \cup X_2) = \Pi_1(X_1) *_{\Pi_1(X_0)} \Pi_1(X_2),$$

where the group on the right hand side is the amalgamated free product defined in [Cohen 1989, Section 1.4].

Remark 2.3. In particular, it follows that $\Pi_1(X_1 \cup X_2)$ is infinite if $\Pi_1(X_1)$ and $\Pi_1(X_2)$ are both nontrivial. For example, see [Cohen 1989, p. 28].

Remark 2.4. Elements in $\Pi_1(X_1 \cup X_2)$ almost never commute except for trivial reasons. Thus follows from the unique representation in [Cohen 1989, p. 28]. For example, if a_1 is in $\Pi_1(X_1)$ but not in $\Pi_1(X_0)$ and a_2 is $\Pi_1(X_2)$ but not in $\Pi_1(X_0)$, then the induced elements of $\Pi_1(X_1 \cup X_2)$ do not commute. Moreover, for similar reasons if $a_1, b_1 \in \Pi_1(X_1)$ do not commute, they will not commute in $\Pi_1(X_1 \cup X_2)$, since the natural inclusion of $\Pi_1(X_1)$ into the amalgamated free product is one-to-one; see [Cohen 1989, p. 27].

Proof of Theorem 2.1. If $\widetilde{H}^i(\Omega, \mathbb{Z})$ is nontrivial for some i, this is proved in the appendix of [Dancer and Yan 1999a]. Here $\widetilde{H}^i_*$ is the reduced cohomology. Note that this covers the case $N \leq 3$, because it is well known that in these dimensions Ω is contractible if it is acyclic.

Thus we may assume Ω is acyclic. Hence if Ω is not contractible, its fundamental group $\Pi_1(\Omega)$ is nontrivial. This follows from [Spanier 1966, Corollary 7.6.24], since it is well known that such an Ω has the homotopy type of a CW complex. Now if $\mathfrak{F}_k$ is a strong deformation retract of Ω^k, the natural inclusion i of $\Pi_1(\mathfrak{F}_k)$ into $\Pi_1(\Omega^k) = \Pi_1(\Omega)^k$ must be an isomorphism of groups. We prove this is impossible essentially by showing that $\Pi_1(\mathfrak{F}_k)$ is much less commutative than $\Pi_1(\Omega)^k$.

First assume $k = 2$. Then $\mathfrak{F}_k$ is simply $\{(x, x) : x \in \Omega\} \sim \Omega$ and the inclusion map induces the natural diagonal map if $\Pi_1(\Omega)$ onto the diagonal of $\Pi_1(\Omega)^2$. This is clearly not onto if $\Pi_1(\Omega)$ is nontrivial. This proves the case $k = 2$.

Now assume $k \geq 3$. First note that $\Pi_1(\mathfrak{F}_k)$ must be infinite (and thus that there is an isomorphism between $\Pi_1(\mathfrak{F}_k)$ and $\Pi_1(\Omega)^k$ implies that $\Pi_1(\Omega)$ is also infinite). This is an exercise in successively applying Van Kampen's theorem to $\mathfrak{F}_k = \bigcup_{1 \leq i < j \leq k} \mathfrak{F}_{kij}$ where $\mathfrak{F}_{kij} := \{(x_1, \dots, x_k) \in \Omega^k : x_i = x_j\}$, which is homeomorphic to Ω^{k-1} and hence has nontrivial fundamental group isomorphic to $\Pi_1(\Omega)^{k-1}$.

Choose distinct elements $g_1, \dots, g_{k-1}$ of $\Pi_1(\Omega)$, where g_1 is the identity e. (This is where we use that $\Pi_1(\Omega)$ is infinite.) We consider the element $a_1 := (g_2, g_2, g_1, \dots, g_{k-1})$ of $\Pi_1(\mathfrak{F}_{k12})$. (Remember $\Pi_1(\mathfrak{F}_{k12}) = \Pi_1(\Omega)^{k-1}$.) This is not in $\Pi_1(\mathfrak{F}_{kij})$ for $i > 1$ or $j > 2$. Similarly, $a_2 := (g_2, g_1, g_2, \dots, g_{k-1})$ is in $\Pi_1(\mathfrak{F}_{k13})$, but not in the other $\Pi_1(\mathfrak{F}_{kij})$. Hence, by our comments above, a_1 and a_2 will not commute in $\Pi_1(\mathfrak{F}_{k12} \cup \mathfrak{F}_{k13})$, which is the amalgamated group $\mathfrak{F}_{k12} *_T \mathfrak{F}_{k13}$ (Van Kampen's theorem) where T is $\Pi_1(\mathfrak{F}_{k12} \cap \mathfrak{F}_{k13})$, which is isomorphic to $\Pi_1(\Omega)^{k-2}$ since

$$\mathfrak{F}_{k12} \cap \mathfrak{F}_{k13} = \{(x_1, \ldots, x_k) \in \Omega^k : x_1 = x_2 = x_3\}.$$

Continuing with many amalgamations, we eventually find that a_1 and a_2 do not commute in $\Pi_1(\mathfrak{F}_k)$. On the other hand they obviously commute in $\Pi_1(\Omega^k) = \Pi_1(\Omega)^k$ since each component commutes in $\Pi_1(\Omega)$. Hence the natural inclusion of $\mathfrak{F}_k$ into Ω^k cannot induce an isomorphism of fundamental groups (since otherwise commutativity would be preserved by the isomorphism). □

Remark 2.5. The proof could be simplified a little by using the idea of a graph of groups as in [Dicks and Dunwoody 1989]. We choose the vertices to correspond to group G^{k-1} and the edges to groups G^{k-2}, where $G = \Pi_1(\Omega)$.

3. Proof of the main results

Proof of Theorem 1.2. This follows immediately by combining Theorem 2.1 with the proof of [Dancer and Yan 1999a, Theorem 1.1], a reference we'll abbreviate as [DY] here. We explain this in a little more detail especially since the ideas occur in the proof of Theorem 1.3.

First by the mountain pass theorem, there is always a solution with one sharp peak. If $k > 1$ and there is no positive solution with k sharp peaks for some small ϵ, the proof of [DY, Theorem 1.1] produces a strong deformation retract of Ω^k minus a neighborhood of $(\partial\Omega \cup \mathfrak{F}_k)$ into a small neighborhood of the fat diagonal $\mathfrak{F}_k$ in Ω^k. We explain this key part a little further below. Since, as in [DY], we can easily deform Ω^k into a suitable compact subset and since we deform a neighborhood of the fat diagonal $\mathfrak{F}_k$ into $\mathfrak{F}_k$, we obtain the required deformation. This contradicts Theorem 2.1. We explain one step above more carefully. First note, as in [DY], that we can strongly deformation retract Ω^k into C^k, where C is a compact subset of Ω, so that the fat diagonal is mapped into itself. Moreover, as in [DY], we also find that we can strongly deformation retract a neighborhood of the fat diagonal in Ω^k into the fat diagonal. Thus it suffices to strongly deformation retract C^k into a neighborhood of the fat diagonal $\mathfrak{F}_k$ in Ω^k and in fact we do not need to define the map very near the fat diagonal (since we can make it the identity there).

We choose R large but fixed and $\mu > 0$ small and fixed. Then as in [DY], let

$$W = \{\boldsymbol{x} := (x_1, \ldots, x_k) \in \Omega^k : d(x_i, \partial\Omega) \geq \mu,\ |x_i - x_j| \geq R\epsilon \text{ when } i \neq j\}.$$

We can easily use the implicit function theorem on this set to reduce (1) to a problem $\nabla K_\epsilon(\boldsymbol{x}) = 0$, where K_ϵ is a smooth function on $W \times [0, \epsilon]$ and the gradient is with respect to $\boldsymbol{x}$. We do this by looking for solutions of (1) of the form $\sum_{i=1}^{k} \phi(\epsilon^{-1}(x - x_i)) + \zeta(\boldsymbol{x}, \epsilon)$, where ζ is small, $\boldsymbol{x} \in W$ and ϕ is the unique positive decaying solution of $-\Delta u = f(u)$ on $\mathbb{R}^N$. Here we have very slightly simplified the argument in [DY]. K_ϵ is called the reduced energy. We then use the differential equation $\dot{\boldsymbol{x}} = -\nabla K_\epsilon(\boldsymbol{x})$ on Ω^k to deform $\{\boldsymbol{x} \in W : K_\epsilon(\boldsymbol{x}) \leq c_1\}$ into $\{\boldsymbol{x} \in \Omega^k : K_\epsilon(\boldsymbol{x}) \leq c_2\}$. Note that there are no k peak solutions implies that K_ϵ has no critical points in Ω^k. Here we choose $c_2 = \epsilon^N(kA - \tau\phi(\epsilon^{-1}d))$ and $c_1 = \epsilon^N(kA + \sigma)$ for suitable chosen small positive d, σ, τ, where d depends on ϵ. We proved two technical results in [DY] that use the asymptotics of K_ϵ. Finally, if $\boldsymbol{x} \in W$ and $|x_i - x_j| = r_\epsilon$ for some i, j, then $K_\epsilon(\boldsymbol{x}) \geq c_2$. Here r_ϵ is a fixed constant. Secondly the flow moves point away from $\partial\Omega^k$ if $K_\epsilon(\boldsymbol{x})$ and $\boldsymbol{x} \in W$. This ensures that the flow does not leave Ω^k and the points do not get close together. This ensures we can deform K_{c_1} into K_{c_2}. As there, we can also arrange so that $K_\epsilon(\boldsymbol{x}) \leq c_1$ on W and thus $K^k \subset K_{c_1} \equiv \{x \in \Omega^k : K_\epsilon(\boldsymbol{x}) \leq c_1\}$ and K_{c_2} is contained in a small neighborhood of the fat diagonal. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. We use exactly the same strategy except the energy levels have to be chosen a little differently and the asymptotics are a little different. If $\delta_1 > 0$ is large, let $c_1 = k\epsilon^N(a + \epsilon^{N-2}\delta_1)$ and $c_2 = k\epsilon^N(a - \epsilon^{N-2}\delta_1)$, where a is defined in (13). We prove that there is a k peak solution whose energy is between c_1 and c_2 by showing that the reduced energy K_ϵ defined in (12) has a critical point in $K_{\epsilon,c_1} \setminus K_{\epsilon,c_2}$. By [Dancer 1995], it suffices to assume that $k > 1$. Let

$$\widetilde{W} = \{\boldsymbol{x} := (x_1, \ldots, x_k) \in \Omega^k : d(x_i, \partial\Omega) \geq \mu,\ |x_i - x_j| \geq \tau \text{ when } i \neq j\}. \tag{6}$$

Here μ and τ are small and positive.

Once again, we use the semiflow of $\dot{\boldsymbol{x}} = -\nabla K_\epsilon(\boldsymbol{x})$ on $K_{\epsilon,c_1} \setminus K_{\epsilon,c_2}$. We prove that if start the flow on $\widetilde{W}$ with $K_\epsilon(x) \leq c_1$ then the flow does not leave Ω^k until $K_\epsilon(x) \leq c_2$. We in fact prove that if two of the x_i get close together and are not close to the boundary, then $\epsilon^{2-2N}(K_\epsilon(x) - kc\epsilon^N)$ is large negative. This means that we only have to worry about some of the x_i getting close to the boundary. With a time rescale, our flow is

$$\dot{\boldsymbol{x}} = -\nabla Z(\boldsymbol{x}) + o(1),$$

unless x gets close to $\partial(\Omega^k)$ or the fat diagonal. Hence if we prove in Lemma 4.1 below that a solution of $\dot{\boldsymbol{x}} = -\nabla Z(\boldsymbol{x})$ starting from a point $\boldsymbol{x}_0 \in \Omega^k$ away from the boundary does not leave Ω^k through the "outer" boundary unless $Z(\xi)$ is large negative, it follows by continuous dependance that the perturbed flow cannot leave Ω^k through the outer boundary and thus can only leave by getting close to the inner boundary where $Z(\boldsymbol{x})$ is large negative and hence $K_\epsilon(\boldsymbol{x}) < c_2$. It is easy to make this uniform on compact subsets of Ω^k.

Hence, provided we prove Lemmas 2.2 and 5.1 we complete the proof in exactly the same way as in the proof of Theorem 1.2. Note that Z is uniformly bounded above on $\widetilde{W}$ and it follows that $\epsilon^{-N}K_\epsilon$ is uniformly bounded above on $\widetilde{W}$. □

4. A technical lemma

Lemma 4.1. *If $\boldsymbol{x}_0 \in \Omega^k$ and $\boldsymbol{x}(t)$ is a solution of $\dot{\boldsymbol{x}} = -\nabla Z(\boldsymbol{x})$ (see (5)) for $0 \le t < T^+(\boldsymbol{x})$ such that $\boldsymbol{x}(t) \in \Omega^k \setminus \mathfrak{F}_k$ for $0 \le t < T^+(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\sup_{0 \le t < T^+(\boldsymbol{x})} Z(\boldsymbol{x}(t)) \le c$, then $\boldsymbol{x}(t)$ stays bounded away from the "outer" boundary of $\Omega^k \setminus \mathfrak{F}_k$ until Z is large negative. Moreover, if $\boldsymbol{x}(t)$ gets close to the "inner" boundary, $Z(\boldsymbol{x}(t))$ is large negative.*

The second part of Lemma 4.1 follows immediately from the first part and Remark 5.2. We in fact prove that if $\boldsymbol{x}(t)$ gets close to the boundary and $Z(\boldsymbol{x}(t))$ is not large negative, then $\boldsymbol{x}(t)$ moves inside in the normal direction.

We now sketch the proof of Lemma 4.1. This is a higher-dimensional analogue of results in [Bartsch et al. 2010] for special values of parameters. We follow the proof there, mainly pointing out the differences, though we do modify it a little to simplify it in our particular case.

First of all, we need an accurate estimate of $H(x, y)$ when the points x and y are close to the boundary. Let us introduce some notation. If $\tau > 0$ is small enough, we define $\Omega_\tau := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \tau\}$. Let $p : \Omega_\tau \to \partial\Omega$ be the projection onto the boundary, that is, $\operatorname{dist}(x, \partial\Omega) = |p(x) - x|$ and let ν_x denote the inward normal at the point $p(x) \in \partial\Omega$. We define $\bar{x} := p(x) - d(x)\nu_x$. With (4) in mind, set

$$\psi(x, y) := H(x, y) - \frac{\gamma_N}{|\bar{x} - y|^{N-2}}.$$

Lemma 4.2. *For any $\tau > 0$ there exists $c_1 > 0$ such that*

$$|\psi(x, y)| + |\nabla\psi(x, y)| \le c_1 \operatorname{dist}(x, \partial\Omega)^{2-N} \quad \text{for } x, y \in \Omega_\tau.$$

Proof. This is an easy modification of the proof of [Bartsch et al. 2010, Lemma 2.2], which is the corresponding result for $N = 2$. □

Remark 4.3. This gives a good first term asymptotic expansion for $\partial H(x, x)/\nu_x$ near $\partial\Omega$. This seems likely to have other uses. The motivation for this result is $\psi \equiv 0$ in the half space case and we expect that locally this is the main approximation term. Note that we can integrate the gradient estimate to obtain an improved estimate for $\psi(x, x)$. This is implicit in the work [Bandle and Flucher 1996].

Proof of Lemma 4.1. As we commented earlier, it suffices to prove the first part. It obvious suffices to prove that if $\boldsymbol{x} = (x_1, \ldots, x_k)$ is near the outer boundary of Ω^k and j gives the minimal value in i of $\operatorname{dist}(x_i, \partial\Omega)$, then $\partial Z(\boldsymbol{x})/\partial\nu^j$, where ν^j is the inward normal derivative in the j-th variable or $Z(\boldsymbol{x})$ is large negative.

We first note if x_j is near $\partial\Omega$, Lemma 4.2 implies that

$$-\mu_1 \operatorname{dist}(x_j, \partial\Omega)^{1-N} \le \frac{\partial H(x, x)}{\partial\nu^j} \le -\mu_2 \operatorname{dist}(x_j, \partial\Omega)^{1-N}$$

near $\partial\Omega$, where $\mu_1, \mu_2 > 0$. This is large negative, so we only have to prove that this dominates terms $\partial G(x_i, x_h)/\partial \nu^j$. This can only be nonzero if i or k is j and then is bounded unless x_i and x_h are close. Thus we only need consider the case when x_h is close to x_i. Since x_i is the point closest to $\partial\Omega$, we have $\langle x_h - x_i, \nu\rangle \le -\delta$ and hence we see that $-\langle\nabla(|x_h - x_i|^{2-N}), \nu\rangle$ is negative or of smaller order than $\partial H(x, x)/\partial\nu$ at least if $|x_h - x_i| \ge \mu\,\mathrm{dist}(x_i, \partial\Omega)$. We still have to estimate $\partial H(x_i, x_h)/\partial\nu$, which to highest order is the inward normal derivative of $|x_h - x_i|^{2-N}$, which is easily seen to be negative. Thus we have the required result unless $|x_h - x_i| = o(\mathrm{dist}(x_i, \partial\Omega))$. We prove in this case that $Z(\boldsymbol{x})$ is large negative. This follows because since $G(x, y) \ge 0$, we have

$$Z(\boldsymbol{x}) \le kH(x_i, x_i) - |x_h - x_i|^{2-N} + H(x_i, x_h),$$

which is large negative since it will be dominated by the middle term. □

5. The reduced energy

We can use a well-known procedure to reduce problem (1) to a finite-dimensional one. It is very similar to arguments in [Bahri et al. 1995; Bartsch et al. 2006; del Pino et al. 2003] for the critical exponent. The solution to the limit problem (7) has the same decay rate as in the critical exponent and hence the arguments are almost identical. In particular, we can follow the argument in [Bartsch et al. 2006]. We sketch the ideas of the proof to obtain the reduced energy estimate.

Assume

$$f(s) = (s^+)^p - (s^+)^q \quad \text{where } \frac{N+2}{N-2} > p > q > \frac{N}{N-2}.$$

We consider the unique radial solution to the limit problem

$$\begin{cases} -\Delta U = f(U) & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U \to 0 & \text{as } |x| \to \infty, \\ U \in C^2(\mathbb{R}^N). \end{cases} \tag{7}$$

For any $x \in \mathbb{R}^N$ and for any $\epsilon > 0$, set

$$U_{\epsilon,x}(y) := U\Big(\frac{y - x}{\epsilon}\Big) \quad \text{for } y \in \mathbb{R}^N.$$

It is clear that $U_{\epsilon,x}$ solves $-\epsilon^2\Delta U_{\epsilon,x} = f(U_{\epsilon,x})$ in $\mathbb{R}^N$. We introduce the projection $PU_{\epsilon,x}$ of U_ϵ in $\mathrm{H}^1_0(\Omega)$ as the solution to the problem

$$\begin{cases} \Delta PU_{\epsilon,x} = \Delta U_{\epsilon,x} & \text{in } \Omega, \\ \Delta PU_{\epsilon,x} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to prove that (see (4))

$$PU_{\epsilon,x}(y) = U_{\epsilon,x}(y) - (\omega_q/\gamma_N)\epsilon^{N-2}H(y,x) + O(\epsilon^N) \tag{8}$$

C^1-uniformly with respect to x in compact sets of Ω. Here

$$\omega_q := \lim_{|y|\to\infty} U(y)|y|^{N-2}. \tag{9}$$

We look for a solution to (1) as

$$u_\epsilon(y) := \sum_{i=1}^{k} PU_{\epsilon,x_i}(y) + \phi_\epsilon(y), \tag{10}$$

where $\boldsymbol{x} := (x_1, \dots, x_k) \in \Omega^k \setminus \mathfrak{F}_k$ and the remainder term ϕ_ϵ belongs to a suitable space. We perform a well-known Liapunov–Schmidt reduction and we reduce the problem to a finite-dimensional one. In particular, we find that u_ϵ as in (10) is a solution to (1), namely a critical point of the functional $J_\epsilon : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \int_\Omega \big(\tfrac{1}{2}\epsilon^2|\nabla u|^2 + F(u)\big)dx, \tag{11}$$

with $F(s) := \int_0^s f(\sigma)d\sigma$, if and only if $\boldsymbol{x}$ is a critical point of the reduced energy

$$K_\epsilon(\boldsymbol{x}) := J_\epsilon\Big(\sum_{i=1}^{k} PU_{\epsilon,x_i} + \phi_\epsilon\Big) \quad \text{for } \boldsymbol{x} \in \Omega^k \setminus \mathfrak{F}_k. \tag{12}$$

A standard argument allows to compute the expansion of the reduced energy. For one peak this is proved in [Dancer and Santra 2010, Lemmas 5.1 and 5.3].

Lemma 5.1. *We have*

$$K_\epsilon(\boldsymbol{x}) = kc\epsilon^N + b\epsilon^{2N-2}Z(\boldsymbol{x}) + o(\epsilon^{2N-2}),$$

C^1-uniformly with respect to x in compact subsets of $\Omega^k \setminus \mathfrak{F}_k$. Here Z is defined in (5),

$$c := \int_{\mathbb{R}^N} \big(\tfrac{1}{2}f(U)U - F(U)\big)dy > 0 \tag{13}$$

and (see (4) and (9))

$$b := \frac{\omega_q}{2\gamma_N}\int_{\mathbb{R}^N} f(U)dy > 0. \tag{14}$$

are positive constants.

Remark 5.2. By Lemma 5.1 it follows immediately from the formula for $Z(\boldsymbol{x})$ that $\epsilon^{-N} K_\epsilon(\boldsymbol{x})$ is large negative if no x_i is close to $\partial\Omega$, $|x_i - x_j| \geq \tau$ whenever $i \neq j$, and $|x_i - x_j| = \tau$ for some $i \neq j$ (provided τ is fixed and small).

Similarly $Z(\boldsymbol{x})$ can only be large negative on $\widetilde{W}$ (see (6)) close to the fat diagonal and hence $K_\epsilon(\boldsymbol{x}) \leq c_2$ implies $\boldsymbol{x} \in \widetilde{W}$ is close to the fat diagonal.

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