

# *Pacific Journal of Mathematics*

**ADDENDUM TO THE ARTICLE  
SUPERCONNECTIONS AND PARALLEL TRANSPORT**

FLORIN DUMITRESCU

## ADDENDUM TO THE ARTICLE SUPERCONNECTIONS AND PARALLEL TRANSPORT

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**We give here an alternate construction for the previously studied parallel transport associated with a superconnection, having the advantage that it is independent of the way the superconnection splits as a connection part plus a bundle-endomorphism valued form.**

Consider, as in Section 4 of [Dumitrescu 2008] (the paper in the title), a superconnection  $\mathbb{A}$  in the sense of Quillen (see [Quillen 1985] and [Berline et al. 1992]) on a  $\mathbb{Z}/2$ -graded vector bundle  $E$  over a manifold  $M$ . That is,

$$\mathbb{A} : \Omega^*(M, E) \rightarrow \Omega^*(M, E)$$

is an odd first-order differential operator satisfying the Leibniz rule

$$\mathbb{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathbb{A}(s),$$

where  $\omega \in \Omega^*(M)$  is a differential form on  $M$  and  $s \in \Gamma(M; E)$  is an arbitrary section of the bundle  $E$  over  $M$ . For such a superconnection we defined in [Dumitrescu 2008] a notion of parallel transport along (families of) superpaths  $c : S \times \mathbb{R}^{1|1} \rightarrow M$  that is compatible under glueing of superpaths. Let us briefly recall this construction. First, we write  $\mathbb{A} = \mathbb{A}_1 + A$ , where  $\mathbb{A}_1 = \nabla$  is the connection part of the superconnection  $\mathbb{A}$  and  $A \in \Omega^*(M, \text{End } E)^{\text{odd}}$  is the linear part of the superconnection. For an arbitrary superpath  $c$  in  $M$ , consider the diagram

$$\begin{array}{ccccc}
 E & & & & c^*E \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 & & \pi^*E & & \\
 \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
 M & \xleftarrow{c} & & & S \times \mathbb{R}^{1|1} \\
 & \swarrow & \downarrow & \searrow & \\
 & & \Pi TM & & \\
 & \swarrow & \downarrow & \searrow & \\
 & & \tilde{c} & & 
 \end{array}$$

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where  $\tilde{c}$  is a canonical lift (defined in [ibid., Section 4.1]) of the path  $c$  to  $\Pi TM$ , the “odd tangent bundle” of  $M$ . Parallel transport along  $c$  was defined by *parallel sections*  $\psi \in \Gamma(c^*E)$  along  $c$  that are solutions to the differential equation

$$(c^*\nabla)_D\psi - (\tilde{c}^*A)\psi = 0.$$

Here  $D = \partial_\theta + \theta\partial_t$  denotes the standard (right-invariant) vector field on  $\mathbb{R}^{1|1}$  [ibid., Section 2.4].

To describe our alternate construction, we first write  $\mathbb{A} = \mathbb{A}_0 + \bar{\mathbb{A}}$ , where  $\mathbb{A}_0$  denotes the zero part of the superconnection and  $\bar{\mathbb{A}}$  the remaining part. Then we define a connection  $\bar{\nabla}$  on the bundle  $\pi^*E$  over  $\Pi TM$  as follows. For pullback sections  $s \in \Gamma(M; E)$  we set

$$\bar{\nabla}_{\mathcal{L}_X}s := \iota_X\bar{\mathbb{A}}s, \quad \bar{\nabla}_{\iota_X}s := 0.$$

Here, for a vector field  $X$  on the manifold  $M$ ,  $\mathcal{L}_X$  and  $\iota_X$  denote the Lie derivative respectively contraction in the  $X$ -direction acting as even respectively odd derivations on  $\Omega^*(M) = \mathcal{C}^\infty(\Pi TM)$ , i.e. as vector fields on  $\Pi TM$ . For arbitrary sections of  $\pi^*E$

$$\Gamma(\Pi TM, \pi^*E) = \Omega^*(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, E)$$

we extend the connection  $\bar{\nabla}$  by the Leibniz rule

$$\bar{\nabla}_{\mathcal{L}_X}(\omega \otimes s) = \mathcal{L}_X\omega \otimes s \pm \omega \otimes \iota_X\bar{\mathbb{A}}s, \quad \bar{\nabla}_{\iota_X}(\omega \otimes s) = \iota_X\omega \otimes s,$$

whenever  $\omega \in \Omega^*(M)$  and  $s \in \Gamma(M; E)$ . These relations are enough to define a connection  $\bar{\nabla}$  on the bundle  $\pi^*E$  over  $\Pi TM$  since the algebra of vector fields on  $\Pi TM$  is generated over  $\mathcal{C}^\infty(\Pi TM)$  by vector fields of the type  $\mathcal{L}_X$  and  $\iota_X$ , where  $X$  denotes an arbitrary vector field on  $M$ , i.e.

$$\text{Vect}(\Pi TM) = \mathcal{C}^\infty(\Pi TM) \langle \mathcal{L}_X, \iota_X \mid X \in \text{Vect}(M) \rangle.$$

Parallel transport along a superpath  $c : S \times \mathbb{R}^{1|1} \rightarrow M$  is then defined by *parallel sections*  $\psi \in \Gamma(c^*E)$  along  $c$  which are solutions to the following differential equation

$$(\tilde{c}^*\bar{\nabla})_D\psi - (c^*\mathbb{A}_0)\psi = 0$$

where the lift  $\tilde{c}$  of  $c$  is defined as before. As in our previous construction, the parallel transport is well-defined [ibid., Proposition 4.2] by this “half-order” differential equation. Moreover, it is compatible under glueing of superpaths; that is, it satisfies properties (i) and (ii) in [ibid., Theorem 4.3]. The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as the  $\bar{\mathbb{A}}$  part of the superconnection  $\mathbb{A}$  which gives rise to the connection  $\bar{\nabla}$  is invariant under such splittings.

Denote by  $\delta$  the de Rham differential on  $\Pi TM$ . If  $\omega$  is a function on  $\Pi TM$ , the 1-form  $\delta\omega$  on  $\Pi TM$  evaluated on the standard odd vector field  $d$  on  $\Pi TM$  gives

$$(\delta\omega)(d) = d\omega,$$

the exterior derivative of  $\omega$ , understood as a function on  $\Pi TM$ . Therefore we have

$$\bar{\nabla}_d s = \bar{\mathbb{A}}s,$$

for any  $s$  a section of the bundle  $E$  over  $M$ . We remark that the connection  $\bar{\nabla}$  is torsion free in the odd directions, i.e.,

$$[\bar{\nabla}_{\iota_X}, \bar{\nabla}_{\iota_Y}] = \bar{\nabla}_{[\iota_X, \iota_Y]}$$

(and both sides are of course equal to zero). Here  $X$  and  $Y$  denote arbitrary vector fields on the manifold  $M$ .

**Remarks.** (1) The two constructions of parallel transport associated to a superconnection presented above coincide when the superconnection on the bundle  $E$  over  $M$  reduces to an ordinary connection (has no linear part). When the manifold  $M$  is just a point, a graded vector bundle with superconnection reduces to a  $\mathbb{Z}/2$ -vector space  $V$  together with an odd endomorphism  $A$  ( $= \mathbb{A}_0$ ) of  $V$ . In this situation the two constructions of parallel transport also coincide, giving rise to the supergroup homomorphism of [Stolz and Teichner 2004, Example 3.2.9]:

$$\mathbb{R}^{1|1} \ni (t, \theta) \longmapsto e^{-tA^2 + \theta A} \in GL(V),$$

encoding the solutions to the half-order differential equation  $D\psi = A\psi$ .

(2) The superconnection can be *recovered* from its associated parallel transport, as was the case with our previous construction. First, one recovers the zero part  $\mathbb{A}_0$  of the superconnection  $\mathbb{A}$  by considering constant superpaths in  $M$ . One then recovers  $\bar{\mathbb{A}}$  by looking at parallel transport along the superpath given by

$$\mathbb{R}^{1|1} \times \Pi TM \rightarrow \mathbb{R}^{0|1} \times \Pi TM \rightarrow M,$$

where the first map is the obvious projection and the second map is the standard superpoint evaluation map. The lift of such a superpath to  $\Pi TM$  is given by the composition

$$\mathbb{R}^{1|1} \times \Pi TM \rightarrow \mathbb{R}^{0|1} \times \Pi TM \rightarrow \Pi TM,$$

where the first map is the projection as before and the second map expresses the flow of the vector field  $d$  on  $\Pi TM$  (since  $d^2 = 0$ , the flow of  $d$  is given by an  $\mathbb{R}^{0|1}$ -action). Given that the push-forward of the vector field  $D$  along the projection map  $\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}$  is the vector field  $d$  on  $\mathbb{R}^{0|1}$  and that  $\bar{\nabla}_d s = \bar{\mathbb{A}}s$ , the parallel transport equation recovers  $\bar{\mathbb{A}}$ . Compare with Section 4.4 of [Dumitrescu 2008],

where we first obtained the connection part by taking an inverse adiabatic limit and afterwards the *linear* part of the superconnection.

### Acknowledgements

The alternate construction presented here is a mere continuation of an idea of Stephan Stolz to interpret a Quillen superconnection on a bundle  $E$  over  $M$  as a connection on the pullback bundle  $\pi^*E$  over  $\Pi TM$ . I would like to thank Peter Teichner for suggesting I write up this addendum.

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