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We give here an alternate construction for the previously studied parallel transport associated with a superconnection, having the advantage that it is independent of the way the superconnection splits as a connection part plus a bundle-endomorphism valued form.

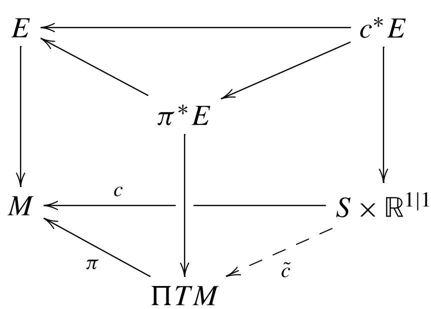
Consider, as in Section 4 of [Dumitrescu 2008] (the paper in the title), a superconnection $\mathbb{A}$ in the sense of Quillen (see [Quillen 1985] and [Berline et al. 1992]) on a $\mathbb{Z}/2$-graded vector bundle E over a *manifold* M. That is,

$$\mathbb{A} : \Omega^*(M, E) \to \Omega^*(M, E)$$

is an odd first-order differential operator satisfying the Leibniz rule

$$\mathbb{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathbb{A}(s),$$

where $\omega \in \Omega^*(M)$ is a differential form on M and $s \in \Gamma(M; E)$ is an arbitrary section of the bundle E over M. For such a superconnection we defined in [Dumitrescu 2008] a notion of parallel transport along (families of) superpaths $c : S \times \mathbb{R}^{1|1} \to M$ that is compatible under glueing of superpaths. Let us briefly recall this construction. First, we write $\mathbb{A} = \mathbb{A}_1 + A$, where $\mathbb{A}_1 = \nabla$ is the connection part of the superconnection $\mathbb{A}$ and $A \in \Omega^*(M, End\ E)^{\text{odd}}$ is the linear part of the superconnection. For an arbitrary superpath c in M, consider the diagram



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where $\tilde{c}$ is a canonical lift (defined in [ibid., Section 4.1]) of the path c to ΠTM, the "odd tangent bundle" of M. Parallel transport along c was defined by *parallel* sections $\psi \in \Gamma(c^*E)$ along c that are solutions to the differential equation

$$(c^*\nabla)_D\psi - (\tilde{c}^*A)\psi = 0.$$

Here $D = \partial_\theta + \theta\partial_t$ denotes the standard (right-invariant) vector field on $\mathbb{R}^{1|1}$ [ibid., Section 2.4].

To describe our alternate construction, we first write $\mathbb{A} = \mathbb{A}_0 + \bar{\mathbb{A}}$, where $\mathbb{A}_0$ denotes the zero part of the superconnection and $\bar{\mathbb{A}}$ the remaining part. Then we define a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM as follows. For pullback sections $s \in \Gamma(M; E)$ we set

$$\bar{\nabla}_{\mathscr{L}_X} s := \iota_X \bar{\mathbb{A}} s, \quad \bar{\nabla}_{\iota_X} s := 0.$$

Here, for a vector field X on the manifold M, $\mathscr{L}_X$ and ι_X denote the Lie derivative respectively contraction in the X-direction acting as even respectively odd derivations on $\Omega^*(M) = \mathscr{C}^\infty(\Pi TM)$, i.e. as vector fields on ΠTM. For arbitrary sections of π^*E

$$\Gamma(\Pi TM, \pi^*E) = \Omega^*(M) \otimes_{\mathscr{C}^\infty(M)} \Gamma(M, E)$$

we extend the connection $\bar{\nabla}$ by the Leibniz rule

$$\bar{\nabla}_{\mathscr{L}_X}(\omega \otimes s) = \mathscr{L}_X\omega \otimes s \pm \omega \otimes \iota_X \bar{\mathbb{A}} s, \quad \bar{\nabla}_{\iota_X}(\omega \otimes s) = \iota_X \omega \otimes s,$$

whenever $\omega \in \Omega^*(M)$ and $s \in \Gamma(M; E)$. These relations are enough to define a connection $\bar{\nabla}$ on the bundle π^*E over ΠTM since the algebra of vector fields on ΠTM is generated over $\mathscr{C}^\infty(\Pi TM)$ by vector fields of the type $\mathscr{L}_X$ and ι_X, where X denotes an arbitrary vector field on M, i.e.

$$Vect(\Pi TM) = \mathscr{C}^\infty(\Pi TM)\langle \mathscr{L}_X, \iota_X \mid X \in \mathrm{Vect}(M)\rangle.$$

Parallel transport along a superpath $c : S \times \mathbb{R}^{1|1} \to M$ is then defined by *parallel sections* $\psi \in \Gamma(c^*E)$ along c which are solutions to the following differential equation

$$(\tilde{c}^*\bar{\nabla})_D\psi - (c^*\mathbb{A}_0)\psi = 0$$

where the lift $\tilde{c}$ of c is defined as before. As in our previous construction, the parallel transport is well-defined [ibid., Proposition 4.2] by this "half-order" differential equation. Moreover, it is compatible under glueing of superpaths; that is, it satisfies properties (i) and (ii) in [ibid., Theorem 4.3]. The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as the $\bar{\mathbb{A}}$ part of the superconnection $\mathbb{A}$ which gives rise to the connection $\bar{\nabla}$ is invariant under such splittings.

Denote by δ the de Rham differential on ΠTM. If ω is a function on ΠTM, the 1-form $\delta\omega$ on ΠTM evaluated on the standard odd vector field d on ΠTM gives

$$(\delta\omega)(d) = d\omega,$$

the exterior derivative of ω, understood as a function on ΠTM. Therefore we have

$$\bar{\nabla}_d s = \bar{\mathbb{A}} s,$$

for any s a section of the bundle E over M. We remark that the connection $\bar{\nabla}$ is torsion free in the odd directions, i.e.,

$$[\bar{\nabla}_{\iota_X}, \bar{\nabla}_{\iota_Y}] = \bar{\nabla}_{[\iota_X, \iota_Y]}$$

(and both sides are of course equal to zero). Here X and Y denote arbitrary vector fields on the manifold M.

Remarks. (1) The two constructions of parallel transport associated to a superconnection presented above coincide when the superconnection on the bundle E over M reduces to an ordinary connection (has no linear part). When the manifold M is just a point, a graded vector bundle with superconnection reduces to a $\mathbb{Z}/2$-vector space V together with an odd endomorphism A $(= \mathbb{A}_0)$ of V. In this situation the two constructions of parallel transport also coincide, giving rise to the supergroup homomorphism of [Stolz and Teichner 2004, Example 3.2.9]:

$$\mathbb{R}^{1|1} \ni (t, \theta) \longmapsto e^{-tA^2 + \theta A} \in GL(V),$$

encoding the solutions to the half-order differential equation $D\psi = A\psi$.

(2) The superconnection can be *recovered* from its associated parallel transport, as was the case with our previous construction. First, one recovers the zero part $\mathbb{A}_0$ of the superconnection $\mathbb{A}$ by considering constant superpaths in M. One then recovers $\bar{\mathbb{A}}$ by looking at parallel transport along the superpath given by

$$\mathbb{R}^{1|1} \times \Pi TM \to \mathbb{R}^{0|1} \times \Pi TM \to M,$$

where the first map is the obvious projection and the second map is the standard superpoint evaluation map. The lift of such a superpath to ΠTM is given by the composition

$$\mathbb{R}^{1|1} \times \Pi TM \to \mathbb{R}^{0|1} \times \Pi TM \to \Pi TM,$$

where the first map is the projection as before and the second map expresses the flow of the vector field d on ΠTM (since $d^2 = 0$, the flow of d is given by an $\mathbb{R}^{0|1}$-action). Given that the push-forward of the vector field D along the projection map $\mathbb{R}^{1|1} \to \mathbb{R}^{0|1}$ is the vector field d on $\mathbb{R}^{0|1}$ and that $\bar{\nabla}_d s = \bar{\mathbb{A}} s$, the parallel transport equation recovers $\bar{\mathbb{A}}$. Compare with Section 4.4 of [Dumitrescu 2008], where we first obtained the connection part by taking an inverse adiabatic limit and afterwards the *linear* part of the superconnection.

Acknowledgements

The alternate construction presented here is a mere continuation of an idea of Stephan Stolz to interpret a Quillen superconnection on a bundle E over M as a connection on the pullback bundle π^*E over ΠTM. I would like to thank Peter Teichner for suggesting I write up this addendum.

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