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*To the memory of Lin Chen.*

**We first study the Chern–Simons partition function of orthogonal quantum group invariants and then propose a new orthogonal Labastida–Mariño–Ooguri–Vafa (LMOV) conjecture as well as a degree conjecture for free energy associated to the orthogonal Chern–Simons partition function. We prove the degree conjecture and some interesting cases of the orthogonal LMOV conjecture. In particular, we provide a formula of the colored Kauffman polynomials for torus knots and links, and applied this formula to verify certain cases of the conjecture at roots of unity except 1. We also derive formulas of Lickorish–Millett type for Kauffman polynomials and relate all these to the orthogonal LMOV conjecture.**

## 1. Introduction

**1.1.** ***Overview.*** Jones's seminal papers [1985; 1987] initiated a new era in knot theory. The HOMFLY polynomial [Freyd et al. 1985] and Kauffman [1990] polynomial for links were subsequently discovered. In the 1990s, Witten, Reshetikhin and Turaev constructed the colored version of these invariants, either by path integrals in physics [Witten 1989], or by the representation theory of quantum groups [Reshetikhin and Turaev 1991; 1990]. These works lead to a unified understanding of quantum group invariants of links.

The colored HOMFLY polynomials, which are associated to the special linear quantum groups, have been studied more carefully after physicists proposed a conjectural relationship between Chern–Simons theory and Gromov–Witten invariants. The Mariño–Vafa formula and the topological vertex [Aganagic et al. 2005; Li et al. 2009; 2003; 2007] are examples illustrating this so-called string duality. The Labastida–Mariño–Ooguri–Vafa conjecture [Labastida and Mariño 2002; Labastida et al. 2000; Ooguri and Vafa 2000] gave highly nontrivial relations between colored HOMFLY polynomials. The first such relation is the classical Lichorish–Millett theorem [1987]. The integers coefficients that appear in the

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LMOV conjecture are called the BPS numbers in string theory, and also related to the integrality in the Gopakumar–Vafa conjecture [1999] for Gromov–Witten invariants [Peng 2007]. By using the cabling technique, Xiao-Song Lin and Hao Zheng [2010] obtained a formula for colored HOMFLY polynomials of torus links in terms of Littlewood–Richardson coefficients, and they were able to check certain cases of the LMOV conjecture for a few (small) torus knots and links. The LMOV conjecture was recently proved by Kefeng Liu and Pan Peng [2010], based on the cabling technique and a careful degree analysis of the cut-join equations.

Actually the LMOV conjecture is part of a bigger picture, the large $N$ duality, proposed by 't Hooft [1974] in the 1970s. Large $N$ duality states that there is a duality between Chern–Simons gauge theory of $S^3$ and topological string theory on the resolved conifold.

In mathematics, the LMOV conjecture predicts that the reformulated invariants (some combination) of colored HOMFLY/Kauffman polynomials are in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$, where $q$ is the quantum deformation number. In this way, these reformulated invariants have an expression similar to the original HOMFLY/Kauffman polynomials, which have variables $q-q^{-1}$, $t$ and $t^{-1}$ with integer coefficients.

**1.2.** ***The orthogonal Labastida–Mariño–Ooguri–Vafa conjecture.*** The study of colored Kauffman polynomials is more difficult. For instance, the definition of the Chern–Simons partition function for the orthogonal quantum groups involves representations of the Brauer centralizer algebras, which admit more complicated orthogonal relations; see [Ram 1991; 1995; 1997]. The orthogonal analog of the cut-join equation [Liu et al. 2003; Liu and Peng 2010] can be found in [Chen 2009].

In this paper, we propose a new conjecture, developed in collaboration with Nicolai Reshetikhin, on the reformulated invariants; ours is the orthogonal quantum group analog of the original LMOV conjecture. Let $\mathcal{L}$ be a link with $L$ components and let $Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L},q,t)$ be the orthogonal Chern–Simons partition function defined in Section 4. Expand the free energy

$$F^{\mathrm{SO}}(\mathcal{L},q,t)=\log Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L},q,t)=\sum_{\vec{\mu}\neq\vec{0}} F_{\vec{\mu}}^{\mathrm{SO}}\, pb_{\vec{\mu}}(\vec{z}\vec{z}).$$

Then the reformulated invariants are defined by

$$g_{\vec{\mu}}(q,t)=\sum_{k\mid\vec{\mu}}\frac{\mu(k)}{k}F_{\vec{\mu}/k}^{\mathrm{SO}}(q^k,t^k).$$

**Conjecture 1.1** (orthogonal LMOV).

$$\frac{\mathfrak{z}_{\vec{\mu}}(q-q^{-1})^2\cdot[g_{\vec{\mu}}(q,t)-g_{\vec{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha}-q^{-\mu_i^\alpha})}\in\mathbb{Z}[q-q^{-1}][t,t^{-1}].$$

**Conjecture 1.2** (degree). *Let $q = e^u$ and $\operatorname{val}_u(F^{\mathrm{SO}}_{\vec{\mu}})$ be the valuation of the variable $u$ and $\ell(\vec{\mu})$ be the sum of the lengths of the partition corresponding to each component of the link $\mathscr{L}$. Then*

$$\operatorname{val}_u(F^{\mathrm{SO}}_{\vec{\mu}}) \geq \ell(\vec{\mu}) - 2.$$

This conjecture is a mathematical formulation of a conjecture of Bouchard, Florea and Mariño [Bouchard et al. 2005], and the integer coefficients on the right hand side of the conjecture above are closely related to BPS numbers in string theory [Bouchard et al. 2005]. More recent progress can be found in [Mariño 2010], which is a refined version of [Bouchard et al. 2005]. The framing version can be found in [Borhade and Ramadevi 2005; Paul et al. 2010]. Our formulation is still quite different from that in [Bouchard et al. 2005; Mariño 2010], because their approach uses representations of Hecke algebra, whereas ours is based on representations of the Birman–Murakami–Wenzl algebra, and uses a type-B Schur function instead of a type-A Schur function as the basis in the orthogonal Chern–Simons partition function.

Theorems that partly answer the orthogonal LMOV conjecture proposed in this paper are listed below. For more precise statements of these theorems, see Sections 5, 7, 8 and 9.

**Theorem 1.3.** *The conjecture is true for all partitions when the link is trivial* (*when it is a disjoint union of unlinked unknots*).

**Theorem 1.4.** *The conjecture is true for partitions of the shape*

$$\vec{\mu} = ((1^{d_1}), (1^{d_2}), \ldots, (1^{d_L})),$$

*where* $(1^{d_\alpha}) = (1, 1, \ldots, 1) \vdash d_\alpha$ *for* $1 \leq \alpha \leq L$.

**Theorem 1.5.** *The conjecture is true if and only if it is true for partitions of the shape* $\vec{\mu} = ((d_1), (d_2), \ldots, (d_L))$.

**Theorem 1.6.** *The conjecture asymptotically holds* (*for all partitions $\vec{\mu}$ and all knots/links*) *as $q$ tends to* 1.

**Theorem 1.7.** *Examples of $\mathscr{L}$ for which the conjecture is true include the torus knots/links $T(2, k)$, where $k$ is odd/even, and each component of the partition $\vec{\mu}$ is of the form* $(1)$, $(1, 1)$ *or* $(2)$; *the two components torus link $T(2, 2k)$ for partition* $(3)$, $(1)$; *and the three components torus link $T(3, 3k)$ for the partition* $(2)$, $(1)$, $(1)$. *These give evidence for the conjecture at nontrivial roots of unity.*

**Theorem 1.8.** *We have the degree estimate*

$$\operatorname{val}_u(F^{\mathrm{SO}}_{\vec{\mu}}) \geq \ell(\vec{\mu}) - 2.$$

In addition, we use the cabling technique developed in [Lin and Zheng 2010] to calculate colored Kauffman polynomials for torus knots and links, which are employed to test the orthogonal LMOV conjecture (Theorem 1.7).

This paper is organized as follows: In Section 2, we review some basic knowledge of partitions, the Birman–Murakami–Wenzl (BMW) algebra and irreducible representation of the Brauer algebra. In Section 3, we review the definition of the quantum group invariants of links and use the cabling formula to simplify the computation of these invariants. As an application of the cabling formula, we obtain colored Kauffman polynomials of all torus knots and links for all partitions (irreducible representations). In Section 4, we define the Chern–Simons partition function for orthogonal quantum groups and the corresponding reformulated invariants. Also, we compute the orthogonal Chern–Simons partition function for disjoint union of unknots (Theorem 1.3). In Section 5, we propose a new orthogonal LMOV conjecture and degree conjecture. Then we test torus knots and links as supporting examples (Theorem 1.7), which can not be treated as special cases of the proof in the following sections. In Section 6, we obtain formulas of Lickorish–Millett type by using skein relations at the intersections of two different link components. This trick is also widely used in Section 7. Anyway, this section is quite independent and such Lickorish–Millett-type formulas can also be treated as applications of the orthogonal LMOV conjecture, which is the starting point of this paper. In Section 7, we prove the equivalence between the vanishing of the first three coefficients of $F_{\vec{\mu}}$ for trivial partitions $\vec{\mu}$ (each component of partitions have only one box), predicted by the degree conjecture, and the Lichorish–Millett type formulas obtained in Section 6. We also prove the orthogonal LMOV conjecture for column-like Young diagram (Theorem 1.4) as a generalization of such Lichorish–Millett type formulas. In Sections 8 and 9, we prove that if the orthogonal LMOV conjecture is valid for the case of rows, then the orthogonal LMOV is valid for all partitions (Theorem 1.5). We also present there the proof of the degree conjecture (Theorem 1.7), which implies that the orthogonal LMOV conjecture asymptotically holds (for all partitions $\vec{\mu}$ and all knots/links) as $q$ tends to 1 (Theorem 1.6).

In Section 10 (the appendix), we first compute explicit expressions of the Chern–Simon partition function for the unknot. We then review an alternative definition of the colored Kauffman polynomial via the Markov trace (skein approach) and test the Hopf link for the orthogonal LMOV conjecture by using this new definition. We also give an explicit computation of the quantum trace for orthogonal quantum groups directly from the universal $R$-matrix. Finally, we list the character table of the Brauer algebra and type-B Schur functions, whose specialization gives colored Kauffman polynomials of the unknot (quantum dimensions) for small partitions. These tables are mainly used to compute colored Kauffman polynomial for torus knots and links. The tables of the integers coefficients predicted by the orthogonal LMOV conjecture are also presented.

## 2. Young diagram and Birman–Murakami–Wenzl algebra

**2.1.** ***Partition and young diagram.*** A *composition* $\mu$ of $n$, denoted by $\mu \models n$, is a finite sequence of positive integers $(\mu_1, \mu_2, \ldots, \mu_\ell)$ such that

$$\mu_1 + \mu_2 + \cdots + \mu_\ell = n.$$

The number of parts in $\mu$ is called the length of $\mu$ and denoted by $\ell = \ell(\mu)$. The size of composition $\mu$ is defined by

$$|\mu| = \sum_{i=1}^{\ell(\lambda)} \mu_i.$$

A *partition* $\lambda$ is a composition such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$

Denote by $\mathcal{P}$ the set of all partitions. We identify a partition with its Young diagram.

If $|\lambda| = d$, we say $\lambda$ is a partition of $d$ and denote this by $\lambda \vdash d$.

We use $m_i(\lambda)$ to denote the number of times that $i$ appears in $\lambda$. Denote the automorphism group of the partition $\lambda$ by $\mathrm{Aut}(\lambda)$.

The order of $\mathrm{Aut}(\lambda)$ is given by

$$|\mathrm{Aut}(\lambda)| = \prod_i m_i(\lambda)!$$

A partition $\lambda$ can also be rewritten in the form

$$(1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots).$$

For instance, we have $(5, 3, 3, 2, 2, 2, 1) = (1^1 2^3 3^2 5^1)$

Associate to a partition $\lambda$ the numbers

$$\mathfrak{z}_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)! \quad \text{and} \quad \kappa_\lambda = \prod_{j=1}^{\ell(\lambda)} \lambda_j(\lambda_j - 2j + 1).$$

**2.2.** ***Partitionable set and infinite series.*** We present here some basic facts about partitionable sets, following the notation of [Liu and Peng 2010].

The concept of partition can be generalized as follows. set. A countable set $(S, +)$ is called a *partitionable* set if

(1) $S$ is totally ordered;

(2) $S$ is an Abelian semigroup with summation "+"; and

(3) the minimum element $\mathbf{0}$ in $S$ is the zero element of the semigroup, that is,

$$\mathbf{0}+a=a=a+\mathbf{0} \quad \text{for any } a\in S.$$

For simplicity, we may briefly write $S$ instead of $(S,+)$.

The following sets are examples of partitionable sets:

(1) The set of all nonnegative integers $\mathbb{Z}_{\geq 0}$.

(2) The set of all partitions $\mathcal{P}$. Let $\lambda,\mu\in\mathcal{P}$. The ordering on $\mathcal{P}$ can be defined by saying $\lambda\geq\mu$ if and only if $|\lambda|>|\mu|$, or $|\lambda|=|\mu|$ and there exists a $j$ such that $\lambda_i=\mu_j$ for $i\leq j-1$ and $\lambda_j>\mu_j$. The summation is taken to be $\cup$ and the zero element is $(0)$.

(3) The set $\mathcal{P}^n$. The order of $\mathcal{P}^n$ is defined similarly as before: Let $\vec{A},\vec{B}\in\mathcal{P}^n$. We say $\vec{A}\geq\vec{B}$ if and only if $\sum_{i=1}^n\sum|A^i|>\sum_{i=1}^n|B^i|$, or $\sum_{i=1}^n|A^i|=\sum_{i=1}^n|B^i|$ and there is a $j$ such that $A^i=B^i$ for $i\leq j-1$ and $A^j>B^j$.

Define

$$\vec{A}\cup\vec{B}=(A^1\cup B^1,A^2\cup B^2,\dots,A^n\cup B^n).$$

The element $((0),(0),\dots,(0))$ is the zero. Then $\mathcal{P}^n$ is a partitionable set.

Let $S$ be a partitionable set. One can define partition with respect to $S$ in a way similarly to that of $\mathbb{Z}_{\geq 0}$, that is, by a finite sequence of nonincreasing nonminimum elements in $S$. We will call it an *$S$-partition*, and $(\mathbf{0})$ the zero $S$-partition. Denote by $\mathcal{P}(S)$ the set of all $S$-partitions.

For an $S$-partition $\Lambda$, we can define the automorphism group of $\Lambda$ similarly to that of a traditional partition. Given $\beta\in S$, denote by $m_\beta(\Lambda)$ the number of times that $\beta$ occurs in the parts of $\Lambda$. We then have

$$\operatorname{Aut}\Lambda=\prod_{\beta\in S}m_\beta(\Lambda)!.$$

Associate to $\Lambda$ the definitions

$$\mathfrak{u}_\Lambda=\frac{\ell(\Lambda)!}{|\operatorname{Aut}\Lambda|}\quad\text{and}\quad\Theta_\Lambda=\frac{(-1)^{\ell(\Lambda)-1}}{\ell(\Lambda)}\mathfrak{u}_\Lambda.$$

The following lemma will be used in Section 4 to deduce the reformulated invariants.

**Lemma 2.1** ([Liu and Peng 2010, Lemma 2.3]). *Let $S$ be a partitionable set. If $f(t)=\sum_{n\geq 0}a_nt^n$, then*

$$f\Big(\sum_{\beta\in S,\beta\neq\mathbf{0}}A_\beta p_\beta(x)\Big)=\sum_{\Lambda\in\mathcal{P}(S)}a_{\ell(\Lambda)}A_\Lambda p_\Lambda(x)\mathfrak{u}_\Lambda,$$

*where*

$$p_\Lambda(x)=\prod_{j=1}^{\ell(\Lambda)} p_{\Lambda_j} \quad \textit{and} \quad A_\Lambda=\prod_{j=1}^{\ell(\Lambda)} A_{\Lambda_j}.$$

*Proof.* Note that

$$\Bigl(\sum_{\beta\in S,\beta\neq\mathbf{0}} \eta_\beta\Bigr)^n=\sum_{\Lambda\in\mathcal{P}(S),\ell(\Lambda)=n}\eta_\Lambda\mathfrak{u}_\Lambda. \qquad \square$$

**2.3.** ***The Birman–Murakami–Wenzl algebra.*** The centralizer algebra

(2-1) $$\mathrm{End}_{U_q(\mathfrak{so}(2N+1))}(V^{\otimes n}) = \{f\in\mathrm{End}(V^{\otimes n}) \mid fx=xf \text{ for all } x\in U_q(\mathfrak{so}(2N+1))\}$$

for the standard representation of $U_q(\mathfrak{so}(2N+1))$ on $V=\mathbb{C}^{2N+1}$ is isomorphic, when $N>n$, to the BMW algebra $C_n$.

Let $\mathbb{C}(t,q)$ be the field of rational functions with two variables. For each positive integer $n$, the BMW algebra is defined to be an algebra $C_n$ over $\mathbb{C}(t,q)$ as follows. The algebra $C_1$ is one dimensional and thus is identified with $\mathbb{C}(t,q)$. For $n>1$, $C_n$ is generated over $\mathbb{C}(t,q)$ by the generators $g_1,g_2,\dots,g_{n-1},e_1,e_2,\dots,e_{n-1}$ and the relations

(A1) $g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1}$ for $1\le i\le n-2$,

(A2) $g_ig_j=g_jg_i$ if $|i-j|\geqslant 2$,

(A3) $e_ig_i=t^{-1}e_i$,

(A4) $e_ig_{i-1}^{\pm1}e_i=t^{\pm1}e_i$, and

(A5) $(q-q^{-1})(1-e_i)=g_i-g_i^{-1}$.

The first two properties are the braiding relations. The following two properties are immediate from the definition above:

(P1) $e_i^2=xe_i$ for $x=1+(t-t^{-1})/(q-q^{-1})$.

(P2) $(g_i-t^{-1})(g_i+q^{-1})(g_i-q)=0$.

When the variables $q$ and $t$ approach 1, with $x=1+(t-t^{-1})/(q-q^{-1})$ fixed, the BMW algebra above specializes to the Brauer algebra $\mathrm{Br}_n$, which is semisimple and isomorphic to the centralizer algebra $\mathrm{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$ if $N>n$; see [Brauer 1937] and also [Weyl 1946]. The BMW algebras are semisimple except possibly when $q$ is a root of unity or $t=q^m$ for some integer $m$. Obviously, the BMW algebra is the deformation of the Brauer algebra.

**2.4.** ***Irreducible representations of Brauer algebras.*** For our purpose, we focus the generic case when the BMW algebras $C_n$ are semisimple. In this situation, the description of the irreducible representations is similar to that of the Brauer algebras $\mathrm{Br}_n$.

Being the centralizer algebra $\mathrm{End}_{\mathfrak{so}(2N+1)} V^{\otimes n}$, $\mathrm{Br}_n$ contains the group algebra $\mathbb{C}[S_n]$ as a direct summand; thus all the irreducible representations of $S_n$ are also irreducible representations of $\mathrm{Br}_n$, labeled by partitions of the integer $n$. Indeed, the set of irreducible representations of $\mathrm{Br}_n$ are bijection with the set of partitions of the integers $n-2k$, where $k = 0, 1, \dots, [n/2]$; see [Ram 1995; Wenzl 1988]. Thus the semisimple algebra $\mathrm{Br}_n$ can be decomposed into a direct sum of simple algebras:

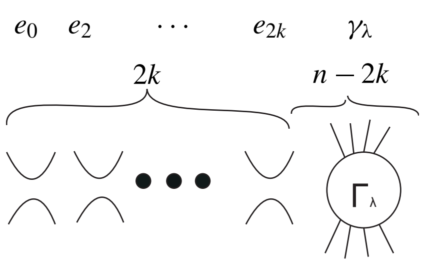
$$\mathrm{Br}_n \cong \bigoplus_{\lambda\vdash n-2k} \bigoplus_{k=0}^{[\frac{n}{2}]} M_{d_\lambda \times d_\lambda}(\mathbb{C}).$$

Beliakova and Blanchet [2001] constructed an explicit basis of the decomposition above. An up-and-down tableau $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a tube of $n$ Young diagrams such that $\lambda_1 = (1)$ and each $\lambda_i$ is obtained by adding or removing one box from $\lambda_{i-1}$. Let $\lambda$ be a partition of $n-2k$. We say $|\Lambda| = \lambda$ if $\lambda_n = \lambda$, and we say an up-and-down tableau $\Lambda$ is of shape $\lambda$. There is a minimal path idempotent $p_\Lambda \in \mathrm{Br}_n$ associated to each $\Lambda$. Then the minimal central idempotent $\pi_\lambda$ of $\mathrm{Br}_n$ corresponding to the irreducible representation labeled by $\lambda$ is given by

$$\pi_\lambda = \sum_{|\Lambda|=\lambda} p_\Lambda.$$

In particular, the dimension of the irreducible representations $d_\lambda$ is the number of up-and-down tableaus of shape $\lambda$. More details can be found in [Beliakova and Blanchet 2001; Wenzl 1988].

The characters table and the orthogonal relations can be found in [Ram 1991; 1995; 1997]. The values of a character of $\mathrm{Br}_n$ are completely determined by its values on the set of elements $e^k \otimes \gamma_\lambda$, where $e$ is the conjugacy class of $e_1, \dots, e_{n-1}$ and $\gamma_\lambda$ is the conjugacy class in $S_{n-2k}$ labeled by the partition $\lambda$ of $n-2k$. The notation $e^k \otimes \gamma_\lambda$ stands for the tangle in the diagram



where $\Gamma_\lambda$ is a diagram in the conjugacy class of $S_{n-2k}$ labeled by a partition $\lambda$ of $n-2k$.

Denote by $\chi_A$ the character of the irreducible representation of $\mathrm{Br}_n$ labeled by a partition $A \vdash n-2k$ for some $k$, and denote by $\chi_B^{S_n}$ the character of the irreducible representation of $S_n$ labeled by a partition $B \vdash n$. It is known that if $A$ is a partition of $n$, then $\chi_A(e^m \otimes \gamma_\lambda) = 0$ for all $m > 0$ and partitions $\lambda \vdash n-2m$, and the characters $\chi_A(\gamma_\mu) = \chi_A^{S_n}(\gamma_\mu)$, for partitions $\mu \vdash n$, coincide with the characters of the permutation group $S_n$ [Ram 1995].

**2.5. *Schur–Weyl duality.*** Both $\mathfrak{so}(2N+1)$ and $\mathrm{Br}_n$ act on the tensor product $V^{\otimes n}$ and their actions commute with each other. As a bimodule, $V^{\otimes n}$ has the decomposition

$$V^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes U_\lambda,$$

where $\lambda$ runs through all the partitions of $n, n-2, n-4, \dots, 0$, and $V_\lambda$ and $U_\lambda$ are the irreducible representations of $\mathfrak{so}(2N+1)$ and $\mathrm{Br}_n$, respectively, labeled by $\lambda$. A similar decomposition holds for the pair $U_q(\mathfrak{so}(2N+1))$ and $C_n$.

A power symmetric function of a sequence of variables $z = (z_i)_{i\in\mathbb{Z}}$ is defined by

$$pb_n(z) = (z_0)^n + \sum_{i=1}^{+\infty}[(z_i)^n + (z_{-i})^n].$$

For a partition $\lambda$,

$$pb_\lambda(z) = \prod_{j=1}^{\ell(\lambda)} pb_{\lambda_j}(z).$$

Denote by $\widehat{Br}_n$ the set of all the characters of $\mathrm{Br}_n$. For each partition $A$, we use $sb_A$ to denote the type-B Schur function associated to $A$ with infinitely many variables $z_0, z_{\pm1}, z_{\pm2}, \dots$, which are completely determined inductively by the system of equations

$$x^k\, pb_\lambda = \sum_{A\in\widehat{Br}_n} \chi_A(e^{\otimes k} \otimes \gamma_\lambda)\, sb_A\,. \tag{2-2}$$

The parameter $x$ is the structure constant in the definition of the Brauer algebra $\mathrm{Br}_n$. The type-B Schur functions are independent of this parameter $x$, as one can see from the character formula of the Brauer algebra, given by [Ram 1995, Theorem 5.1]. If $A$ is a partition of $n$, then $sb_A$ is a symmetric polynomial of degree $n$ (not necessarily homogeneous).

Throughout this paper, we fix the following notations for partition set $\mathcal{P}^L$, where $L$ is the number of components of the link $\mathcal{L}$.

For $\vec{\mu} = (\mu^1, \mu^2, \dots, \mu^L) \in \mathcal{P}^L$, let

$$|\vec{\mu}| = (|\mu^1|, |\mu^2|, \dots, |\mu^L|) \in \mathbb{Z}^L \tag{2-3}$$

and define

$$\|\vec{\mu}\| = \sum_{\alpha=1}^{L} |\mu^\alpha|. \tag{2-4}$$

Write

$$\ell(\vec{\mu}) = \sum_{\alpha=1}^{L} \ell(\mu^\alpha) \tag{2-5}$$

for the sum of the length of each partition.

We write $pb_{\vec{\mu}}(\vec{z}) = \prod_{\alpha=1}^{L} pb_{\mu^\alpha}(z_\alpha)$, where $z_\alpha = (z_{\alpha,i})_{i\in\mathbb{Z}}$.

Let $\widehat{Br}_{|\vec{\mu}|}$ denote the set $\widehat{Br}_{|\mu^1|} \times \cdots \times \widehat{Br}_{|\mu^L|}$. Then $\chi_{\vec{A}}(\gamma_{\vec{\mu}}) = \prod_{\alpha=1}^{L} \chi_{A^\alpha}(\gamma_{\mu^\alpha})$ for the character $\chi_{A^\alpha}$ of $\mathrm{Br}_{|\mu^\alpha|}$ labeled by $A^\alpha$, a partition of $|\mu^\alpha| - 2k^\alpha$, and the conjugacy class $\gamma_{\mu^\alpha}$ of $\mathrm{Br}_{d_\alpha}$ labeled by $\mu^\alpha$.

## 3. Colored Kauffman polynomials and cabling formula

**3.1.** ***Colored Kauffman polynomials (orthogonal quantum group invariants) and cabling technique.*** Let $B_m$ be the braid group of $m$ strands that is generated by $\sigma_1, \dots, \sigma_{m-1}$ with defining relations

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1. \end{cases} \tag{3-1}$$

Every link can be represented by the closure of some element in braid group $B_m$. This kind of braid representation is not unique. We fix such a braid representation, and then we define the quantum group invariants of link via this braid. Finally we will see that such a definition is independent of the choice of the braid representation.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $U_q(\mathfrak{g})$ be the corresponding quantized enveloping algebra.

The ribbon category structure associated with $U_q(\mathfrak{g})$ is given by the following data:

(1) Associated to each pair of $U_q(\mathfrak{g})$-modules $V$ and $W$ is an isomorphism

$$\check{\mathcal{R}}_{V,W} : V \otimes W \to W \otimes V$$

such that

$$\check{\mathcal{R}}_{U\otimes V,W} = (\check{\mathcal{R}}_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \check{\mathcal{R}}_{V,W}),$$
$$\check{\mathcal{R}}_{U,V\otimes W} = (\mathrm{id}_V \otimes \check{\mathcal{R}}_{U,W})(\check{\mathcal{R}}_{U,V} \otimes \mathrm{id}_W)$$

for $U_q(\mathfrak{g})$-modules $U$, $V$, $W$.

Given $f \in \mathrm{Hom}_{U_q(\mathfrak{g})}(U, \widetilde{U})$ and $g \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$, one has the naturality condition

$$(g \otimes f) \circ \check{\mathscr{R}}_{U,V} = \check{\mathscr{R}}_{\widetilde{U},\widetilde{V}} \circ (f \otimes g).$$

(2) There exists an element $K_{2\rho} \in U_q(\mathfrak{g})$, called the enhancement of $\check{\mathscr{R}}$, such that

$$K_{2\rho}(v \otimes w) = K_{2\rho}(v) \otimes K_{2\rho}(w)$$

for any $v \in V$ and $w \in W$. Here $\rho$ is the half-sum of all positive roots of $\mathfrak{g}$.

Moreover, for every $z \in \mathrm{End}_{U_q(\mathfrak{g})}(V, W)$ with $z = \sum_i x_i \otimes y_i$, $x_i \in \mathrm{End}(V)$ and $y_i \in \mathrm{End}(W)$, one has the quantum trace

$$\mathrm{tr}_W(z) = \sum_i \mathrm{tr}(y_i K_{2\rho}) \cdot x_i \in \mathrm{End}_{U_q(\mathfrak{g})}(V)$$

(3) For any $U_q(\mathfrak{g})$-module $V$, the ribbon structure $\theta_V : V \to V$ associated to $V$ satisfies

$$\theta_V^{\pm 1} = \mathrm{tr}_V\, \check{\mathscr{R}}^{\pm 1}_{V,V}.$$

The ribbon structure also satisfies the naturality condition

$$x \cdot \theta_V = \theta_{\widetilde{V}} \cdot x$$

for any $x \in \mathrm{Hom}_{U_q(\mathfrak{g})}(V, \widetilde{V})$.

Let $\mathscr{L}$ be a link with components $\mathscr{K}_\alpha$ for $\alpha = 1, \ldots, L$, represented by the closure of $\beta \in B_m$. We associate each $\mathscr{K}_\alpha$ an irreducible representation $V_{A^\alpha}$ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ labeled by highest weight $A^\alpha$. In the sense of [Ram 1995], these irreducible representations can be labeled by partitions. Abusing notation, we use the $A^\alpha$ to denote those partitions. Let $i_1, \ldots, i_m$ be integers such that $i_k = \alpha$ if the $k$-th strand of $\beta$ belongs to the $\alpha$-th component of $\mathscr{L}$.

Let $U$ and $V$ be two $U_q(\mathfrak{g})$-modules labeling two outgoing strands of the crossing. The braidings $\check{\mathscr{R}}_{U,V}$ and $\check{\mathscr{R}}^{-1}_{V,U}$ are assigned as in following figure.


$\check{\mathscr{R}}_{U,V}$ $\qquad$ $\check{\mathscr{R}}^{-1}_{V,U}$

The assignment above will give a representation of $B_m$ on $U_q(\mathfrak{g})$-module $V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}$. Namely, for any generator $\sigma_j \in B_m$, define

$$h(\sigma_j) = \mathrm{id}_{V_{A^{i_1}}} \otimes \cdots \otimes \check{\mathcal{R}}_{V_{A^{i_{j+1}}}, V_{A^{i_j}}} \otimes \cdots \otimes \mathrm{id}_{V_{A^{i_m}}},$$

and

$$h(\sigma_j^{-1}) = \mathrm{id}_{V_{A^{i_1}}} \otimes \cdots \otimes \check{\mathcal{R}}^{-1}_{V_{A^{i_j}}, V_{A^{i_{j+1}}}} \otimes \cdots \otimes \mathrm{id}_{V_{A^{i_m}}},$$

Therefore, any link $\mathcal{L}$ will provide an isomorphism

$$h(\beta) \in \mathrm{End}_{U_q(\mathfrak{g})}(V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}).$$

The representation of the braid group $B_n$ on $V^{\otimes n}$ factors through the BMW algebra $C_n$ by sending $\sigma_j$ to $g_j \in C_n$. By abuse of notation, we still denote this via $g_j = h(\sigma_j)$.

The quantum trace

$$\mathrm{tr}_{V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}} h(\beta)$$

defines the framing-dependent link invariant of link $\mathcal{L}$.

In order to eliminate the framing dependency, we make the refinement [Lin and Zheng 2010]

$$W^{\mathfrak{so}(2N+1)}_{V_{A^1}, \dots, V_{A^L}}(\mathcal{L}; q) = \theta_{V_{A^1}}^{-w(\mathcal{K}_1)} \cdots \theta_{V_{A^L}}^{-w(\mathcal{K}_L)} \mathrm{tr}_{V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}}(h(\beta)),$$

where $w(\mathcal{K}_\alpha)$ is the writhe number of $\mathcal{K}_\alpha$ in $\beta$, that is, the number of positive crossing minus the number of negative crossings.

The quantity above is invariant under the Markov moves, and hence is an invariant of the underlying link $\mathcal{L}$.

Quantum group invariants of links can be defined over any complex simple Lie algebra $\mathfrak{g}$. However, in this paper, we mainly consider the quantum group invariants of links defined over $\mathfrak{so}(2N+1)$. More generally, one can also include the case for $\mathfrak{so}(2N)$ and $\mathfrak{sp}(2N)$; however, we will not do so, since the quantum group invariants associated to these Lie algebras all give the colored Kauffman polynomials. To distinguish $U_q(\mathfrak{so}(2N+1))$ from the quantum group corresponding to the spin group, we only consider those representations parametrized by the highest weights in the root lattice of the Lie group $\mathrm{SO}(2N+1)$, instead of the spin group. These highest weights are, similar to the case of $\mathfrak{sl}_N$, partitions of length at most $N$, that is, $\{\mu | \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N \geqslant 0\}$.

Let's consider $U_q(\mathfrak{so}(2N+1))$, the quantized universal enveloping algebra of orthogonal Lie algebra $\mathfrak{so}(2N+1)$.

The ribbon category structure is defined by letting $\check{R} = P_{12}\mathcal{R}$ for the universal $\mathcal{R}$-matrix above, and taking $K_{2\rho}$ to be $q^{-\rho^*}$. The operator $P_{12} : V \otimes W \to W \otimes V$

switches the two components, and $\rho^*$ denotes the element in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponding to $\rho$.

The positive roots of $\mathfrak{so}(2N+1)$ are given by $\vartheta_i \pm \vartheta_j$ for $1 \leqslant i < j \leqslant N$ and $\vartheta_1, \vartheta_2, \ldots, \vartheta_N$, where $\vartheta_i$ has eigenvalue $x_i$ when acting on the matrix element

$$\mathrm{diag}\{-x_N, -x_{N-1}, \ldots, -x_1, 0, x_1, \ldots, x_{N-1}, x_N\}$$

in the Cartan subalgebra. The sum of the positive roots is given by

$$2\rho = \sum_{i=1}^{N} \vartheta_i + \sum_{1 \leqslant i < j \leqslant N} [(\vartheta_i - \vartheta_j) + (\vartheta_i + \vartheta_j)] = \sum_{i=1}^{N} (2N+1-2i)\vartheta_i,$$

and

$$K_{2\rho} = \mathrm{diag}\{q^{1-2N}, q^{3-2N}, \ldots, q^{-3}, q^{-1}, 1, q, q^3 \cdots, q^{2N-3}, q^{2N-1}\}.$$

Alternatively, we can write

$$K_{2\rho}(v_i) = \begin{cases} q^{2i-1-2N} v_i & \text{if } 1 \leq i \leq N, \\ v_i & \text{if } i = N+1, \\ q^{2i-3-2N} v_i & \text{if } N+2 \leq i \leq 2N+1. \end{cases}$$

The natural representation of $U_q(\mathfrak{so}(2N+1))$ on $V$ has universal matrix $\check{\mathcal{R}}$ acting on $V \otimes V$ by [Turaev 1988]

$$\begin{aligned} \check{\mathcal{R}} = q \sum_{i \neq N+1} E_{i,i} \otimes E_{i,i} + E_{N+1,N+1} \otimes E_{N+1,N+1} + \sum_{j} \sum_{i \neq j, i \neq 2N+2-j} E_{j,i} \otimes E_{i,j} \\ + q^{-1} \sum_{i \neq N+1} E_{2N+2-i,i} \otimes E_{i,2N+2-i} + (q - q^{-1}) \sum_{i<j} E_{i,i} \otimes E_{j,j} \\ - (q - q^{-1}) \sum_{i<j} q^{\bar{i} - \bar{j}} E_{2N+2-j,i} \otimes E_{j,2N+2-i}, \end{aligned}$$

where $E_{i,j}$ is the $(2N+1) \times (2N+1)$ matrix with

$$(E_{i,j})_{kl} = \begin{cases} 1 & \text{if } (k,l) = (i,j), \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad \bar{i} = \begin{cases} i + \frac{1}{2} & \text{if } 1 \leq i \leq N, \\ i & \text{if } i = N+1, \\ i - \frac{1}{2} & \text{if } N+2 \leq i \leq 2N+1. \end{cases}$$

To compute the quantum invariants more explicitly, we will first introduce the representation theory of the BMW algebra.

From now on, we only restrict ourselves in the case when the BMW algebra $C_n$ is semisimple and $N$ is large. The representations of $C_n$ can be described in the same way as the Brauer algebra $\mathrm{Br}_n$. The semisimplicity implies that the representation

$V^{\otimes n}$ of $C_n$ admits a direct sum decomposition

$$V^{\otimes n} = \bigoplus_{\lambda \in \widehat{Br}_n} d_\lambda \cdot V_\lambda.$$

The multiplicities $d_\lambda$ are all positive integers. In particular, any irreducible representation $V_A$ of $U_q(\mathfrak{so}(2N+1))$ appears as a direct summand of $V^{\otimes r}$ for integers $r = |A|, |A|+2, |A|+4, \ldots$. By Schur's lemma,

$$C_n \cong \operatorname{End}_{U_q(\mathfrak{so}(2N+1))} V^{\otimes n} \cong \bigoplus_{\lambda \in \widehat{Br}_n} C_\lambda,$$

where $C_\lambda = \operatorname{End}_{U_q(\mathfrak{so}(2N+1))}(d_\lambda V_\lambda)$ is isomorphic to the $d_\lambda \times d_\lambda$ matrix algebra, labeled by the characters $\widehat{Br}_n$ of $\mathrm{Br}_n$ as the decomposition of $V^{\otimes n}$.

A *minimal idempotent* $p \in C_n$ satisfies $p^2 = p$ and the action of $U_q(\mathfrak{so}(2N+1))$ on the subspace $p \cdot V^{\otimes n}$ is an irreducible representation. Another description of $p$ is that there exist exactly one $\lambda \in \widehat{Br}_n$ such that the restriction of $p$ to $C_\lambda$ is nonzero, and it's a diagonalizable matrix with exactly one eigenvalue 1 and all others 0.

Let $y$ be an element in $C_n$, and denote by $\zeta_n^\lambda(y)$ the normal (or say, nonquantum) trace of its $\lambda$ component via the isomorphism above. Since $y$ and all the idempotents are elements in $C_n$, they are finite linear combinations of products of the generators $g_i$ and $e_i$, which imply $\zeta_n^\lambda(y)$ is, in general, a rational function of $q$ and $t$.

It is not hard to get the following identity from the Turaev's [1988] construction of universal matrix $\check{\mathcal{R}}$ (see Section 10 for details):

$$\theta_V = q^{2N} \cdot \mathrm{id}_V,$$

where $V$ is the standard representation of $U_q(\mathfrak{so}(2N+1))$ on $\mathbb{C}^{2N+1}$.

More generally, we have the following lemma obtained by Reshetikhin [1987].

**Lemma 3.1.** *For each partition $\lambda \vdash n-2f$ with $\ell(\lambda) \le N$, one has*

$$\theta_{V_\lambda} = q^{\kappa_\lambda + 2N(n-2f)} \cdot \mathrm{id}_{V\lambda},$$

*where $\kappa_\lambda = \prod_{j=1}^{\ell(\lambda)} \lambda_j(\lambda_j - 2j + 1)$.*

This result can be understand in the following way. First we have $\theta_V = q^{2N} \cdot \mathrm{id}_V$. A result of Aiston and Morton ([1998, Theorem 5.5], compare with [Lin and Zheng 2010, Theorem 4.1]) states that

$$\theta_{V_\lambda} = q^{\kappa_\lambda + nN - n^2/N} \cdot \mathrm{id}_{V\lambda}.$$

Lin and Zheng use a different normalization for universal $\check{\mathcal{R}}$-matrices, and thus have

$$q^{1/N}\theta_V = q^N \cdot \mathrm{id}_V$$

and also a different corresponding normalization for $h : \mathbb{C}B_n \to C_n(V)$ factoring through the Hecke algebra $\mathscr{H}_n(q)$ via

$$q^{1/N}\sigma_i \mapsto g_i \mapsto q^{1/N} h_V(\sigma_i).$$

Then we translate their normalization to ours, that is,

$$\begin{aligned} &\sigma_i \mapsto g_i \mapsto h(\sigma_i), \\ &\theta_V = q^N \cdot \mathrm{id}_V, \qquad \theta_{V_\lambda} = q^{\kappa_\lambda + nN} \cdot \mathrm{id}_{V\lambda}. \end{aligned}$$

Then it is quite easy to get $\theta_{V_\lambda} = q^{\kappa_\lambda + 2N(n-2f)} \cdot \mathrm{id}_{V\lambda}$.

Now we can write down the explicit formula for the orthogonal quantum group invariants as

$$W^{\mathrm{SO}}_{A^1,\dots,A^L}(\mathscr{L}; q) = q^{-\sum_{\alpha=1}^{L} \kappa_{A^\alpha} w(\mathscr{K}_\alpha) - 2N \sum_{\alpha=1}^{L} |A^\alpha| w(\mathscr{K}_\alpha)} \cdot \mathrm{tr}_{V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}}(h(\beta)) \tag{3-2}$$

for all sufficiently integers $N$. In particular, when the link is trivial with $L$ components, the quantum group invariant is this product of quantum dimensions:

$$W^{\mathrm{SO}}_{A^1,\dots,A^L}(\bigcirc^L; q) = \prod_{\alpha=1}^{L} \dim_q(V_{A^\alpha}). \tag{3-3}$$

The quantum dimension is computed in [Wenzl 1990], a calculation we quote here. Let $\lambda$ be a partition. We also identify it with the corresponding Young diagram. For each pair of positive integers $(i, j)$, define

$$h(i, j) = \lambda_i + \lambda'_j - i - j + 1$$

to be the *hook length*, where $\lambda'$ is the transposed Young diagram of $\lambda$. Also define

$$d(i, j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \le j, \\ -\lambda'_i - \lambda'_j + i + j - 1 & \text{if } i > j. \end{cases}$$

**Theorem 3.2** [Wenzl 1990]. *Let $\lambda$ be a Young diagram with $m$ rows and let $\mathscr{Q}_\lambda(t, q)$ be the rational function given by*

$$\mathscr{Q}_\lambda(t, q) = \prod_{(j,j)\in\lambda} \left(1 + \frac{tq^{\lambda_j - \lambda'_j} - t^{-1}q^{\lambda'_j - \lambda_j}}{[h(j, j)]_q}\right) \prod_{\substack{(i,j)\in\lambda \\ i\neq j}} \frac{tq^{d(i,j)} - t^{-1}q^{-d(i,j)}}{[h(i, j)]_q}. \tag{3-4}$$

*Then the quantum trace* $\dim_q V_\lambda$ *of the representation of $U_q(\mathfrak{so}(2N+1))$ corresponding to $\lambda$ is equal to $\mathscr{Q}_\lambda(q^{2N}, q)$ for all $N > |\lambda|$.*

In the expression above, if we fix $t$ and let $q$ tends to 1, the pole order of $\mathscr{Q}_\lambda(t, q)$ is $|\lambda|$, the number of boxes in the Young diagram. The poles order at $q = 1$ of the quantum group invariant of unknots in (3-3) is $\|\vec{A}\| = \sum_{\alpha=1}^{L} |A^\alpha|$.

The special value

$$sb_A(q^{1-2N}, q^{3-2N}, \dots, q^{-1}, 1, q, \dots, q^{2N-3}, q^{2N-1}) = \mathfrak{Q}_\lambda(q, q^{2N})$$

is the quantum dimension $\dim_q(V_A)$, denoted also by $sb_A(q,t)$. Here we only evaluate the function in the variables

$$z_{-N}, z_{1-N}, \dots, z_{-1}, z_0, z_1, \dots, z_{N-1}, z_N,$$

and set all the remaining variables equal to zero.

The quantum dimension of small partitions can be found in Section 10, where we use the symbol $sb_A(q,t)$ for the type-B Schur function.

Similar to the type-$A$ Schur function, the type-$B$ Schur function has the expansion

$$sb_\lambda(z_{-N}, z_{1-N}, \dots, z_{-1}, z_0, z_1, \dots, z_{N-1}, z_N) = \sum_{\mu \models n, \ell(\mu) \le 2N+1} \dim(p_\lambda V^{\otimes n} \cap M^\mu) \cdot \prod_{i=-N}^{N} z_i^{\mu(i+N+1)},$$

where $M^\mu$, called the permutation module, is defined by

$$M^\mu = \{ v \in V^{\otimes n} \mid H_i(v) = q^{\mu_i - \mu_{i+1}} v \text{ for } i = 1, 2, \dots, N-1 \text{ and } H_N(v) = q^{\mu_N} v \},$$

and $\dim(p_\lambda V^{\otimes n} \cap M^\mu)$ is called the *Kostka number*.

It is normally very hard to calculate these quantum group invariants. Anyway, we can simplify the computation a lot with the help of the cabling technique.

The following lemma reduces the study of quantum group invariants of arbitrary representations to the study of the links and minimal idempotents.

**Lemma 3.3** [Lin and Zheng 2010, Lemma 3.3]. *Let $\beta \in B_m$ be a braid and let $p_\alpha \in C_{d_\alpha}$ for $\alpha = 1, \dots, L$ be $L$ minimal idempotents corresponding to the irreducible representations $V_{A^1}, \dots, V_{A^L}$, where $A^\alpha$ denotes the partition of $|A^\alpha| = d_\alpha$ labeling $V_{A^\alpha}$. Let $\vec{d} = (d_1, \dots, d_L)$ and let $i_1, \dots, i_m$ be integers such that $i_k = \alpha$ if the $k$-th strand of $\beta$ belongs to the $\alpha$-th component of $\mathcal{L}$. Let $\beta_{\vec{d}}$ be the cabling braid of $\beta$, replacing the $k$-th strand of $\beta$ by $d_{i_k}$ parallel ones. Then*

$$\operatorname{tr}_{V_{A^{i_1}} \otimes \cdots \otimes V_{A^{i_m}}} h(\beta) = \operatorname{tr}_{V^{\otimes n}}[h(\beta_{\vec{d}}) \cdot (p_{i_1} \otimes \cdots \otimes p_{i_m})], \tag{3-5}$$

*where $n = d_{i_1} + d_{i_2} + \cdots + d_{i_m}$.*

One immediately gets the following lemmas proved in [Lin and Zheng 2010] and reformulated into the setting of the orthogonal group.

**Lemma 3.4.** *For any element $y \in C_n$,*

$$\operatorname{tr}_{V^{\otimes n}} y = \sum_{k=0}^{[n/2]} \sum_{\lambda \vdash n-2k} \zeta_n^\lambda(y) \cdot sb_\lambda(q^{1-2N}, q^{3-2N}, \dots, q^{-1}, 1, q, \dots, q^{2N-3}, q^{2N-1}). \tag{3-6}$$

*For any braid $\beta \in B_m$, take $y = h(\beta_{\vec{d}}) \cdot (p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_m})$, where the closure of $\beta$ is the link $\mathcal{L}$. The setting is same as that in Lemma 3.3, so that after replacing $q^{2N}$ by $t$, we have*

$$W^{\mathrm{SO}}_{\vec{A}}(\mathcal{L}, q, t) = q^{-\sum_{\alpha=1}^{L} \kappa_{A^\alpha} w(\mathcal{K}_\alpha)} t^{-\sum_{\alpha=1}^{L} |A^\alpha| w(\mathcal{K}_\alpha)} \cdot \sum_{k=0}^{[n/2]} \sum_{\lambda \vdash n-2k} \zeta_n^\lambda(h(\beta_{\vec{d}}) \cdot (p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_m})) \cdot \mathcal{Q}_\lambda(q, t), \tag{3-7}$$

*where $n = |A^{i_1}| + \cdots + |A^{i_m}|$.*

If $A^1, \dots, A^L$ are all the natural representations of $\mathfrak{so}(2N+1)$ on $\mathbb{C}^{2N+1}$, that is, they are all equal to (1), the invariant becomes

$$W^{\mathrm{SO}}_{A^1,\dots,A^L}(\mathcal{L}, q, t) = t^{2\,lk(\mathcal{L})}\Big(1 + \frac{t - t^{-1}}{q - q^{-1}}\Big) K_{\mathcal{L}}(q, t),$$

where $lk(\mathcal{L})$ is the linking number of $\mathcal{L}$ for the Kauffman polynomial $K_{\mathcal{L}}(q, t)$, where we normalized the Kauffman polynomials such that $K_{\bigcirc}(q, t) = 1$. The orthogonal group invariants $W^{\mathrm{SO}}_{A^1,\dots,A^L}(\mathcal{L}; q, t)$ for general $A^\alpha$ are also called colored Kauffman polynomials.

**3.2.** ***An explicit formula of colored Kauffman polynomials for torus links.*** The coefficients $\zeta_n^\lambda(h(\beta_{\vec{d}}) \cdot (p_{i_1} \otimes p_{i_2} \otimes \cdots \otimes p_{i_m}))$ in (3-7) are usually hard to compute. However, they are computable for torus links. The torus link $T(r, k)$ is the closure of $(\delta_r)^k = (\sigma_1 \cdots \sigma_{r-1})^k$. It is a knot if and only if $(r, k) = 1$. For example, $T(2, 3)$ is the trefoil knot, and $T(2, 2)$ is the Hopf link. We develop the method in this subsection based on the work [Lin and Zheng 2010].

**Lemma 3.5.** *For each partition $\lambda \vdash (n - 2f)$ where $f = 0, 1, \dots, [n/2]$, we have*

$$h((\delta_n)^n) \cdot p_\lambda = q^{\kappa_\lambda - 4fN} \cdot p_\lambda = q^{\kappa_\lambda} t^{-2f} \cdot p_\lambda. \tag{3-8}$$

*Proof.* Again write $V$ for the standard representation of $U_q(\mathfrak{so}(2N+1))$ on the vector space $\mathbb{C}^{2N+1}$.

From Lemma 3.1, for each partition $\lambda \vdash n - 2f$ with $\ell(\lambda) \le N$, one has

$$\theta_{V_\lambda} = q^{\kappa_\lambda + 2N(n-2f)} \cdot \mathrm{id}_{V_\lambda}.$$

Substitute this into the formula

$$(\theta_V^{\otimes n} \cdot h((\delta_n)^n) \cdot p_\lambda = \theta_{V_\lambda} \cdot p_\lambda$$

proved in [Lin and Zheng 2010, Lemma 3.2] and the result follows. □

In the following, we assume $z_0 = 1$ and $z_{-n} z_n = 1$ for all positive integers $n = 1, 2, \ldots, N$, that is, the matrix $\operatorname{diag}(z_{-N}, z_{1-N}, \ldots z_{-1}, z_0, z_1, \ldots, z_{N-1}, z_N)$ is a generic element in the maximal torus of $\mathrm{SO}(2N+1, \mathbb{C})$. Let the constants $\tilde{c}^{\lambda}_{\vec{A}}$ be the rational number determined by the equations

$$\prod_{\alpha=1}^{L} sb_{A^\alpha}(z^r_{-N}, \ldots, z^r_{-1}, z^r_0, z^r_1, \ldots, z^r_N) = \sum_{f=0}^{[rn/2]} \sum_{\lambda \vdash rn-2f} \tilde{c}^{\lambda}_{\vec{A}} \cdot sb_\lambda(z_{-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_N). \tag{3-9}$$

**Theorem 3.6.** *Let $\mathscr{L}$ be the torus link $T(rL, kL)$ with $r$ and $k$ relatively prime. Suppose $A^\alpha$ is a partition of $d_\alpha$ for each $\alpha = 1, 2, \ldots, L$ and $n = d_1 + d_2 + \cdots + d_L$. Then*

$$W^{\mathrm{SO}}_{\vec{A}}(\mathscr{L}, q, t) = q^{-kr\sum_{\alpha=1}^{L} \kappa_{A^\alpha}} \cdot t^{-k(r-1)n} \cdot \sum_{f=0}^{[nr/2]} \sum_{\lambda \vdash (rn-2f)} \tilde{c}^{\lambda}_{\vec{A}} \cdot q^{k\kappa_\lambda/r} t^{-2fk/r} \cdot sb_\lambda(q, t). \tag{3-10}$$

Theorem 3.6 gives an explicit formula for the orthogonal quantum group invariants (colored Kauffman polynomials) of torus links in terms of constants $\tilde{c}^{\lambda}_{\vec{A}}$. Sebastien Stevan [2011] generalized this result to all classic gauge groups and cable knots. In Section 5, we use this formula to verify certain cases of Conjecture 5.1. The proof of Theorem 3.6 follows from the cabling formula (3-7), Lemma 3.5 and the following lemma.

**Lemma 3.7.** *Let $n = \|\vec{A}\|$, where $A^\alpha \vdash d_\alpha$, and let $r$ and $k$ be two relatively prime positive integers. Take $\beta \in B_{rn}$ to be the braid obtained by cabling the $(iL + j)$-th strand of $(\delta_{rL})^{kL}$ to $|A^j|$ parallel ones. For each partition $\lambda \vdash (rn - 2f)$, where $f = 0, 1, 2, \ldots, [rn/2]$, we have*

$$\zeta^{\lambda}_{rn}(h(\beta) \cdot (p_{A^1} \otimes \cdots \otimes p_{A^L})^{\otimes r}) = \tilde{c}^{\lambda}_{\vec{A}} \cdot q^{-k\sum_{\alpha=1}^{L} \kappa_{A^\alpha} + k\kappa_\lambda/r} t^{-2kf/r}. \tag{3-11}$$

*Proof.* Write $p_{\vec{A}} = p_{A^1} \otimes \cdots \otimes p_{A^L}$ and let $\pi_\lambda$ be the unit of $C_\lambda$. Obviously $\pi_\lambda$ is in the center of $C_{rn}$. A slightly nonobvious fact is that $h(\beta)$ commutes with $p_{\vec{A}}^{\otimes n}$, which follows from the naturality of $\check{\mathscr{R}}$. Let

$$x_\lambda = \pi_\lambda \cdot h(\beta) \cdot p_{\vec{A}}^{\otimes r} \tag{3-12}$$

be a matrix in $C_\lambda$, whose trace is

$$\mathrm{tr}(x_\lambda) = \zeta^\lambda(h(\beta)\cdot p_{\vec{A}}^{\otimes r}). \tag{3-13}$$

The cabling of torus link has the nice property

$$h(\beta^r) = h((\delta_{rn})^{krn})\cdot (h((\delta_{d_1})^{-kd_1})\otimes\cdots\otimes h((\delta_{d_L})^{-kd_L}))^{\otimes r}. \tag{3-14}$$

Lemma 3.5 then implies

$$x_\lambda^r = \pi_\lambda\cdot h(\beta^r)\cdot p_{\vec{A}}^{\otimes r} = q^{-kr\sum_{\alpha=1}^L \kappa_{A^\alpha}+k\kappa_\lambda} t^{-2kf}\cdot \pi_\lambda\cdot p_{\vec{A}}^{\otimes r}. \tag{3-15}$$

Thus the eigenvalues of $x_\lambda$ are either 0 or $q^{-kr\sum_{\alpha=1}^L \kappa_{A^\alpha}+k\kappa_\lambda/r} t^{-2kf/r}$ times a $r$-th root of unity. Together with the fact that $\mathrm{tr}(x_\lambda)\in\mathbb{Q}(q,t)$, we see that

$$\mathrm{tr}(x_\lambda) = a^\lambda\cdot q^{-k\sum_{\alpha=1}^L \kappa_{A^\alpha}+k\kappa_\lambda/r} t^{-2kf/r}$$

for some $a^\lambda\in\mathbb{Q}$ independent of $q$ and $t$.

It remains to compute this rational number $a^\lambda$. On passing to the limit $q\to 1$ and $t\to 1$, the element $h(\beta)$ reduces to a permutation $\tau\in S_{rn}$ acting cyclically on the $V^{\otimes n}$-factors of $V^{\otimes rn} = V^{\otimes n}\otimes\cdots\otimes V^{\otimes n}$:

$$\begin{aligned}
&\sum_{f=0}^{[rn/2]}\sum_{\lambda\vdash(rn-2f)} a^\lambda\cdot sb_\lambda(z_{-N}, z_{1-N},\dots,z_{-1},z_0,z_1,\dots,z_{N-1},z_N)\\
&\quad=\sum_{f=0}^{[rn/2]}\sum_{\lambda\vdash(rn-2f)} a^\lambda \sum_{\substack{\mu\models rn,\\ \ell(\mu)\le 2N+1}} \dim(p_\lambda V^{\otimes rn}\cap M^\mu)\cdot\prod_{i=-N}^N z_i^{\mu(i+N+1)}\\
&\quad=\sum_{\substack{\mu\models rn,\\ \ell(\mu)\le 2N+1}} \mathrm{tr}(\tau|_{p_{\vec{A}}^{\otimes r}V^{\otimes rn}\cap M^\mu})\cdot\prod_{i=-N}^N z_i^{\mu(i+N+1)}\\
&\quad=\sum_{\substack{\mu\models n,\\ \ell(\mu)\le 2N+1}} \dim(p_{\vec{A}}V^{\otimes n}\cap M^\mu)\cdot\prod_{i=-N}^N z_i^{r\mu(i+N+1)}\\
&\quad=\prod_{\alpha=1}^L\Big[\sum_{\substack{\ell(\mu)\le 2N+1,\\ \mu\models n_\alpha}} \dim(p_{A^\alpha}V^{\otimes n_\alpha}\cap M^\mu)\cdot\prod_{i=-N}^N z_i^{r\mu(i+N+1)}\Big]\\
&\quad=\prod_{\alpha=1}^L sb_{A^\alpha}(z_{-N}^r, z_{1-N}^r,\dots,z_{-1}^r,z_0^r,z_1^r,\dots,z_{N-1}^r,z_N^r).
\end{aligned}$$

Compare with (3-9), we have $a^\lambda = \tilde{c}_{\vec{A}}^\lambda$. □

**Remark 3.1.** Similar computations starting with $U_q(\mathfrak{sp}(2N))$ and $U_q(\mathfrak{so}(2N))$ lead to the same theorem for Kauffman polynomials. Thus, together with the type-A analog proved in [Lin and Zheng 2010, Theorem 5.1], these computations provide formulas of quantum group invariants of torus links associated to simple Lie-algebras of type A, B, C and D.

## 4. Orthogonal Chern–Simons partition function

**4.1.** ***Partition function.*** The *orthogonal Chern–Simons partition function* of $\mathcal{L}$ is defined by

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L};q,t)=\sum_{\vec{\mu}\in\mathcal{P}^L}\frac{pb_{\vec{\mu}}(\vec{z})}{\mathfrak{z}_{\vec{\mu}}}\cdot\sum_{\vec{A}\in\widehat{Br}_{|\vec{\mu}|}}\chi_{\vec{A}}(\gamma_{\vec{\mu}})W_{\vec{A}}^{\mathrm{SO}}(\mathcal{L};q,t).\tag{4-1}$$

This definition is motivated from physicists' path integral approach [Borhade and Ramadevi 2005], and it is different from the definition given by [Borhade and Ramadevi 2005, Equation (4.10)]. Unlike the $\mathrm{SU}(N)$ Chern–Simons partition function, the $Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L};q,t)$ above cannot be simplified to

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L};q,t)=\sum_{\vec{A}\in\mathcal{P}^L}W_{\vec{A}}^{\mathrm{SO}}(\mathcal{L};q,t)\,sb_{\vec{A}}(\vec{z})\tag{4-2}$$

because orthogonality of type-A Schur functions fails in the type-B case [Ram 1991; 1995; 1997].

Define the *free energy* as

$$F^{\mathrm{SO}}(\mathcal{L};q,t)=\log Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L};q,t).\tag{4-3}$$

The partition function of unknots with $L$ components can be computed explicitly (See Proposition 10.2 for details.) In fact we have the following expression for the free energy:

$$F^{\mathrm{SO}}(\bigcirc^L;q,t)=\sum_{n=1}^{+\infty}\frac{1}{n}\cdot\left(1+\frac{t^n-t^{-n}}{q^n-q^{-n}}\right)\cdot\sum_{\alpha=1}^{L}pb_n(z_\alpha).\tag{4-4}$$

**4.2.** ***Reformulated invariants.*** The reformulated link invariants are rational functions $g_{\vec{A}}(q,t)\in\mathbb{C}(q,t)$ determined by the expansion

$$F^{\mathrm{SO}}(\mathcal{L};q,t)=\sum_{d=1}^{\infty}\sum_{\vec{\mu}\neq\vec{0}}\frac{1}{d}g_{\vec{\mu}}(q^d,t^d)\prod_{\alpha=1}^{L}pb_{\mu^\alpha}((z_\alpha)^d).\tag{4-5}$$

As in [Labastida and Mariño 2002], define the operator $\psi_d$ by

$$\psi_d\circ F(q,t;pb(\vec{z}))=F(q^d,t^d;pb(\vec{z}^d)).\tag{4-6}$$

Then define the *plethystic exponential* [Getzler and Kapranov 1998]

$$\operatorname{Exp}(F) = \exp\Bigl(\sum_{k=1}^{+\infty} \frac{\psi_k}{k} \circ F\Bigr) \tag{4-7}$$

and its inverse

$$\operatorname{Log}(F) = \sum_{k=1}^{+\infty} \frac{\mu(k)}{k} \log(\psi_k \circ F), \tag{4-8}$$

where $\mu(k)$ is the Möbius function. In terms of these operators, we can write

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathscr{L}; q, t) = \operatorname{Exp}\Bigl(\sum_{\vec{\mu}\neq\vec{0}} g_{\vec{\mu}}(q, t) \prod_{\alpha=1}^{L} pb_{\mu^\alpha}(z_\alpha)\Bigr). \tag{4-9}$$

We expand the partition function as

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathscr{L}; q, t) = 1 + \sum_{\vec{\mu}\neq\vec{0}} Z_{\vec{\mu}}^{\mathrm{SO}} pb_{\vec{\mu}}(\vec{z}), \tag{4-10}$$

where

$$Z_{\vec{\mu}}^{\mathrm{SO}}(\mathscr{L}; q, t) = \sum_{\vec{A}\in\widehat{Br}_{|\vec{\mu}|}} \frac{\chi_{\vec{A}}(\gamma_{\vec{\mu}})}{\mathfrak{z}_{\vec{\mu}}} W_{\vec{A}}^{\mathrm{SO}}(\mathscr{L}; q, t),$$

and expand the free energy as

$$F^{\mathrm{SO}}(\mathscr{L}; q, t) = \sum_{\vec{\mu}\neq\vec{0}} F_{\vec{\mu}}^{\mathrm{SO}} pb_{\vec{\mu}}(\vec{z}). \tag{4-11}$$

From Lemma 2.1 (which is [Liu and Peng 2010, Lemma 2.3]), we have

$$F_{\vec{\mu}}^{\mathrm{SO}} = \sum_{\substack{\Lambda\in\mathcal{P}(\mathcal{P}^L),\\ |\Lambda|=\vec{\mu}}} \frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!}{\ell(\Lambda)|\operatorname{Aut}\Lambda|} Z_{\Lambda}^{\mathrm{SO}}. \tag{4-12}$$

Clearly $F_{\vec{\mu}}^{\mathrm{SO}}$ is a rational function of $q$ and $t$. The reformulated invariants then can be defined by

$$g_{\vec{\mu}}(q, t) = \sum_{k|\vec{\mu}} \frac{\mu(k)}{k} F_{\vec{\mu}/k}^{\mathrm{SO}}(q^k, t^k),$$

where $\mu(k)$ is the Möbius function.

## 5. Orthogonal Labastida–Mariño–Ooguri–Vafa conjecture

**5.1.** ***Orthogonal LMOV conjecture.*** Now we can state the main conjecture of this paper, which is the analog of LMOV conjecture for orthogonal Chern–Simons theory.

**Conjecture 5.1** (orthogonal LMOV). *The rational function $g_{\vec{\mu}}(q,t) \in \mathbb{Q}(q,t)$ has the property that*

$$\frac{z_{\vec{\mu}}(q-q^{-1})^2 \cdot [g_{\vec{\mu}}(q,t) - g_{\vec{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha} - q^{-\mu_i^\alpha})} \in \mathbb{Z}[q-q^{-1}][t,t^{-1}].$$

We may write the (conjectured) polynomial above as

$$\frac{z_{\vec{\mu}}(q-q^{-1})^2 \cdot [g_{\vec{\mu}}(q,t) - g_{\vec{\mu}}(q,-t)]}{2\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha} - q^{-\mu_i^\alpha})} = \sum_{g\in\mathbb{Z}_+/2}\sum_{\beta\in\mathbb{Z}} N_{\vec{\mu},g,\beta}(q-q^{-1})^{2g}t^\beta.$$

The integers $N_{\vec{\mu},g,\beta}$ (or their linear combinations, depending on a choice of basis) are explained as BPS numbers in string theory [Bouchard et al. 2005; Mariño 2010], and these numbers should coincide with the BPS numbers calculated by Gromov–Witten theory; see for example [Pandharipande 2002; Peng 2007]. Physicists predict that the Gromov–Witten theory of orientifolds is dual to the type-B Chern–Simons theory [Bouchard et al. 2005], that is, the partition functions of these two theories coincide up to some normalization. Thus the integers $N_{\vec{\mu},g,\beta}$ are conjecturally equal to some linear combinations of intersection numbers on the moduli space of stable maps from curves into unoriented manifolds. However, a mathematical construction of such moduli space is still lacking.

**Remark 5.1.** Actually the antisymmetrization $\frac{1}{2}(g_{\vec{\mu}}(q,t) - g_{\vec{\mu}}(q,-t))$ is not necessary for some knots/links. Thus if we expand

$$\frac{z_{\vec{\mu}}(q-q^{-1})^2 g_{\vec{\mu}}(q,t)}{\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell(\mu^\alpha)}(q^{\mu_i^\alpha} - q^{-\mu_i^\alpha})},$$

then we may get more integer coefficients. Readers may find that the proof of most theorems except for some cases of Theorem 1.7 are still valid for this expansion.

**Remark 5.2.** Physicists Bouchard, Florea and Mariño [Bouchard et al. 2005] and more recently Mariño [2010] have similar conjectures for a different partition functions. It seems none of these definitions are equivalent to the definition given here. However, it is pointed out by Mariño that the reformulated invariants may coincide for some examples of torus knots. But we obtain different integer invariants for torus links. Thus the relation between these conjectures are still unclear. It shows that antisymmetrization process is not necessary for some links and knots. Anyway, in next subsection, we will leave the integer coefficient invariants of torus knots and links before antisymmetrization for interested readers to investigate, together with the relationship between the conjecture proposed in [Bouchard et al. 2005; Mariño 2010] and ours.

To describe the behavior of the reformulated invariants near $q=1$, let $q=e^u$ and embed $\mathbb{Q}(q,t)$ into $\mathbb{Q}(t)((u))$. Denote by $\mathrm{val}_u(F^{\mathrm{SO}}_{\vec{\mu}})$ the valuation of $F^{\mathrm{SO}}_{\vec{\mu}}$ in the valuation field $\mathbb{Q}(t)((u))$. This valuation is the same as the zero order of the rational function $F^{\mathrm{SO}}_{\vec{\mu}}$ at $q=1$.

**Conjecture 5.2** (degree). *The valuation* $\mathrm{val}_u(F^{\mathrm{SO}}_{\vec{\mu}})$ *is greater than or equal to* $\ell(\vec{\mu})-2$.

Conjecture 5.2 claims that all the coefficients of lower degree vanish. It is not a consequence of Conjecture 5.1. We will see later that this vanishing is closely related to formulas of Lickorish–Millett type. This kind of degree conjecture is also an important part of [Liu and Peng 2010]. We will prove Conjecture 5.2 in Sections 8 and 9.

**5.2.** ***Torus links as examples supporting the main conjecture.*** In this subsection, we verify the orthogonal LMOV conjecture by testing torus links and knots for small partitions.

Several examples of torus links and knots of type $T(2,k)$ suggest that the antisymmetrization of the reformulated invariants $g_{\vec{\mu}}(q,t)$ in Conjecture 5.1 is necessary. In the following, we will denote $q-q^{-1}$ by $z$ for simplicity. We compute the colored Kauffman polynomials for these examples in Section 10 (the appendix). For tables of integer coefficients $N_{\vec{\mu},g,\beta}$ of these torus links and knots, please refer to Section 10.

**Example 1.** Taking $r=1$, the torus link $T(2,2k)$ has 2 components.

**Case 1A.** Consider the partition (1), (1) for link $T(2,2k)$

Denote by $W_{(n)}(\text{unknot})$ by $W_{(n)}$ in the following computations, where $n\in\mathbb{Z}_{\geq 0}$. It is easy to verify that

$$\begin{aligned}
\mathfrak{z}_{(1),(1)}g_{(1),(1)} &= W_{(1),(1)}-W_{(1)}^2\\
&= q^{2k}\,sb_{(2)}+q^{-2k}\,sb_{(1,1)}+t^{-2k}-sb_{(1)}^2\\
&= \left(\frac{q^{2k+1}+q^{-2k-1}}{q^1+q^{-1}}-1\right)t^2+\left(\frac{q^{2k-1}+q^{-2k+1}}{q+q^{-1}}-1\right)t^{-2}\\
&\qquad -\left(\frac{q^k-q^{-k}}{q-q^{-1}}\right)^2+t^{-2k}.
\end{aligned}$$

Thus all the integer invariant numbers $N_{\vec{\mu},g,\beta}$ equal 0.

For the following cases, please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$.

**Case 1B.** Consider the partition (1, 1), (1) for link $T(2, 2k)$:

$$z_{(1,1),(1)}g_{(1,1),(1)} = W_{(2),(1)} + W_{(1,1),(1)} + W_{(1)} - 2W_{(1),(1)}W_{(1)} - (W_{(2)} + W_{(1,1)} + 1)W_{(1)} + 2W_{(1)}^3$$

It is interesting that the rational function

$$\frac{(q-q^{-1})^2}{(q-q^{-1})^3} z_{(1,1),(1)}g_{(1,1),(1)}(q,t)$$

is already in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$, without antisymmetrization.

The conjectural prediction on $g_{(1,1),(1)}$ is also proved in Section 7. Next we compute $g_{(2),(1)}(T(2,2k))$, which will not be covered by any proof in following sections.

**Case 1C.** Consider the partition (2), (1) for link $T(2, 2k)$:

$$z_{(2),(1)}g_{(2),(1)} = (W_{(2),(1)} - W_{(1,1),(1)} + W_{(1)}) - W_{(1)}(W_{(2)} - W_{(1,1)} + 1).$$

The rational function

$$\frac{(q-q^{-1})^2}{(q-q^{-1})(q^2-q^{-2})} z_{(2),(1)}g_{(2),(1)}(q,t)$$

is also in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ without antisymmetrization.

Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$ after antisymmetrization.

The behavior of $g_{(2),(2)}(T(2,2k);q,t)$ is much different from the three examples above. It is the first example that the multicover contribution must be taken into account.

**Case 1D.** Consider the partition (2), (2) for link $T(2, 2k)$:

$$z_{(2),(2)}g_{(2),(2)} = W_{(2),(2)} - 2W_{(2),(1,1)} + W_{(1,1),(1)} - W_{(2)}^2 - W_{(1,1)}^2 + 2W_{(2)}W_{(1,1)} - 2W_{(1),(1)}(q^2,t^2) + 2W_{(1)}^2(q^2,t^2).$$

The rational function

$$\frac{(q-q^{-1})^2}{(q^2-q^{-2})^2} z_{(2),(2)}g_{(2),(2)}(q,t)$$

is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ and antisymmetrization is necessary here.

**Case 1E.** Consider the partition (3), (1) for link $T(2, 2k)$:

$$z_{(3),(1)}g_{(3),(1)} = (W_{(3),(1)} - W_{(2,1),(1)} + W_{(1,1,1),(1)}) - W_{(1)}(W_{(3)} - W_{(2,1)} + W_{(1,1,1)}).$$

The rational function
$$\frac{(q-q^{-1})^2}{(q^3-q^{-3})(q-q^{-1})} z_{(3),(1)} g_{(3),(1)}(q,t)$$
is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ and antisymmetrization is necessary.

**Example 2.** Consider the torus knots $T(2,k)$, where $k$ is an odd integer.

**Case 2A.** Consider the partition $(1,1)$ for link $T(2,k)$

The following calculation provides an example of the case proved in Section 7. We have
$$z_{(1,1)} g_{(1,1)} = W_{(2)} + W_{(1,1)} - W_{(1)}^2(q,t) + 1$$

The rational function
$$\frac{(q-q^{-1})^2}{(q-q^{-1})^2} z_{(1,1)} g_{(1,1)}(q,t)$$
is already in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$ without antisymmetrization.

**Case 2B.** Consider the partition $(2)$ for link $T(2,k)$. We have
$$z_{(2)} g_{(2)} = W_{(2)} - W_{(1,1)} - W_{(1)}(q^2,t^2) + 1.$$
The rational function
$$\frac{(q-q^{-1})^2}{(q^2-q^{-2})} z_{(2)} g_{(2)}(q,t)$$
is not in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$. Please see Section 10 for the table of integers $N_{\vec{\mu},g,\beta}$ after antisymmetrization.

**Example 3.** Taking $r=1$, the torus link $T(3,3k)$ has 3 components.

Consider the partition $(2),(1),(1)$ for the link $T(3,3k)$.

Denote $W_{(1),(1)}(T(2,2k))$ simply by $W_{(1),(1)}$ in the following computations.

We have
$$\begin{aligned} z_{(2),(1),(1)} g_{(2),(1),(1)} = {} & W_{(2),(1),(1)} - W_{(1,1),(1),(1)} - W_{(1),(1)} W_{(2)} + W_{(1),(1)} W_{(1,1)} \\ & - 2W_{(2),(1)} W_{(1)} + 2W_{(1,1),(1)} W_{(1)} + 2W_{(2)} W_{(1)}^2 - 2W_{(1,1)} W_{(1)}^2 \end{aligned}$$
and rational function
$$\frac{(q-q^{-1})^2}{(q^2-q^{-2})(q-q^{-1})^2} z_{(2),(1),(1)} g_{(2),(1),(1)}(q,t)$$
is in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$.

Until now, we have seen the orthogonal LMOV conjecture is valid for the knots $T(2,k)$ and $T(3,3k)$ when $k$ is small.

In fact, we can prove it for arbitrary $k \in \mathbb{Z}_{>0}$.

For instance, we investigate torus knot $T(2,k)$ for odd integer number $k$ with partition (2). We can express

$$\mathrm{z}_{(2)}(g_{(2)}(q,t)-g_{(2)}(q,-t))/2$$

in terms of *pb* polynomials instead of *sb* polynomials. After simplification, we have

$$\begin{aligned}
&\frac{\mathrm{z}_{(2)}(g_{(2)}(q,t)-g_{(2)}(q,-t))}{2}\\
&=t^{-2k}\frac{t-t^{-1}}{q-q^{-1}}\biggl(\frac{(q^{2k}-q^{-2k})}{(q-q^{-1})(q^3-q^{-3})}\bigl(-(t^2+t^{-2})(q^{2k}+q^{-2k}-q^2-q^{-2})\\
&\qquad\qquad +(q^4+q^{-4})(q^{2k}+q^{-2k})-q^4-2-q^{-4}\bigr)\\
&\qquad\qquad +\frac{t^{-k}(q^{4k}-q^{-4k})}{q^2-q^{-2}}(-t(q^{k+1}-q^{-k-1})+t^{-1}(q^{k-1}-q^{-k+1}))\biggr),
\end{aligned}$$

By a tedious discussion on the residue of $k$ modulo 6, one can see that the rational function

$$\frac{(q-q^{-1})^2}{2(q^2-q^{-2})}(g_{(2)}(q,t)-g_{(2)}(q,-t))$$

is in the ring $\mathbb{Z}[t,t^{-1}][q-q^{-1}]$. Actually, all these examples can be proved in this way.

## 6. Formulas of Lickorish–Millett type

The Skein relations of Kauffman polynomials are

(1) $\langle\mathscr{L}_+\rangle-\langle\mathscr{L}_-\rangle=z(\langle\mathscr{L}_{||}\rangle-\langle\mathscr{L}_=\rangle)$, where $\mathscr{L}_+$, $\mathscr{L}_-$, $\mathscr{L}_{||}$ and $\mathscr{L}_=$ stand for positive crossing, negative crossing, vertical resolution and horizontal resolution respectively,

(2) $\langle\mathscr{L}^{+\mathrm{kink}}\rangle=t\langle\mathscr{L}\rangle$ and $\langle\mathscr{L}^{-\mathrm{kink}}\rangle=t^{-1}\langle\mathscr{L}\rangle$.

The variable $z$ is our $q-q^{-1}$ in previous sections, and the Kauffman brackets are given by

$$K_{\mathscr{L}}(z,t)=t^{-w(\mathscr{L})}\langle\mathscr{L}\rangle,$$

where the writhe number of link $w(\mathscr{L})=2\,lk(\mathscr{L})+\sum_{\alpha=1}^{L}w(\mathcal{K}_\alpha)$, with the normalization $K_{\bigcirc}(z,t)=1$ for the unknot $\bigcirc$. In terms of quantum group invariants, we have

$$W^{\mathrm{SO}}_{(1)^L}(\mathscr{L})=(1+(t-t^{-1})/z)t^{-\sum_{\alpha=1}^{L}w(\mathcal{K}_\alpha)}\langle\mathscr{L}\rangle.$$

The Kauffman polynomials admit the expansions

$$K_{\mathscr{L}}(z,t)=\sum_{g\geq 0}\tilde{p}^{\mathscr{L}}_{g+1-L}(t)z^{g+1-L}\quad\text{and}\quad\langle\mathscr{L}\rangle=\sum_{g\geq 0}p^{\mathscr{L}}_{g+1-L}(t)z^{g+1-L}$$

with respect to variable $z$. The classical Lickorish–Millett formula [1987] reads

$$\tilde{p}^{\mathcal{L}}_{1-L}(t) = t^{-2\,lk(\mathcal{L})}(t-t^{-1})^{L-1}\prod_{\alpha=1}^{L}\tilde{p}^{\mathcal{K}_\alpha}_0(t)$$

and so

$$p^{\mathcal{L}}_{1-L}(t) = (t-t^{-1})^{L-1}\prod_{\alpha=1}^{L}p^{\mathcal{K}_\alpha}_0(t),$$

which gives a concrete description of $\tilde{p}^{\mathcal{L}}_{1-L}(t)$, the coefficient of the lowest degree terms of $K_{\mathcal{L}}(z,t)$, in terms of invariants of the subknots $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_L$ of $\mathcal{L}$. In the following theorem, we provide explicit formulas for $p^{\mathcal{L}}_{2-L}(t)$ and $p^{\mathcal{L}}_{3-L}(t)$, which are regarded as higher Lickorish–Millett relations. These formulas can be proved purely by skein relations. Through resolving intersections at different link components, it is not hard to prove the following. Also, these formulas can be directly deduced from Conjecture 5.2 (see Section 7). Kanenobu [2006] got some relationships (nonexplicit) between these terms.

**Theorem 6.1.** *Let $\mathcal{L}_{1,2}$ be the sublink of $\mathcal{L}$ which composed of components $K_1$ and $K_2$. The coefficients $p^{\mathcal{L}}_{2-L}(t)$ and $p^{\mathcal{L}}_{3-L}(t)$ are given by the formulas*

$$\begin{aligned}
p^{\mathcal{L}}_{2-L}(t) &= (L-1)(t-t^{-1})^{L-2}p^{\mathcal{K}_1}_0(t)\cdots p^{\mathcal{K}_L}_0(t)\\
&\qquad\qquad + (t-t^{-1})^{L-1}(p^{\mathcal{K}_1}_1(t)p^{\mathcal{K}_2}_0(t)\cdots p^{\mathcal{K}_L}_0(t) + perm);\\
p^{\mathcal{L}}_{3-L}(t) &= \binom{L-1}{2}(t-t^{-1})^{L-3}p^{\mathcal{K}_1}_0(t)\cdots p^{\mathcal{K}_L}_0(t)\\
&\qquad\qquad + (t-t^{-1})^{L-2}(p^{\mathcal{L}_{1,2}}_1(t)p^{\mathcal{K}_3}_0(t)\cdots p^{\mathcal{K}_L}_0(t) + perm)\\
&\qquad\qquad - (L-2)(t-t^{-1})^{L-1}(p^{\mathcal{K}_1}_2(t)p^{\mathcal{K}_2}_0(t)\cdots p^{\mathcal{K}_L}_0(t) + perm).
\end{aligned}$$

*Proof.* The formulas in the theorem are obvious when $L=1$, and the formula for $p^{\mathcal{L}}_{3-L}(t)$ is also valid for $L=2$. We proceed by induction. Let $\mathcal{L}$ be a link with $L+1$ components. The main idea is to use skein relations at the intersection points of different components of the $\mathcal{L}$ until the component $\mathcal{K}_{L+1}$ splits from the link.

First we apply the skein relation at the crossings between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ until there is no intersection between them. We need to apply the skein relation $(n^+_{1,L+1}+n^-_{1,L+1})/2$ times, where $n^+_{1,L+1}$ and $n^-_{1,L+1}$ denote the number of positive and negative crossings between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$, respectively. Thus the linking number between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ is $lk(\mathcal{L}_{1,L+1}) = (n^+_{1,L+1}-n^-_{1,L+1})/2$.

From the calculation, one can see that using the skein relation at a positive crossing will lead similar result. Thus without loss of generality, we can assume $n^-_{1,L+1}>0$ and apply the skein relation at a negative crossing first:

$$\langle\mathcal{L}_+\rangle - \langle\mathcal{L}_-\rangle = z(\langle\mathcal{L}_{(1\|L+1),2,\ldots,L}\rangle - \langle\mathcal{L}_{(1=L+1),2,\ldots,L}\rangle),$$

where $\mathcal{L}_-$ is the original link $\mathcal{L}$, $(1 \parallel L+1)$ (respectively $(1 = L+1)$) is the new knot component derived from $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ by taking vertical (respectively horizontal) lines as its resolution at the intersection of $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ in the new links $\mathcal{L}_{(1\parallel L+1),2,\dots,L}$ (respectively $\mathcal{L}_{(1=L+1),2,\dots,L}$). Both new links have $L$ components, while $\mathcal{L}_+$ is the link obtained simply by changing the sign of the chosen crossing, and thus has the same $L+1$ components as the original link $\mathcal{L} = \mathcal{L}_-$.

Taking a few leading terms in the skein relation formula, we get

$$\begin{aligned}(p_{-L}^{\mathcal{L}_+}(t)z^{-L} + p_{1-L}^{\mathcal{L}_+}(t)z^{1-L} + p_{2-L}^{\mathcal{L}_+}(t)z^{2-L}) \\ - (p_{-L}^{\mathcal{L}}(t)z^{-L} + p_{1-L}^{\mathcal{L}}(t)z^{1-L} + p_{2-L}^{\mathcal{L}}(t)z^{2-L}) \\ = z(p_{1-L}^{\mathcal{L}_{(1\|L+1),2,\dots,L}}(t)z^{1-L} - p_{1-L}^{\mathcal{L}_{(1=L+1),2,\dots,L}}(t)z^{1-L})\end{aligned}$$

and comparing the coefficients, we find

(1) $p_{-L}^{\mathcal{L}_+}(t) = p_{-L}^{\mathcal{L}}(t)$ (this one gives the formula for $p_{1-L}^{\mathcal{L}}$, which we don't use),

(2) $p_{1-L}^{\mathcal{L}_+}(t) = p_{1-L}^{\mathcal{L}}(t)$,

(3) $p_{2-L}^{\mathcal{L}_+}(t) - p_{2-L}^{\mathcal{L}}(t) = p_{1-L}^{\mathcal{L}_{(1\|L+1),2,\dots,L}} - p_{1-L}^{\mathcal{L}_{(1=L+1),2,\dots,L}}$.

By the Lickorish–Millett formula,

$$\begin{aligned}p_{1-L}^{\mathcal{L}_{(1\|L+1),2,\dots,L}} &= (t-t^{-1})^{L-1} p_0^{\mathcal{K}_2}(t)\cdots p_0^{\mathcal{K}_L}(t)\, p_0^{\mathcal{K}_{(1\|L+1)}}(t), \\ p_{1-L}^{\mathcal{L}_{(1=L+1),2,\dots,L}} &= (t-t^{-1})^{L-1} p_0^{\mathcal{K}_2}(t)\cdots p_0^{\mathcal{K}_L}(t)\, p_0^{\mathcal{K}_{(1=L+1)}}(t),\end{aligned}$$

where $\mathcal{K}_{(1\|L+1)}$ (respectively $\mathcal{K}_{(1=L+1)}$) is the knot derived from the sublink $\mathcal{L}_{1,L+1}$ by taking vertical (respectively horizontal) lines as its resolution at the chosen crossing. Thus

$$p_{2-L}^{\mathcal{L}_+} - p_{2-L}^{\mathcal{L}} = (t-t^{-1})^{L-1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}(p_0^{\mathcal{K}_{(1\|L+1)}} - p_0^{\mathcal{K}_{(1=L+1)}}). \tag{6-1}$$

We play a trick here to find the expression for $p_0^{\mathcal{K}_{(1\|L+1)}}$ and $p_0^{\mathcal{K}_{(1=L+1)}}$. Consider the sublink $\mathcal{L}_{1,L+1}$ of $\mathcal{L}$, which has only two components $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$. Then use the skein relation at exactly the same crossing as we did in the original link $\mathcal{L}$. The same argument leads to

$$p_1^{(\mathcal{L}_{1,L+1})_+} - p_1^{\mathcal{L}_{1,L+1}} = p_0^{\mathcal{K}_{(1\|L+1)}} - p_0^{\mathcal{K}_{(1=L+1)}},$$

which substituted back gives

$$p_{2-L}^{\mathcal{L}_+} - p_{2-L}^{\mathcal{L}} = (t-t^{-1})^{L-1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L}(p_1^{(\mathcal{L}_{1,L+1})_+} - p_1^{\mathcal{L}_{1,L+1}}). \tag{6-2}$$

In the equation above, $p_{2-L}^{\mathcal{L}}$ is expressed in terms of invariants of $\mathcal{L}_+$ and some simple terms. Then we apply the skein relations $\mathcal{L}_+$ at other intersection points between $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ until these two components become unlinked. We cancel all the middle states in this procedure, and finally we reach

$$p_{2-L}^{\mathcal{L}^{(1)}} - p_{2-L}^{\mathcal{L}} = (t-t^{-1})^{L-1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} \bigl(p_1^{\mathcal{L}^{(1)}_{1,L+1}} - p_1^{\mathcal{L}_{1,L+1}}\bigr). \tag{6-3}$$

Here $\mathcal{L}^{(1)}$ is the final state of $\mathcal{L}$, in which $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ are unlinked, and $\mathcal{L}^{(1)}_{1,L+1}$ is the corresponding final state of $\mathcal{L}_{1,L+1}$ under the same procedure of skein relations. In $\mathcal{L}^{(1)}_{1,L+1}$, the two components $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$ are unlinked too, that is, $\mathcal{L}^{(1)}_{1,L+1}$ is the disjoint union of two knots $\mathcal{K}_1$ and $\mathcal{K}_{L+1}$:

$$W^{\mathrm{SO}}_{(1),(1)}(\mathcal{L}^{(1)}_{1,L+1}) = W^{\mathrm{SO}}_{(1)}(\mathcal{K}_1) W^{\mathrm{SO}}_{(1)}(\mathcal{K}_{L+1}).$$

By the definition of Kauffman polynomials,

$$W^{\mathrm{SO}}(\mathcal{L}) = \Bigl(1 + \frac{t-t^{-1}}{z}\Bigr) t^{-\sum_{\alpha=1}^{L} w(\mathcal{K}_\alpha)} \langle \mathcal{L} \rangle,$$

so for all links $\mathcal{L}$, we have

$$\langle \mathcal{L}^{(1)}_{1,L+1} \rangle = \Bigl(1 + \frac{t-t^{-1}}{z}\Bigr) \langle \mathcal{K}_1 \rangle \langle \mathcal{K}_{L+1} \rangle.$$

Up to the third leading terms, we have

$$\begin{aligned} &p_{-1}^{\mathcal{L}^{(1)}_{1,L+1}} z^{-1} + p_0^{\mathcal{L}^{(1)}_{1,L+1}} + p_1^{\mathcal{L}^{(1)}_{1,L+1}} z \\ &\qquad = \Bigl(1 + \frac{t-t^{-1}}{z}\Bigr)(p_0^{\mathcal{K}_1} + p_1^{\mathcal{K}_1} z + p_2^{\mathcal{K}_1} z^2)(p_0^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_{L+1}} z + p_2^{\mathcal{K}_{L+1}} z^2) \end{aligned}$$

and comparing the coefficients, we have

$$\begin{aligned} p_{-1}^{\mathcal{L}^{(1)}_{1,L+1}} &= (t-t^{-1}) p_0^{\mathcal{K}_1} p_0^{\mathcal{K}_{L+1}}, \\ p_0^{\mathcal{L}^{(1)}_{1,L+1}} &= p_0^{\mathcal{K}_1} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_0^{\mathcal{K}_1} p_1^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_{L+1}}), \\ p_1^{\mathcal{L}^{(1)}_{1,L+1}} &= p_0^{\mathcal{K}_1} p_1^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_0^{\mathcal{K}_1} p_2^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_1} p_1^{\mathcal{K}_{L+1}} + p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_{L+1}}). \end{aligned}$$

In summary, we now have

$$\begin{aligned} &p_{2-L}^{\mathcal{L}^{(1)}} - p_{2-L}^{\mathcal{L}} \\ &\quad = (t-t^{-1})^{L-1} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_1^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1} p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} \\ &\qquad + (t-t^{-1})^{L} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_2^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L} p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} p_1^{\mathcal{K}_{L+1}} \\ &\qquad\quad + (t-t^{-1})^{L} p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} - (t-t^{-1})^{L-1} p_1^{\mathcal{L}_{1,L+1}} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L}. \end{aligned}$$

Next perform all the procedures above for the link $\mathcal{L}^{(1)}$ to obtain the final state $\mathcal{L}^{(1)(2)}$ in which the components $\mathcal{K}_2$ and $\mathcal{K}_{L+1}$ are unlinked.

Repeat this process totally $L$ times until $\mathcal{K}_{L+1}$ is not linked to the sublink $\mathcal{L}_{1,\dots,L}$ of $\mathcal{L}$; the result is given by

$$\begin{aligned}
&p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} - p_{2-L}^{\mathcal{L}}\\
&= L(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_1^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L-1}(p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_0^{\mathcal{K}_{L+1}}\\
&\qquad + L(t-t^{-1})^{L} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_2^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L}(p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_1^{\mathcal{K}_{L+1}}\\
&\quad + (t-t^{-1})^{L}(p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_0^{\mathcal{K}_{L+1}} - (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,L+1}} p_0^{K_2}\cdots p_0^{\mathcal{K}_L} + \mathrm{perm}).
\end{aligned}$$

Since the link $\mathcal{L}^{(1)\cdots(L)}$ is the disjoint union of the sublink $\mathcal{L}_{1,\dots,L}$ of $\mathcal{L}$ and the knot $\mathcal{K}_{L+1}$,

$$W_{(1)^{L+1}}^{\mathrm{SO}}(\mathcal{L}^{(1)\cdots(L)}) = W_{(1)^L}^{\mathrm{SO}}(\mathcal{L}_{1,\dots,L}) W_{(1)}^{\mathrm{SO}}(\mathcal{K}_{L+1}).$$

Again, this can be rewritten in the form

$$\langle \mathcal{L}^{(1)\cdots(L)}\rangle = \Big(1+\frac{t-t^{-1}}{z}\Big)\langle \mathcal{L}_{1,\dots,L}\rangle\langle \mathcal{K}_{L+1}\rangle.$$

Up to third leading terms, we have

$$\begin{aligned}
&p_{-L}^{\mathcal{L}^{(1)\cdots(L)}} z^{-L} + p_{-L}^{\mathcal{L}^{(1)\cdots(L)}} z^{1-L} + p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} z^{2-L}\\
&= \Big(1+\frac{t-t^{-1}}{z}\Big)(p_{1-L}^{\mathcal{L}_{1,\dots,L}} z^{1-L} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} z^{2-L} + p_{3-L}^{\mathcal{L}_{1,\dots,L}} z^{3-L})(p_0^{\mathcal{K}_{L+1}} + p_1^{\mathcal{K}_{L+1}} z + p_2^{\mathcal{K}_{L+1}} z^2).
\end{aligned}$$

Comparing the coefficients, we have

$$\begin{aligned}
p_{-L}^{\mathcal{L}^{(1)\cdots(L)}} &= (t-t^{-1}) p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}},\\
p_{1-L}^{\mathcal{L}^{(1)\cdots(L)}} &= p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}),\\
p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} &= p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}\\
&\qquad + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_2^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{3-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}).
\end{aligned}$$

We now can finish the proof by induction. Be careful that our link $\mathcal{L}$ has $L+1$ components. The sublink $\mathcal{L}_{1,\dots,L}$ has $L$ components, and by induction

$$p_{2-L}^{\mathcal{L}_{1,\dots,L}} = (L-1)(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} + (t-t^{-1})^{L}(p_1^{\mathcal{K}_1} p_0^{\mathcal{K}_2}\cdots p_0^{\mathcal{K}_L} + \mathrm{perm}),$$

so

$$\begin{aligned}
p_{2-(L+1)}^{\mathcal{L}}(t) &= p_{1-L}^{\mathcal{L}^{(1)\cdots(L)}}\\
&= p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}})\\
&= L(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^{L}(p_1^{\mathcal{K}_1}\cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} + \mathrm{perm}).
\end{aligned}$$

This finishes the proof of the first part of the theorem.

Now we have enough results to prove the second part. We have seen that

$$\begin{aligned}
p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} &= p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}} \\
&\qquad + (t-t^{-1})(p_{1-L}^{\mathcal{L}_{1,\dots,L}} p_2^{\mathcal{K}_{L+1}} + p_{2-L}^{\mathcal{L}_{1,\dots,L}} p_1^{\mathcal{K}_{L+1}} + p_{3-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}) \\
&= L(t-t^{-1})^{L-1} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_1^{\mathcal{K}_{L+1}} + (L-1)(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} \\
&\quad + (t-t^{-1})^{L-1}(p_1^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_0^{\mathcal{K}_{L+1}} + (t-t^{-1})^L p_0^{\mathcal{K}_1} \cdots p_0^{K_L} p_2^{\mathcal{K}_{L+1}} \\
&\quad + (t-t^{-1})^L (p_1^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_1^{\mathcal{K}_{L+1}} + (t-t^{-1}) p_{3-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}.
\end{aligned}$$

Combined with the expression of $p_{2-L}^{\mathcal{L}^{(1)\cdots(L)}} - p_{2-L}^{\mathcal{L}}$, we get an expression for $p_{2-L}^{\mathcal{L}}$ in terms of sublinks:

$$\begin{aligned}
p_{2-L}^{\mathcal{L}} &= (L-1)(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_0^{\mathcal{K}_{L+1}} - (L-1)(t-t^{-1})^L p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_L} p_2^{\mathcal{K}_{L+1}} \\
&\quad - (t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) p_0^{\mathcal{K}_{L+1}} \\
&\quad + (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,L+1}} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_L} + \mathrm{perm}) + (t-t^{-1}) p_{3-L}^{\mathcal{L}_{1,\dots,L}} p_0^{\mathcal{K}_{L+1}}.
\end{aligned}$$

Since the sublink $\mathcal{L}_{1,\dots,L}$ of $\mathcal{L}$ contains $L$ components, by induction we have

$$\begin{aligned}
p_{3-L}^{\mathcal{L}_{1,\dots,L}}(t) &= \binom{L-1}{2}(t-t^{-1})^{L-3} p_0^{\mathcal{K}_1}(t) \cdots p_0^{\mathcal{K}_L}(t) \\
&\quad + (t-t^{-1})^{L-2}(p_1^{\mathcal{L}_{1,2}}(t) p_0^{\mathcal{K}_3}(t) \cdots p_0^{\mathcal{K}_L}(t) + \mathrm{perm}) \\
&\quad - (L-2)(t-t^{-1})^{L-1}(p_2^{\mathcal{K}_1}(t) p_0^{\mathcal{K}_2}(t) \cdots p_0^{\mathcal{K}_L}(a) + \mathrm{perm}).
\end{aligned}$$

Here the permutation only involves the first $L$ components. Later, when computing the invariants of $\mathcal{L}$, the permutations will also include the $(L+1)$-st component. Since the content is self-evident, we will not mention this issue again. Substituting the induction above formula into the expression for $\mathcal{L}$ finishes the proof of the second part of the theorem:

$$\begin{aligned}
p_{2-L}^{\mathcal{L}} &= \binom{L}{2}(t-t^{-1})^{L-2} p_0^{\mathcal{K}_1} \cdots p_0^{\mathcal{K}_{L+1}} \\
&\quad + (t-t^{-1})^{L-1}(p_1^{\mathcal{L}_{1,2}} p_0^{\mathcal{K}_3} \cdots p_0^{\mathcal{K}_{L+1}} + \mathrm{perm}) \\
&\quad - (L-1)(t-t^{-1})^L (p_2^{\mathcal{K}_1} p_0^{\mathcal{K}_2} \cdots p_0^{\mathcal{K}_{L+1}} + \mathrm{perm}). \quad \square
\end{aligned}$$

## 7. The proof of the conjecture for the column diagram

In the last section, we provide two formulas of Lickorish–Millett type. In general, similar computations lead to expressions for $p_n^{\mathcal{L}}(t)$ in terms of invariants of sublinks of $\mathcal{L}$. Each additional component of $\mathcal{L}$ gives rise to two such relations; thus we expect that there should be $2L-2$ such relations, that is, we should be able to describe $p_n^{\mathcal{L}}(t)$ for $1-L \le n \le L$ by sublinks of $\mathcal{L}$.

When the index $n$ increases, the expression become messy. To give a unified treatment, we formulate the problem in terms of the partition function $Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L}; q, t)$ and free energy $F^{\mathrm{SO}}(\mathcal{L}; q, t)$. Recall that we write

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathcal{L}; q, t) = 1 + \sum_{\vec{\mu} \neq \vec{0}} Z_{\vec{\mu}}^{\mathrm{SO}} pb_{\vec{\mu}} \quad \text{and} \quad F^{\mathrm{SO}}(\mathcal{L}; q, t) = \sum_{\vec{\mu} \neq \vec{0}} F_{\vec{\mu}}^{\mathrm{SO}} pb_{\vec{\mu}},$$

where $\vec{\mu} = (\mu^1, \dots, \mu^L)$ for partitions $\mu^1, \dots, \mu^L$. In this section, we mainly focus on the situation when all the $\mu^i$ are columnlike partitions. We first look at the first situation in which all the $\mu^i$ are partitions 1. We may simply denote such $\vec{\mu}$ by $(1)^L = (1), \dots, (1)$ since the partition of 1 is unique and there is no ambiguity. The coefficients $Z_{(1)^L}^{\mathrm{SO}} = W_{(1)^L}^{\mathrm{SO}}$.

Let $\Delta$ be a subset of the set $[L] := \{1, \dots, L\}$. Write $\mathcal{L}_\Delta$ for the sublink of $\mathcal{L}$ comprising only the components with labels in $\Delta$. For example, when $\Delta = \{1, 2\}$, $\mathcal{L}_\Delta$ is the link $\mathcal{L}_{1,2}$ discussed in the previous section. We also denote by $\Delta$ the partition $\vec{\mu} = (\mu^1, \dots, \mu^L)$ such that $\mu^i = (1)$ if $i \in \Delta$, and 0 otherwise. The convention in the definition of quantum group invariants is $W_\Delta^{\mathrm{SO}}(\mathcal{L}) := W_{(1)^{|\Delta|}}^{\mathrm{SO}}(\mathcal{L}_\Delta)$. The formula (4-12) then can be written as

$$F_{(1)^L}^{\mathrm{SO}}(\mathcal{L}) = \sum_{r=1}^{L} \frac{(-1)^{r-1}}{r} \sum_{\Delta_1, \dots, \Delta_r} \prod_{i=1}^{r} W_{\Delta_i}^{\mathrm{SO}}(\mathcal{L}), \tag{7-1}$$

where the second sum is over all nonempty subsets $\Delta_1, \dots, \Delta_r$ that form a partition of the set $[L]$. We have seen that $F_{(1)^L}^{\mathrm{SO}}(\mathcal{L}) \in \mathbb{Q}(t)((z))$ for $z = q - q^{-1}$ has an expansion

$$F_{(1)^L}^{\mathrm{SO}}(\mathcal{L}) = \sum_{i \geq -L} a_i(t) z^i.$$

Conjecture 5.2 predicts that $\mathrm{val}_z\, F_{(1)^L}^{\mathrm{SO}}(\mathcal{L}) \geq L - 2$, that is, $a_{-L} = a_{1-L} = \cdots = a_{L-3} = 0$. We now prove $a_{-L} = a_{1-L} = a_{2-L} = 0$ by the classical Lichorish–Millett theorem and the two formulas derived in last section.

**Theorem 7.1.** *Expand $F_{(1)^L}^{\mathrm{SO}}(\mathcal{L})$ as above. Then we have the vanishing result $a_{-L} = a_{1-L} = a_{2-L} = 0$ if $L \geq 3$. In other words,*

$$\mathrm{val}_u(F_{(1)^L}^{\mathrm{SO}}(\mathcal{L})) = \mathrm{val}_z(F_{(1)^L}^{\mathrm{SO}}(\mathcal{L})) \geq 3 - L.$$

*In the case $L = 2$, the second formula in Theorem 6.1 is empty; thus we only have $a_{-2} = a_{-1} = 0$ and $\mathrm{val}_z(F_{(1)^2}^{\mathrm{SO}}(\mathcal{L})) \geq 0$.*

*Proof.* We prove the theorem for $a_{1-L}$ when $L \geqslant 2$ by calculating (7-1). The proofs for $a_{-L}$ and $a_{2-L}$ are similar and we leave them to the reader.

$$W_{\Delta}^{\mathrm{SO}}(\mathscr{L}) \cong \Big(1+\frac{t-t^{-1}}{z}\Big)t^{-\sum_{\alpha\in\Delta} w(\mathscr{K}_\alpha)} \cdot \big(p_{1-|\Delta|}^{\mathscr{L}_\Delta} z^{1-|\Delta|} + p_{2-|\Delta|}^{\mathscr{L}_\Delta} z^{2-|\Delta|} + p_{3-|\Delta|}^{\mathscr{L}_\Delta} z^{3-|\Delta|}\big) \pmod{z^{3-|\Delta|}}.$$

Denote by $[z^n]f$ the coefficient of $z^n$ in $f \in \mathbb{Q}(t)((z))$.

$$a_{1-L} = t^{-\sum_{\alpha=1}^{L} w(\mathscr{K}_\alpha)} \sum_{r=1}^{L} \frac{(-1)^{r-1}}{r} \sum_{\Delta_1,\dots,\Delta_r} [z^{1-L}]\Big(1+\frac{t-t^{-1}}{z}\Big)^r \prod_{i=1}^{r} \langle \mathscr{L}_{\Delta_i}\rangle.$$

For each possible collection $\Delta_1, \dots, \Delta_r$,

$$\begin{aligned}
&[z^{1-L}]\Big(1+\frac{t-t^{-1}}{z}\Big)^r \prod_{i=1}^{r}\langle\mathscr{L}_{\Delta_i}\rangle \\
&= r(t-t^{-1})^{r-1}\prod_{i=1}^{r} p_{1-|\Delta_i|}^{\mathscr{L}_{\Delta_i}} + (t-t^{-1})^r \sum_{i=1}^{r} p_{1-|\Delta_1|}^{\mathscr{L}_{\Delta_1}} \cdots \widehat{p_{1-|\Delta_i|}^{\mathscr{L}_{\Delta_i}}} \cdots p_{1-|\Delta_r|}^{\mathscr{L}_{\Delta_r}} \cdot p_{2-|\Delta_i|}^{\mathscr{L}_{\Delta_i}} \\
&= L(t-t^{-1})^{L-1}\prod_{\alpha=1}^{L} p_0^{K_\alpha}(t) + (t-t^{-1})^L \sum_{j=1}^{L} p_1^{K_j} \prod_{\substack{i=1\\ i\neq j}}^{L} p_0^{K_i}
\end{aligned}$$

has the same contribution. We need to count the number of these collections. Let $\Lambda$ be a partition of $L$ of length $r$. The number of collections $\{\Delta_1, \dots, \Delta_r\}$ with $\{|\Delta_1|, \dots, |\Delta_r|\}$ equal to the partition $\Lambda$ is given by $\frac{r!}{|\mathrm{Aut}\,\Lambda|} \cdot \frac{L!}{\Lambda_1!\cdots\Lambda_r!}$; hence

$$a_{1-L} = t^{-\sum_{\alpha=1}^{L} w(\mathscr{K}_\alpha)}\Big(L(t-t^{-1})^{L-1}\prod_{\alpha=1}^{L} p_0^{K_\alpha}(t) + (t-t^{-1})^L \sum_{j=1}^{L} p_1^{K_j} \prod_{\substack{i=1,\\ i\neq j}}^{L} p_0^{K_i}\Big) \cdot \sum_{\Lambda\vdash L} \frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!}{\ell(\Lambda)|\mathrm{Aut}\,\Lambda|} \cdot \frac{L!}{\Lambda!}$$

which is zero by the following Lemma 7.2 since $L \geq 2$. □

**Lemma 7.2.** *Assume $d_\alpha \geq 1$ for $\alpha = 1, 2, \dots, L$ and the sum $d = d_1 + \cdots + d_L$ is strictly greater than* 1 (*that is, if all $d_i = 1$, then we assume $L > 1$*). *Then*

$$\sum_{\vec{\lambda}\vdash\vec{d}} \frac{(-1)^{\ell(\vec{\lambda})-1}\ell(\vec{\lambda})!}{\ell(\vec{\lambda})|\mathrm{Aut}\,\vec{\lambda}|\prod_{\alpha=1}^{L}\prod_{j=1}^{\ell(\lambda^\alpha)}\lambda_j^\alpha!} = 0. \tag{7-2}$$

*Proof.* Let $\vec{t}=(t_1,\dots,t_L)$ and $|\vec{t}|=t_1+\cdots+t_L$ in the trivial equality

$$|\vec{t}|=\log(\exp(|\vec{t}|))=\log\Big(1+\sum_{n=1}^{+\infty}|\vec{t}|^n/n!\Big),$$

so we have

$$t_1+\cdots+t_L=\log\Big(1+\sum_{\vec{\beta}\in\mathbb{Z}^L_{\geq 0},\vec{\beta}\neq 0}\vec{t}^{\vec{\beta}}/\vec{\beta}!\Big),$$

where we have adopted the notation $\vec{t}^{\vec{\beta}}=\prod_{\alpha=1}^L t_\alpha^{\beta_\alpha}$ and $\vec{\beta}!=\prod_{\alpha=1}^L\beta_\alpha!$. Expand the logarithm as

$$t_1+\cdots+t_L=\sum_{\vec{\beta}\in\mathbb{Z}^L_{\geq 0}}\vec{t}^{\vec{\beta}}\sum_{\vec{\lambda}\vdash\vec{\beta}}\frac{(-1)^{\ell(\vec{\lambda})-1}}{\ell(\vec{\lambda})|\mathrm{Aut}\,\vec{\lambda}|\prod_{\alpha=1}^L\lambda^\alpha!}.$$

Comparing the coefficients of the term $t_1^{d_1}\cdots t_L^{d_L}$ gives the vanishing formula. $\square$

We remark that the vanishing of $a_{1-L}$ and $a_{2-L}$ also imply the formulas for $p^{\mathcal{L}}_{2-L}$ and $p^{\mathcal{L}}_{3-L}$ proved in last section. The approach in the previous section has the merit that it produces explicit expressions, while the statement in terms of free energy can give a uniform treatment containing all the relations of Lickorish–Millett type, as in the following theorem.

**Theorem 7.3.** *Under the same notation as above, we have the vanishing result* $a_{-L}=a_{1-L}=\cdots=a_{L-3}=0$. *In other words*, $\mathrm{val}_z(F^{\mathrm{SO}}_{(1)^L}(\mathcal{L}))\geq L-2$. *Indeed, we have*

$$(q-q^{-1})^{2-L}F^{\mathrm{SO}}_{(1)^L}(\mathcal{L})\in\mathbb{Z}[t,t^{-1}][q-q^{-1}].$$

*As a corollary, Conjecture 5.1 is true for partitions* $\vec{\mu}=(1,1,\dots,1)$.

*Proof.* We prove the theorem by induction. When $L=1$, $\mathcal{L}$ is a knot, and

$$F^{\mathrm{SO}}_{(1)^1}(\mathcal{L})=W^{\mathrm{SO}}_{(1)}(\mathcal{L})=\Big(1+\frac{t-t^{-1}}{z}\Big)t^{-w(\mathcal{L})}\langle\mathcal{L}\rangle=\Big(1+\frac{t-t^{-1}}{z}\Big)K_{\mathcal{L}}$$

since the Kauffman polynomial of $\mathcal{L}$ obviously has $z$-valuation equal to $-1=L-2$. The theorem thus holds for knots.

Now assume $\mathcal{L}$ is a link with $L>1$ components $\mathcal{K}_1,\dots,\mathcal{K}_L$. We first deal with the simple case when $\mathcal{L}$ is the disjoint union of the $\mathcal{K}_\alpha$. Then for any partition $\Delta_1,\dots,\Delta_r$ of the set $[L]$, the product $\prod_{i=1}^r W^{\mathrm{SO}}_{\Delta_i}(\mathcal{L})=\prod_{\alpha=1}^L W^{\mathrm{SO}}_{(1)}(\mathcal{K}_\alpha)$ is independent of the partition. Again let $\Lambda$ be a partition of $L$ of length $r$. The number of collections $\{\Delta_1,\dots,\Delta_r\}$ with $\{|\Delta_1|,\dots,|\Delta_r|\}$ equal to the partition $\Lambda$ is given by $\frac{r!}{|\mathrm{Aut}\,\Lambda|}\cdot\frac{L!}{\Lambda_1!\cdots\Lambda_r!}$; hence

$$F^{\mathrm{SO}}_{(1)^L}(\mathscr{L})=\prod_{\alpha=1}^{L}W^{\mathrm{SO}}_{(1)}(\mathscr{K}_\alpha)\cdot\sum_{\Lambda\vdash L}\frac{(-1)^{\ell(\Lambda)-1}\ell(\Lambda)!L!}{\ell(\Lambda)|\mathrm{Aut}\,\Lambda|\Lambda!}=0.$$

There is another way to see this directly. If the link $\mathscr{L}$ is the disjoint union of the $\mathscr{K}_\alpha$, then the free energy $F^{\mathrm{SO}}(\mathscr{L},pb(z_1),\dots,pb(z_L))$ is the sum of the free energies $F^{\mathrm{SO}}(\mathscr{K}_\alpha;pb(z_\alpha))$. The expansion of such a sum $F^{\mathrm{SO}}(\mathscr{L})$ with respect to $pb_{\vec{\mu}}$ does not contain terms of the form $\prod_{\alpha=1}^{L}p_1(z_\alpha)$. Thus the theorem is true for links that are disjoint unions of knot components.

Finally, consider the Skein relation

$$\langle\mathscr{L}_+\rangle-\langle\mathscr{L}_-\rangle=z(\langle\mathscr{L}_{||}\rangle-\langle\mathscr{L}_=\rangle),$$

where $\langle\mathscr{L}_+\rangle$ and $\langle\mathscr{L}_-\rangle$ are two links that coincide everywhere except at one crossing $P$ between two different components $\mathscr{K}_a$ and $\mathscr{K}_b$ of the link $\mathscr{L}$ for $1\le a<b\le L$. The link $\langle\mathscr{L}_{||}\rangle$ (respectively $\langle\mathscr{L}_=\rangle$) is the link by replacing the crossing $P$ by two parallel vertical (respectively horizontal) lines. Both $\langle\mathscr{L}_{||}\rangle$ and $\langle\mathscr{L}_=\rangle$ have $L-1$ components. Let's compute the difference

$$F^{\mathrm{SO}}_{(1)^L}(\mathscr{L}_+)-F^{\mathrm{SO}}_{(1)^L}(\mathscr{L}_-)=\sum_{r=1}^{L}\frac{(-1)^{r-1}}{r}\sum_{\Delta_1,\dots,\Delta_r}\Big(\prod_{i=1}^{r}W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_+)-\prod_{i=1}^{r}W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_-)\Big).$$

The summation is again over all partitions $\Delta_1,\dots,\Delta_r$ of the set $[L]$. An important observation is that $\prod_{i=1}^{r}W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_+)-\prod_{i=1}^{r}W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_-)=0$ if $a$ and $b$ are not in the same set $\Delta_i$ for some $i$, because in this situation the sublinks $\mathscr{L}_{+,\Delta_i}$ coincide with the sublinks $\mathscr{L}_{-,\Delta_i}$. In particular, this is the case if $r=L$. The difference above can be simplified as

$$\begin{aligned}
&F^{\mathrm{SO}}_{(1)^L}(\mathscr{L}_+)-F^{\mathrm{SO}}_{(1)^L}(\mathscr{L}_-)\\
&=\sum_{r=1}^{L-1}\frac{(-1)^{r-1}}{r}\sum_{i=1}^{r}\sum_{\Delta_1,\dots,\Delta_r;a,b\in\Delta_i}(W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_+)-W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_-))\prod_{j=1,j\neq i}^{r}W^{\mathrm{SO}}_{\Delta_j}(\mathscr{L}_+)\\
&=t^{2\,lk(\mathscr{K}_a,\mathscr{K}_b)-1}\sum_{r=1}^{L-1}\frac{(-1)^{r-1}}{r}\sum_{i=1}^{r}\sum_{\substack{\Delta_1,\dots,\Delta_r,\\ a,b\in\Delta_i}}z\cdot(W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_{||})-W^{\mathrm{SO}}_{\Delta_i}(\mathscr{L}_=))\\
&\qquad\cdot\prod_{j=1,j\neq i}^{r}W^{\mathrm{SO}}_{\Delta_j}(\mathscr{L}_+)\\
&=zt^{2\,lk(\mathscr{K}_a,\mathscr{K}_b)-1}\cdot(F^{\mathrm{SO}}_{(1)^{L-1}}(\mathscr{L}_{||})-F^{\mathrm{SO}}_{(1)^{L-1}}(\mathscr{L}_=)).
\end{aligned}$$

By induction, both

$$z^{2-(L-1)}\cdot F^{\mathrm{SO}}_{(1)^{L-1}}(\mathscr{L}_{||})\quad\text{and}\quad z^{3-L}\cdot F^{\mathrm{SO}}_{(1)^{L-1}}(\mathscr{L}_=)$$

are in the ring $\mathbb{Z}[t, t^{-1}][z]$. Thus if the theorem is true for the link $\mathcal{L}_+$ if and only if it is true for the link $\mathcal{L}_-$.

For a general link $\mathcal{L}$ not necessarily a disjoint union, onc can change crossings between different components of $\mathcal{L}$ until it becomes a disjoint union of $L$ knots. Since the theorem is true for disjoint unions, it is true for $\mathcal{L}$. □

The results of Section 6 can be viewed as applications of Theorem 7.3 combined with some combinatorial identities like Lemma 7.2.

To study the cases of partitions with more boxes, we first develop the cabling technique. Let $\beta$ be a braid of which the closure is the link $\mathcal{L}$. For each $\vec{d} \in \mathbb{Z}_+^L$, denote by $\beta_{\vec{d}}$ the braid obtained by cabling the $k$-th strand of $\beta$ to $d_\alpha$ parallel ones if it is in the $\alpha$-th component of $\mathcal{L}$. The partition function of $\mathcal{L}$ and the Kauffman polynomials are related by the following lemma.

**Lemma 7.4.** *Assume $\beta$ is of writhe zero on every component. Then the partition function of $\mathcal{L}$ is related to the Kauffman polynomial of the cabling link by*

$$W_{(1)^d}^{\mathrm{SO}}(\beta_{\vec{d}}) = \sum_{\vec{A} \in \widehat{Br}_{\vec{d}}} \chi_{\vec{A}}(\mathrm{id}) W_{\vec{A}}^{\mathrm{SO}}(\mathcal{L}; q, t) = \vec{d}! \cdot Z^{\mathrm{SO}}_{((1^{d_1}),\ldots,(1^{d_L}))}(\mathcal{L}),$$

*where $d = \sum_{\alpha=1}^L d_\alpha$ and $\vec{d}! = \mathfrak{z}_{((1^{d_1}),\ldots,(1^{d_L}))} = d_1! \cdots d_L!$.*

*Proof.* Take $\beta$ to be the link of zero writhe on every component. Then the cabling link $\beta_{\vec{d}}$ is also of zero writhe on every component, and the quantum group invariants $W_{\vec{A}}$ are equal to the trace of

$$\beta_{\vec{d}} \cdot (p_{A^1} \otimes \cdots \otimes p_{A^L}) \tag{7-3}$$

in the Birman–Murakami–Wenzl algebra $C_M$ for $M = d_1 r_1 + \cdots + d_L r_L$, and $p_{A^\alpha}$ is the minimal idempotent in $C_{d_\alpha}$ corresponding to the irreducible representation numbered by the partition $A^\alpha$. Apparently, each $p_{A^\alpha}$ should appear $r_i$ times in the tensor above. However, the naturality of the universal $\mathcal{R}$-matrices plus the trace property will move all $p_{A^\alpha}$ to the same strand, and thus one $p_{A^\alpha}$ for each $\alpha = 1, 2, \ldots, L$ is enough.

The expansion coefficients $Z^{\mathrm{SO}}_{(1^{d_1},\ldots,1^{d_L})}(\mathcal{L})$ of the partition function can be calculated directly:

$$\begin{aligned} \vec{d}! \cdot Z^{\mathrm{SO}}_{(1^{d_1},\ldots,1^{d_L})}(\mathcal{L}) &= \sum_{\vec{A} \in \widehat{Br}_{\vec{d}}} \chi_{\vec{A}}(\mathrm{id}) W_{\vec{A}}^{\mathrm{SO}}(\mathcal{L}; q, t) \\ &= \sum_{\vec{A} \in \widehat{Br}_{\vec{d}}} \chi_{\vec{A}}(\mathrm{id}) \operatorname{tr}_{V^M}(\beta_{\vec{d}} \cdot (p_{A^1} \otimes \cdots \otimes p_{A^L})) \\ &= \operatorname{tr}_{V^M}(\beta_{\vec{d}}) = W^{\mathrm{SO}}_{(1)^d}(\beta_{\vec{d}}; q, t). \end{aligned}$$

We have used that for a semisimple algebra, the dimension of an irreducible representation $\chi_{A^i}(\mathrm{id})$ is the same as the multiplicity of $A^i$ in the semisimple decomposition of the algebra. So

$$\sum_{\vec{A}\in\widehat{Br}_{\vec{d}}} \chi_{\vec{A}}(\mathrm{id})(p_{A^1}\otimes\cdots\otimes p_{A^L}) = \mathrm{id}$$

in the third equality. □

**Remark 7.1.** A similar formula holds for the HOMFLY polynomials and can be proved in the same way.

**Theorem 7.5.** *Suppose $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{P}^L$ is a partition such that $\mu^\alpha = (1, 1, \dots, 1) \vdash d_\alpha$ for each $\alpha = 1, \dots, L$. Then*

$$\vec{d}!(q-q^{-1})^{2-d}\cdot F_{\vec{\mu}}(\mathcal{L}, q, t) \in \mathbb{Z}[t, t^{-1}][q-q^{-1}].$$

*In particular, the Conjecture 5.1 (the orthogonal LMOV conjecture) is valid for such columnlike partitions.*

*Proof.* We will use the symbol $(1)^{\vec{d}}$ to denote the partition $\vec{\mu}$ in the theorem. Let $\beta$ be a braid whose closure is the link $\mathcal{L}$ with zero writhe. Let $\beta_{\vec{d}}$ be the cabling braid as in Section 3. The calculation in Lemma 7.4 in fact shows that

$$Z^{\mathrm{SO}}_{(1)^{\vec{d}}}(\mathcal{L}) = \frac{1}{\vec{d}!} W^{\mathrm{SO}}_{(1)^d}(\beta_{\vec{d}}),$$

which reduces the situation back to the Kauffman case. A more careful observation is the cabling equality

$$F^{\mathrm{SO}}_{(1)^{\vec{d}}}(\mathcal{L}) = \frac{1}{\vec{d}!} F^{\mathrm{SO}}_{(1)^d}(\beta_{\vec{d}}),$$

which, together with Theorem 7.3, finishes the proof.

We now prove the cabling equality by comparing both sides. The left side is

$$\sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\vec{A}_1,\dots,\vec{A}_r} Z^{\mathrm{SO}}_{\vec{A}_1}(\mathcal{L})\cdots Z^{\mathrm{SO}}_{\vec{A}_r}(\mathcal{L}) = \sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\vec{A}_1,\dots,\vec{A}_r}\frac{W^{\mathrm{SO}}_{(1)^{\|\vec{A}_1\|}}(\beta_{|\vec{A}_1|})\cdots W^{\mathrm{SO}}_{(1)^{\|\vec{A}_r\|}}(\beta_{|\vec{A}_r|})}{\vec{A}_1!\cdots\vec{A}_r!},$$

where the summation is over all partitions $(\vec{A}_1, \dots, \vec{A}_r)$ sharing length with the partition $(1)^{\vec{d}}$. As $\vec{A}_i$ must be of the form $((1^{a_i^1}), (1^{a_i^2}), \dots, (1^{a_i^L}))$, with $\sum_{i=1}^r a_i^\alpha = d_\alpha$ for every $\alpha = 1, 2, \dots, L$, we have

$$|\vec{A}_i| = (a_i^1, a_i^2, \dots, a_i^L), \quad \|\vec{A}_i\| = a_i^1 + a_i^2 + \cdots + a_i^L, \quad \vec{A}_i! = a_i^1! a_i^2! \cdots a_i^L!$$

as in the introduction. Again $\beta$ is the braid with zero writhe on every component representing the link $\mathcal{L}$, and $\beta_{|\vec{A}_i|}$ is the cabling link.

The right side is

$$\frac{1}{\vec{d}!}\sum_{r=1}^{d}\frac{(-1)^{r-1}}{r}\sum_{\Delta_1,\ldots,\Delta_r}\prod_{i=1}^{r}W^{\mathrm{SO}}_{\Delta_i}(\beta_{\vec{d}}),$$

where $\Delta_1,\ldots,\Delta_r$ are nonempty sets that form a partition of the set $[d]$. Each $\Delta_i$ can be further decomposed into a partition $\Xi_i^1,\Xi_i^2,\ldots,\Xi_i^L$, such that elements in $\Xi_i^\alpha$ labeling the components in $\beta_{\vec{d}}$ arise from the cabling of the $\alpha$-th component of $\mathcal{L}$. Write $a_i^\alpha=|\Xi_i^\alpha|$ for the number of elements in $\Xi_i^\alpha$, which can be zero. Then the vectors $\vec{A}_i$ defined by

$$\vec{A}_i=((1^{a_i^1}),(1^{a_i^2}),\ldots,(1^{a_i^L}))$$

become one term in the summation on the left side. Furthermore, for each fixed such $\vec{A}_i$, there are $\prod_{\alpha=1}^{L}\frac{d_\alpha!}{a_1^\alpha!\cdots a_r^\alpha!}$ possible partition sets $\Xi_i^\alpha$. The equality holds. $\square$

## 8. The case of rows implies the conjecture

In this section, we discuss the case for a general partition $\vec{\mu}$, and reduce it to the case of rectangular ones.

We first define an equivalence relation on the BMW algebra $C_n$: Two elements $x,y\in C_n$ are equivalent, denoted by $x\sim y$, if $\mathrm{tr}(xz)=\mathrm{tr}(yz)$ for all central elements $z\in C_n$. Obviously, if two elements $x$ and $y$ are conjugate, say if there exists an invertible element $g\in C_n$ such that $gxg^{-1}=y$, then $x\sim y$. Since the algebra $C_n$ is semisimple, two idempotents $p_1$ and $p_2$ are equivalent if and only if they give isomorphic representations of $C_n$.

Let $p_\lambda$ be a minimal path idempotent in $C_n$. Write $m_\mu=\sum_\lambda\chi_\lambda(\gamma_\mu)p_\lambda$, and also regard this as an element in the Grothendieck group of representations of the BMW algebra. The branching rule [Beliakova and Blanchet 2001] for the BMW algebra is

$$p_\lambda\otimes 1=\sum_{\lambda'}p_{\lambda'},$$

where the summation is over all partitions $\lambda'$ that either add one box to $\lambda$ or remove one box from $\lambda$. Since the characters $\chi_A(\gamma_\mu)$ of the Brauer algebra are all integers, repeated use of the branching rule leads to a decomposition of the tensor product of minimal idempotents:

$$m_{(\mu_1)}\otimes m_{(\mu_2)}\otimes\cdots\otimes m_{(\mu_\ell)}\sim\sum_A b_A p_A, \tag{8-1}$$

where the sum is over all possible partitions $A$ and the multiplicities $b_A$ are all integers. Furthermore, the integers $b_A$ are uniquely determined by this equivalence relation, by multiplying both sides by the minimal central idempotents $\pi_A$ of $C_n$.

**Lemma 8.1.** *The integers $b_A$ satisfy $b_A = \chi_A(\gamma_\mu)$ for the characters of Brauer algebras.*

*Proof.* Since the BMW algebras are deformations of the Brauer algebras, they share the same branching rules. Specialize (8-1) to the Brauer algebras by fixing $x = 1 + (t - t^{-1})/(q - q^{-1})$ and let $t$ and $q$ go to 1. Then using the isomorphism $\mathrm{Br}_n \cong \mathrm{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$ for $x = 2N + 1$, we get

$$\widetilde{m}_{(\mu_1)} \otimes \widetilde{m}_{(\mu_2)} \otimes \cdots \otimes \widetilde{m}_{(\mu_\ell)} \sim \sum_A b_A \tilde{p}_A, \tag{8-2}$$

where

$$\widetilde{m}_{(\mu_i)} = \sum_{A \in \widehat{Br}_{\mu_i}} \chi_A(\gamma_{(\mu_i)}) \tilde{p}_A$$

and $\tilde{p}_A$ is a minimal idempotent in $\mathrm{End}_{\mathfrak{so}(2N+1)}(V^{\otimes n})$. Regard (8-2) as an equality in the Grothendieck group of (finite dimensional representations) of the Lie group $\mathrm{SO}(2N + 1)$. The character is given by

$$\begin{aligned}
&\prod_{i=1}^{\ell}\Bigl(\sum_{h=0}^{[\mu_i/2]} \sum_{\lambda \vdash \mu_i - 2h} \chi_\lambda(\gamma_{(\mu_i)})\, sb_\lambda(z_{-N}, z_{1-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_{N-1}, z_N)\Bigr) \\
&\quad = \prod_{i=1}^{\ell} p_{\mu_i}(z_{-N}, z_{1-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_{N-1}, z_N) \\
&\quad = p_\mu(z_{-N}, z_{1-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_{N-1}, z_N) \\
&\quad = \sum_{h=0}^{[|\mu|/2]} \sum_{\lambda \vdash |\mu| - 2h} \chi_\lambda(\gamma_\mu)\, sb_\lambda(z_{-N}, z_{1-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_{N-1}, z_N).
\end{aligned}$$

Thus the two elements $\widetilde{m}_{(\mu_1)} \otimes \widetilde{m}_{(\mu_2)} \otimes \cdots \otimes \widetilde{m}_{\mu_\ell}$ and $\widetilde{m}_\mu$ are equal in the Grothendieck group of $\mathrm{SO}(2N + 1)$, which determines the integers $b_A = \chi_A(\gamma_\mu)$, that is, we have

$$m_{(\mu_1)} \otimes m_{(\mu_2)} \otimes \cdots \otimes m_{(\mu_\ell)} \sim m_\mu. \qquad \square$$

Let $\vec{\ell}=(\ell_1,\dots,\ell_L)$, and let $\mathscr{L}_{\vec{\ell}}$ be closure of the cabling braid $\beta_{\vec{\ell}}$, which is obtained by cabling the $\alpha$-th component of $\beta$ into $\ell_\alpha$ parallel ones. Then we have

$$\begin{aligned}
\mathrm{z}_{\vec{\mu}}\cdot Z_{\vec{\mu}}(\mathscr{L}) &= \sum_{\vec{A}}\chi_{\vec{A}}(\gamma_{\vec{\mu}})\,\mathrm{tr}(\beta_{\vec{\ell}}\cdot p_{\vec{A}})\\
&= \mathrm{tr}(\beta_{\vec{\ell}}\cdot m_{\vec{\mu}}) = \mathrm{tr}\Big(\beta_{\vec{\ell}}\cdot\bigotimes_{\alpha=1}^{L}(m_{(\mu_1^\alpha)}\otimes\cdots\otimes m_{(\mu_{\ell_\alpha}^\alpha)})\Big)\\
&= \Big(\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell_\alpha}\mu_i^\alpha\Big)\cdot Z_{(\mu_1^1),(\mu_2^1),\dots,(\mu_{\ell_1}^1)(\mu_1^2),(\mu_2^2),\dots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\dots,(\mu_{\ell_L}^L)}(\mathscr{L}_{\vec{\ell}})
\end{aligned}$$

and

$$\begin{aligned}
\text{(8-3)}\quad & \mathrm{z}_{\vec{\mu}}\cdot F_{\vec{\mu}}(\mathscr{L})\\
&= \Big(\prod_{\alpha=1}^{L}\prod_{i=1}^{\ell_\alpha}\mu_i^\alpha\Big)\cdot F_{(\mu_1^1),(\mu_2^1),\dots,(\mu_{\ell_1}^1),(\mu_1^2),(\mu_2^2),\dots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\dots,(\mu_{\ell_L}^L)}(\mathscr{L}_{\vec{\ell}}).
\end{aligned}$$

The partition $(\mu_1^1),(\mu_2^1),\dots,(\mu_{\ell_1}^1),(\mu_1^2),(\mu_2^2),\dots,(\mu_{\ell_2}^2),\dots,(\mu_1^L),(\mu_2^L),\dots,(\mu_{\ell_L}^L)\in\mathscr{P}^{|\ell(\vec{\mu})|}$ has the property that each component is of length one. In particular, it is rectangular, and we have the following theorem.

**Theorem 8.2.** *Conjecture 5.1 is true for all partitions $\vec{\mu}$ if and only if it is true for rectangular one $\vec{\mu}$, if and only if it is true for $\vec{\mu}=(\mu^1,\dots,\mu^L)$ such that each $\mu^\alpha=(d_\alpha)$ is of length one.*

Equation (8-3) together with Proposition 9.2 implies Conjecture 5.2 (the degree conjecture), that is, the degree estimate at $q=1$ is valid for all partitions $\vec{\mu}$.

**Theorem 8.3.** *Conjecture 5.2 is true for all links and all partitions.*

Theorem 8.3 implies that Conjecture 5.1 is "true at $q=1$" (Theorem 1.6), that is, the left hand side of Conjecture 5.1 is regular at $q-1$. The situation at other roots of unity seems to be more difficult. Some torus knots and links examples are verified in Section 5, which can be treated as the conjecture at roots of unity besides 1.

## 9. Estimation of degree

Call a partition $\lambda\vdash n$ rectangular if the Young diagram of $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_\ell)$ is rectangular, that is, $\lambda_1=\lambda_2=\cdots=\lambda_\ell$. A rectangular partition is determined by its length $\ell$ and its size $n$.

Let $\delta_n=\sigma_1\sigma_2\cdots\sigma_{n-1}$ be a braid in $B_n$. Let $\ell$ be an integer dividing $n$ and write $a=n/\ell$. Then the braid $(\delta_n)^\ell$ is associated to the rectangular partition $\lambda=(a,a,\dots,a)\vdash n$. It is easy to see that $h((\delta_n)^n)$ is in the center of $C_n$. Let $A$ be a partition of $n-2f$ for some integer $f$, and let $\pi_A$ be a minimal central idempotent in $C_n$, and let $p_A$ be a minimal idempotent such that $p_A\pi_A = p_A$. Under the isomorphism

$$C_n \cong \bigoplus_{A\in\widehat{Br}_n} M_{d_A\times d_A}(\mathbb{C}),$$

the product $h((\delta_n)^n)\cdot\pi_A$ is a scalar matrix at the block corresponding to $A$, and zero at other places. From Section 3, we know that $h((\delta_n)^n)\cdot\pi_A = q^{\kappa_A}t^{-2f}\pi_A$, which implies that the eigenvalues of $h((\delta_n)^\ell)\cdot\pi_A$ are either 0 or $q^{\kappa_A/a}t^{-2f/a}$ times $n$-th roots of unity. We conclude that $\operatorname{tr}(h((\delta_n)^\ell)\cdot p_A) = b_A\cdot q^{\kappa_A/a}t^{-2f/a}$ for some rational number $b_A$. Taking the specialization $q,t\to 1$, we obtain the value $b_A = \chi_A(\gamma_\lambda)$ for the character $\chi_A$ of Brauer algebra.

Now we compute $Z_{\vec\lambda}(\mathscr{L},q,t)$ for rectangular partition $\vec\lambda$. Write $\vec\lambda = (\lambda^1,\ldots,\lambda^L)$ such that $\lambda^\alpha = (a_\alpha,\ldots,a_\alpha) = ({a_\alpha}^{\ell_\alpha})$ for each $\alpha = 1,2,\ldots,L$. Our goal in this section is to estimate the $u$ degree of $F_{\vec\lambda}(\mathscr{L};,q,t)$ for $u=\log q$ and a rectangular partition $\vec\lambda$.

**Definition 9.1.** Let $\vec\tau = (\tau_1,\ldots,\tau_L)\in\mathbb{C}^L$ be a vector. We define the framing dependent link invariants

$$W_{\vec A}(\mathscr{L},q,t,\vec\tau) := W_{\vec A}(\mathscr{L},q,t)q^{\sum_{\alpha=1}^L \kappa_{A^\alpha}\tau_\alpha}t^{\sum_{\alpha=1}^L |A^\alpha|\tau_\alpha},$$

and the framing-dependent partition function by

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau) = \sum_{\vec\mu\in\mathscr{P}^L}\frac{pb_{\vec\mu}}{\mathfrak{z}_{\vec\mu}}\cdot\sum_{\vec A\in\widehat{Br}_{|\vec\mu|}}\chi_{\vec A}(\gamma_{\vec\mu})W_{\vec A}^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau). \tag{9-1}$$

Similarly we define the free energy $F^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau) = \log Z_{\mathrm{CS}}^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau)$ and the coefficients $F_{\vec\mu}^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau)$ and $W_{\vec\mu}^{\mathrm{SO}}(\mathscr{L};q,t,\vec\tau)$ as before, replacing the link invariants by the framing dependent invariants. The specialization $\vec\tau=0$ gives the framing independent invariants.

We compute the partition functions at the special values $\tau_\alpha = w_\alpha + 1/a_\alpha$ for $w_\alpha\in\mathbb{Z}$ by taking a braid $\beta(\vec w)$ with writhe number $w_\alpha$ on each component $\mathscr{K}_\alpha$ of $\mathscr{L}$. Let $\mathscr{L}_{\vec n,\vec w}^{\mathrm{twist}}$ be the closure of the product of the cabling braid $\beta_{\vec n}(\vec w)$ of $\beta(\vec w)$ and the braid $(\omega_{\lambda^1}\otimes\cdots\otimes\omega_{\lambda^L})$, where $n_\alpha = a_\alpha\ell_\alpha$ and $\omega_{\lambda^\alpha} = (\delta_{n_\alpha})^{\ell_\alpha}$. The diagrams below provide an example that illustrates the twisted cabling process in the case $a=2$, $\ell=1$ and $n=a\ell=2$. Suppose $\mathscr{L}$ is a braid in Figure 1(a), which represents a knot with writhe number $w=4$. Figure 1(b), is obtained by cabling each component into two strands. The twist $\omega_\lambda$ is then as in the bottom of Figure 1(c), which adds a crossing to Figure 1(b). The final twisted cabling link $\mathscr{L}_{\vec n,\vec w}^{\mathrm{twist}}$ is the closure of the braid in Figure 1(d).

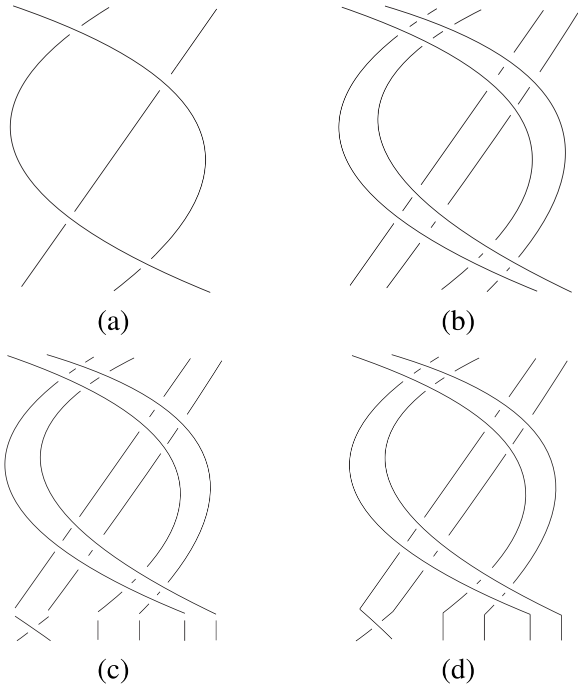



**Figure 1**

The link $\mathscr{L}_{\vec{n},\vec{w}}^{\text{twist}}$ has $\ell(\vec{\lambda}) = \ell_1 + \ell_2 + \cdots + \ell_L$ components, and there are $\ell_i$ components of writhe $w_\alpha a_\alpha^2 + a_\alpha - 1$. We have

$$\begin{aligned} Z_{\vec{\lambda}}^{\text{SO}}(\mathscr{L}; q, t, \vec{\tau}) &= \frac{1}{\mathfrak{z}_{\vec{\lambda}}} \cdot \sum_{\vec{A} \in \widehat{Br}_{|\vec{\lambda}|}} \chi_{\vec{A}}(\gamma_{\vec{\lambda}}) W_{\vec{A}}^{\text{SO}}(\mathscr{L}; q, t, \vec{\tau}) \\ &= \frac{1}{\mathfrak{z}_{\vec{\lambda}}} \cdot \operatorname{tr}\Big(\beta_{\vec{n}}(\vec{w}) \cdot \sum_{\vec{A} \in \widehat{Br}_{|\vec{\lambda}|}} \chi_{\vec{A}}(\gamma_{\vec{\lambda}}) q^{\sum_{\alpha=1}^{L} \frac{\kappa_{A^\alpha}}{a_\alpha}} t^{\sum_{\alpha=1}^{L} \frac{|A^\alpha|}{a_\alpha}} \cdot p_{\vec{A}}\Big) \\ &= \frac{t^{\sum_{\alpha=1}^{L} \ell_\alpha}}{\mathfrak{z}_{\vec{\lambda}}} \cdot \operatorname{tr}(\beta_{\vec{n}}(\vec{w}) \cdot (\omega_{\lambda^1} \otimes \cdots \otimes \omega_{\lambda^L})) \\ &= \frac{t^{\sum_{\alpha=1}^{L} a_\alpha \ell_\alpha (w_\alpha a_\alpha + 1)}}{\mathfrak{z}_{\vec{\lambda}}} \cdot W_{(1)^{\ell(\vec{\lambda})}}(\mathscr{L}_{\vec{n},\vec{w}}^{\text{twist}}, q, t). \end{aligned}$$

As in the proof of Theorem 7.5, we get

$$F_{\vec{\lambda}}^{\text{SO}}(\mathscr{L}; q, t, \vec{\tau}) = \frac{t^{\sum_{\alpha=1}^{L} a_\alpha \ell_\alpha (w_\alpha a_\alpha + 1)}}{\prod_{\alpha=1}^{L} \ell_\alpha ! a_\alpha^{\ell_\alpha}} \cdot F_{(1)^{\ell(\vec{\lambda})}}^{\text{SO}}(\mathscr{L}_{\vec{n},\vec{w}}^{\text{twist}}; q, t). \tag{9-2}$$

In particular, we get the following proposition.

**Proposition 9.1.** *For a rectangular partition $\vec{\lambda}$ such that $\mu^\alpha = (a_\alpha{}^{\ell_\alpha})$, and any tube of integers $\vec{w} = (w_1, \dots, w_L)$, we have*

$$(q-q^{-1})^{2-\ell(\vec{\lambda})} \cdot F^{\mathrm{SO}}_{\vec{\lambda}}(\mathcal{L}; q, t, \vec{\tau}) \in \mathbb{Q}[q-q^{-1}][t, t^{-1}]$$

*for $\vec{\tau} = (w_1 + 1/a_1, w_2 + 1/a_2, \dots, w_L + 1/a_L)$.*

Consider the embedding $\mathbb{Q}(q)[t, t^{-1}] \hookrightarrow \mathbb{Q}[\![T]\!]((u))$ via the change of variables $q = e^u$ and $t = e^T$, we can expand the rational function $F^{\mathrm{SO}}_{\vec{\lambda}}(\mathcal{L}; q, t, \vec{\tau})$ into a formal power series in variables $u$ and $T$ as

$$F^{\mathrm{SO}}_{\vec{\lambda}}(\mathcal{L}; e^u, e^T, \vec{\tau}) = \sum_{k=0}^{\infty} \sum_{i \geq -\|\vec{\lambda}\|} P_{k,i}(\vec{\tau}) T^k u^i$$

with coefficients $P_{k,i} \in \mathbb{Q}[\tau^1, \dots, \tau^L]$.

The proposition above implies that the coefficients $P_{k,i}(\vec{\tau})$ for $i < \ell(\vec{\lambda}) - 2$ vanish when each $\tau_k - 1/a_k$ takes arbitrary integer values, which is possible only when the polynomials $P_{k,i}(\vec{\tau})$ for $i < \ell(\vec{\lambda}) - 2$ are zero polynomials (a lattice is Zariski dense). Now specializing to the framing $\tau_1 = \tau_2 = \cdots = \tau_L = 0$ leads to the following theorem.

**Proposition 9.2.** *Let $\vec{\lambda}$ be a rectangular partition. Then the formal power series $F^{\mathrm{SO}}_{\vec{\lambda}}(\mathcal{L}; e^u, t)$ and $g_{\vec{\lambda}}(\mathcal{L}; e^u, t)$ in the valuation field $\mathbb{Q}(t)(\!(u)\!)$ has $u$-valuation greater or equal to $\ell(\vec{\lambda}) - 2$.*

## 10. Appendix

**10.1. *The case of the unknot.*** In this appendix, we calculate $F^{\mathrm{SO}}(\bigcirc^L; q, t)$. We only deal with the case of unknot, that is, $L = 1$, since the general case $L \geq 1$ can be done exactly the same way, except that the notation will be more complicated.

**Proposition 10.1.**
$$\sum_{A \in \widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) W^{\mathrm{SO}}_A(\bigcirc; q, t)) = \prod_{i=1}^{\ell(\mu)} \Big[1 + \frac{t^{\mu_i} - t^{-\mu_i}}{q^{\mu_i} - q^{-\mu_i}}\Big].$$

*Proof.* Let $t = q^{2N}$ and compare with the quantum group definition of the colored Kauffman polynomials,

$$\begin{aligned}\sum_{A \in \widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) W^{\mathrm{SO}}_A(\bigcirc; q, q^{2N})) &= \sum_{A \in \widehat{Br}_{|\mu|}} \chi_A(\gamma_\mu) \operatorname{tr}_{V_A}(K_{2\rho}) \\ &= pb_\mu(q^{1-2N}, q^{3-2N}, \dots, q^{-1}, 1, q, \dots, q^{2N-3}, q^{2N-1}) \\ &= \prod_{i=1}^{\ell(\mu)} \Big[1 + \frac{t^{\mu_i} - t^{-\mu_i}}{q^{\mu_i} - q^{-\mu_i}}\Big].\end{aligned}$$

Since both sides of the equation in the proposition are rational functions in $t$, and they agree for arbitrary sufficiently large $N$, they must coincide. □

**Proposition 10.2.**

$$Z_{\mathrm{CS}}^{\mathrm{SO}}(\bigcirc; q, t) = \exp\Big(\sum_{k=1}^{+\infty} \frac{1}{k}\Big(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}\Big) pb_k\Big). \tag{10-1}$$

*Proof.*
$$\begin{aligned} Z_{\mathrm{CS}}^{\mathrm{SO}}(\bigcirc; q, t) &= \sum_{\lambda\in\mathscr{P}} \frac{1}{\mathfrak{z}_\lambda} \cdot \prod_{i=1}^{\ell(\lambda)} \Big[1 + \frac{t^{\lambda_i} - t^{-\lambda_i}}{q^{\lambda_i} - q^{-\lambda_i}}\Big] \cdot pb_\lambda \\ &= \sum_{n=1}^{+\infty} \sum_{\lambda_1,\dots,\lambda_n=1}^{+\infty} \frac{1}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i} \prod_{i=1}^{n} \Big[1 + \frac{t^{\lambda_i} - t^{-\lambda_i}}{q^{\lambda_i} - q^{-\lambda_i}}\Big] \cdot pb_\lambda \\ &= \sum_{n=1}^{+\infty} \frac{1}{n!} \Big[\sum_{k=1}^{+\infty} \frac{1}{k}\Big(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}\Big) pb_k\Big]^n, \end{aligned}$$

which equals the stated result. □

So we get the free energy expressed as

$$F^{\mathrm{SO}}(\bigcirc; q, t) = \sum_{k=1}^{+\infty} \frac{1}{k}\Big(1 + \frac{t^k - t^{-k}}{q^k - q^{-k}}\Big) pb_k\,.$$

**Remark 10.1.** This expression appeared in [Borhade and Ramadevi 2005], where it was computed from the path integral definition of the Chern–Simons partition function. Our derivation is based on our mathematical definition in terms of quantum group invariants and representations of the Brauer algebra.

**10.2.** ***An alternative definition of colored Kauffman polynomials via Markov trace and Hopf link.*** The quantum group approach to the knot/link theory has produce a lot of invariants via the representation theory. However, the calculations are usually very complicated. Fortunately, only quantum traces are essentially used. This enable us to find a combinatorial method instead of the quantum group one. Birman and Wenzl [1989] and Wenzl [1988] introduced a Markov trace definition. We will briefly introduce their construction here.

There is a well-defined *Markov trace* tr on the union of BMW algebra $C_n$ with the following properties.

(1) $\mathrm{tr}(h(\alpha_1)h(\alpha_2)) = \mathrm{tr}(h(\alpha_2)h(\alpha_1))$ for any $\alpha_i \in B_n$.

(2) $\mathrm{tr}(h(\beta)g_n^{\pm 1}) = (t^{\pm 1}/x)\,\mathrm{tr}(h(\beta))$ for any $\beta \in B_n$.

(3) $\mathrm{tr}(1) = 1$.

(4) $\mathrm{tr}(h(\beta)) = x^{1-n}\langle\hat{\beta}\rangle = x^{1-n}t^{w(\hat{\beta})}K(\hat{\beta}, q, t)$, where $\hat{\beta}$ is the link obtained by closing the braid $\beta \in B_n$ and $K(\mathscr{L}, q, t)$ is the classical Kauffman polynomial of the link $\mathscr{L}$.

First normalize the trace by setting

$$\mathrm{Tr}(\xi) = x^n \cdot \mathrm{tr}(\xi) \quad \text{for } \xi \in C_n.$$

Let $\mathscr{L}$ be a link with $L$ components $\mathscr{K}_\alpha$ for $\alpha = 1, \dots, L$, represented by the closure of $\beta \in B_m$. We associate to each $\mathscr{K}_\alpha$ an irreducible representation $V_{A^\alpha}$ of the quantized universal enveloping algebra $U_q(\mathfrak{so}(2N+1))$. Let $p_\alpha \in C_{d_\alpha}$ for $\alpha = 1, \dots, L$ be $L$ minimal idempotents corresponding to the irreducible representations $V_{A^1}, \dots, V_{A^L}$, where $A^\alpha$ denotes the partition of $|A^\alpha| = d_\alpha$ labeling $V_{A^\alpha}$. Let $\vec{d} = (d_1, \dots, d_L)$ and let $i_1, \dots, i_m$ be integers such that $i_k = \alpha$ if the $k$-th strand of $\beta$ belongs to the $\alpha$-th component of $\mathscr{L}$. Let $\beta_{\vec{d}}$ be the cabling braid of $\beta$, replacing the $k$-th strand of $\beta$ by $d_{i_k}$ parallel ones. Then

$$\text{(10-2)}\quad W^{\mathrm{SO}}_{\vec{A}}(\mathscr{L}; q, t) = q^{-\sum_{\alpha=1}^{L}\kappa_{A^\alpha}w(\mathscr{K}_\alpha)}t^{-\sum_{\alpha=1}^{L}|A^\alpha|w(\mathscr{K}_\alpha)} \cdot \mathrm{Tr}(h(\beta_{\vec{d}}) \cdot (p_{i_1} \otimes \cdots \otimes p_{i_m})).$$

Now we look at a concrete example to illustrate this method.

Let $\mathscr{L}$ be the Hopf link, represented by the braid $\beta = g_1^2$. Set $z = q - q^{-1}$. It's easy to get

$$W^{\mathrm{SO}}_{(1),(1)}(\mathscr{L}) = x((t - t^{-1})/z + 1 + z(t - t^{-1})).$$

Let $\mu$ be a partition of 2, and let $A$ be a partition of 2 or 0 labeling the irreducible representations of Brauer algebra $\mathrm{Br}_2$. The character table reads $\chi_{(1,1)}(\gamma_{(2)}) = -1$ and $\chi_{(2)}(\gamma_{(2)}) = \chi_{(1,1)}(\gamma_{(1,1)}) = \chi_{(1,1)}(\gamma_{(1,1)}) = 1$. The representation labeled by $A = (2)$ is the trivial representation.

We want to compute $W^{\mathrm{SO}}_{(1),(2)}(\mathscr{L})$. The minimal idempotents (studied by [Beliakova and Blanchet 2001]) in $C_2$ are

$$p_{(2)} = \Big(\frac{q^{-1} + g_2}{q + q^{-1}}\Big)(1 - x^{-1}e_2), \quad p_{(1)^2} = \Big(\frac{q - g_2}{q + q^{-1}}\Big)(1 - x^{-1}e_2), \quad p_\phi = x^{-1}e_2,$$

where $\phi$ is the empty partition.

Denote the cabling of $\beta$ by $\beta_{1,2}$, which is given by $h(\beta_{1,2}) = g_1 g_2^2 g_1$.

By using the definition of BMW algebras $C_n$ and the properties of Markov trace. We have the following formulas for the twisted cabling braids

$$\begin{aligned}\mathrm{tr}(h(\beta_{1,2})\cdot g_2) &= \mathrm{tr}(g_1 g_2^2 g_1 g_2)\\ &= \frac{t}{x^2}\Big(\frac{t-t^{-1}}{z} + 1 + (3t - 2t^{-1} - t^{-3})z + (1 - t^{-2})z^2 + (t - t^{-1})z^3\Big),\end{aligned}$$

where we used property (P2) of the BMW algebra $C_n$ as well as the classic Kauffman polynomial of the trefoil knot and the Hopf link.

Similarly, we have

$$\operatorname{tr}(h(\beta_{1,2})) = \frac{(x+zt-zt^{-1})^2}{x^2} \quad \text{and} \quad \operatorname{tr}(h(\beta_{1,2})\cdot e_2) = \frac{1}{x},$$

where $h(\beta_{1,2})\cdot e_2$ is actually the image of a link of the disjoint union of two unknots.

Since

$$p_{(2)} - p_{(1)^2} + p_\phi = \frac{-z}{q+q^{-1}} + \frac{2g}{q+q^{-1}} + \frac{(z+q+q^{-1}-2t^{-1})e_2}{x(q+q^{-1})},$$

we have

$$\begin{aligned} 2Z_{(1),(2)}(\mathscr{L};q,t) &= W_{(1),(2)}(\mathscr{L};q,t) - W_{(1),(1,1)}(\mathscr{L};q,t) + W_{(1),(0)}(\mathscr{L};q,t) \\ &= x^3 \operatorname{tr}\big(h(\beta_{1,2})\cdot(p_{(1)}\otimes(p_{(2)}-p_{(1)^2}+p_\phi))\big) \\ &= \frac{x}{q+q^{-1}}\Big(\frac{t^2-t^{-2}}{z} + (q+q^{-1}) + (t^2-t^{-2})(z^3+4z)\Big) \end{aligned}$$

and

$$2Z_{(1)}(\bigcirc)Z_{(2)}(\bigcirc) = \frac{x}{q+q^{-1}}\Big(q+q^{-1}+\frac{t^2-t^{-2}}{z}\Big),$$

Thus we have

$$\begin{aligned} 2F_{(1),(2)}(\mathscr{L},q,t) &= 2Z_{(1),(2)}(\mathscr{L}) - 2Z_{(1)}(\bigcirc)Z_{(2)}(\bigcirc) \\ &= (q+q^{-1})(t^2-t^{-2})[z+(t-t^{-1})] \end{aligned}$$

and

$$\frac{2(q-q^{-1})^2F_{(1),(2)}}{(q-q^{-1})(q^2-q^{-2})} = (t^2-t^{-2})[(t-t^{-1})+z] \in \mathbb{Z}[z][t,t^{-1}]$$

as predicted in the Conjecture 5.1. Actually this example has already been discussed in Case 1C in Section 5.

**10.3.** ***Character tables of Brauer algebras and type-B Schur functions.*** Here are some character tables for Brauer algebras. Write $pb_\lambda = \sum_{A\in\widehat{Br}_{|\lambda|}} \chi_A(\gamma_\lambda)\, sb_A$, and we compute the character table by the following formula, which is [Ram 1995, Theorem 5.1]:

$$\chi_\lambda(\gamma_\mu) = \sum_{\substack{\nu\vdash,|\mu| \\ \nu\supset\lambda}} \Big(\sum_{\beta\text{ even}} c^\nu_{\lambda\beta}\Big)\chi_\nu^{S_{|\mu|}}(\gamma_\mu).$$

where the $c^\nu_{\lambda\beta}$ are called the Littlewood–Richardson coefficients and defined via type-A Schur functions as

$$s_\alpha s_\beta = \sum_{|\gamma|=|\alpha|+|\beta|} c^\gamma_{\alpha\beta} s_\gamma$$

Combining the formulas above, we obtain the expressions

$$\begin{aligned}
pb_{(1)} &= sb_{(1)}, \\
pb_{(2)} &= sb_{(2)} - sb_{(1,1)} + 1, \\
pb_{(1,1)} &= sb_{(2)} + sb_{(1,1)} + 1, \\
pb_{(3)} &= sb_{(3)} - sb_{(2,1)} + sb_{(1,1,1)}, \\
pb_{(2,1)} &= sb_{(3)} - sb_{(1,1,1)} + sb_{(1)}, \\
pb_{(1,1,1)} &= sb_{(3)} + 2\, sb_{(2,1)} + sb_{(1,1,1)} + 3\, sb_{(1)},
\end{aligned}$$

and conversely we can also express functions $sb$ in terms of $pb$.

**10.4.** ***Colored Kauffman polynomials of torus links/knots and tables of integer coefficients $N_{\vec{\mu},g,\beta}$.*** The torus link $\mathcal{L} = T(rL, kL)$ has $L$ components if $(r, k) = 1$. We compute the orthogonal quantum group invariants by the following formula proved in Theorem 3.6:

$$W_{\vec{A}}^{\mathrm{SO}}(\mathcal{L}; q, t) = q^{-kr \sum_{\alpha=1}^{L} \kappa_{A^\alpha}} t^{-k(r-1)n} \sum_{f=0}^{[rn/2]} \sum_{\lambda \vdash rn-2f} \tilde{c}_{\vec{A}}^{\lambda} q^{\frac{k\kappa_\lambda}{r}} t^{-\frac{2fk}{r}} sb_\lambda(q, t) \tag{10-3}$$

The explicit formula for these type-B Schur functions $sb_\lambda(q, t)$ are computed in the subsection above.

Recall the definition of the constants $\tilde{c}_{\vec{A}}^{\lambda}$ by the formula

$$\prod_{\alpha=1}^{L} sb_{A^\alpha}(z^r) = \sum_{f=0}^{[rn/2]} \sum_{\lambda \vdash rn-2f} \tilde{c}_{\vec{A}}^{\lambda}\, sb_\lambda(z).$$

For example, in the case $r = 2$ and $L = 1$, we have

| $\tilde{c}$ | (0) | (2) | (1, 1) | (4) | (3, 1) | (2, 2) | (2, 1, 1) | (1, 1, 1, 1) |
|---|---|---|---|---|---|---|---|---|
| (1) | 1 | 1 | −1 | × | × | × | × | × |
| (2) | 1 | 1 | −1 | 1 | −1 | 1 | 0 | 0 |
| (1, 1) | 1 | 1 | −1 | 0 | 0 | 1 | −1 | 1 |

In this subsection, we provide tables for the values of the integers $N_{\vec{\mu},g,\beta}$ in the formula

$$\frac{\mathfrak{z}_{\vec{\mu}}(q - q^{-1})^2 \cdot [g_{\vec{\mu}}(q, t) - g_{\vec{\mu}}(q, -t)]}{2 \prod_{\alpha=1}^{L} \prod_{i=1}^{\ell(\mu^\alpha)} (q^{\mu_i^\alpha} - q^{-\mu_i^\alpha})} = \sum_{g \in \mathbb{Z}_+/2} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu},g,\beta} z^{2g} t^\beta.$$

**Example 4.** Taking $r = 1$, the torus link $T(2, 2k)$ has 2 components. By (10-3), we have the following results expressed in the tables.

**Case 4A.** For $T(2, 2k)$ with partitions (1), (1), we have $N_{\vec{\mu},g,\beta} = 0$.

**Case 4B.** For $T(2, 2k)$ with partitions (1, 1), (1), we have

| $k=1$ | $\beta=-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $g=0$ | $-1$ | $3$ | $-3$ | $1$ |

and

| $k=2$ | $\beta=-5$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|---|
| $g=0$ | $-4$ | $4$ | $12$ | $-20$ | $8$ |
| $1$ | $-1$ | $1$ | $3$ | $-9$ | $6$ |
| $2$ | $0$ | $0$ | $0$ | $-1$ | $1$ |

**Case 4C.** For $T(2, 2k)$ with partitions (2), (1), we have

| $k=1$ | $\beta=-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $g=0$ | $1$ | $-1$ | $-1$ | $1$ |

and

| $k=2$ | $\beta=-5$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|---|
| $g=0$ | $2$ | $-2$ | $2$ | $-6$ | $4$ |
| $1$ | $1$ | $-1$ | $1$ | $-5$ | $4$ |
| $2$ | $0$ | $0$ | $0$ | $-1$ | $1$ |

**Case 4D.** For $T(2, 2k)$ with partitions (2), (2), we have

| $k=1$ | $\beta=-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $g=1/2$ | $-2$ | $2$ | $-2$ | $2$ |
| $3/2$ | $-1$ | $1$ | $-1$ | $1$ |

and

| $k=2$ | $\beta=-5$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|---|
| $g=1/2$ | $-8$ | $4$ | $20$ | $-36$ | $20$ |
| $3/2$ | $-24$ | $20$ | $40$ | $-96$ | $60$ |
| $5/2$ | $-22$ | $21$ | $29$ | $-97$ | $69$ |
| $7/2$ | $-8$ | $8$ | $9$ | $-47$ | $38$ |
| $9/2$ | $-1$ | $1$ | $1$ | $-11$ | $10$ |
| $11/2$ | $0$ | $0$ | $0$ | $-1$ | $1$ |

**Case 4E.** For $T(2, 2k)$ with partitions (3), (1), we have

| $k=1$ | $\beta=-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $g=1/2$ | $-1$ | $0$ | $0$ | $1$ |

and

| $k=2$ | $\beta=-5$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|---|
| $g=1/2$ | $-4$ | $4$ | $0$ | $-8$ | $8$ |
| $3/2$ | $-5$ | $5$ | $0$ | $-14$ | $14$ |
| $5/2$ | $-1$ | $1$ | $0$ | $-7$ | $7$ |
| $7/2$ | $0$ | $0$ | $0$ | $-1$ | $1$ |

**Example 5.** The torus knots $T(2, k)$, where $k$ is an odd integer. Again we compute the following tables by (10-3).

**Case 5A.** For $T(2, k)$ with partition (1, 1), we have

| $k=3$ | $\beta=-11$ | $-9$ | $-7$ | $-5$ | $-3$ |
|---|---|---|---|---|---|
| $g=1/2$ | $36$ | $-132$ | $180$ | $-108$ | $24$ |
| $3/2$ | $105$ | $-377$ | $453$ | $-207$ | $26$ |
| $5/2$ | $112$ | $-450$ | $494$ | $-165$ | $9$ |
| $7/2$ | $54$ | $-275$ | $286$ | $-66$ | $1$ |
| $9/2$ | $12$ | $-90$ | $91$ | $-13$ | $0$ |
| $11/2$ | $1$ | $-15$ | $15$ | $-1$ | $0$ |
| $13/2$ | $0$ | $-1$ | $1$ | $0$ | $0$ |

**Case 5B.** For $T(2,k)$ with partition (2), we have

| $k=3$ | $\beta=-11$ | $-9$ | $-7$ | $-5$ | $-3$ |
|---|---|---|---|---|---|
| $g=1/2$ | $-6$ | $26$ | $-42$ | $30$ | $-8$ |
| $3/2$ | $-35$ | $125$ | $-161$ | $85$ | $-14$ |
| $5/2$ | $-56$ | $210$ | $-238$ | $91$ | $-7$ |
| $7/2$ | $-36$ | $165$ | $-174$ | $46$ | $-1$ |
| $9/2$ | $-10$ | $66$ | $-67$ | $11$ | $0$ |
| $11/2$ | $-1$ | $13$ | $-13$ | $1$ | $0$ |
| $13/2$ | $0$ | $1$ | $-1$ | $0$ | $0$ |

**Example 6.** Taking $r=1$, the torus link $T(3,3k)$ has 3 components. By (10-3), we have the following tables for the torus link $T(3,3k)$ with partitions (2), (1), (1):

| $k=1$ | $\beta=-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $g=1/2$ | $2$ | $-2$ | $-10$ | $10$ |
| $3/2$ | $0$ | $0$ | $-6$ | $6$ |
| $5/2$ | $0$ | $0$ | $-1$ | $1$ |

and

| $k=2$ | $\beta=-5$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|---|
| $g=1/2$ | $16$ | $-48$ | $176$ | $-336$ | $192$ |
| $3/2$ | $12$ | $-68$ | $452$ | $-1036$ | $640$ |
| $5/2$ | $2$ | $-38$ | $494$ | $-1406$ | $948$ |
| $7/2$ | $0$ | $-10$ | $286$ | $-1056$ | $780$ |
| $9/2$ | $0$ | $-1$ | $91$ | $-467$ | $377$ |
| $11/2$ | $0$ | $0$ | $15$ | $-121$ | $106$ |
| $13/2$ | $0$ | $0$ | $1$ | $-17$ | $16$ |
| $15/2$ | $0$ | $0$ | $0$ | $-1$ | $1$ |

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