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RENORMALIZED WEIGHTED VOLUME AND CONFORMAL FRACTIONAL LAPLACIANS

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We give a notion of renormalized weighted volume in the setting of conformal geometry following the ideas of Fefferman and Graham. Indeed, it is a precise term in the asymptotic expansion near the boundary for a weighted volume related to the conformal fractional Laplacian operator and fractional-order Q-curvature.

1. Introduction

The relation between Poincaré–Einstein metrics and conformal objects on the boundary has aroused a lot of interest, in some sense motivated by what is called in physics the anti-de Sitter/conformal field theory correspondence, or AdS/CFT correspondence. Since the appearance of the work now published as [Fefferman and Graham 2012], there has been a great deal of literature on the ambient and Poincaré metrics. In particular, the notions of renormalized volume and area introduced in the physics literature are now important objects of study in the area of geometrical analysis and conformal geometry.

On the other hand, for $\gamma \in (0, n/2)$ one can consider the conformal fractional Laplacian (Paneitz) operator P_γ defined on the boundary of a conformally compact Einstein manifold X^{n+1}, as introduced in [Graham and Zworski 2003; Mazzeo and Melrose 1987] coming from scattering theory. In the Euclidean case, P_γ is just the standard fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$, but in the general case it is a nonlocal conformally covariant operator of fractional order.

When γ is an integer, say $\gamma = k$, the P_k are the conformally invariant powers of the Laplacian constructed by Graham, Jenne, Mason and Sparling [Graham et al. 1992]; they are local operators. In particular, when $k = 1$ we have the well-known conformal Laplacian,

$$P_1 = -\Delta + \frac{n-2}{4(n-1)} R,$$

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and when $k=2$, the Paneitz operator [Paneitz 2008]

$$P_2=(-\Delta)^2+\delta\,(a_n Rg+b_n\mathrm{Ric})\,d+\frac{n-4}{2}Q_2.$$

As pointed out in [Chang and González 2011], the conformal fractional Laplacian can be characterized as the Dirichlet-to-Neumann operator for a divergence-type, second-order degenerate elliptic equation with a weight in the Muckenhoupt class $\mathcal{A}_2$. This characterization allows one to study nonlocal operators by using the available tools for elliptic equations.

The associated fractional-order curvature Q_γ defined on the boundary of a conformally compact Einstein manifold can be introduced as $Q_\gamma=P_\gamma 1$, and it satisfies an important conformally covariant property. Q_1 is just the scalar curvature. However, for noninteger powers γ, the geometrical properties of Q_γ are not yet well understood. See the related works [Qing and Raske 2006; González et al. 2012; González and Qing 2010; Guillarmou and Qing 2010], for instance.

The notion of renormalized volume was first investigated by the physicists in relation to the AdS/CFT correspondence. It was considered in [Fefferman and Graham 2002] (see also [Graham 2000; Chang et al. 2007] for good surveys with many explicit examples). Given an asymptotically hyperbolic manifold X^{n+1} with boundary M^n and defining function ρ, one may compute the asymptotic expansion of the volume of the region $\{\rho>\epsilon\}$. The renormalized volume is defined as one very specific term in this asymptotic expansion. When the dimension n is odd, the renormalized volume is a conformal invariant of the conformally compact structure, and it can be calculated as the conformal primitive of the Q-curvature coming from the scattering operator (this is the case $\gamma=n/2$). In that case that n is even, the picture is more complex, and one can show the that the renormalized volume is one term of the Chern–Gauss–Bonnet formula in higher dimensions; see [Chang et al. 2006].

The aim of this note is to give a weighted version for the renormalized volume, and to find its relation to the fractional curvature Q_γ, for values $\gamma\in(0,1)$. The volume in this case is computed with respect to a very specific weight function ρ^* that will be introduced later in Lemma 2.2. This weight function is adapted to each fractional-order problem, and it is interpreted as the defining function that in some sense straightens out the coordinates of $M\times(0,\delta)$. We show:

Theorem 1.1. *Let (X^{n+1},g^+) be a conformally compact Einstein manifold, and ρ a defining function for $M^n=\partial X$, such that in a neighborhood $M\times(0,\delta)$ the metric is written in normal form*

$$g^+=\rho^{-2}(d\rho^2+g_\rho),$$

with $g_\rho = \hat{h} + O(\rho^2)$, for some $\hat{h}$ in the conformal infinity. Then we have the following asymptotic expansion for the weighted volume when $\epsilon \to 0$:

$$\begin{aligned}\text{vol}_{g^+,\gamma}(\{\rho > \epsilon\}) &:= \int_{\{\rho>\epsilon\}} (\rho^*)^{\frac{n}{2}-\gamma}\, \text{dvol}_{g^+} \\ &= \left(\tfrac{n}{2}+\gamma\right)^{-1} \text{vol}(M)\, \epsilon^{-\frac{n}{2}-\gamma} + V_\gamma \epsilon^{-\frac{n}{2}+\gamma} + O(\epsilon^{-\frac{n}{2}-\gamma+2}),\end{aligned} \tag{1}$$

where the weight ρ^* is the special defining function found in Lemma 2.2. Moreover, the term $V_\gamma := V_\gamma[g^+, \hat{h}]$ can be precisely computed as

$$V_\gamma = \frac{1}{d_\gamma\left(\frac{n}{2}-\gamma\right)} \int_M Q_\gamma[g^+, \hat{h}]\, \text{dvol}_{\hat{h}}\,. \tag{2}$$

Remark. We define V_γ as the *renormalized weighted volume*. Contrary to the usual definition of renormalized volume, V_γ is not a conformal invariant in the class $[\hat{h}]$; however, it is interesting to set up a fractional-order Yamabe type problem for (2), and this problem has been partially solved in [González and Qing 2010]. For the critical power $\gamma = n/2$, the renormalized weighted volume will correspond to the standard notion of renormalized volume.

Remark. Equation (1) can be interpreted as a first variation formula for Q_γ, which can shed some light on the geometrical interpretation for a fractional-order nonlocal curvature Q_γ.

For integer values of γ, the renormalized volume for the Paneitz operator P_k was already considered by Chang and Fang [2008], who studied a class of variational functionals, which in locally conformally flat manifolds is deeply related to the symmetric functions of the eigenvalues of the Schouten tensor.

Another interesting connection, pointed out by T. Rivière, has to do with the renormalized area for a complete minimal surface in $\mathbb{H}^3$, considered in [Alexakis and Mazzeo 2010]. This notion is equivalent to the classical Willmore energy of the surface. In some sense, this corresponds, in our case, to the values $n = 1$ and $\gamma = \frac{1}{2}$, which are critical for the problem.

From another point of view, given a smooth manifold compact manifold X^{n+1} with boundary M, endowed with a smooth metric $\bar{g}$, one can ask if there exists an analogous construction. In [Chang and González 2011] it was shown that for exponents $\gamma \in \left(0, \frac{1}{2}\right)$ it is possible to construct a fractional-order operator P_γ on M through an extension problem with respect to the metric $\bar{g}$, while for $\gamma \in \left(0, \frac{1}{2}\right)$, nonvanishing mean curvature is an obstruction to the existence of such an operator.

The same picture appears when computing the renormalized weighted volume. For the case $\gamma \in \left(0, \frac{1}{2}\right)$ we have the analogue of Theorem 1.1, while for $\gamma \in \left(\frac{1}{2}, 1\right)$ nonvanishing mean curvature would create a different term in the asymptotic expansion (1). These results are summarized in Theorem 4.2 in the last section.

The inspiration for these results came from an apparently unrelated problem. Caffarelli and Souganidis [2010] study some Bence–Merriman–Osher-type algorithms (originating in [Merriman et al. 1994]) corresponding to the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$. They show convergence to moving fronts, with two different behaviors depending on the value of γ: when $0<\gamma<\frac{1}{2}$, the normal velocity of the interface depends on a fractional-order mean curvature H_γ, but in the case $\frac{1}{2}<\gamma<1$, the resulting interface moves simply by (suitable scaled) mean curvature flow. We try to obtain a similar result in a geometric setting: the moving fronts of Caffarelli and Souganidis are replaced in our case by the level sets of a weight ρ^* that measures the distance to the boundary, and we get the same dichotomy.

The fractional mean curvature H_γ is a nonlocal, fractional-order curvature for the boundary of a compact set in Euclidean space, defined by means of a singular integral. It was been considered in [Caffarelli et al. 2010; Caffarelli and Valdinoci 2011], for instance, but so far there has not been a clear picture of its geometrical meaning. Of course, the natural question is to find the relation between the curvatures H_γ and Q_γ, at least when $\gamma\in\left(0,\frac{1}{2}\right)$.

These types of results indicate that, when $\gamma<\frac{1}{2}$, the operator presents very strong nonlocal behavior, and does not depend as much on the local geometry.

Structure of the paper. In Section 2 we review the necessary concepts of scattering theory, the construction of the fractional Paneitz operator through an extension problem and the notion of renormalized volume. In Section 3 we consider the renormalized weighted volume in the conformally compact Einstein setting in order to give the proof of Theorem 1.1. Finally, in Section 4 we extend this notion to any compact manifold with smooth boundary and give the proof of Theorem 4.2.

2. Background on the conformal fractional Laplacian

We review the definition of the conformal fractional Laplacian as the scattering operator in a conformally compact Einstein manifold. For an introduction, see for instance the first sections of [Chang and González 2011], and the references therein; here we give a brief summary.

Let M be a compact manifold of dimension n with a metric $\hat{h}$. Let X^{n+1} be a smooth manifold of dimension $n+1$ with boundary M. A function ρ is a defining function of ∂X in X if $\rho>0$ in X, $\rho=0$ on ∂X, and $d\rho\neq 0$ on ∂X. We say that g^+ is a conformally compact metric on X with conformal infinity $(M,[\hat{h}])$ if there exists a defining function ρ such that the manifold $(\bar{X},\bar{g})$ is compact for $\bar{g}=\rho^2g^+$, and $\bar{g}|_M\in[\hat{h}]$. If, in addition (X^{n+1},g^+) is a conformally compact manifold and $\mathrm{Ric}[g^+]=-ng^+$, then we call (X^{n+1},g^+) a conformally compact Einstein manifold.

Given a conformally compact, asymptotically hyperbolic manifold (X^{n+1}, g^+) and a representative $\hat{h}$ in $[\hat{h}]$ on the conformal infinity M, there is a uniquely defining function ρ such that, on $M \times (0, \delta)$ in X, g^+ has the normal form

$$g^+ = \rho^{-2}(d\rho^2 + g_\rho), \tag{3}$$

where g_ρ is a one-parameter family of metrics on M satisfying $g_\rho|_M = \hat{h}$. Moreover, g_ρ has an asymptotic expansion containing only even powers of ρ, at least up to degree n. For the rest of the paper, we assume that the metric g^+ is written in this normal form.

It is known [Graham and Zworski 2003; Mazzeo and Melrose 1987] that, given $f \in \mathbb{C}^\infty(M)$ and $s \in \mathbb{C}$, the eigenvalue problem

$$-\Delta_{g^+} u - s(n-s)u = 0 \quad \text{in } X, \tag{4}$$

has a solution of the form

$$u = F\rho^{n-s} + G\rho^s, \quad F, G \in \mathscr{C}^\infty(X), \quad F|_{\rho=0} = f, \tag{5}$$

for all $s \in \mathbb{C}$ unless $s(n-s)$ belongs to the pure point spectrum of $-\Delta_{g^+}$. The scattering operator on M is defined as $S(s)f = G|_M$; it is a meromorphic family of pseudodifferential operators in the half-plane $\operatorname{Re} s > n/2$. The values $s = n/2, n/2+1, n/2+2, \ldots$ are simple poles of finite rank, known as the trivial poles; $S(s)$ may have other poles, but for the rest of the paper we will always assume that we are not in those exceptional cases.

Given a conformally compact Einstein manifold (X, g^+) with conformal infinity $(M, [\hat{h}])$, we define the conformally covariant fractional powers of the Laplacian on M as follows: for $s = \frac{n}{2} + \gamma$, $\gamma \in \left(0, \frac{n}{2}\right)$, $\gamma \notin \mathbb{N}$, we set

$$P_\gamma^{\hat{h}} := P_\gamma[g^+, \hat{h}] = d_\gamma S\left(\frac{n}{2} + \gamma\right), \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{6}$$

With this choice of multiplicative factor, the principal symbol of P_γ is exactly the principal symbol of the fractional Laplacian $(-\Delta_{\hat{h}})^\gamma$.

The operators $P_\gamma[g^+, \hat{h}]$ satisfy an important conformal covariance property. Indeed, for a conformal change of metric

$$\hat{h}_w = w^{\frac{4}{n-2\gamma}} \hat{h}, \quad w > 0, \tag{7}$$

we have

$$P_\gamma[g^+, \hat{h}_w]\varphi = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma[g^+, \hat{h}]\,(w\varphi),$$

for all smooth φ on M.

We define the Q_γ-curvature of the metric associated to the functional P_γ to be

$$Q_\gamma^{\hat h} := Q_\gamma[g^+, \hat h] = P_\gamma[g^+, \hat h](1). \tag{8}$$

In particular, for a change of metric as (7), we obtain the equation for the Q_γ-curvature:

$$P_\gamma^{\hat h}(w) = w^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma^{\hat h_w}.$$

Next, we consider the characterization of the fractional Paneitz operator P_γ on a manifold M through an extension problem for a degenerate elliptic equation, in the spirit of Caffarelli and Silvestre [2007]. Indeed, we have:

Theorem 2.1 [Chang and González 2011]. *Let (X, g^+) be any conformally compact Einstein manifold, and let M be its boundary, with defining function ρ satisfying* (3). *Then, given $f \in \mathscr{C}^\infty(M)$ and $\gamma \in (0,1)$, the Poisson problem* (4)–(5) *for $s = \frac{n}{2} + \gamma$ is equivalent to the extension problem*

$$\begin{aligned} -\operatorname{div}(\rho^a \nabla U) + E(\rho) U = 0 \quad & \text{in } (X, \bar g), \\ U = f \quad & \text{on } M, \end{aligned} \tag{9}$$

where

$$\bar g = \rho^2 g^+, \quad U = \rho^{s-n} u, \quad a = 1 - 2\gamma, \quad s = \frac{n}{2} + \gamma,$$

and the derivatives in (9) *are taken with respect to the metric $\bar g$. The low-order term is given by*

$$E(\rho) = -\Delta_{\bar g}\big(\rho^{\frac{a}{2}}\big)\rho^{\frac{a}{2}} + \big(\gamma^2 - \tfrac14\big)\,\rho^{-2+a} + \frac{n-1}{4n} R_{\bar g}\rho^a. \tag{10}$$

We have the following expression for the fractional conformal Laplacian (6):

$$P_\gamma[g^+, \hat h] f = \frac{d_\gamma}{2\gamma} \lim_{\rho\to 0} \big(\rho^a \partial_\rho U\big).$$

Before we continue, we remind the reader of how to compute the $Q_\gamma[g^+, \hat h]$-curvature, as defined in (8), for $\gamma \in \big(0, \frac{n}{2}\big) \setminus \mathbb{N}$, $s = \frac{n}{2} + \gamma$. We set $f \equiv 1$, and find the solution to the Poisson problem

$$\begin{cases} -\Delta_{g^+} v - s(n-s) v = 0 \quad \text{in} X, \\ v = F\rho^{n-s} + G\rho^s, \quad F = 1 + O(\rho^2), G = h + O(\rho^2). \end{cases} \tag{11}$$

Then

$$Q_\gamma[g^+, \hat h] = d_\gamma h.$$

Next, we construct the special defining function ρ^* that will be needed in the definition of weighed volume. From the results in [Chang and González 2011; González and Qing 2010], one can see that it is possible to find some ρ^* satisfying that the zero-order term $E(\rho^*)$ in Equation (9) vanishes so that the extension problem is a pure divergence equation. More precisely,

Lemma 2.2. *Let (X, g^+) be a conformally compact Einstein manifold with conformal infinity $(M, [\hat{h}])$. For each $\gamma \in (0, 1)$, there exists another defining function ρ^* on X, satisfying $\rho^* = \rho + O(\rho^{2\gamma+1})$, and such that for the term E defined in* (10) *we have*

$$E(\rho^*) \equiv 0.$$

The metric $g^* = (\rho^*)^2 g^+$ satisfies $g^*|_{\rho=0} = \hat{h}$ and has asymptotic expansion

$$g^* = (d\rho^*)^2 \left[1 + O((\rho^*)^{2\gamma})\right] + \hat{h}\left[1 + O((\rho^*)^{2\gamma})\right].$$

In addition, if U is a solution of

$$\begin{aligned} -\operatorname{div}\left((\rho^*)^a \nabla U\right) &= 0 \quad \text{in } (X, g^*), \\ U &= f \quad \text{on } M, \end{aligned}$$

then

$$P_\gamma[g^+, \hat{h}]f = \frac{d_\gamma}{2\gamma} \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} U + f\, Q_\gamma[g^+, \hat{h}],$$

This defining function ρ^* is related to the eigenfunctions of $-\Delta_{g^+}$, and is constructed as follows: given $\gamma \in (0, 1)$, solve the Poisson problem (11). Then we simply set

$$\rho^* := v^{\frac{1}{n-s}}. \tag{12}$$

3. A notion of renormalized weighted volume

Before we give the proof of Theorem 1.1, we recall the original notion of renormalized volume, for n odd, as introduced by Fefferman and Graham [2002]. Given a conformally compact Einstein manifold (X, g^+) with defining function ρ, we can write the expansion

$$\operatorname{vol}_{g^+}(\{\rho > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \ldots + c_{n-1}\epsilon^{-1} + V + o(1).$$

We call the constant term $V := V_{n/2}[X, g^+]$ the renormalized volume for (X, g^+). It is independent of the choice of $\hat{h}$ in the class $[\hat{h}]$. It can be computed as follows: for each $s \in \mathbb{C}$, consider the solution u_s of (4) with boundary data $f \equiv 1$. Set

$$v = -\frac{d}{ds}\bigg|_{s=n} u_s.$$

Then v solves

$$-\Delta_{g^+} v = n \quad \text{in } X,$$

and has the asymptotic behavior

$$v = \log \rho + A + B\rho^n$$

in a neighborhood of M, where A, B are functions even in ρ, and $A|_{\rho=0} = 0$.

If n is odd, then $B|_M$ is determined by the choice of a Poincaré metric g^+ and a representative metric $\hat{h}$. Moreover,

$$B|_M = -\frac{d}{ds} S(s) 1 \bigg|_{s=n}.$$

The $Q_{n/2}$-curvature is defined as

$$Q_{n/2} = Q_{n/2}[g^+, \hat{h}] = d_{n/2} B|_M = -d_{n/2} \frac{d}{ds} S(s) 1 \bigg|_{s=n},$$

where the constant $d_{n/2}$ is written as in (6). This quantity is globally determined and depends in general on the extension X.

If $\hat{h}_w = e^{2w} \hat{h}$, then $Q_{n/2}$ satisfies the transformation law

$$e^{nw} Q_{n/2}[g^+, \hat{h}_w] = Q_{n/2}[g^+, \hat{h}] + d_{n/2} S(n) w.$$

Moreover,

$$V_{n/2}[X, g^+] = \frac{1}{d_{n/2}} \int_M Q_{n/2}[g^+, \hat{h}] \, dv_{\hat{h}}.$$

Now we are ready for the proof of Theorem 1.1. Let (X^{n+1}, g^+) be a conformally compact Einstein manifold with conformal infinity $(M^n, [\hat{h}])$. We write the metric in normal form, i.e., $g^+ = \bar{g}/\rho^2$, for $\bar{g} = d\rho^2 + g_\rho$, at least in a neighborhood $M \times (0, \delta)$. Fix $\gamma \in (0, 1)$ and $s = n/2 + \gamma$. Let v be the solution of the eigenvalue problem (11). By construction, v has the precise asymptotic behavior

$$v = \rho^{n-s}(1 + O(\rho^2)) + \rho^s (h + O(\rho^2)), \quad h = (d_\gamma)^{-1} Q_\gamma [g^+, \hat{h}]. \tag{13}$$

On the one hand, we integrate by parts,

$$\begin{aligned} I_1 := -\int_{\{\rho>\epsilon\}} \Delta_{g^+} v \, \mathrm{dvol}_{g^+} &= \epsilon^{1-n} \int_{\{\rho=\epsilon\}} \partial_\rho v \, \mathrm{dvol}_{g_\epsilon} \\ &= (n-s)\epsilon^{-s} \int_{\{\rho=\epsilon\}} \big(1+O(\epsilon^2)\big) \, \mathrm{dvol}_{g_\epsilon} + s\epsilon^{-n+s} \int_{\{\rho=\epsilon\}} \big(h+O(\epsilon^2)\big) \, \mathrm{dvol}_{g_\epsilon}, \end{aligned} \tag{14}$$

where we have used the asymptotic expansion for v given in (13).

Now we check that the expansion for the volume element is just

$$\mathrm{dvol}_{g_\epsilon} = \mathrm{dvol}_{\hat{h}} + O(\epsilon^2),$$

because $g_\rho = \hat{h} + O(\rho^2)$, and use that

$$Q_\gamma = P_\gamma 1 = d_\gamma h, \quad s = \frac{n}{2} + \gamma,$$

to arrive at

$$I_1 = \left(\tfrac{n}{2} - \gamma\right) \epsilon^{-\frac{n}{2}-\gamma} \operatorname{vol}(M) + \frac{1}{d_\gamma} \left(\tfrac{n}{2} + \gamma\right) \epsilon^{-\frac{n}{2}+\gamma} \int_M Q_\gamma \, \mathrm{dvol}_{\hat{h}} + O(\epsilon^{-\frac{n}{2}-\gamma+2}).$$

On the other hand, we recall the definition of the special defining function ρ^* from (12). Then

$$I_2 := \int_{\{\rho>\epsilon\}} v \, \mathrm{dvol}_{g^+} = \int_{\{\rho>\epsilon\}} (\rho^*)^{n-s} \, \mathrm{dvol}_{g^+} .$$

We remind the reader that v is a solution of (11), so that

$$-\Delta_{g^+} v - s(n-s)v = 0. \tag{15}$$

From (15), putting together I_1 and I_2, we obtain

$$\begin{aligned}&\int_{\{\rho>\epsilon\}} (\rho^*)^{\frac{n}{2}-\gamma} \, \mathrm{dvol}_{g^+} \\ &= \left(\tfrac{n}{2} + \gamma\right)^{-1} \epsilon^{-\frac{n}{2}-\gamma} \operatorname{vol}(M) + \frac{1}{d_\gamma} \left(\tfrac{n}{2} - \gamma\right)^{-1} \epsilon^{-\frac{n}{2}+\gamma} \int_M Q_\gamma \, \mathrm{dvol}_{\hat{h}} + O(\epsilon^{-\frac{n}{2}-\gamma+2}),\end{aligned}$$

as desired.

4. Weighted normalized volume in a general setting

In this section we change our point of view and consider a more general problem. Given any smooth compact manifold X with boundary and a metric $\bar{g}$, it is possible to give a notion of the conformal Paneitz operator P_γ with respect to the metric $\bar{g}$ and its associated curvature in this setting. We have (see also [Guillarmou and Guillopé 2007] for the case $\gamma = \frac{1}{2}$):

Proposition 4.1 [Chang and González 2011]. *Let $(X^{n+1}, \bar{g})$ be a compact smooth manifold with boundary. Fix $\gamma \in (0, 1)$ and suppose that U is the solution to the boundary value problem* (9). *Then one can construct the conformal fractional Laplacian as follows*:

- *For $\gamma \in (0, \frac{1}{2})$,*

$$P_\gamma[g^+, \hat{h}]f = -d^*_\gamma \lim_{\rho\to 0} \rho^a \partial_\rho U, \tag{16}$$

where

$$d^*_\gamma = \frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}.$$

- *For* $\gamma = \frac{1}{2}$,
$$P_{1/2}[g^+, \hat{h}]f = -\lim_{\rho\to 0} \partial_\rho U + \frac{n-1}{2} Hf,$$
where $H := \frac{1}{2n} Tr_{\hat{h}}(h^{(1)})$ *is the mean curvature of* M.
- *For* $\gamma \in \left(\frac{1}{2}, 1\right)$, (16) *still holds if* $H = 0$.

We review this construction: Let $(\bar{X}, \bar{g})$ be a compact smooth manifold of dimension $n+1$ with boundary M of dimension n. Let X be the interior of $\bar{X}$. Let ρ be a geodesic defining function for M and $\hat{h} := \bar{g}|M$. It is possible to find a solution for the singular Yamabe problem to produce an asymptotically hyperbolic metric g^+ in X, conformal to $\bar{g}$, of negative constant scalar curvature $R_{g^+} = -n(n+1)$, and with a very precise polyhomogeneous expansion. Classical references are [Aviles and McOwen 1988; Mazzeo 1991; Andersson et al. 1992]. More precisely, if write the metric in X as $\bar{g} = d\rho^2 + \bar{g}_\rho$, where g_ρ is a one-parameter family of metrics on M satisfying $g_\rho|_{\rho=0} = \hat{h}$, then

$$g^+ = \frac{\bar{g}(1 + \rho\alpha + \rho^n \beta)}{\rho^2}, \tag{17}$$

where $\alpha \in \mathscr{C}^\infty(\bar{X})$, $\beta \in \mathscr{C}^\infty(X)$ and β has a polyhomogeneous expansion

$$\beta(\rho, x) = \sum_{i=0}^{\infty} \sum_{j=0}^{N_i} \beta_{ij} \rho^i (\log \rho)^j \tag{18}$$

near the boundary, $N_i \in \mathbb{N} \cup \{0\}$ and $\beta_{ij} \in \mathscr{C}^\infty(\bar{X})$. Here we note that the log terms do not appear in the first terms of the expansion, so, for our purposes and because $\gamma \in (0, 1)$, they can be ignored. We define

$$\frac{1}{\hat{\rho}^2} := \frac{1 + \rho\alpha + \rho^n \beta}{\rho^2}, \tag{19}$$

so that (17) is rewritten as

$$g^+ = \frac{\bar{g}}{\hat{\rho}^2},$$

On the other hand, note that $\bar{g}_\rho$ may not have only even terms in its expansion. However, once the boundary metric $\hat{h} := \bar{g}_\rho|_{\rho=0} = \bar{g}|_M$ is fixed, we can find a boundary-defining function $y = \rho + O(\rho^2)$ such that

$$g^+ = \frac{dy^2 + g_y}{y^2}$$

near M, where g_y is a one-parameter family of metrics on M such that $g_y|_{y=0} = \hat{h}$, with the regularity of $\rho\alpha + \rho^n\beta$. The main property of g_y is that, if we make the expansion $g_y = g^{(0)} + g^{(1)}y + O(y^2)$, then $g^{(0)} = \hat{h}$ and $\mathrm{trace}_{g^{(0)}} g^{(1)} = 0$. We set $\tilde{g} = dy^2 + g_y$, so

$$g^+ = \frac{\tilde{g}}{y^2}.$$

The scattering operator can be solved on any smooth asymptotically hyperbolic manifold (X, g^+). First, solve the Poisson equation

$$-\Delta_{g^+} u - s(n-s)u = 0. \tag{20}$$

For each $f \in \mathscr{C}^\infty(M)$, there exists a solution of the form

$$u = y^{n-s}F + y^s G, \quad F = f + O(y^2), \quad G = h + O(y).$$

Then, for $s = \frac{n}{2} + \gamma$, we define the conformal fractional Laplacian in this setting as

$$P_\gamma[g^+, \hat{h}]f = d_\gamma h$$

and the fractional-order curvature

$$Q_\gamma[g^+, \hat{h}] := P_\gamma[g^+, \hat{h}]1. \tag{21}$$

In our case, we do have some log terms in the expansion (18). However, they appear at order n, and consequently, they do not change the first terms in the asymptotic expansion for u.

Let H be the mean curvature of ∂X as a boundary of the $(n+1)$-manifold $(\bar{X}, \bar{g})$. If we make the expansion $\bar{g}_\rho = \bar{g}^{(0)} + \bar{g}^{(1)}y + O(y^2)$, $\bar{g}^{(0)} = \hat{h}$, it is easy to check that

$$H := \frac{1}{2n}\mathrm{trace}_{\hat{h}}(\bar{g}^{(1)}). \tag{22}$$

It was shown in [Chang and González 2011] that $\hat{\rho}$, ρ and y are related by

$$\hat{\rho} = y(1 - Hy + O(y^2)), \quad \rho = y\left[1 + \left(-H + \frac{\alpha}{2}\right)y + O(y^2)\right]. \tag{23}$$

Let v be the solution of the eigenvalue problem (20) with Dirichlet data $f \equiv 1$. Then v has an asymptotic expansion

$$v = y^{n-s}[1 + O(y^2)] + y^s[h + O(y)] \tag{24}$$

where $Q_\gamma = d_\gamma h$. As in (12), we set $\rho^* = v^{1/(n-s)}$. This is the weight we will be considering.

We are ready to define a weighted version of volume for a compact manifold $(\bar{X}, \bar{g})$ with respect to a defining function ρ. Let $\gamma \in (0, 1)$. For each $\epsilon > 0$, we set

$$\mathrm{vol}_{g^+,\gamma}(\{\rho > \epsilon\}) := \int_{\{\rho > \epsilon\}} (\rho^*)^{\frac{n}{2}-\gamma}\, \mathrm{dvol}_{g^+}. \tag{25}$$

Our main result is the study of its asymptotic behavior when $\epsilon \to 0$:

Theorem 4.2. *Let* $(X, \bar{g})$ *be a compact* $(n+1)$*-dimensional smooth manifold with boundary, and let* $\hat{h}$ *be the restriction of the metric* $\bar{g}$ *to the boundary* $M := \partial X$. *Let* g^+ *be the asymptotically hyperbolic metric on* X *and* ρ *be the geodesic boundary defining function constructed in* (17). *Let* H *be the mean curvature of* M *as defined in* (22), *and* Q_γ *the fractional order curvature given by* (21). *Then the weighted volume* (25) *has an asymptotic expansion in* ϵ *given by*

- *If* $\gamma < \frac{1}{2}$, *or if* $\gamma > \frac{1}{2}$ *but* $\int_M \Psi = 0$, *then*

$$\mathrm{vol}_{g^+,\gamma}(\{\rho > \epsilon\}) = \epsilon^{-\frac{n}{2}-\gamma}\left[\left(\tfrac{n}{2}+\gamma\right)^{-1}\mathrm{vol}(M) + \epsilon^{2\gamma}V_\gamma + \textit{ higher-order terms}\right]$$

where

$$V_\gamma[g^+, \hat{h}] := \frac{1}{d_\gamma}\frac{1}{\frac{n}{2}-\gamma}\int_M Q_\gamma[g^+, \hat{h}]\,\mathrm{dvol}_{\hat{h}}\,.$$

- *However, if* $\gamma > \frac{1}{2}$, *and* $\int_M \Psi = 0$, *then*

$$\mathrm{vol}_{g^+,\gamma}(\{\rho > \epsilon\}) = \epsilon^{-\frac{n}{2}-\gamma}\left[\left(\tfrac{n}{2}+\gamma\right)^{-1}\mathrm{vol}(M) + \epsilon W_0 + \textit{ higher-order terms}\right]$$

for

$$W_0 := \left(\tfrac{n}{2}+\gamma\right)^{-1}\int_M \Psi\,\mathrm{dvol}_{\hat{h}}\,.$$

The quantity Ψ *is defined in* (26) *and appears naturally in the proof.*

Proof. Let v be the solution of the eigenvalue equation (20) with Dirichlet data $f \equiv 1$, and integrate this relation in the set $\{\rho > \epsilon\}$. First, we know that $g^+ = \bar{g}/\hat{\rho}^2$ and that $\bar{g} = d\rho^2 + \bar{g}_\rho$. Then, integration by parts gives that

$$I_1 := -\int_{\{\rho>\epsilon\}} \Delta_{g^+} v\,\mathrm{dvol}_{g^+} = \int_{\{\rho=\epsilon\}} \hat{\rho}^{1-n}\partial_\rho v\,\mathrm{dvol}_{\bar{g}_\epsilon}\,.$$

Now we check that the Taylor expansion for the volume element. Write the Taylor expansion of the metric $\bar{g}$ in coordinates y:

$$\bar{g}_\rho = \hat{h} + \bar{g}^{(1)}y + O(y^2).$$

Then

$$\det(\bar{g}_\rho) = \det(\hat{h})\left(1 + y\,\mathrm{trace}_{\hat{h}}\bar{g}^{(1)} + O(y^2)\right) = \det(\hat{h})\left(1 + y2nH + O(y^2)\right),$$

where we have used (22) for the last equality. Next, if $\rho = \epsilon$, then $y = \epsilon(1 + O(\epsilon))$, so

$$\mathrm{dvol}_{\bar{g}_\rho|_{\rho=\epsilon}} = \mathrm{dvol}_{\hat{h}}\left(1 + nH\epsilon + O(\epsilon^2)\right).$$

Now we write the expansion of v from (24) in the variable ρ, using the second equation in (23):

$$v = \rho^{n-s}\left[1+(n-s)\left(H-\frac{\alpha}{2}\right)\rho + O(\rho^2)\right] + \rho^s\left[h + O(\rho)\right].$$

Moreover, from (19),

$$\hat{\rho}^{1-n} = \rho^{1-n}\left[1+(n-1)\frac{\alpha}{2}\rho + O(\rho^2)\right].$$

Then, substituting all the terms in I_1 when $\rho = \epsilon$,

$$\begin{aligned} I_1 = \epsilon^{-s}(n-s)\operatorname{vol}(M) & \\ &+ (n-s)\epsilon^{-s+1}\int_M \left[(2n-s+1)H + (s-2)\frac{\alpha}{2}\right]\mathrm{dvol}_{\hat{h}} + O(\epsilon^{-s+2}) \\ &+ s\epsilon^{s-n}\int_M h\,\mathrm{dvol}_{\hat{h}} + O(\epsilon^{s-n+1}). \end{aligned}$$

We would like to find an asymptotic expansion for I_1. As in (14), the main order in the expansion will be ϵ^{-s}, $s = \frac{n}{2}+\gamma$. However, for the next order in the expansion will come from a competition between ϵ^{-n+s}, $-n+s = -\frac{n}{2}+\gamma$ and ϵ^{-s+1}, $-s+1 = -\frac{n}{2}-\gamma+1$, which gives the dichotomy $\gamma > \frac{1}{2}$ or $\gamma < \frac{1}{2}$.

Use that

$$Q_\gamma = P_\gamma 1 = d_\gamma h, \quad s = \frac{n}{2}+\gamma,$$

to arrive at

$$\begin{aligned} I_1 = \left(\tfrac{n}{2}-\gamma\right)\epsilon^{-\frac{n}{2}-\gamma}\operatorname{vol}(M) + \left(\tfrac{n}{2}-\gamma\right)\epsilon^{-\frac{n}{2}-\gamma+1}\int_M \Psi\,\mathrm{dvol}_{\hat{h}} & \\ + \frac{1}{d_\gamma}\left(\tfrac{n}{2}+\gamma\right)\epsilon^{-\frac{n}{2}+\gamma}\int_M Q_\gamma\,\mathrm{dvol}_{\hat{h}}, & \end{aligned}$$

where we write

$$\Psi := \left[(2n-s+1)H + (s-2)\frac{\alpha}{2}\right], \tag{26}$$

plus some higher-order terms in ϵ that we do not care to write.

On the other hand, we use the explicit formula for the special defining function ρ^*, i.e., $\rho^* = v^{1/(n-s)}$, so we get

$$I_2 := \int_{\{\rho>\epsilon\}} v\,\mathrm{dvol}_{g^+} = \int_{\{\rho>\epsilon\}} (\rho^*)^{n-s}\,\mathrm{dvol}_{g^+}$$

From (20), putting together I_1 and I_2, we obtain

$$\begin{aligned} \int_{\{\rho>\epsilon\}} (\rho^*)^{\frac{n}{2}-\gamma}\,\mathrm{dvol}_{g^+} = \left(\tfrac{n}{2}+\gamma\right)^{-1}\epsilon^{-\frac{n}{2}-\gamma}\operatorname{vol} M + \left(\tfrac{n}{2}+\gamma\right)^{-1}\epsilon^{-\frac{n}{2}-\gamma+1}\int_M \Psi\,\mathrm{dvol}_{\hat{h}} & \\ + \frac{1}{d_\gamma}\left(\tfrac{n}{2}-\gamma\right)^{-1}\epsilon^{-\frac{n}{2}+\gamma}\int_M Q_\gamma\,\mathrm{dvol}_{\hat{h}} + \text{h.o.t.} & \end{aligned}$$

Now look at the expansion: the first-order term is clear. However, the next order term depends on the value of γ. If $\gamma < \frac{1}{2}$, then this term is just

$$V_\gamma := \frac{1}{d_\gamma} \frac{1}{\frac{n}{2} - \gamma} \int_M Q_\gamma \, \mathrm{dvol}_{\hat{h}},$$

and the same happens if $\int_M \Psi \, \mathrm{dvol}_{\hat{h}} = 0$. In this case we can write

$$\int_{\{\rho > \epsilon\}} (\rho^*)^{\frac{n}{2} - \gamma} \, \mathrm{dvol}_{g^+} = \epsilon^{-\frac{n}{2} - \gamma} \left[\left(\tfrac{n}{2} + \gamma \right)^{-1} \mathrm{vol}\, M + \epsilon^{2\gamma} V_\gamma + \text{h.o.t.} \right].$$

However, if $\gamma > \frac{1}{2}$, and $\int_M \Psi \, \mathrm{dvol}_{\hat{h}} \neq 0$, then the coefficient of the second-order term is

$$W_0 := \left(\tfrac{n}{2} + \gamma \right)^{-1} \int_M \Psi \, \mathrm{dvol}_{\hat{h}}$$

and we write

$$\int_{\{\rho > \epsilon\}} (\rho^*)^{\frac{n}{2} - \gamma} \, \mathrm{dvol}_{g^+} = \epsilon^{-\frac{n}{2} - \gamma} \left[\left(\tfrac{n}{2} + \gamma \right)^{-1} \mathrm{vol}(M) + \epsilon W_0 + \text{h.o.t.} \right]. \qquad \square$$

Remark. When the starting point is an asymptotically hyperbolic manifold (X, g^+) with defining function $\hat{\rho}$ in the case $\gamma > \frac{1}{2}$ we simply have

$$W_0 = \left(\tfrac{n}{2} + \gamma \right)^{-1} (2n - s + 1) \int_M H,$$

which is just the integral of the mean curvature. This is perhaps the most natural setting for the problem.

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