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Let $X_0^\star(k, n, s)$ denote the sum of all multiple zeta-star values of weight k, depth n and height s. Kaneko and Ohno conjectured that, for any positive integers m, n, s with $m, n \geq s$, the difference

$$(-1)^m X_0^\star(m+n+1, n+1, s) - (-1)^n X_0^\star(m+n+1, m+1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients. We give a proof of this conjecture.

1. Introduction

Given a sequence $\boldsymbol{k} = (k_1, \ldots, k_n)$ of positive integers with $k_1 > 1$, the weight $\mathrm{wt}(\boldsymbol{k})$, depth $\mathrm{dep}(\boldsymbol{k})$ and height $\mathrm{ht}(\boldsymbol{k})$ are defined by

$$\mathrm{wt}(\boldsymbol{k}) = k_1 + \cdots + k_n, \qquad \mathrm{dep}(\boldsymbol{k}) = n, \qquad \mathrm{ht}(\boldsymbol{k}) = \#\{i \mid k_i \geq 2\},$$

respectively. For such a sequence $\boldsymbol{k}$, there are two well-studied real numbers: the multiple zeta value $\zeta(\boldsymbol{k})$, defined by

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

and the multiple zeta-star value $\zeta^\star(\boldsymbol{k})$, defined by

$$\zeta^\star(\boldsymbol{k}) = \zeta^\star(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

We call the values $\zeta(\boldsymbol{k})$ and $\zeta^\star(\boldsymbol{k})$ with weight $\mathrm{wt}(\boldsymbol{k})$, depth $\mathrm{dep}(\boldsymbol{k})$ and height $\mathrm{ht}(\boldsymbol{k})$.

The well-known Ohno–Zagier relation [2001] is a class of relations about the sums of multiple zeta values of fixed weight, depth and height. For integers k, n, s with $k \geq n+s$ and $n \geq s \geq 1$, we denote by $X_0(k, n, s)$ the sum of all multiple zeta values of weight k, depth n and height s. The Ohno–Zagier relation says that

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$$X_0(k,n,s)\in\mathbb{Q}[\zeta(2),\zeta(3),\zeta(5),\dots].$$

More explicitly, Ohno and Zagier gave the generating function expression

$$\sum_{\substack{k\geq n+s\\ n\geq s\geq 1}} X_0(k,n,s)u^{k-n-s}v^{n-s}t^{s-1} = \frac{1}{uv-t}\Big\{1-\exp\Big(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}(u^n+v^n-\alpha^n-\beta^n)\Big)\Big\},$$

where α and β are determined by $\alpha+\beta=u+v$ and $\alpha\beta=t$. In [Li 2010], we showed that the Ohno–Zagier relation can be deduced from the regularized double shuffle relation. In [Li 2008], we generalized the concept height to i-height, studied sums of multiple zeta values of fixed weight, depth and general height, and expressed a kind of generating function of these sums in terms of generalized hypergeometric functions.

Similarly, we denote by $X_0^\star(k,n,s)$ the sum of all multiple zeta-star values of weight k, depth n and height s for integers k,n,s with $k\geq n+s$ and $n\geq s\geq 1$. The authors of [Aoki et al. 2008] considered a generating function $\Phi_0^\star(u,v,t)$ of sums $X_0^\star(k,n,s)$, where

$$\Phi_0^\star(u,v,t)=\sum_{k\geq n+s,n\geq s\geq 1} X_0^\star(k,n,s)u^{k-n-s}v^{n-s}t^{2s-2}.$$

It was proved there that $\Phi_0^\star(u,v,t)$ can be expressed by a special value of the generalized hypergeometric function ${}_3F_2$ as

$$\Phi_0^\star(u,v,t)=\frac{1}{(1-v)(1-\beta)}\,{}_3F_2\Big(\begin{matrix}1-\beta,1-\beta+u,1\\2-v,2-\beta\end{matrix};1\Big), \tag{1}$$

where α,β are determined by $\alpha+\beta=u+v$ and $\alpha\beta=uv-t^2$, and the generalized hypergeometric function ${}_3F_2$ is defined as (see [Bailey 1935])

$${}_3F_2\Big(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\\beta_1,\beta_2\end{matrix};z\Big)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{n!(\beta_1)_n(\beta_2)_n}z^n,$$

with the Pochhammer symbol $(a)_n$ given by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & \text{if } n=0,\\ a(a+1)\cdots(a+n-1) & \text{if } n>0.\end{cases}$$

Similarly to [Li 2008], the authors of [Aoki et al. 2011] considered a kind of generating function of sums of multiple zeta-star values of fixed weight, depth and general height, and represented this generating function via generalized hypergeometric functions.

Since the generating function $\Phi_0^\star(u,v,t)$ is represented by ${}_3F_2$ as in (1), it is expected that in general $X_0^\star(k,n,s)$ can't be written as a polynomial of zeta values with rational coefficients. While in [2010] Kaneko and Ohno considered some kind of duality of multiple zeta-star values, and proposed the following conjecture.

Conjecture [Kaneko and Ohno 2010]. For any positive integers m, n, s satisfying $m, n \geq s$, we have

$$(-1)^m X_0^\star(m+n+1, n+1, s) - (-1)^n X_0^\star(m+n+1, m+1, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

Kaneko and Ohno showed that this is true for $s=1$. Using the result of [Aoki et al. 2008] about the generating function $\Phi_0^\star(u, v, 0)$, Yamazaki [2010] gave another proof of this case. Note that the Kaneko–Ohno theorem for their conjecture in the case $s=1$ can be restated as

$$(2)\quad u\,\Phi_0^\star(-u, v, 0) - v\,\Phi_0^\star(-v, u, 0) = \frac{1}{u} - \frac{1}{v} + \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)}\Big(\big(\Gamma(v)\Gamma(1-v)\big)^2 - \big(\Gamma(u)\Gamma(1-u)\big)^2\Big).$$

The purpose of this paper is to give a proof of the Kaneko–Ohno conjecture. In fact, similarly to (2), we give an expression of $u\,\Phi_0^\star(-u, v, t) - v\,\Phi_0^\star(-v, u, t)$ by gamma functions in Theorem 2.2. Our proof is based on the expression of $\Phi_0^\star(u, v, t)$ given in [Aoki et al. 2008], and hence is similar to the one of [Yamazaki 2010] for the special case $s=1$.

In Section 2, we state our main result and give some corollaries. In Section 3, we prepare a result about generalized hypergeometric series ${}_3F_2$. In the last section, we give the proof of the main theorem.

2. Statement of the main result

Main theorem. As in Section 1, we denote by $X_0^\star(k, n, s)$ the sum of all multiple zeta-star values of weight k, depth n and height s for integers k, n, s with $k \geq n+s$ and $n \geq s \geq 1$. Let $\Phi_0^\star(u, v, t)$ be the generating function defined by

$$\Phi_0^\star(u, v, t) = \sum_{k \geq n+s, n \geq s \geq 1} X_0^\star(k, n, s)\, u^{k-n-s} v^{n-s} t^{2s-2}.$$

For variables u, v, t, we define a and b by the conditions $a+b = -u+v$ and $ab = -uv - t^2$. Equivalently, we have

$$a, b = \frac{-u+v \pm \sqrt{(u+v)^2 + 4t^2}}{2}.$$

After that we define the function $A(u, v, a, b)$ by

$$(3)\quad A(u, v, a, b) = \frac{1}{2\pi}\left\{\frac{\cos \pi u}{\sin \pi v} - \frac{\cos \pi v}{\sin \pi u} + \cos \pi(a-b)(\cot \pi u - \cot \pi v)\right\}.$$

Note that $A(u,v,a,b)=A(u,v,b,a)$, which shall play an important role in the proof of our main theorem. We can express $A(u,v,a,b)$ by gamma functions as in the following lemma.

Lemma 2.1. *We have*

$$A(u,v,a,b)=\frac{1}{\Gamma(u+a)\Gamma(1-u-a)}\left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(a)\Gamma(1-a)}+\frac{\Gamma(u)\Gamma(1-u)}{\Gamma(b)\Gamma(1-b)}\right), \tag{4}$$

$$A(u,v,a,b)=\frac{1}{\Gamma(u+b)\Gamma(1-u-b)}\left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(b)\Gamma(1-b)}+\frac{\Gamma(u)\Gamma(1-u)}{\Gamma(a)\Gamma(1-a)}\right). \tag{5}$$

Proof. Equation (5) follows from (4) and the fact $A(u,v,a,b)=A(u,v,b,a)$. Using the well-known reflection formula

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s},$$

we find that the right-hand side of Equation (4) becomes

$$\frac{\sin\pi(u+a)}{\pi}\left(\frac{\sin\pi a}{\sin\pi v}+\frac{\sin\pi b}{\sin\pi u}\right),$$

which is equal to

$$\frac{1}{2\pi}\left(\frac{\cos\pi u-\cos\pi(v+a-b)}{\sin\pi v}+\frac{\cos\pi(u+a-b)-\cos\pi v}{\sin\pi u}\right).$$

Now it is easy to finish the proof. □

The main theorem of this paper is this:

Theorem 2.2. *We have*

$$\begin{aligned} u\,\Phi_0^\star(-u,v,t)-v\,\Phi_0^\star(-v,u,t) \\ =\frac{u-v}{ab}+A(u,v,a,b)\,\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}. \end{aligned} \tag{6}$$

Some remarks. By the definition of the generating function $\Phi_0^\star(u,v,t)$, it is easy to see that

$$\begin{aligned} &u\,\Phi_0^\star(-u,v,t)-v\,\Phi_0^\star(-v,u,t) \\ &=\sum_{\substack{m,n\geq s\\ s\geq 1}}(-1)^s\Big((-1)^m X_0^\star(m+n+1,n+1,s)-(-1)^n X_0^\star(m+n+1,m+1,s)\Big) \\ &\qquad\qquad\times u^{m+1-s}v^{n+1-s}t^{2s-2} \\ &\qquad+\sum_{n\geq s\geq 1}(-1)^{n+s}X_0^\star(n+s,s,s)\,(u^{n+1-s}-v^{n+1-s})\,t^{2s-2}. \end{aligned} \tag{7}$$

Since we have the expansion

$$\Gamma(1-x)=\exp\Bigl(\gamma x+\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}x^n\Bigr),$$

where γ is Euler's constant, we know that Theorem 2.2 indeed implies the Kaneko–Ohno conjecture.

Corollary 2.3. *For any positive integers m, n, s with $m, n \geq s$, the difference*

$$(-1)^m X_0^{\star}(m+n+1, n+1, s)\;-\;(-1)^n X_0^{\star}(m+n+1, m+1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients.

Taking into account to the second term of the right-hand side of (7), we have another corollary.

Corollary 2.4. *For any positive integers k, s with $k \geq 2s$, the sum $X_0^{\star}(k, s, s)$ can be expressed as a polynomial of zeta values with rational coefficients.*

Note that this is an immediate consequence of the symmetric sum formula for multiple zeta-star values (see [Hoffman 1992, Theorem 2.1]).

Letting $t=0$ in Theorem 2.2, we can derive (2). In fact, in this case, we can assume that $a=-u$ and $b=v$. For $A(u, v, a, b)$, we use the equivalent equation (4). Then using Theorem 2.2, we get (2).

3. A result about generalized hypergeometric series ${}_3F_2$

To prove the main theorem of this paper, we introduce the following result.

Proposition 3.1. *For $a, b, c \in \mathbb{C}$ with sufficient small real parts, we have*

$$(8)\quad {}_3F_2\Bigl(\begin{matrix}a, b, c\\ a+b, 1+c\end{matrix};1\Bigr)
=\frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)}\bigl(\psi(1+c-b)-\psi(a)-\psi(b)-\gamma\bigr)
-\frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)}\sum_{n=1}^{\infty}\frac{(a)_n(1-b)_n}{n\,n!\,(1+c-b)_n},$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function.

To save space, from now on we will denote the special value

$${}_3F_2\Bigl(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{matrix};1\Bigr)\quad\text{by}\quad {}_3F_2\Bigl(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{matrix}\Bigr).$$

To prove the proposition, we need two transformation formulas. The first one, from [Bailey 1935, Section 3.8, (1), p. 21], is

$$(9)\quad {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\right)=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}\,{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\beta_2-\alpha_3\\ \alpha_1+\alpha_2-\beta_1+1,\beta_2\end{matrix}\right)$$
$$+\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1+\alpha_2-\beta_1)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}$$
$$\times\,{}_3F_2\left(\begin{matrix}\beta_1-\alpha_1,\beta_1-\alpha_2,\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3\\ \beta_1-\alpha_1-\alpha_2+1,\beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right),$$

provided that $\mathrm{Re}(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)>0$ and $\mathrm{Re}(\alpha_3-\beta_1+1)>0$. The second one, from [Bailey 1935, Examples 7, p. 98] is

$$(10)\quad {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\right)$$
$$=\frac{\Gamma(\beta_2)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\,{}_3F_2\left(\begin{matrix}\beta_1-\alpha_1,\beta_1-\alpha_2,\alpha_3\\ \beta_1,\beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right),$$

provided that $\mathrm{Re}(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)>0$ and $\mathrm{Re}(\beta_2-\alpha_3)>0$.

Proof of Proposition 3.1. Taking a parameter ε such that $|\varepsilon|$ is sufficient small, we have

$${}_3F_2\left(\begin{matrix}a,b,c\\ a+b,1+c\end{matrix}\right)=\lim_{\varepsilon\to 0}{}_3F_2\left(\begin{matrix}a,b,c\\ a+b+\varepsilon,1+c-\varepsilon\end{matrix}\right).$$

Now we consider the series

$${}_3F_2\left(\begin{matrix}a,b,c\\ a+b+\varepsilon,1+c-\varepsilon\end{matrix}\right).$$

Applying (9), we get

$$(11)\quad {}_3F_2\left(\begin{matrix}a,b,c\\ a+b+\varepsilon,1+c-\varepsilon\end{matrix}\right)$$
$$=\frac{\Gamma(a+b+\varepsilon)\Gamma(\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)}\,{}_3F_2\left(\begin{matrix}a,b,1-\varepsilon\\ 1-\varepsilon,1+c-\varepsilon\end{matrix}\right)$$
$$+\frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1-\varepsilon)\Gamma(1+c)}\,{}_3F_2\left(\begin{matrix}a+\varepsilon,b+\varepsilon,1\\ 1+\varepsilon,1+c\end{matrix}\right).$$

To the first ${}_3F_2$-series in the right-hand side of (11), we apply the Gaussian summation formula (see [Bailey 1935, Section 1.3, (1)])

$$\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{n!(\beta)_n}=\frac{\Gamma(\beta)\Gamma(\beta-\alpha_1-\alpha_2)}{\Gamma(\beta-\alpha_1)\Gamma(\beta-\alpha_2)}$$

for $\mathrm{Re}(\beta-\alpha_1-\alpha_2)>0$, and apply (10) to the second ${}_3F_2$-series in the right-hand side of (11), we obtain

$$
{}_3F_2\left(\begin{matrix} a,b,c \\ a+b+\varepsilon,\,1+c-\varepsilon \end{matrix}\right)
= \frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)}
- \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)}\,{}_3F_2\left(\begin{matrix} \varepsilon,\,1-b,\,a+\varepsilon \\ 1+\varepsilon,\,1+c-b \end{matrix}\right).
$$

To the ${}_3F_2$-series in the right-hand side of the above equation, we split it into two terms as $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$. Then we see that

$$
{}_3F_2\left(\begin{matrix} a,b,c \\ a+b+\varepsilon,\,1+c-\varepsilon \end{matrix}\right)
$$

is equal to

$$
\frac{1}{\varepsilon}\left(\frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)}
- \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)}\right)
- \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a+\varepsilon)_n(1-b)_n}{(n+\varepsilon)n!(1+c-b)_n}.
$$

Finally, let ε go to 0 to finish the proof. For the first two lines of the above expression, we use l'Hôpital's rule and the fact that $\psi(1)=-\gamma$. □

4. Proof of the main theorem

In this section, we prove Theorem 2.2.

Lemma 4.1. *Let α and β be determined by $\alpha+\beta=u+v$ and $\alpha\beta=uv-t^2$. We have*

$$
\Phi_0^{\star}(u,v,t) = \frac{\Gamma(\beta-\alpha)\Gamma(1-\beta)\Gamma(v)\Gamma(1-v)}{\Gamma(1-\alpha)\Gamma(1+u-\alpha)\Gamma(1+\alpha-u)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(1-\beta)_n}{n!(1+\alpha-\beta)_n}\frac{\alpha-u}{n+\alpha-u}
+ \frac{\Gamma(\alpha-\beta)\Gamma(1-\alpha)\Gamma(v)\Gamma(1-v)}{\Gamma(1-\beta)\Gamma(1+u-\beta)\Gamma(1+\beta-u)} \sum_{n=0}^{\infty} \frac{(\beta)_n(1-\alpha)_n}{n!(1+\beta-\alpha)_n}\frac{\beta-u}{n+\beta-u}.
$$

Proof. A result of Aoki, Kombu, and Ohno [Aoki et al. 2008] about the generating function $\Phi_0^{\star}(u,v,t)$ gives that

$$\Phi_0^\star(u,v,t)=\frac{\Gamma(\beta-\alpha)\Gamma(1-v)}{\Gamma(1-\alpha)\Gamma(1+u-\alpha)}\int_0^1 s^{-\beta}(1-s)^{v-1}{}_2F_1\left(\begin{matrix}\alpha,\alpha-u\\1+\alpha-\beta\end{matrix};s\right)ds$$
$$+\frac{\Gamma(\alpha-\beta)\Gamma(1-v)}{\Gamma(1-\beta)\Gamma(1+u-\beta)}\int_0^1 s^{-\alpha}(1-s)^{v-1}{}_2F_1\left(\begin{matrix}\beta,\beta-u\\1+\beta-\alpha\end{matrix};s\right)ds.$$

Here ${}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix};s\right)$ is the Gaussian hypergeometric function, given by

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};s\right)=\sum_{n=0}^\infty\frac{(a)_n(b)_n}{n!(c)_n}s^n.$$

Hence, we have

$$\int_0^1 s^{-\beta}(1-s)^{v-1}{}_2F_1\left(\begin{matrix}\alpha,\alpha-u\\1+\alpha-\beta\end{matrix};s\right)ds=\sum_{n=0}^\infty\frac{(\alpha)_n(\alpha-u)_n}{n!(1+\alpha-\beta)_n}\int_0^1 s^{n-\beta}(1-s)^{v-1}ds$$
$$=\sum_{n=0}^\infty\frac{(\alpha)_n(\alpha-u)_n}{n!(1+\alpha-\beta)_n}\frac{\Gamma(1+n-\beta)\Gamma(v)}{\Gamma(1+n+v-\beta)}.$$

Now it is easy to finish the proof. □

Recall that we have defined a and b by $a+b=-u+v$ and $ab=-uv-t^2$. Using Lemma 4.1, we immediately get the following result.

Lemma 4.2. *We have*

$$u\,\Phi_0^\star(-u,v,t)-v\,\Phi_0^\star(-v,u,t)=F(u,v,a,b)+F(u,v,b,a),$$

where $F(u,v,a,b)$ is defined by

$$\frac{\Gamma(b-a)}{\Gamma(1-u-a)\Gamma(1+u+a)}$$
$$\times\left(\frac{u\Gamma(v)\Gamma(1-v)\Gamma(1-b)}{\Gamma(1-a)}\sum_{n=0}^\infty\frac{(a)_n(1-b)_n}{n!(1+a-b)_n}\frac{u+a}{n+u+a}\right.$$
$$\left.-\frac{v\Gamma(u)\Gamma(1-u)\Gamma(1+a)}{\Gamma(1+b)}\sum_{n=0}^\infty\frac{(1+a)_n(-b)_n}{n!(1+a-b)_n}\frac{u+a}{n+u+a}\right).$$

Since we have

$$\sum_{n=0}^\infty\frac{(a)_n(1-b)_n}{n!(1+a-b)_n}\frac{u+a}{n+u+a}=\sum_{n=0}^\infty\frac{(a)_n(-b)_n}{n!(a-b)_n}\frac{(a-b)(u+a)(n-b)}{-b(n+a-b)(n+u+a)},$$

and

$$\frac{n-b}{(n+a-b)(n+u+a)}=\frac{-a}{u+b}\frac{1}{n+a-b}+\frac{v}{u+b}\frac{1}{n+u+a},$$

we get

$$\sum_{n=0}^{\infty}\frac{(a)_n(1-b)_n}{n!(1+a-b)_n}\frac{u+a}{n+u+a}$$
$$=\frac{a(u+a)\Gamma(1+a-b)}{b(u+b)\Gamma(1+a)\Gamma(1-b)}-\frac{v(a-b)}{b(u+b)}\,{}_3F_2\left(\begin{matrix}a,-b,u+a\\a-b,1+u+a\end{matrix}\right).$$

In the above, we have used Gaussian summation formula for Gaussian hypergeometric function at unit argument. Similarly, we have

$$\sum_{n=0}^{\infty}\frac{(1+a)_n(-b)_n}{n!(1+a-b)_n}\frac{u+a}{n+u+a}$$
$$=\frac{b(u+a)\Gamma(1+a-b)}{a(u+b)\Gamma(1+a)\Gamma(1-b)}+\frac{u(a-b)}{a(u+b)}\,{}_3F_2\left(\begin{matrix}a,-b,u+a\\a-b,1+u+a\end{matrix}\right).$$

Hence we get the following lemma.

Lemma 4.3. *We have*

$$F(u,v,a,b)=F_1(u,v,a,b)+F_2(u,v,a,b),$$

where

$$F_1(u,v,a,b)$$
$$=\frac{(u+a)\Gamma(b-a)\Gamma(1+a-b)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)}\left(\frac{u\Gamma(v)\Gamma(1-v)}{b\Gamma(a)\Gamma(1-a)}-\frac{v\Gamma(u)\Gamma(1-u)}{a\Gamma(b)\Gamma(1-b)}\right),$$

$$F_2(u,v,a,b)=\frac{uv(a-b)\Gamma(b-a)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)}$$
$$\times\left(\frac{\Gamma(v)\Gamma(1-v)\Gamma(-b)}{\Gamma(1-a)}-\frac{\Gamma(u)\Gamma(1-u)\Gamma(a)}{\Gamma(1+b)}\right){}_3F_2\left(\begin{matrix}a,-b,u+a\\a-b,1+u+a\end{matrix}\right).$$

Now we compute $F_1(u,v,a,b)+F_1(u,v,b,a)$ and $F_2(u,v,a,b)+F_2(u,v,b,a)$.

Lemma 4.4. *The sum $F_1(u,v,a,b)+F_1(u,v,b,a)$ equals*

$$\frac{u-v}{ab}+\frac{(a-b)uv}{ab(u+a)(u+b)}\Gamma(b-a)\Gamma(1+a-b)A(u,v,a,b).$$

Proof. Using the reflection formula for gamma function, we see that

$$F_1(u,v,a,b)+F_1(u,v,b,a)=\Gamma(b-a)\Gamma(1+a-b)$$
$$\times\left\{\frac{\sin\pi(u+a)}{\pi(u+b)}\left(\frac{u\sin\pi a}{b\sin\pi v}-\frac{v\sin\pi b}{a\sin\pi u}\right)-\frac{\sin\pi(u+b)}{\pi(u+a)}\left(\frac{u\sin\pi b}{a\sin\pi v}-\frac{v\sin\pi a}{b\sin\pi u}\right)\right\}.$$

The term in braces in this expression is

$$\frac{1}{2\pi(u+b)}\Big(\frac{u(\cos\pi u-\cos\pi v\cos\pi(a-b)+\sin\pi v\sin\pi(a-b))}{b\sin\pi v} - \frac{v(\cos\pi u\cos\pi(a-b)-\sin\pi u\sin\pi(a-b)-\cos\pi v)}{a\sin\pi u}\Big) - \frac{1}{2\pi(u+a)}\Big(\frac{u(\cos\pi u-\cos\pi v\cos\pi(b-a)+\sin\pi v\sin\pi(b-a))}{a\sin\pi v} - \frac{v(\cos\pi u\cos\pi(b-a)-\sin\pi u\sin\pi(b-a)-\cos\pi v)}{b\sin\pi u}\Big). \tag{12}$$

Picking up the common factors, and noting the identities

$$\begin{aligned}
\frac{1}{b(u+b)}-\frac{1}{a(u+a)}&=\frac{v(a-b)}{ab(u+a)(u+b)},\\
\frac{1}{a(u+b)}-\frac{1}{b(u+a)}&=\frac{u(b-a)}{ab(u+a)(u+b)},\\
\frac{u}{b(u+b)}+\frac{v}{a(u+b)}+\frac{u}{a(u+a)}+\frac{v}{b(u+a)}&=\frac{2(v-u)}{ab},
\end{aligned}$$

we see that (12) becomes

$$\frac{uv(a-b)}{ab(u+a)(u+b)}A(u,v,a,b)+\frac{(v-u)\sin\pi(a-b)}{ab\pi},$$

which finishes the proof. □

Lemma 4.5. *The sum $F_2(u,v,a,b)+F_2(u,v,b,a)$ equals*

$$\frac{(b-a)uv}{ab(u+a)(u+b)}\Gamma(b-a)\Gamma(1+a-b)A(u,v,a,b) + A(u,v,a,b)\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}.$$

Proof. Applying Proposition 3.1 to the ${}_3F_2$-series in $F_2(u,v,a,b)$, we find that $F_2(u,v,a,b)$ becomes

$$\frac{(a-b)\Gamma(a-b)\Gamma(b-a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)\Gamma(1-u-a)}\left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(a)\Gamma(1-a)}-\frac{\Gamma(u)\Gamma(1-u)}{\Gamma(-b)\Gamma(1+b)}\right) \times\left(\psi(1+v)-\psi(a)-\psi(-b)-\gamma-\sum_{n=1}^{\infty}\frac{(a)_n(1+b)_n}{n\,n!(1+v)_n}\right),$$

which is just

$$A(u,v,a,b)\,\frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \times\Big(\psi(1+v)-\psi(a)-\psi(-b)-\gamma-\sum_{n=1}^{\infty}\frac{(a)_n(1+b)_n}{n\,n!\,(1+v)_n}\Big).$$

Hence, using the equality $A(u,v,a,b)=A(u,v,b,a)$, we find that

$$\begin{aligned}F_2(u,v,a,b)&+F_2(u,v,b,a)\\ &=A(u,v,a,b)\,\frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}\\ &\quad\times\Big\{\sum_{n=1}^{\infty}\Big(\frac{(1+a)_n(b)_n}{n\,n!\,(1+v)_n}-\frac{(a)_n(1+b)_n}{n\,n!\,(1+v)_n}\Big)+\psi(b)-\psi(-b)+\psi(-a)-\psi(a)\Big\}.\end{aligned}$$

It is easy to see that

$$\sum_{n=1}^{\infty}\Big(\frac{(1+a)_n(b)_n}{n\,n!\,(1+v)_n}-\frac{(a)_n(1+b)_n}{n\,n!\,(1+v)_n}\Big)=\frac{b-a}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!\,(1+v)_n},$$

which equals

$$\frac{b-a}{ab}\,\frac{\Gamma(1+u)\Gamma(1+v)}{\Gamma(1+u+a)\Gamma(1+u+b)}+\frac{1}{b}-\frac{1}{a}$$

by the Gaussian summation formula. Applying the formulas

$$\psi(-x)-\psi(x)-\frac{1}{x}=\pi\cot\pi x,$$

and

$$\pi\cot\pi a-\pi\cot\pi b=\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}{\Gamma(b-a)\Gamma(1+a-b)},$$

we finish the proof. □

Proof of Theorem 2.2. The theorem follows from Lemmas 4.2, 4.3, 4.4 and 4.5. □

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