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The natural filtrations of odd Hamiltonian superalgebras and special odd Hamiltonian superalgebras of formal vector fields are proved to be invariant under their automorphism group respectively, by determining the set of ad-quasi-nilpotent elements. Thereby, the automorphism groups of these Lie superalgebras are determined.

1. Introduction

As is well known, filtration structures and automorphism groups play an important role in the classification of modular Lie algebras (see [Jin 1992; Strade and Farnsteiner 1988; Wilson 1975]) and nonmodular Lie superalgebras (see [Kac 1977; Kac 1998; Scheunert 1979]), respectively. Cartan-type Lie algebras and Lie superalgebras possess natural filtration structures. The natural filtrations of the infinite-dimensional Lie algebras $L(m)$ and $\hat{L}(m)$ were proved to be invariant in [Rudakov 1986], where $L = W$, S, H or K. The natural filtrations of the general Lie superalgebra and special Lie superalgebra of formal vector fields were proved to be invariant in [Zhang and Liu 2004]. The invariance of natural filtrations of Cartan-type Lie algebras or Lie superalgebras provides a useful method of determining intrinsic properties and automorphism groups (see [Wilson 1971; Zhang and Liu 2004]).

We consider the infinite-dimensional odd Hamiltonian superalgebra HO and special odd Hamiltonian superalgebra SHO of formal vector fields, which are involved in [Kac 1998]. The corresponding results of Cartan-type Lie algebras are generalized and Jin's methods are used (see [Jin 1992]). Denote by $\{X_i\}_{i\geq -1}$

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the natural filtration of X. By determining the ad-quasi-nilpotent elements in the even part and the subalgebras generated by certain ad-quasi-nilpotent elements, we prove that the natural filtration of X is invariant under automorphisms in the following sense: If φ is an automorphism of X, then $\varphi(X_i) \subseteq X_i$ for every $i \geq -1$. Besides, we prove that every automorphism of X is continuous and can be induced from an automorphism of $\Lambda(n,n)$. Finally, we prove that the automorphism group of X is isomorphic to the admissible automorphism group of the base superalgebra $\Lambda(n,n)$.

This paper is arranged as follows. In Section 2, we recall the necessary definitions concerning Lie superalgebras of Cartan type HO and SHO of formal vector fields. In Section 3, we characterize the ad-quasi-nilpotent elements with certain properties, and prove the invariance of their natural filtrations. In Section 4, we determine the automorphism groups of Lie superalgebras of Cartan type HO and SHO of formal vector fields.

2. Preliminaries

In this paper, $\mathbb{F}$ denotes an algebraically closed field of characteristic zero, and n is a positive integer greater than 3. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denote the ring of integers modulo 2. Let $\mathrm{P}(n) = \mathbb{F}[[x_1, \dots, x_n]]$ denote the ring of formal power series in n variables over field $\mathbb{F}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$, we abbreviate $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ to $x^{(\alpha)}$, and put $|\alpha| = \sum_{i=1}^n \alpha_i$. Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_{n+1}, x_{n+2}, \dots, x_{2n}$. Denote by $\Lambda(n,n)$ the tensor product $\mathrm{P}(n) \otimes_{\mathbb{F}} \Lambda(n)$. Then $\Lambda(n,n)$ is a noncommutative linearly compact topological superalgebra with a fundamental system $\{(\Lambda_1)^j\}_{j\geq 1}$ of neighborhoods of 0, where $(\Lambda_1)^j = \mathrm{span}_{\mathbb{F}}\{x_{i_1} \cdots x_{i_k} \mid j \leq k\}$. In particular, $(\Lambda_1)^1$ is the ideal of $\Lambda(n,n)$ generated by $\{x_1, \dots, x_{2n}\}$ (see [Kac 1998]). For $g \in \mathrm{P}(n)$ and $f \in \Lambda(n)$, we abbreviate $g \otimes f$ to gf.

Put $Y_0 = \{1, 2, \dots, n\}$, $Y_1 = \{n+1, \dots, 2n\}$ and $Y = Y_0 \cup Y_1$. Let

$$\mathbb{B}_k = \{\langle i_1, i_2, \dots, i_k \rangle \mid n+1 \leq i_1 < i_2 < \cdots < i_k \leq 2n\}$$

and $\mathbb{B}(n) = \bigcup_{k=0}^n \mathbb{B}_k$, where $\mathbb{B}_0 = \varnothing$. Given $u = \langle i_1, i_2, \dots, i_k \rangle \in \mathbb{B}_k$, set $|u| = k$, $\{u\} = \{i_1, i_2, \dots, i_k\}$ and $x^u = x_{i_1} x_{i_2} \cdots x_{i_k}$ (with the convention that $|\varnothing| = 0$ and $x^{\varnothing} = 1$). Then $\{x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}_0^n, u \in \mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of the infinite-dimensional superalgebra $\Lambda(n,n)$. Clearly, $\Lambda(n,n)$ has a $\mathbb{Z}$-grading structure $\Lambda(n,n) = \bigoplus_{i\geq 0} \Lambda(n,n)_{[i]}$, where

$$\Lambda(n,n)_{[i]} = \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)} x^u \mid |\alpha| + |u| = i, \alpha \in \mathbb{N}_0^n, u \in \mathbb{B}(n)\}.$$

An arbitrary element $f \in \Lambda(n,n)$ can be uniquely decomposed into $f=\sum_{i=0}^{\infty} f_i$, where $f_i \in \Lambda(n,n)_{[i]}$. The continuation of the addition of topological algebra $\Lambda(n,n)$ allows us to get the sum of infinite nonzero elements of $\Lambda(n,n)$. Set $\Lambda(n,n)_j = \bigoplus_{i\geq j} \Lambda(n,n)_{[i]}$. Then $\{\Lambda(n,n)_j\}_{j\geq 0}$ is a filtration of $\Lambda(n,n)$. Clearly, $\Lambda(n,n)_j = \{(\Lambda_1)^j\}$, where $j \in \mathbb{N}_0$.

Let $D_1, D_2, \dots, D_{2n}$ be the linear transformations of $\Lambda(n,n)$ such that

$$D_i(x^{(\alpha)}x^u) = \begin{cases} x^{(\alpha-\varepsilon_i)}x^u & \text{if } i \in Y_0, \\ x^{(\alpha)} \cdot (\partial x^u/\partial x_i) & \text{if } i \in Y_1. \end{cases}$$

Then D_i is a derivation of superalgebra $\Lambda(n,n)$ for every $i \in Y$. Let $\operatorname{Der}\Lambda(n,n)$ be the Lie superalgebra consisting of all continuous derivations of $\Lambda(n,n)$. Then $\operatorname{Der}\Lambda(n,n) = W(n,n)$, where $W(n,n) = \{\sum_{i=1}^{2n} f_i D_i \mid f_i \in \Lambda(n,n)\}$, and we call $W(n,n)$ the general superalgebra of formal vector fields (see [Kac 1998]). Clearly, $W(n,n)$ has a $\mathbb{Z}$-grading structure $W(n,n) = \bigoplus_{i\geq -1} W(n,n)_{[i]}$, where $W(n,n)_{[i]} = \operatorname{span}_{\mathbb{F}}\{f D_j \mid f \in \Lambda(n,n)_{[i+1]},\ j \in Y\}$. Let

$$W(n,n)_j = \bigoplus_{i\geq j} W(n,n)_{[i]}.$$

Then $\{W(n,n)_j\}_{j\geq -1}$ is called the natural filtration of $W(n,n)$. Therefore, $W(n,n)$ is a linearly compact topological Lie superalgebra with $\{W(n,n)_j\}_{j\geq -1}$ as a fundamental system of neighborhoods of 0.

If $\deg f$ appears in some expression in this paper, we always regard f as a $\mathbb{Z}_2$-homogenous element and $\deg f$ as the $\mathbb{Z}_2$-degree of f. Then $\deg D_i = \mu(i)$, where

$$\mu(i) = \begin{cases} \bar{0} & \text{if } i \in Y_0, \\ \bar{1} & \text{if } i \in Y_1. \end{cases}$$

The following formula holds in $W(n,n)$ (see [Zhang 1997]):

$$[f D_i, g D_j] = f D_i(g) D_j - (-1)^{\deg f D_i \deg g D_j} g D_j(f) D_i,$$

where $f, g \in \Lambda(n,n)$ and $i, j \in Y$.

Put

$$i' = \begin{cases} i+n & \text{if } i \in Y_0, \\ i-n & \text{if } i \in Y_1. \end{cases}$$

Let $T_H : \Lambda(n,n) \to W(n,n)$ be the linear mapping such that

$$T_H(f) = \sum_{i=1}^{2n} (-1)^{\mu(i)\deg f} D_i(f) D_{i'}. \tag{1}$$

Put $HO(n) = \{T_H(f) \mid f \in \Lambda(n,n)\}$. Then $HO(n)$ is an infinite-dimensional Lie superalgebra (see [Kac 1998]), called the odd Hamiltonian superalgebra of formal vector fields. For $f, g \in \Lambda(n,n)$ the equation

$$[T_H(f), T_H(g)] = T_H\big(T_H(f)(g)\big) \tag{2}$$

holds (see [Kac 1998]). Clearly, the algebra $HO(n)$ has a $\mathbb{Z}$-grading structure $HO(n) = \bigoplus_{i\geq -1} HO(n)_{[i]}$, where $HO(n)_{[i]} = \{T_H(f) \mid f \in \Lambda(n,n)_{[i+2]}\}$. Set $HO(n)_i = HO(n) \cap W(n,n)_i$. Then $\{HO(n)_i\}_{i\geq -1}$ is called the natural filtration of $HO(n)$.

Let $HO(n,n)$ be the $\mathbb{Z}_2$-graded space $\Lambda(n,n)/\mathbb{F}\cdot 1$ with reversed parity, that is, $HO(n,n) = HO(n,n)_{\bar{0}} + HO(n,n)_{\bar{1}}$, where

$$HO(n,n)_\theta = \operatorname{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \in \Lambda(n,n)_{\theta+\bar{1}} \mid |\alpha|+|u| \geq 1\}, \quad \theta \in \mathbb{Z}_2.$$

We denote by $\mathrm{p}(y)$ the $\mathbb{Z}_2$-degree of the element y of $HO(n,n)$ to distinguish it from the $\mathbb{Z}_2$-degree in $\Lambda(n,n)$. By (2), we can define a Lie multiplication in $HO(n,n)$ by

$$[y,z] = \sum_{i=1}^{2n} (-1)^{\mu(i)\,\mathrm{p}(y)+\mu(i)} D_i(y) D_{i'}(z). \tag{3}$$

Clearly, Lie superalgebra $HO(n,n)$ is isomorphic to $HO(n)$.

Let $\Delta = \sum_{i=1}^{n} D_i D_{i'}$ be a linear mapping on $\Lambda(n,n)$, let

$$\Lambda^{\Delta}(n,n) = \{f \in \Lambda(n,n) \mid \Delta f = 0\},$$

and let $\overline{SHO}(n,n) = \Lambda^{\Delta}(n,n)/\mathbb{F}\cdot 1$. Then $\overline{SHO}(n,n)$ is a $\mathbb{Z}_2$-graded subspace of $HO(n,n)$. For $f, g \in \Lambda(n,n)$ we have

$$\Delta\big(T_H(f)(g)\big) = (-1)^{\deg f+\bar{1}} T_H(f)(\Delta g) - (-1)^{\deg f \deg g + \deg f} T_H(g)(\Delta f);$$

see [Kac 1998]. Therefore, with the multiplication defined in (3), $\overline{SHO}(n,n)$ is a subalgebra of $HO(n,n)$. Set

$$SHO(n,n) = \operatorname{span}_{\mathbb{F}}\big\{[x^{(\alpha)}, x^u] \,\big|\, \alpha \in \mathbb{N}_0^n,\ u \in \mathbb{B}(n),\ |\alpha|+|u| \geq 3\big\}. \tag{4}$$

Then $SHO(n,n)$ is an infinite-dimensional subalgebra of $\overline{SHO}(n,n)$, called the special odd Hamiltonian superalgebra of formal vector fields (see [Kac 1998]). Clearly, $SHO(n,n)$ has a $\mathbb{Z}$-grading structure $SHO(n,n) = \bigoplus_{i\geq -1} SHO(n,n)_{[i]}$, where

$$SHO(n,n)_{[i]} = \operatorname{span}_{\mathbb{F}}\big\{[x^{(\alpha)}, x^u] \,\big|\, \alpha \in \mathbb{N}_0^n,\ u \in \mathbb{B}(n),\ |\alpha|+|u| = i+4\big\}.$$

Set $SHO(n,n)_i = SHO(n,n) \cap W(n,n)_i$. Then $\{SHO(n,n)_i\}_{i\geq -1}$ is called the natural filtration of $SHO(n,n)$.

Set $SHO(n) = T_H(SHO(n,n))$. Clearly, with the multiplication defined in (2), $SHO(n)$ is a Lie superalgebra that is isomorphic to $SHO(n,n)$. For the sake of simplicity, we always write SHO for $SHO(n,n)$ or $SHO(n)$.

In the following, we simply write HO for $HO(n)$, and let X denote the Lie superalgebra HO or SHO.

3. Invariant subalgebras and natural filtrations

Lemma 3.1. *Suppose that $y \in X_{[-1]} \cap X_{\bar{0}}$ and that $y \neq 0$. Then*

$$y(\Lambda(n,n)_{[r]}) = \Lambda(n,n)_{[r-1]}$$

for all $r \in \mathbb{N}$; hence $y(\Lambda(n,n)) = \Lambda(n,n)$.

Proof. We first prove that $\Lambda(n,n)_{[r-1]} \subseteq y(\Lambda(n,n)_{[r]})$. Write $y = \sum_{j=1}^{n} c_j D_j$, where $c_j \in \mathbb{F}$. Then there exists at least one nonzero element in $\{c_1, \dots, c_n\}$. Let $c_{1j} = c_j$, where $1 \leq j \leq n$. Then there exist $c_{lj} \in \mathbb{F}$, where $2 \leq l \leq n$, $1 \leq j \leq n$, such that the matrix $(c_{ij})_{1 \leq i,j \leq n}$ is invertible. Let $(a_{ij})_{1 \leq i,j \leq n} = (c_{ij})^{-1}_{1 \leq i,j \leq n}$. Note that $(1, 0, \dots, 0)\,(c_{ij})_{1 \leq i,j \leq n} = (c_1, c_2, \dots, c_n)$. It follows that

$$(1, 0, \dots, 0) = (c_1, c_2, \dots, c_n)\,(a_{ij})_{1 \leq i,j \leq n}. \tag{5}$$

Let $h_j = \sum_{i=1}^{n} a_{ij} x_i$, where $j \in Y_0$, and let $h_k = x_k$, where $k \in Y_1$. Then the set $\{h_1, h_2, \dots, h_{2n}\}$ is an $\mathbb{F}$-basis of $\Lambda(n,n)_{[1]}$. Therefore, for every $r \in \mathbb{N}$, we have

$$\Lambda(n,n)_{[r-1]} = \operatorname{span}_{\mathbb{F}}\{h_1^{\alpha_1} \cdots h_n^{\alpha_n} h_{i_1} \cdots h_{i_k}\},$$

where $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $\langle i_1, \dots, i_k \rangle \in \mathbb{B}_k$ and $\sum_{i=1}^{n} \alpha_i + k = r - 1$. Noting that $\alpha_1 \in \mathbb{N}_0$, we see that $\alpha_1 + 1 \neq 0$, since $\operatorname{char}\mathbb{F} = 0$. By (5), we have $y(h_j) = \delta_{j1}$, where $j \in Y_0$. Consequently, we have

$$y((\alpha_1 + 1)^{-1} h_1 h_1^{\alpha_1} \cdots h_n^{\alpha_n} h_{i_1} \cdots h_{i_k}) = h_1^{\alpha_1} \cdots h_n^{\alpha_n} h_{i_1} \cdots h_{i_k}.$$

Thus $\Lambda(n,n)_{[r-1]} \subseteq y(\Lambda(n,n)_{[r]})$. The reverse inclusion can be verified trivially.

Suppose that $f = \sum_{s \geq 0} f_s$ is an arbitrary element of $\Lambda(n,n)$, where f_s is in $\Lambda(n,n)_{[s]}$. According to the results above, for every $s \in \mathbb{N}_0$, there exists a g_{s+1} in $\Lambda(n,n)_{[s+1]}$ such that $f_s = y(g_{s+1})$. Since y is continuous, it follows that $f = \sum_{s \geq 0} y(g_{s+1}) = y(\sum_{s \geq 0} g_{s+1}) \in y(\Lambda(n,n))$. Thus $\Lambda(n,n) = y(\Lambda(n,n))$. □

Lemma 3.2. *Suppose that $y \in X_{[-1]} \cap X_{\bar{0}}$ and that $y \neq 0$. Then $[y, X_{[r]}] = X_{[r-1]}$ for all $r \in \mathbb{N}_0$.*

Proof. It suffices to show that $X_{[r-1]} \subseteq [y, X_{[r]}]$. Consider the case of HO. Suppose that $T_H(f)$ is an element of $HO_{[r-1]}$, where $f \in \Lambda(n,n)_{[r+1]}$. Then by Lemma 3.1 there exists $g \in \Lambda(n,n)_{[r+2]}$ such that $y(g) = f$, which combined with (2) yields that $T_H(f) = [y, T_H(g)] \in [y, HO_{[r]}]$.

Consider the case of SHO. Suppose that $[x^{(\alpha)}, x^u]$ is a standard basis element of $SHO_{[r-1]}$, and assume that $y = x_{i'}$, where $i \in Y_0$. Then by (3), we have that $-[x^{(\alpha)}, x^u] = [x_{i'}, [x^{(\alpha+\varepsilon_i)}, x^u]]$ is in $[y, SHO_{[r]}]$. □

Lemma 3.3. *Suppose that $y \in X_{-1} \cap X_{\bar{0}} \setminus X_0$ and that $y \neq 0$. Then $[y, X] = X$.*

Proof. It suffices to show that $X \subseteq [y, X]$. Suppose that $y = \sum_{i \geq -1} y_i$, where $y_i \in X_{[i]}$, and suppose that $z = \sum_{i \geq -1} z_i$ is an arbitrary element of X, where $z_i \in X_{[i]}$. Then by Lemma 3.2, we can inductively pick $w_j \in X_{[j]}$, such that $[y_{-1}, w_0] = z_{-1}$ when $j = 0$, and $[y_{-1}, w_j] = z_{j-1} - \sum_{i=0}^{j-1} [y_i, w_{j-1-i}]$ when $j > 0$. For arbitrary $k \in \mathbb{N}_0$, direct calculations show that

$$\left[y_{-1}, \sum_{j=0}^{k} w_j\right] = [y_{-1}, w_0] + \sum_{j=1}^{k} [y_{-1}, w_j] = \sum_{j=-1}^{k-1} z_j - \sum_{0 \leq i+j \leq k-1} [y_i, w_j], \tag{6}$$

and

$$\begin{aligned} \left[\sum_{i \geq 0} y_i, \sum_{j=0}^{k} w_j\right] &= \left[\sum_{i=0}^{k-1} y_i, \sum_{j=0}^{k} w_j\right] + \left[\sum_{i \geq k} y_i, \sum_{j=0}^{k} w_j\right] \\ &= \sum_{0 \leq i+j \leq k-1} [y_i, w_j] + \sum_{i+j \geq k} [y_i, w_j]. \end{aligned} \tag{7}$$

Combining (6) and (7), we have

$$\begin{aligned} \left[y, \sum_{j=0}^{k} w_j\right] &= \left[y_{-1}, \sum_{j=0}^{k} w_j\right] + \left[\sum_{i \geq 0} y_i, \sum_{j=0}^{k} w_j\right] \\ &= \sum_{j=-1}^{k-1} z_j + \sum_{i+j \geq k} [y_i, w_j] \in \sum_{j=-1}^{k-1} z_j + X_k. \end{aligned}$$

Noting that $X_k = \bigcap_{i=0}^{k} X_i$, we see that $\left[y, \sum_{j=0}^{k} w_j\right] \equiv \sum_{j=-1}^{k-1} z_j \pmod{\bigcap_{i=0}^{k} X_i}$. Let $w = \sum_{j \geq 0} w_j$. Then $[y, w] = [y, \sum_{j \geq 0} w_j] \equiv \sum_{j \geq -1} z_j \pmod{\bigcap_{i \geq 0} X_i}$, whence $[y, w] = z$. Thus $X \subseteq [y, X]$. $\square$

For an element y of Lie superalgebra L, we call y ad-nilpotent if there exists a positive integer t such that $(\operatorname{ad} y)^t (L) = 0$. We call y ad-quasi-nilpotent if $\bigcap_{t=1}^{\infty} (\operatorname{ad} y)^t (L) = 0$ (see [Humphreys 1972; Jin 1992]). Obviously, ad-nilpotent elements are ad-quasi-nilpotent elements. In particular, D_i is an ad-nilpotent element of X for every $i \in Y_1$.

Let J be a subset of L. Put

$$\mathrm{qn}_L(J) := \{y \in J \mid y \text{ is an ad-quasi-nilpotent element of } L\}.$$

In the following, we simply write $\mathrm{qn}(J)$ for $\mathrm{qn}_X(J)$, and denote by $\mathrm{Qn}(J)$ the subalgebra of X generated by $\mathrm{qn}(J)$. It is clear that $X_1 \subseteq \mathrm{qn}(X)$. In the following, we will determine the ad-quasi-nilpotent elements of $X_{[0]}$, and prove the invariance of natural filtration of X.

We denote by $\mathrm{M}_{2n}(\Lambda(n, n))$ the $\mathbb{F}$-algebra consisting of all $2n \times 2n$ matrices over $\Lambda(n, n)$, denote by $\mathrm{pr}_{[0]}$ the projection of $\Lambda(n, n)$ on $\Lambda(n, n)_{[0]}$, and denote by pr_1 the projection on $\Lambda(n,n)_1$. For $(a_{ij})_{i,j\in Y}\in \mathrm{M}_{2n}(\Lambda(n,n))$, we also denote

$$\mathrm{pr}_{[0]}:(a_{ij})_{i,j\in Y}\mapsto (\mathrm{pr}_{[0]}(a_{ij}))_{i,j\in Y} \quad\text{and}\quad \mathrm{pr}_1:(a_{ij})_{i,j\in Y}\mapsto (\mathrm{pr}_1(a_{ij}))_{i,j\in Y}.$$

Lemma 3.4. *Suppose that $h_1,h_2,\dots,h_{2n}\in\Lambda(n,n)_1$ with $\deg(h_j)=\mu(j)$ such that the matrix $(\mathrm{pr}_{[0]}(D_ih_j))_{i,j\in Y}$ is invertible. Then there exists an automorphism σ of $\Lambda(n,n)$ such that*

$$\sigma(x_i)=h_i \quad \text{for all } i\in Y. \tag{8}$$

Proof. Let $\sigma:\Lambda(n,n)\to\Lambda(n,n)$ be an even endomorphism such that (8) holds. Note that the natural filtration of $\Lambda(n,n)$ is invariant under σ. Then σ induces a linear transformation σ_i of $\Lambda(n,n)_i/\Lambda(n,n)_{i+1}$ for every $i\geq 0$. We first use induction on k to show that σ_k is bijective. Since the matrix of σ_1 with respect to $\mathbb{F}$-basis $\{x_1+\Lambda(n,n)_2,\dots,x_{2n}+\Lambda(n,n)_2\}$ is just $(\mathrm{pr}_{[0]}(D_ih_j))_{i,j\in Y}$, we see that σ_1 is bijective. Suppose that $k>1$ and $x^{(\alpha)}x^u$ is an element of $\Lambda(n,n)_{[k]}$. Then we can write $x^{(\alpha)}x^u=f_jf_{k-j}$, where $f_j\in\Lambda(n,n)_{[j]}$ and $f_{k-j}\in\Lambda(n,n)_{[k-j]}$ with $1\leq j<k$. By induction, there exist $f'_j\in\Lambda(n,n)_{[j]}$ and $f'_{k-j}\in\Lambda(n,n)_{[k-j]}$ such that $\sigma(f'_j)\equiv f_j \pmod{\Lambda(n,n)_{j+1}}$ and $\sigma(f'_{k-j})\equiv f_{k-j} \pmod{\Lambda(n,n)_{k-j+1}}$, whence

$$\sigma(f'_jf'_{k-j})=\sigma(f'_j)\sigma(f'_{k-j})\equiv f_jf_{k-j}=x^{(\alpha)}x^u \pmod{\Lambda(n,n)_{k+1}}.$$

Thus σ_k is surjective. Note that since $\Lambda(n,n)_k/\Lambda(n,n)_{k+1}$ is finite-dimensional, it follows that σ_k is bijective.

We next prove that σ is bijective. Suppose that $f\in\ker(\sigma)\cap\Lambda(n,n)_i$ for any $i\geq 0$. Then $\sigma_i(f+\Lambda(n,n)_{i+1})=0$. It follows that $f\in\Lambda(n,n)_{i+1}$, since σ_i is injective. Thus $\ker(\sigma)\subseteq\bigcap_{j\geq i}\Lambda(n,n)_j=0$, and σ is injective. Suppose that $g=g_0+g_1\in\Lambda(n,n)$, where $g_0\in\mathbb{F}$, $g_1\in\Lambda(n,n)_1$. Since σ_1 is surjective, there exists $g'_1\in\Lambda(n,n)_1$ such that $\sigma_1(g'_1+\Lambda(n,n)_2)=g_1+\Lambda(n,n)_2$. It follows that $g_2:=g_1-\sigma(g'_1)\in\Lambda(n,n)_2$. Note that σ_i is surjective for every $i\geq 0$. Then we can inductively pick $g'_i\in\Lambda(n,n)_i$, and define $g_{i+1}\in\Lambda(n,n)_{i+1}$ by

$$g_{i+1}:=g_i-\sigma(g'_i). \tag{9}$$

Let $g'=g_0+\sum_{i\geq 0}g'_i$. Since σ is continuous, it follows from (9) that

$$\sigma(g')=\sigma(g_0)+\sum_{i\geq 0}\sigma(g'_i)=g_0+\sum_{i\geq 1}(g_i-g_{i+1})=g_0+g_1=g,$$

whence σ is surjective. □

Let ρ be the corresponding representation with respect to $X_{[0]}$-module $X_{[-1]}$, that is, $\rho(y)=\mathrm{ad}\,y|_{X_{[-1]}}$ for all $y\in X_{[0]}$. It is easy to see that ρ is faithful. For $y\in X_{[0]}$, we also denote by $\rho(y)$ the matrix of $\rho(y)$ relative to the fixed ordered

$\mathbb{F}$-basis $\{D_1, D_2, \ldots, D_{2n}\}$. Denote by $\mathrm{gl}(n,n)$ the general linear Lie superalgebra of $2n \times 2n$ matrices over $\mathbb{F}$ (see [Scheunert 1979]).

Lemma 3.5. *Suppose that A is an invertible matrix of $\mathrm{gl}(n,n)$, and $y \in W(n,n)_{[0]}$. Then there exists an automorphism φ of $W(n,n)$ such that $\rho(\varphi(y)) = A\rho(y)A^{-1}$.*

Proof. Suppose that $A = \big(a_{ij}\big)_{1\le i,j\le 2n}$, and let $h_j = \sum_{i=1}^{2n} a_{ij}x_i$, where $1 \le j \le 2n$. Then $\{h_1, h_2, \ldots, h_{2n}\}$ is an $\mathbb{F}$-basis of $\Lambda(n,n)_{[1]}$. Note that $D_i(h_j) = a_{ij}$ for all $i, j \in Y$. It follows from Lemma 3.4 that there exists $\sigma \in \mathrm{Aut}\,\Lambda(n,n)$ such that $\sigma(x_i) = h_i$ for all $i \in Y$. Clearly, $\sigma \in \mathrm{Aut}(\Lambda(n,n) : W(n,n))$.

Let $\varphi : W(n,n) \to W(n,n)$ be the linear mapping such that $z \mapsto \sigma z\sigma^{-1}$ for all z in $W(n,n)$. Then φ is an automorphism of $W(n,n)$. We claim that φ is the desired automorphism. Suppose that $A^{-1} = (c_{ij})_{1\le i,j\le 2n}$, and let $y = \sum_{s,r\in Y} b_{sr}x_sD_r$ be an arbitrary element of $W(n,n)_{[0]}$, where $b_{sr} \in \mathbb{F}$. Then $\rho(y) = (b_{sr})_{1\le s,r\le 2n}$. Noting that $(\varphi y)(\sigma x_i) = \sigma(yx_i)$, we see that $\varphi(y) = \sum_{t,j\in Y} \sigma(yx_t)D_j$. Thus

$$\varphi(y) = \sum_{t,j\in Y} c_{tj}\sigma\Big(\sum_{s\in Y} b_{st}x_s\Big)D_j = \sum_{t,j,s\in Y} b_{st}c_{tj}\sigma(x_s)D_j$$
$$= \sum_{t,j,s\in Y} b_{st}c_{tj}h_sD_j = \sum_{t,j,s,k\in Y} a_{ks}b_{st}c_{tj}x_kD_j.$$

It follows that $\rho(\varphi(y)) = A\rho(y)A^{-1}$. □

Lemma 3.6. *Suppose that $y \in W(n,n)_{[0]}$. Then* $\mathrm{ad}\,y$ *is a nilpotent linear transformation of $W(n,n)_{[r]}$ for every $r \ge -1$ if and only if $\rho(y)$ is a nilpotent matrix.*

Proof. If $\mathrm{ad}\,y|_{W(n,n)_{[-1]}}$ is nilpotent, then the definition of ρ shows that $\rho(y)$ is a nilpotent matrix. Conversely, suppose that $\rho(y)$ is nilpotent. By Lemma 3.5, it suffices to consider the case when $\rho(y)$ is a strictly upper triangular matrix. Suppose that $y = \sum_{i,j\in Y, i<j} a_{ij}x_iD_j$, where $a_{ij} \in \mathbb{F}$.

We first prove that $\mathrm{ad}\,x_iD_j$ is nilpotent linear transformation of $W(n,n)_{[r]}$ for every $r \ge -1$ when $i < j$. For any standard basis element $x^{(\alpha)}x^uD_k$ of $W(n,n)_{[r]}$, where $\alpha \in \mathbb{N}_0^n$, $u \in \mathbb{B}(n)$ and $k \in Y$, two cases arise.

Case 1. $j \in Y_0$. If $k \ne i$, then

$$(\mathrm{ad}\,x_iD_j)^t(x^{(\alpha)}x^uD_k) = x_i^tD_j^t(x^{(\alpha)})x^uD_k = 0$$

when $t \ge r+2$. If $k = i$, then

$$(\mathrm{ad}\,x_iD_j)^t(x^{(\alpha)}x^uD_i) = x_i^tD_j^t(x^{(\alpha)})x^uD_i - tx_i^{t-1}D_j^{t-1}(x^{(\alpha)})x^uD_j = 0$$

when $t \ge r+3$.

Case 2. $j \in Y_1$. If $k \ne i$, then

$$(\mathrm{ad}\,x_iD_j)^t(x^{(\alpha)}x^uD_k) = (\mathrm{ad}\,x_iD_j)^{t-2}(x_i^2x^{(\alpha)}D_j^2(x^u)D_k) = 0$$

when $t \geq 2$, and if $k = i$, then

$$(\operatorname{ad} x_i D_j)^t (x^{(\alpha)} x^u D_i) = l(\operatorname{ad} x_i D_j)^{t-3} (x_i^2 x^{(\alpha)} D_j^2(x^u) D_j) = 0$$

when $t \geq 3$, where $l = 1$ or $l = -1$.

Therefore $(\operatorname{ad} x_i D_j)^t (x^{(\alpha)} x^u D_k) = 0$ when $t \geq r+4$. Let $f_k = \sum_{\alpha,u} c_{\alpha,u} x^{(\alpha)} x^u$ be an arbitrary element of $\Lambda(n,n)_{[r+1]}$, where $c_{\alpha,u} \in \mathbb{F}$, $k \in Y$. Then for any $t \geq r+4$,

$$(\operatorname{ad} x_i D_j)^t (f_k D_k) = \sum_{\alpha,u} (\operatorname{ad} x_i D_j)^t (c_{\alpha,u} x^{(\alpha)} x^u D_k) = 0,$$

since $(\operatorname{ad} x_i D_j))^t$ is continuous. Consequently, we see that $\operatorname{ad} x_i D_j|_{W(n,n)_{[r]}}$ is nilpotent for every $r \geq -1$ when $i < j$.

Note that the set $\{\pm x_i D_j, 0 \mid i < j\}$ is closed under the multiplication of $W(n,n)$, and the Lie superalgebra $\operatorname{span}_{\mathbb{F}}\{\pm x_i D_j, 0 \mid i < j\}$ is finite-dimensional. It follows from [Strade and Farnsteiner 1988, Theorem 1.3.1] that $\operatorname{ad} y|_{W(n,n)_{[r]}}$ is nilpotent. □

Lemma 3.7. *Suppose that $y \in X_{[0]}$. Then* ad y *is a nilpotent linear transformation of $X_{[r]}$ for every $r \geq -1$ if and only if $\rho(y)$ is a nilpotent matrix.*

Proof. Clearly, $\rho(y)$ is a nilpotent matrix when $\operatorname{ad} y|_{X_{[-1]}}$ is nilpotent. Conversely, suppose that $\rho(y)$ is a nilpotent matrix. Then by Lemma 3.6, ad y is a nilpotent linear transformation of $W(n,n)_{[r]}$ for every $r \geq -1$. Since X is a subalgebra of $W(n,n)$, it follows that $\operatorname{ad} y|_{X_{[r]}}$ is nilpotent. □

Lemma 3.8. *Suppose that $y = y_0 + y_1 \in \operatorname{qn}(X_0)$, where $y_0 \in X_{[0]}$, $y_1 \in X_1$. Then $\rho(y_0)$ is a nilpotent matrix, and hence, $y_0 \in \operatorname{qn}(X_{[0]})$.*

Proof. Let $X_{(i)} = X/X_{i+1}$ for every $i \geq -1$. Then $X_{(i)} \cong \bigoplus_{j \leq i} X_{[j]}$. For every $i \geq -1$, let τ_i be the endomorphism on $X_{(i)}$ satisfying $\tau_i(z) \equiv [y, z] \pmod{X_{i+1}}$ for all $z \in X_{(i)}$. Assume that $\rho(y_0)$ is not a nilpotent matrix. Then τ_i is not nilpotent for every $i \geq -1$. Let $X_{(i)} = U_i \oplus V_i$ be the Fitting decomposition of $X_{(i)}$ with respect to τ_i, where $U_i \neq 0$, $\tau_i|_{U_i}$ is invertible, $\tau_i|_{V_i}$ is nilpotent. Since $X_{(i)} = X_{(i+1)}/X_{[i+1]}$, $\tau_i = \tau_{i+1} \pmod{X_{i+1}}$ and $\tau_{i+1}(X_{[i+1]}) \subseteq (X_{[i+1]})$, it follows that

$$X_{(i)} = (U_{i+1} + X_{[i+1]}/X_{[i+1]}) \oplus (V_{i+1} + X_{[i+1]}/X_{[i+1]}).$$

is also the Fitting decomposition of $X_{(i)}$ with respect to τ_i, and by the uniqueness of the Fitting decomposition, we get $U_i = U_{i+1} + X_{[i+1]}/X_{[i+1]}$. This implies that U_i is the projection of U_{i+1} on $X_{(i)}$. Set

$$\overline{U} = \{z \in X \mid \text{ the projection of } z \text{ on } X_{(i)} \text{ belongs to } U_i \text{ for all } i \geq -1\}.$$

By the completeness of X, the set $\overline{U}$ is nonempty, and its projection on $X_{(i)}$ is U_i for each $i \geq -1$. It follows that $[y, \overline{U}] = \overline{U}$. So $\bigcap_{t=0}^{\infty} (\operatorname{ad} y)^t (X) \supseteq \overline{U} \neq 0$, contradicting the hypothesis that y is ad-quasi-nilpotent. Therefore, $\rho(y_0)$ is a nilpotent matrix, which combined with Lemma 3.7 yields $y_0 \in \mathrm{qn}(X_{[0]})$. □

Proposition 3.9. *One has* $\mathrm{qn}(X_0) = AN_0 \cap X_{[0]} + X_1$, *where*

$$AN_0 = \{y \mid y \in W(n,n)_{[0]} \text{ such that } \rho(y) \text{ is a nilpotent matrix}\}.$$

Proof. By Lemma 3.8, it suffices to show that $AN_0 \cap X_{[0]} + X_1 \subseteq \mathrm{qn}(X_0)$. Suppose that $y_0 \in AN_0 \cap X_{[0]}$, and suppose that $y_1 \in X_1$. Let $y = y_0 + y_1$. Then $\rho(y_0)$ is a nilpotent matrix. According to Lemma 3.7, there exists a positive integer t_i such that $(\mathrm{ad}\, y_0)^{t_i}(X_{[i]}) = 0$ for each $i \geq -1$. This implies that $(\mathrm{ad}\, y_0)^{t_i}(X_i) \subseteq X_{i+1}$. Consequently, we have $(\mathrm{ad}\, y)^{t_{-1}+\cdots+t_k}(X_0) \subseteq X_{k+1}$ for any $k \geq -1$. It follows that $\bigcap_{t=1}^{\infty} (\mathrm{ad}\, y)^t(X_0) \subseteq \bigcap_{k=1}^{\infty} X_k = 0$, whence $y \in \mathrm{qn}(X_0)$. □

Lemma 3.10. $\mathrm{Qn}(X_{\bar{0}}) = \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}}$, *and then* $\mathrm{Qn}(X_{\bar{0}}) \subseteq X_0 \cap X_{\bar{0}}$.

Proof. Note that $X_1 \subseteq \mathrm{Qn}(X)$. It follows that $X_1 \cap X_{\bar{0}} \subseteq \mathrm{Qn}(X_{\bar{0}})$. Consequently, we have $\mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}} \subseteq \mathrm{Qn}(X_{\bar{0}})$.

Conversely, suppose that $y = y_{-1} + y_0 \in \mathrm{qn}(X_{\bar{0}})$, where $y_{-1} \in X_{[-1]} \cap X_{\bar{0}}$, and $y_0 \in X_0 \cap X_{\bar{0}}$. Assume that $y_{-1} \neq 0$. It follows from Lemma 3.3 that $[y, X] = X$, which implies that y is not an ad-quasi-nilpotent element of X, a contradiction. Thus $y \in X_0 \cap X_{\bar{0}}$.

Now we can write $y = y_0 + y_1$, where $y_0 \in X_{[0]} \cap X_{\bar{0}}$, and $y_1 \in X_1 \cap X_{\bar{0}}$. By Lemma 3.8, we have $y_0 \in \mathrm{qn}(X_{[0]} \cap X_{\bar{0}}) \subseteq \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}})$. It follows that $y \in \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}}$. Thus

$$\mathrm{qn}(X_{\bar{0}}) \subset \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}}. \tag{10}$$

The right-hand side of (10) is a subalgebra of $X_{\bar{0}}$. Then

$$\mathrm{Qn}(X_{\bar{0}}) \subset \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}}.$$

Thus the lemma holds. □

Lemma 3.11. *Suppose that* $i, j \in Y$ *with* $i \neq j'$. *Then* $T_H(x_i x_j) \in \mathrm{qn}(X_{[0]})$.

Proof. It suffices to show that $\mathrm{ad}\, T_H(x_i x_j)$ is a nilpotent linear transformation of $W(n,n)_{[t]}$ for every $t \geq -1$. Suppose that $x^{(\alpha)} x^u D_k$ is a standard basis element of $W(m,n)_{[t]}$, where $t \geq -1$. To simplify our proof for the lemma, we only verify the case $i \in Y_0$, $j \in Y_1$, and $k \neq i, j$ as the proofs for the other cases are similar and hence omitted.

An induction on l shows that

$$\begin{aligned}(\mathrm{ad}\, T_H(x_i x_j))^l(x^{(\alpha)} x^u D_k) \\ = (-1)^{l-1} l x_i^{l-1} D_{j'}^{l-1}(x^{(\alpha)}) x_j D_{i'}(x^u) D_k + (-1)^l x_i^l D_{j'}^l(x^{(\alpha)}) x^u D_k = 0.\end{aligned}$$

This yields that $(\mathrm{ad}\, T_H(x_i x_j))^l(x^{(\alpha)} x^u D_k) = 0$ when $l \geq t+3$. □

Let I_n denote the identity matrix of size $2n \times 2n$, and let e_{ij} denote the $2n \times 2n$ matrix whose (i, j)-entry is 1 and whose other entries are 0. Set

$$\tilde{p}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^{\mathrm{T}} \end{pmatrix} \in \mathrm{gl}(n, n) \;\middle|\; B = -B^{\mathrm{T}}, C = C^{\mathrm{T}} \right\},$$

$$p(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^{\mathrm{T}} \end{pmatrix} \in \tilde{p}(n) \;\middle|\; \mathrm{tr}\, A = 0 \right\},$$

where A^{T} is the transpose of A. Then $\tilde{p}(n)$, $p(n)$ are subalgebras of $\mathrm{gl}(n, n)$ (see [Kac 1998]). Clearly,

$$\tilde{p}(n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{pmatrix} \in \mathrm{gl}(n, n) \right\},$$

$$p(n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{pmatrix} \in \mathrm{gl}(n, n) \;\middle|\; \mathrm{tr}\, A = 0 \right\}.$$

Lemma 3.12. (1) $\rho(T_H(x_i x_j)) = (-1)^{\mu(i')} e_{i'j} - (-1)^{\mu(i')\mu(j)} e_{j'i}$ *for all* $i, j \in Y$, *hence* $\rho(HO_{[0]}) = \tilde{p}(n)$ *and* $\rho(HO_{[0]} \cap HO_{\bar{0}}) = \tilde{p}(n)_{\bar{0}}$.

(2) $\rho(T_H(x_i x_j)) = (-1)^{\mu(i')} e_{i'j} - (-1)^{\mu(i')\mu(j)} e_{j'i}$ *for all* $i, j \in Y$ *with* $i \neq j'$, $\rho(T_H(x_k x_{k'} - x_l x_{l'})) = e_{kk} - e_{k'k'} - e_{ll} + e_{l'l'}$ *for all* $k, l \in Y_0$ *with* $k \neq l$, *hence* $\rho(SHO_{[0]}) = p(n)$ *and* $\rho(SHO_{[0]} \cap SHO_{\bar{0}}) = p(n)_{\bar{0}}$.

Proof. (1) Direct calculation shows that $\mathrm{ad}\, T_H(x_i x_j)(D_j) = (-1)^{\mu(i')} D_{i'}$, and $\mathrm{ad}\, T_H(x_i x_j)(D_i) = -(-1)^{\mu(i')\mu(j)} D_{j'}$. It follows that

$$\rho(T_H(x_i x_j)) = (-1)^{\mu(i')} e_{i'j} - (-1)^{\mu(i')\mu(j)} e_{j'i}.$$

Note that $HO_0 = \mathrm{span}_{\mathbb{F}}\{T_H(x_i x_j) \mid i, j \in Y\}$. Consequently, (1) holds.

(2) The proof is similar to that of (1), hence omitted. □

Lemma 3.13. $\rho(\mathrm{Qn}(X_{[0]} \cap X_{\bar{0}})) = p(n)_{\bar{0}}$.

Proof. Suppose that $y \in \mathrm{qn}(X_{[0]} \cap X_{\bar{0}})$. Then $\rho(y)$ is a nilpotent matrix by Lemma 3.8. It follows from Lemma 3.12 that $\rho(y) = \mathrm{diag}(A, -A^{\mathrm{T}})$, where A and $-A^{\mathrm{T}}$ are $n \times n$ nilpotent matrices. This shows that $\mathrm{tr}\, A = 0$, that is $\rho(y) \in p(n)_{\bar{0}}$. Thus $\rho(\mathrm{qn}(X_{[0]} \cap X_{\bar{0}})) \subseteq p(n)_{\bar{0}}$.

Conversely, set

$$R = \{T_H(x_i x_j) \mid i \in Y_0,\ j \in Y_1,\ \text{with } i \neq j'\}.$$

Then $R \subseteq \mathrm{qn}(X_{[0]} \cap X_{\bar{0}})$, by Lemma 3.11, whence $\rho(R) \subseteq \rho(\mathrm{qn}(X_{[0]} \cap X_{\bar{0}}))$. Noting that $\rho(T_H(x_i x_j)) = e_{j'i} - e_{i'j}$ for all $i \in Y_0$, $j \in Y_1$ with $i \neq j'$, by Lemma 3.12, we see that $\rho(R)$ generates $p(n)_{\bar{0}}$. Thus $p(n)_{\bar{0}} \subseteq \rho(\mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}))$. □

Note that $\mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}})) = \{y \in X_{\bar{0}} \mid [y, \mathrm{Qn}(X_{\bar{0}})] \subseteq \mathrm{Qn}(X_{\bar{0}})\}$. Clearly, the set $\mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$ is invariant under automorphisms of X.

Proposition 3.14. $X_0 \cap X_{\bar{0}} = \mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$. *In particular*, $X_0 \cap X_{\bar{0}}$ *is an invariant subalgebra.*

Proof. Note that $[\tilde{p}(n)_{\bar{0}}, p(n)_{\bar{0}}] = p(n)_{\bar{0}}$. It follows from Lemmas 3.12 and 3.13 that

$$[\rho(X_{[0]} \cap X_{\bar{0}}), \rho(\mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}))] = [\tilde{p}(n)_{\bar{0}}, p(n)_{\bar{0}}] = \rho(\mathrm{Qn}(X_{[0]} \cap X_{\bar{0}})),$$

whence

$$[X_{[0]} \cap X_{\bar{0}}, \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}})] = \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}), \tag{11}$$

since ρ is faithful. By Lemma 3.10 and (11), we have

$$\begin{aligned}[X_0 \cap X_{\bar{0}}, \mathrm{Qn}(X_{\bar{0}})] &= [X_{[0]} \cap X_{\bar{0}} + X_1 \cap X_{\bar{0}}, \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}}] \\ &\subseteq [X_{[0]} \cap X_{\bar{0}}, \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}})] + X_1 \cap X_{\bar{0}} \\ &\subseteq \mathrm{Qn}(X_{[0]} \cap X_{\bar{0}}) + X_1 \cap X_{\bar{0}} = \mathrm{Qn}(X_{\bar{0}}).\end{aligned}$$

Thus $X_0 \cap X_{\bar{0}} \subseteq \mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$.

Conversely, suppose that $y = y_{-1} + y_0 \in \mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$, where $y_{-1} \in X_{[-1]} \cap X_{\bar{0}}$, $y_0 \in X_0 \cap X_{\bar{0}}$. We want to show that $y_{-1} = 0$. Assume $y_{-1} = \sum_{t=1}^{n} a_t D_t \neq 0$, where $a_t \in \mathbb{F}$. Then we can pick $k \in Y_0$ such that $a_k \neq 0$, and then pick $j_k \in Y_1$ such that $j_k \neq k'$. From Lemma 3.11 we see that $T_H(x_k x_{j_k}) \in \mathrm{Qn}(X_{\bar{0}})$, which combined with our hypothesis that $y \in \mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$, yields $[y, T_H(x_k x_{j_k})] \in \mathrm{Qn}(X_{\bar{0}})$, whence $[y, T_H(x_k x_{j_k})] \in X_0 \cap X_{\bar{0}}$ by Lemma 3.10. On the other hand, a direct calculation shows that

$$[y, T_H(x_k x_{j_k})] = [y_{-1}, T_H(x_k x_{j_k})] + [y_0, T_H(x_k x_{j_k})] = -a_k D_{j_k'} + [y_0, T_H(x_k x_{j_k})].$$

Since $[y_0, T_H(x_k x_{j_k})] \in X_0 \cap X_{\bar{0}}$, we see that $a_k = 0$, contradicting our assumption, thus $y_{-1} = 0$. So $y = y_0 \in X_0 \cap X_{\bar{0}}$, proving $\mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}})) \subseteq X_0 \cap X_{\bar{0}}$. Since $\mathrm{Nor}_{X_{\bar{0}}}(\mathrm{Qn}(X_{\bar{0}}))$ is invariant, we see that $X_0 \cap X_{\bar{0}}$ is invariant. □

Set $\Omega = \{y \in \mathrm{qn}(X_0 \cap X_{\bar{0}}) \mid [y, X_0 \cap X_{\bar{0}}] \subseteq \mathrm{qn}(X_0 \cap X_{\bar{0}})\}$. Then Proposition 3.14 shows that Ω is invariant under automorphisms of X.

Proposition 3.15. $X_1 \cap X_{\bar{0}} = \Omega$. *In particular*, $X_1 \cap X_{\bar{0}}$ *is an invariant subalgebra.*

Proof. We only verify the case of SHO, as the proof for HO is similar and hence omitted. Suppose that $y = y_0 + y_1$ is an arbitrary element of Ω, where y_0 is in $SHO_{[0]} \cap SHO_{\bar{0}}$ and y_1 is in $SHO_1 \cap SHO_{\bar{0}}$. Suppose that

$$y_0 = \sum_{i,j \in Y_0, i \neq j} a_{ij} T_H(x_i x_{j'}) + \sum_{i=1}^{n-1} a_{ii} T_H(x_i x_{i'} - x_{i+1} x_{(i+1)'}),$$

where $a_{ij} \in \mathbb{F}$. Let $b_{ij} = a_{ij}$, where $i, j \in Y_0$ with $i \neq j$, and let $b_{ii} = a_{ii} - a_{i-1,i-1}$, where $i \in Y_0$, and $a_{00} = a_{nn} = 0$. Let

$$l = \min\left\{i \in Y_0 \;\middle|\; b_{ij_0} \neq 0 \text{ for some } j_0 \in Y\right\},$$
$$t = \min\left\{j \in Y_0 \;\middle|\; b_{i_0 j} \neq 0 \text{ for some } i_0 \in Y\right\}.$$

We first consider the case $l \leq t$. Set $k = \max\{j \in Y_0 \mid b_{lj} \neq 0\}$. Then $l \leq t \leq k$, and $b_{lk} \neq 0$. If $l = k$, then

$$y_0 = \sum_{i=l+1}^{n} \sum_{j=l, i\neq j}^{n} a_{ij} T_H(x_i x_{j'}) + \sum_{i=l}^{n-1} a_{ii} T_H(x_i x_{i'} - x_{i+1} x_{(i+1)'}),$$

and

$$\rho(y_0) = \sum_{i=l}^{n} a_{ii}(e_{ii} - e_{i+1 i+1}) - \sum_{i=l}^{n} a_{ii}(e_{i'i'} - e_{(i+1)'(i+1)'}) + \sum_{i=l+1}^{n} \sum_{j=l, i\neq j}^{n} a_{ij}(e_{ji} - e_{i'j'}) = \begin{pmatrix} a_{ll}A_{ll} & * \\ 0 & * \end{pmatrix},$$

where A_{ll} is the $l \times l$ matrix whose (l, l)-entry is 1 and 0 elsewhere. Since $a_{ll} \neq 0$, we conclude that $\rho(y_{[0]})$ is not a nilpotent matrix. It follows from Lemma 3.8 that $y \notin \mathrm{qn}(SHO_{\bar{0}})$, contradicting the hypothesis that $y \in \Omega$. Therefore $l < k$ and

$$y_0 = \sum_{j=t, j\neq l}^{k} a_{lj} T_H(x_l x_{j'}) + \sum_{i=l+1}^{n} \sum_{j=t, i\neq j}^{n} a_{ij} T_H(x_i x_{j'}) + \sum_{i=t}^{n} a_{ii} T_H(x_i x_{i'} - x_{i+1} x_{(i+1)'}).$$

A direct calculation shows that

$$\begin{aligned}
\rho\big([T_H(x_k x_{l'}), y_0]\big) &= [e_{lk} - e_{k'l'}, \rho(y_0)] \\
&= a_{lk}(e_{ll} - e_{l'l'} - e_{kk} + e_{k'k'}) - \sum_{j=t, j\neq l}^{k-1} a_{lj}(e_{jk} - e_{k'j'}) \\
&\quad + \sum_{i=l+1, i\neq k}^{n} a_{ik}(e_{li} - e_{i'l'}) + (a_{kk} - a_{k-1,k-1} - \delta_{lt} a_{ll})(e_{lk} - e_{k'l'}) \\
&= \begin{pmatrix} a_{lk}A_{ll} & * \\ 0 & * \end{pmatrix},
\end{aligned}$$

so $\rho([T_H(x_k x_{l'}), y_0])$ is not a nilpotent matrix. Since

$$[T_H(x_k x_{l'}), y] = [T_H(x_k x_{l'}), y_0] + [T_H(x_k x_{l'}), y_1],$$

it follows from Lemma 3.8 that $[T_H(x_k x_{l'}), y] \notin \mathrm{qn}(SHO_0)$, contradicting our hypothesis that $y \in \Omega$.

We now consider the case $l > t$. Set $k = \max\{i \in Y_0 \mid b_{it} \neq 0\}$. Then $t < l \leq k$, $b_{kt} \neq 0$ and

$$y_0 = \sum_{i=l}^{k} a_{it} T_H(x_i x_{t'}) + \sum_{i=l}^{n} \sum_{j=t+1, i \neq j}^{n} a_{ij} T_H(x_i x_{j'}) + \sum_{i=l}^{n} a_{ii} T_H(x_i x_{i'} - x_{i+1} x_{(i+1)'}).$$

A direct computation shows that

$$\begin{aligned}\rho\big([T_H(x_t x_{k'}), y_0]\big) &= [e_{kt} - e_{t'k'}, \rho(y_0)] \\ &= -a_{kt}(e_{tt} - e_{t't'} - e_{kk} + e_{k'k'}) + \sum_{i=l}^{k-1} a_{it}(e_{ki} - e_{i'k'}) \\ &\quad - \sum_{j=t+1, j \neq k}^{n} a_{kj}(e_{jt} - e_{t'j'}) + (a_{k-1,k-1} - a_{k,k})(e_{kt} - e_{t'k'}) \\ &= \begin{pmatrix} -a_{kt} B_{tt} & 0 \\ * & * \end{pmatrix},\end{aligned}$$

where B_{tt} is the $t \times t$ matrix whose (t, t)-entry is 1 and 0 elsewhere. It follows that $\rho([T_H(x_t x_{k'}), y_0])$ is not a nilpotent matrix, whence $[T_H(x_t x_{k'}), y] \notin \mathrm{qn}(SHO_0)$ by Lemma 3.8, a contradiction which yields $y_0 = 0$. Therefore $y = y_1 \in SHO_1 \cap SHO_{\bar{0}}$, proving $\Omega \subseteq SHO_1 \cap SHO_{\bar{0}}$.

Conversely, noting that $SHO_1 \subseteq \mathrm{qn}(SHO)$, we see that

$$\begin{aligned}&[SHO_1 \cap SHO_{\bar{0}}, SHO_0 \cap SHO_{\bar{0}}] \\ &\qquad \subseteq SHO_1 \cap SHO_{\bar{0}} \subseteq SHO_0 \cap SHO_{\bar{0}} \cap \mathrm{qn}(SHO) = \mathrm{qn}(SHO_0 \cap SHO_{\bar{0}}).\end{aligned}$$

Thus $SHO_1 \cap SHO_{\bar{0}} \subseteq \Omega$. Since Ω is invariant, we see that $SHO_1 \cap SHO_{\bar{0}}$ is invariant. □

Lemma 3.16. $[X_{\bar{1}}, X_1 \cap X_{\bar{0}}] = X_0 \cap X_{\bar{1}}$.

Proof. It suffices to show that $X_0 \cap X_{\bar{1}} \subseteq [X_{\bar{1}}, X_1 \cap X_{\bar{0}}]$. We first consider the case of HO. Suppose that $T_H(x^{(\alpha)} x^u) \in HO_0 \cap HO_{\bar{1}}$, where $\alpha \in \mathbb{N}_0^n$, $u \in \mathbb{B}(n)$. Note that if $|u| \neq n$, then there exists $k \in Y_1 \setminus \{u\}$ such that

$$T_H(x^{(\alpha)} x^u) = [D_k, T_H(x^{(\alpha)} x_k x^u)] \in [HO_{\bar{1}}, HO_1 \cap HO_{\bar{0}}],$$

and if $|u| = n$, then

$$-T_H(x^{(\alpha)} x^u) = [T_H(x_{i'} x_j), T_H(x_i x^{(\alpha)} D_j(x^u))] \in [HO_{\bar{1}}, HO_1 \cap HO_{\bar{0}}],$$

for all $i \in Y_0$, $j \in Y_1$.

Next, we consider the case of SHO. Suppose that $[x^{(\alpha)}, x^u] \in SHO_0 \cap SHO_{\bar{1}}$, where $\alpha \in \mathbb{N}_0^n$, $u \in \mathbb{B}(n)$. If $|u| \neq n$, then there exists $k \in Y_1$ such that $k \notin \{u\}$. It follows from (3) that

$$-[x^{(\alpha)}, x^u] = [x_{k'}, [x^{(\alpha)}, x_k x^u]] \in [SHO_{\bar{1}}, SHO_1 \cap SHO_{\bar{0}}].$$

If $|u| = n$, then by the hypothesis, we see that $|u|$ is even. It follows that

$$[x^{(\alpha)}, x^u] = [x_{3'} \cdots x_{n'}, [x^{(\alpha+\varepsilon_3)}, x_{3'} x_{1'} x_{2'}]] \in [SHO_{\bar{1}}, SHO_1 \cap SHO_{\bar{0}}]. \qquad \square$$

Theorem 3.17. *The natural filtration of X is invariant under automorphisms of X.*

Proof. By Proposition 3.15, we see that $X_1 \cap X_{\bar{0}}$ is invariant under automorphisms of X. It follows from Lemma 3.16 that $X_0 \cap X_{\bar{1}}$ is invariant, which combined with Proposition 3.14 yields that X_0 is invariant. Noting that

$$X_{[i]} = \{y \in X \mid [y, X] \subseteq X_{[i-1]}\}$$

for every $i \geq 1$, we see that $X_{[i]}$ is invariant. Thus X is invariant. $\square$

4. The automorphism group of X

Lemma 4.1. *Suppose that $\varphi_X \in \operatorname{Aut} X$. Then*:

(1) *φ_X is a continuous automorphism.*

(2) *There exists an $\mathbb{F}$-basis $\{e_1, e_2, \dots, e_{2n}\}$ of $X_{[-1]}$ such that $\varphi_X(D_i) \equiv e_i \pmod{X}$.*

Proof. (1) By Theorem 3.17, we have $\varphi_X(X_i) \subseteq X_i$ for every $i \geq -1$. It follows that $X_i \subseteq \varphi_X^{-1}(X_i)$. On the other hand, noting that $\varphi_X^{-1} \in \operatorname{Aut} X$, we see that $\varphi_X^{-1}(X_i) \subseteq X_i$. Consequently, φ_X is a continuous automorphism.

(2) By Theorem 3.17, φ_X can induce an $\mathbb{F}$-isomorphism $\overline{\varphi}_X$ of the quotient spaces

$$\overline{\varphi}_X : X/X_0 \to X/X_0,$$

such that $\overline{\varphi}_X(y + X_0) = \varphi_X(y) + X_0$ for all $y \in X$. Since $\{D_i + X_0 \mid i \in Y\}$ is an $\mathbb{F}$-basis of X/X_0, it follows that $\{\varphi_X(D_i) + X_0 \mid i \in Y\}$ is an $\mathbb{F}$-basis of X/X_0. Then for every $i \in Y$, there exists $e_i \in X_{[-1]}$ such that $\varphi_X(D_i) + X_0 = e_i + X_0$. Thus (2) holds. $\square$

Proposition 4.2. *Suppose that $\varphi, \psi \in \operatorname{Aut} X$. If $\varphi|_{X_{[-1]}} = \psi|_{X_{[-1]}}$, then $\varphi = \psi$.*

Proof. We first use induction on k to show that $\varphi|_{X_{[k]}} = \psi|_{X_{[k]}}$, where $k \geq -1$. The result is obvious for $k = -1$. We assume it for $k - 1$. Suppose that $y \in X_{[k]}$, and let $z = \varphi(y) - \psi(y)$. Then

$$[z, \psi(D_i)] = [\varphi(y) - \psi(y), \psi(D_i)] = \varphi([y, D_i]) - \psi([y, D_i]) = 0$$

for every $i \in Y$. By Lemma 4.1 (2), we can write $\psi(D_i) = e_i + w_i$, where $\{e_1, \dots, e_{2n}\}$ is an $\mathbb{F}$-basis of $X_{[-1]}$ and $w_i \in X_0$. It follows that $[z, e_i + w_i] = 0$.

Suppose that $e_i = \sum_{j\in Y} a_{ij} D_j$, where $a_{ij} \in \mathbb{F}$, and let $(c_{ij})_{1\le i,j\le 2n} = (a_{ij})^{-1}_{1\le i,j\le 2n}$. Then $[z, -w_i] = [z, e_i] = \sum_{j\in Y} a_{ij}[z, D_j]$, whence

$$[z, D_l] = \sum_{i=1}^{2n} c_{li}[z, -w_i] \quad \text{for all } l \in Y. \tag{12}$$

Noting that $y \in X_0$, we see that $z \in X_0$. Thus we can write $z = \sum_{j\ge 0} z_j$, where $z_j \in X_{[j]}$. Applying (12), we have $[z, D_l] \in X_0$, thus $[z_0, D_l] \in X_0 \cap X_{[-1]} = 0$ for all $l \in Y$. This yields $z_0 = 0$. Now we can write $z = \sum_{j\ge 1} z_j$. Repeating the argument above, we can see that $z_j = 0$ for each $j \ge 1$. It follows that $z = 0$, whence $\varphi|_{X_{[k]}} = \psi|_{X_{[k]}}$. Hence $\varphi = \psi$, by Lemma 4.1 (1). □

Given $\sigma \in \operatorname{Aut}\Lambda(n,n)$ and $D \in \operatorname{Der}\Lambda(n,n)$, we set $D^\sigma = \sigma D \sigma^{-1}$. Then $\tilde{\sigma} : D \mapsto D^\sigma$ is an automorphism of $\operatorname{Der}\Lambda(n,n)$. Let

$$\operatorname{Aut}(\Lambda(n,n) : X) = \{\sigma \in \operatorname{Aut}\Lambda(n,n) \mid |\tilde{\sigma}(X) \subseteq X\}.$$

Then $\operatorname{Aut}(\Lambda(n,n) : X)$ is a subgroup of $\operatorname{Aut}\Lambda(n,n)$, and it is called the admissible automorphism group of $\Lambda(n,n)$ relative to X. Obviously, the morphism $\Phi : \operatorname{Aut}(\Lambda(n,n) : X) \to \operatorname{Aut} X$ given by $\sigma \mapsto \tilde{\sigma}|_X$ is a homomorphism of groups.

Lemma 4.3. (1) *Suppose that $A \in \mathrm{M}_{2n}(\Lambda(n,n))$. Then $\mathrm{pr}_{[0]}(A)$ is invertible if and only if A is invertible.*

(2) *Suppose that $\{e_1, \dots, e_{2n}\}$ is a $\Lambda(n,n)$-basis of $W(n,n)$. Let $\mathrm{pr}_{[-1]}$ be the projection of $W(n,n)$ onto $W(n,n)_{[-1]}$. Then $\{\mathrm{pr}_{[-1]} e_1, \dots, \mathrm{pr}_{[-1]} e_{2n}\}$ is an $\mathbb{F}$-basis of $W(n,n)_{[-1]}$.*

(3) *Suppose that φ is an automorphism of X, and suppose that $\{y_i \mid i \in Y\} \subset X$ is a $\Lambda(n,n)$-basis of $W(n,n)$. Then $\{\varphi(y_i) \mid i \in Y\}$ is also a $\Lambda(n,n)$-basis of $W(n,n)$.*

(4) *The natural filtration of $\Lambda(n,n)$ is invariant under automorphisms of $\Lambda(n,n)$.*

Proof. (1) We first prove that A is invertible when $\mathrm{pr}_{[0]} A$ is invertible. Set

$$\mathrm{P}(n)_1 = \{f \in \mathrm{P}(n) \mid \mathrm{pr}_{[0]}(f) = 0\} \quad \text{and}$$
$$T = \operatorname{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid D_i(x^u) \ne 0 \text{ for some } i \in Y\}.$$

Then we can write $A = \mathrm{pr}_{[0]}(A) + B + C$, where $B \in \mathrm{M}_{2n}(\mathrm{P}(n)_1)$, $C \in \mathrm{M}_{2n}(T)$. Let $D = \mathrm{pr}_{[0]}(A) + B$. Since $\mathrm{P}(n)$ is commutative, we see that $\det D$ is well defined. Note that $\mathrm{pr}_{[0]}(\det D) = \det(\mathrm{pr}_{[0]} D) \ne 0$, we can write $\det D = a + f$, where $0 \ne a \in \mathbb{F}$, $f \in \mathrm{P}(n)_1$. Put $g = a^{-1}\big(\sum_{i=0}^{\infty}(-1)^i (a^{-1} f)^i\big)$. A direct calculation shows that $g \det D = 1$. It follows that $\det D$ is invertible, whence D is invertible.

Let E be the inverse of D. Since $C \in \mathrm{M}_{2n}(T)$, we have $CE \in \mathrm{M}_{2n}(T)$, which combined with the fact that the product of any $n+1$ elements of T is 0 yields that

CE is nilpotent. Thus $I + CE$ is invertible. Consequently, we have

$$AE(I+CE)^{-1} = (C+D)E(I+CE)^{-1} = (CE+DE)(I+CE)^{-1} = I.$$

Therefore A is invertible.

Using the fact that $\mathrm{pr}_{[0]}(AB) = \mathrm{pr}_{[0]}(A)\,\mathrm{pr}_{[0]}(B)$ for arbitrary matrices A and B in $\mathrm{M}_{2n}(\Lambda(n,n))$, we can prove the converse implication.

(2) Suppose that $(D_1, \ldots, D_{2n})^{\mathrm{T}} = A(e_1, \ldots, e_{2n})^{\mathrm{T}}$, where $A \in \mathrm{M}_{2n}(\Lambda(n,n))$. Then $(D_1, \ldots, D_{2n})^{\mathrm{T}} = \mathrm{pr}_{[0]}\,A(\mathrm{pr}_{[-1]}(e_1), \ldots, \mathrm{pr}_{[-1]}(e_{2n}))^{\mathrm{T}}$. Since $\{D_1, \ldots, D_{2n}\}$ is an $\mathbb{F}$-basis of $W(n,n)_{[-1]}$, it follows that $\{\mathrm{pr}_{[-1]}(e_1), \ldots, \mathrm{pr}_{[-1]}(e_{2n})\}$ is an $\mathbb{F}$-basis of $W(n,n)_{[-1]}$.

(3) By Theorem 3.17, φ induces canonically $\overline{\varphi} \in \mathrm{gl}(X/X_0)$. Denote by $\bar{y}_i$ the image of y_i under the canonically map $X \to X/X_0$. Then $\{\bar{y}_i \mid i \in Y\}$ is an $\mathbb{F}$-basis of X/X_0. Assume that

$$(\varphi(y_1), \ldots, \varphi(y_{2n}))^{\mathrm{T}} = A(D_1, \ldots, D_{2n})^{\mathrm{T}},$$

where $A \in \mathrm{M}_{2n}(\Lambda(n,n))$. Decomposing $A = \mathrm{pr}_{[0]}\,A + \mathrm{pr}_1\,A$, we obtain

$$(\overline{\varphi}(\overline{y}_1), \ldots, \overline{\varphi}(\overline{y}_{2n}))^{\mathrm{T}} = (\overline{\varphi(y_1)}, \ldots, \overline{\varphi(y_{2n})})^{\mathrm{T}} = \mathrm{pr}_{[0]}\,A(\overline{D}_1, \ldots, \overline{D}_{2n})^{\mathrm{T}}.$$

This implies that $\mathrm{pr}_{[0]}\,A$ is invertible. It follows from (1) that A is invertible. Therefore $\{\varphi(y) \mid i \in Y\}$ is a $\Lambda(n,n)$-basis of $W(n,n)$.

(4) Since $\mathrm{Der}\,\Lambda(n,n) = W(n,n)$, we have $\mathrm{Aut}\,\Lambda(n,n) = \mathrm{Aut}(\Lambda(n,n) : W(n,n))$. By [Zhang and Liu 2004, Theorem 2.12], the natural filtration of $W(n,n)$ is invariant under $\mathrm{Aut}\,W(n,n)$. Note that for every $i \in Y$, $\tilde{\sigma}(fD_i) = (\sigma f)(\tilde{\sigma}D_i)$, where $\sigma \in \mathrm{Aut}\,\Lambda(n,n)$ and $f \in \Lambda(n,n)$, which implies the desired result. □

Theorem 4.4. *The map* $\Phi : \mathrm{Aut}(\Lambda(n,n) : X) \to \mathrm{Aut}\,X$ *given by* $\sigma \mapsto \tilde{\sigma}|_X$ *is an isomorphism.*

Proof. It suffices to show that Φ is bijective. Assume that $\sigma \in \mathrm{Aut}(\Lambda(n,n) : X)$ is an element such that $\tilde{\sigma}|_X = 1|_X$. We first use induction on $|\alpha| + |u|$ to show that $\sigma(x^{(\alpha)}x^u) = x^{(\alpha)}x^u$, where $x^{(\alpha)}x^u$ is a standard basis element of $\Lambda(n,n)$, $\alpha \in \mathbb{N}_0^n$, $u \in \mathbb{B}(n)$. If $|\alpha| + |u| = 1$, then $x^{(\alpha)}x^u = x_i$ for some $i \in Y$. Since for every $k \in Y$

$$D_k(\sigma(x_i)) = (\tilde{\sigma}(D_k))(\sigma(x_i)) = \sigma D_k \sigma^{-1}\sigma(x_i) = \sigma D_k(x_i) = \sigma(\delta_{ik}) = \delta_{ik} = D_k(\delta_{ik}),$$

it follows that $D_k(\sigma(x_i) - x_i) = 0$, which combined with Lemma 4.3(4) yields $\sigma(x_i) = x_i$. If $|\alpha| + |u| > 1$, then by induction

$$D_k(\sigma(x^{(\alpha)}x^u) - x^{(\alpha)}x^u) = (\tilde{\sigma}D_k)\sigma(x^{(\alpha)}x^u) - D_k(x^{(\alpha)}x^u) = 0,$$

for every $k \in Y$. Thus $\sigma(x^{(\alpha)}x^u) - x^{(\alpha)}x^u \in \mathbb{F} \cap \Lambda(n,n)_1 = 0$. Consequently, $\sigma = 1$ and Φ is injective.

We next prove that Φ is surjective. Suppose that φ is in $\operatorname{Aut} X$. Then by Lemma 4.3 (3), $\{\varphi(D_1), \dots, \varphi(D_{2n})\}$ is a $\Lambda(n,n)$-basis of $W(n,n)$. Therefore, we can suppose that $\varphi(T_H(x_i x_j)) = \sum_{t=1}^{2n} h_{ijt}\varphi(D_t)$, where $i, j \in Y$ with $i \neq j'$, and $h_{ijt} \in \Lambda(n,n)$. Applying Lemma 4.3 (2), we have $h_{ijt} \in \Lambda(n,n)_1$. Thus

$$\varphi\big([D_k, T_H(x_i x_j)]\big) = \Big[\varphi(D_k), \sum_{t=1}^{2n} h_{ijt}\varphi(D_t)\Big] = \sum_{t=1}^{2n}(\varphi(D_k)(h_{ijt}))\varphi(D_t). \tag{13}$$

On the other hand,

$$\begin{aligned}\varphi\big([D_k, T_H(x_i x_j)]\big) &= \varphi[D_k, (-1)^{\mu(i)\mu(j')}x_j D_{i'} + (-1)^{\mu(j)}x_i D_{j'}] \\ &= (-1)^{\mu(i)\mu(j')}\delta_{kj}\varphi(D_{i'}) + (-1)^{\mu(j)}\delta_{ki}\varphi(D_{j'}).\end{aligned} \tag{14}$$

In particular, by letting $i = 1$ and $j \in Y \setminus \{1'\}$ in equations (13) and (14), one sees that $\varphi(D_k)(h_{1j1'}) = \delta_{kj} + \delta_{k1}\delta_{j1}$ for all $k \in Y$. Similarly, by letting $i = 2'$ and $j = 1'$, we obtain $\varphi(D_k)(h_{2'1'2}) = \delta_{k1'}$. Let $h_1 = \frac{1}{2}h_{111'}$, $h_{1'} = h_{2'1'2}$ and $h_j = h_{1j1'}$ for $j \in Y \setminus \{1, 1'\}$. Then $h_j \in \Lambda(n,n)_1$ with $\deg(h_j) = \mu(j)$, and

$$\varphi(D_i)(h_j) = \delta_{ij} \quad \text{for all } i, j \in Y. \tag{15}$$

Suppose that $\varphi(D_i) = \sum_{t=1}^{2n} f_{it} D_t$, where $f_{it} \in \Lambda(n,n)$. It follows from (15) that

$$(\delta_{ij})_{i,j\in Y} = (\varphi(D_i)h_j)_{i,j\in Y} = (f_{ij})_{i,j\in Y}(D_i h_j)_{i,j\in Y}.$$

This implies that $(D_i h_j)_{i,j\in Y}$ is invertible, whence $(\mathrm{pr}_{[0]}(D_i h_j))_{i,j\in Y}$ is invertible by Lemma 4.3(1). Consequently, there exists $\sigma \in \operatorname{Aut}\Lambda(n,n)$ such that $\sigma(x_i) = h_i$ by Lemma 3.4, which combined with (15) yields

$$(\tilde{\sigma} D_i - \varphi D_i)(h_j) = \sigma(D_i x_j) - \delta_{ij} = 0$$

for all $i, j \in Y$. Since $h_1, h_2, \dots, h_{2n}$ generate $\Lambda(n,n)$, we see that $\tilde{\sigma} D_i = \varphi D_i$, whence $\tilde{\sigma}|_X = \varphi$ by Proposition 4.2. □

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