Pacific Journal of Mathematics

THE HEISENBERG ULTRAHYPERBOLIC EQUATION: THE BASIC SOLUTIONS AS DISTRIBUTIONS

ANTHONY C. KABLE

Volume 258 No. 1 July 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 258, No. 1, 2012

THE HEISENBERG ULTRAHYPERBOLIC EQUATION: THE BASIC SOLUTIONS AS DISTRIBUTIONS

ANTHONY C. KABLE

Tempered distributions are associated to the basic solutions of the Heisenberg ultrahyperbolic equations and the properties of these distributions are investigated. For almost all values of the parameter, a fundamental solution for the Heisenberg ultrahyperbolic operator is expressed in terms of these distributions.

1. Introduction and outline

Let $d \geq 1$ and denote by N the Heisenberg group of dimension $2d+1$. We write the elements of N as (x, y, t), where x is a 1-by-d real row vector, y is a d-by-1 real column vector, and $t \in \mathbb{R}$, so that the group operation is

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + xy').$$

For each $z \in \mathbb{C}$, the associated Heisenberg ultrahyperbolic operator is

$$\square_z = \Delta + \big(\mathbb{E}_x + (z + z_0)\big)\frac{\partial}{\partial t},$$

where

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i \partial y_i}$$

is the Euclidean ultrahyperbolic operator,

$$\mathbb{E}_x = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}$$

is the Euler operator with respect to x, and $z_0 = d/2$. By a complex change of variables, the Heisenberg ultrahyperbolic operators may be made to coincide with the Heisenberg Laplacians. For this reason, there are many formal similarities between the two families of operators, although their analytic properties are naturally rather different.

MSC2010: primary 22E30; secondary 35R03, 22E25.
Keywords: Kelvin transform, conformally invariant operator, Heisenberg group.

Each of the operators $\Box_z$ admits the group $\mathrm{SL}(d+2,\mathbb{R})$ as a group of conformal symmetries. More precisely, this group can be realized in four different ways as a group of conformal symmetries of $\Box_z$, with the four realizations being indexed by two sign parameters $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Amongst the conformal symmetries thus obtained is an analogue of the classical Kelvin transform, which acts on functions on subsets of N by

$$\big(\mathbb{K}(z,\varepsilon_1,\varepsilon_2)\varphi\big)(x,y,t) = |t-xy|_{\varepsilon_1}^{-(z+z_0)}\,|t|_{\varepsilon_2}^{z-z_0}\varphi\Big(\frac{x}{t-xy}, \frac{y}{t}, -\frac{1}{t}\Big),$$

where we use the notation $|u|_+^z = |u|^z$ and $|u|_-^z = \mathrm{sgn}(u)|u|^z$ for $z\in\mathbb{C}$ and $u\in\mathbb{R}^\times$. Unlike the classical Kelvin transform, which has order two, this operator has order four. Its inverse is

$$\big(\mathbb{K}(z,\varepsilon_1,\varepsilon_2)^{-1}\varphi\big)(x,y,t) = \varepsilon_1\varepsilon_2\,|t-xy|_{\varepsilon_1}^{-(z+z_0)}\,|t|_{\varepsilon_2}^{z-z_0}\varphi\Big(-\frac{x}{t-xy}, -\frac{y}{t}, -\frac{1}{t}\Big),$$

and the conformal property of $\Box_z$ with respect to $\mathbb{K}(z,\varepsilon_1,\varepsilon_2)$ is expressed by the equation

$$\mathbb{K}(z,\varepsilon_1,\varepsilon_2)\circ\Box_z\circ\mathbb{K}(z,\varepsilon_1,\varepsilon_2)^{-1} = c\,\Box_z,$$

with the conformal factor $c(x,y,t) = t(t-xy)$. This identity follows from the general theory developed in [Barchini et al. 2009] and [Kable 2011a]. The reader may find further discussion of it in [Kable 2011b]. Of course, the identity may also be verified by computation.

The conformal identity implies that the inverse Kelvin transform maps the solution space of the equation $\Box_z f = 0$ into itself. (There is no need at present to be careful about the domains of the solutions.) In particular, since the constant function 1 is visibly a solution to the ultrahyperbolic equation, so also is the function $\varphi_0(z,\varepsilon_1,\varepsilon_2) = \mathbb{K}(z,\varepsilon_1,\varepsilon_2)^{-1}1$. These are the solutions that we refer to as basic solutions. They are analogues of the radial solution $1/r^{2-n}$ of the Euclidean Laplacian in dimension n. Explicitly, we have

$$\varphi_0(z,\varepsilon_1,\varepsilon_2) = \varepsilon_1\varepsilon_2\,|t-xy|_{\varepsilon_1}^{-(z+z_0)}\,|t|_{\varepsilon_2}^{z-z_0}$$

on the set $\{(x,y,t)\in N \mid t(t-xy)\neq 0\}$. Note that the sum of the exponents in this expression is $-d$, so that the basic solutions are assuredly singular on at least part of the hypersurface $t(t-xy)=0$. The aim of this work is to interpret the basic solutions as tempered distributions on N and to compute $\Box_z\varphi_0(z,\varepsilon_1,\varepsilon_2)$ in the distributional sense.

For comparison, we first recall the situation in the case of the Heisenberg Laplacian. Here the basic solution φ_0 is a locally integrable function of moderate growth, singular only at the identity in N. Thus it defines a tempered distribution in the usual way and the distribution $\Box_z\varphi_0$ is supported at the identity. By using this and the behavior of $\square_z\varphi_0$ under the action of a suitable subgroup of the conformal group, it is easy to see that $\square_z\varphi_0$ is a multiple of δ_0, the Dirac delta at the identity. The constant of proportionality was computed by Folland and Stein [1974, Section 6]. It depends on the parameter z and vanishes for certain exceptional values of z. Except for these values of z, the fundamental solution of the Heisenberg Laplacian is a multiple of the basic solution.

Returning now to the Heisenberg ultrahyperbolic equation, the first difficulty that we must address is that $\varphi_0(z,\varepsilon_1,\varepsilon_2)$ is almost never a locally integrable function on N and so it does not give rise to a tempered distribution directly. We resolve this problem by introducing a two parameter family $T(s_1,s_2,\varepsilon_1,\varepsilon_2)$ of tempered distributions. These distributions are associated to locally integrable functions of moderate growth provided that $\operatorname{re} s_1$ and $\operatorname{re} s_2$ are positive. They are then defined in general by analytic continuation (Proposition 2.1). Formally, we have

$$T(z-z_0,-(z+z_0),\varepsilon_1,\varepsilon_2)=\varepsilon_1\varepsilon_2\frac{1}{\Gamma\big(1+(z-z_0)\big)\Gamma\big(1-(z+z_0)\big)}\varphi_0(z,\varepsilon_2,\varepsilon_1),$$

and this equation may be taken literally as an identity of distributions provided that we restrict to an open set whose closure lies in the complement of the hypersurface $t(t-xy)=0$. In light of this, the problem of computing $\square_z\varphi_0(z,\varepsilon_2,\varepsilon_1)$ may be reinterpreted precisely as the problem of computing

$$\square_z T(z-z_0,-(z+z_0),\varepsilon_1,\varepsilon_2)$$

as a distribution. The result is that

$$\square_z T(z-z_0,-(z+z_0),\varepsilon,\varepsilon)=0$$

and

$$\square_z T(z-z_0,-(z+z_0),\varepsilon,-\varepsilon)=a_\varepsilon(z)\delta_0,$$

where δ_0 denotes the Dirac delta at the identity and $a_\varepsilon(z)$ is an elementary function. (The precise value of $a_\varepsilon(z)$ is given in Theorem 3.12.) In particular, for most z, $T(z-z_0,-(z+z_0),\varepsilon,-\varepsilon)$ is a multiple of a fundamental solution for the Heisenberg ultrahyperbolic operator. The situation in this regard is explained after Corollary 3.15.

Although these results are easy to state, their proofs are a little lengthy, and so it may be helpful to provide a brief guide to them. Let us write

$$S(z,\varepsilon_1,\varepsilon_2)=\square_z T(z-z_0,-(z+z_0),\varepsilon_1,\varepsilon_2).$$

The first step, taken in Proposition 2.3, is to find polynomials that annihilate the distribution $S(z,\varepsilon_1,\varepsilon_2)$. This is a more precise version of obtaining a restriction on the support of $S(z,\varepsilon_1,\varepsilon_2)$, and allows us to conclude that $S(z,\varepsilon_1,\varepsilon_2)$ is the corestriction of a tempered distribution $D(z,\varepsilon_1,\varepsilon_2)$ that is supported on the cone

$xy = 0$ in the hyperplane $t = 0$. Next, the symmetry properties of this distribution with respect to the automorphism group of the cone are determined in Corollary 2.6. By appealing to the classification of distributions that are supported on light cones and invariant under indefinite orthogonal groups (due originally to de Rham and subsequently reconsidered by a number of authors), we are able to determine $D(z, \varepsilon_1, \varepsilon_2)$ up to an overall factor depending on z, ε_1, and ε_2 in Theorem 2.9. In some cases, symmetry considerations show that this factor is zero; in other cases, less information is forthcoming. This concludes Section 2.

Section 3 is devoted to determining the factors in the remaining cases. In principle, one simply has to compute both sides on a suitably chosen Schwartz function. However, the practical difficulties are substantial. The points at which we are required to evaluate the distributions are deep into the region where they are defined by analytic continuation and, even in their initial region of convergence, the relevant expressions involve integrals of higher transcendental functions that do not seem to be known. Thus we must take a more oblique approach, and this is done in Theorems 3.12 and 3.14. The main point is that the hyperplane $t = 0$ is a prehomogeneous vector space under the action of a certain subgroup of the conformal group. We introduce, and study in detail, a function of two variables that reduces to the classical (local) zeta function of this prehomogeneous vector space when one of the variables is specialized to zero. The required factors can be expressed as integrals of this function with respect to the second variable and, once enough information is obtained about the analytic properties of the function, this allows the factors to be evaluated. Further information about the strategy is included in the proof of Theorem 3.12 and the surrounding discussion.

Although we do not pursue this aspect of things in the present work, the reader should note that the distributions $T(s_1, s_2, \varepsilon_1, \varepsilon_2)$ are intimately related to the standard integral intertwining operators for a family of degenerate principal series representations of the conformal group. Similarly, the operators $\square_z$ may be interpreted as differential intertwining operators for this family. In this framework,

$$\square_z T(z - z_0, -(z + z_0), \varepsilon_1, \varepsilon_2)$$

may be interpreted as the composition of a differential and an integral intertwining operator. The analytic properties of integral intertwining operators have received a great deal of attention, but much less is known about the analytic properties of differential intertwining operators. The author hopes to pursue this in the future.

This work forms a part of a broader investigation of the properties of conformally invariant systems of differential equations. [Barchini et al. 2009; Kable 2011a; Kable 2011b] are also parts of the same program. In keeping with his background as an algebraist, the author normally denotes the result of applying the differential operator D to the function f by $D \bullet f$, and this notation is used in the articles just cited, as well as other articles on the same subject not referred to here. The referee has suggested that this notation is uncongenial to analysts and, since the work reported here is mostly analytic, the notation Df has been adopted instead. The author would like to thank the referee for this and several other helpful suggestions.

2. The tempered distributions

Take $s_1, s_2 \in \mathbb{C}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. When $\operatorname{re} s_1 \geq 0$ and $\operatorname{re} s_2 \geq 0$, the function $|t|_{\varepsilon_1}^{s_1} |t - xy|_{\varepsilon_2}^{s_2}$ is locally integrable and of moderate growth and so may be thought of as a tempered distribution on N. When $\operatorname{re} s_1 \geq 0$ and $\operatorname{re} s_2 \geq 0$, we define a tempered distribution $T(s_1, s_2, \varepsilon_1, \varepsilon_2)$ by

$$T(s_1, s_2, \varepsilon_1, \varepsilon_2) = \frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+d+1)} |t|_{\varepsilon_1}^{s_1} |t - xy|_{\varepsilon_2}^{s_2},$$

where Γ denotes the gamma function.

Proposition 2.1. *The family of tempered distributions* $(s_1, s_2) \mapsto T(s_1, s_2, \varepsilon_1, \varepsilon_2)$ *has an analytic continuation to* $\mathbb{C}^2$ *for all* $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. *This continuation satisfies*

$$\square_{z_0+s_1} T(s_1, s_2, \varepsilon_1, \varepsilon_2) = T(s_1 - 1, s_2, -\varepsilon_1, \varepsilon_2)$$

and

$$\square_{-(z_0+s_2)} T(s_1, s_2, \varepsilon_1, \varepsilon_2) = -T(s_1, s_2 - 1, \varepsilon_1, -\varepsilon_2)$$

for all $(s_1, s_2) \in \mathbb{C}^2$ *and* $\varepsilon_1, \varepsilon_2 \in \{\pm\}$.

Proof. Two useful identities for the $|\cdot|_\varepsilon^s$ symbol, namely

$$\frac{d}{du} |u|_\varepsilon^s = s |u|_{-\varepsilon}^{s-1} \quad \text{and} \quad u |u|_\varepsilon^s = |u|_{-\varepsilon}^{s+1},$$

will be used repeatedly below. Our first step will be to establish the two identities in the statement when $\operatorname{re} s_1$ and $\operatorname{re} s_2$ are sufficiently large. Under this assumption, we have

$$\frac{\partial}{\partial t} \bigl(|t|_{\varepsilon_1}^{s_1} |t - xy|_{\varepsilon_2}^{s_2} \bigr) = s_1 |t|_{-\varepsilon_1}^{s_1-1} |t - xy|_{\varepsilon_2}^{s_2} + s_2 |t|_{\varepsilon_1}^{s_1} |t - xy|_{-\varepsilon_2}^{s_2-1},$$

and, after introducing the appropriate normalizing factors into this relation, it may be written as

$$(s_1 + s_2 + d) \frac{\partial}{\partial t} T(s_1, s_2, \varepsilon_1, \varepsilon_2) = T(s_1 - 1, s_2, -\varepsilon_1, \varepsilon_2) + T(s_1, s_2 - 1, \varepsilon_1, -\varepsilon_2). \tag{2-1}$$

We also have

$$\Delta\big(|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}\big)=|t|_{\varepsilon_1}^{s_1}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j\partial y_j}\big(|t-xy|_{\varepsilon_2}^{s_2}\big)=-s_2\,|t|_{\varepsilon_1}^{s_1}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\big(x_j\,|t-xy|_{-\varepsilon_2}^{s_2-1}\big)$$
$$=-s_2\,|t|_{\varepsilon_1}^{s_1}\sum_{j=1}^{d}\big(|t-xy|_{-\varepsilon_2}^{s_2-1}-(s_2-1)x_jy_j\,|t-xy|_{\varepsilon_2}^{s_2-2}\big)$$
$$=-s_2d\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}+s_2(s_2-1)\,|t|_{\varepsilon_1}^{s_1}(xy)\,|t-xy|_{\varepsilon_2}^{s_2-2}$$
$$=-s_2d\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}-s_2(s_2-1)\,|t|_{\varepsilon_1}^{s_1}(t-xy)\,|t-xy|_{\varepsilon_2}^{s_2-2}+s_2(s_2-1)\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{\varepsilon_2}^{s_2-2}$$
$$=-s_2d\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}-s_2(s_2-1)\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}+s_2(s_2-1)\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{\varepsilon_2}^{s_2-2}$$
$$=-s_2(s_2+d-1)\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}+s_2(s_2-1)\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{\varepsilon_2}^{s_2-2},$$

and, by a similar computation,

$$\mathbb{E}_x\big(|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}\big)=s_2\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}-s_2\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{-\varepsilon_2}^{s_2-1},$$

so that

$$(\mathbb{E}_x-s_2)\big(|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}\big)=-s_2\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}$$

and

$$\frac{\partial}{\partial t}(\mathbb{E}_x-s_2)\big(|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}\big)=-s_2(s_1+1)\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1}-s_2(s_2-1)\,|t|_{-\varepsilon_1}^{s_1+1}\,|t-xy|_{\varepsilon_2}^{s_2-2}.$$

By adding the first of the formulas in the previous sentence to the last, we obtain

$$\Big(\Delta+\frac{\partial}{\partial t}(\mathbb{E}_x-s_2)\Big)\big(|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{\varepsilon_2}^{s_2}\big)=-s_2(s_1+s_2+d)\,|t|_{\varepsilon_1}^{s_1}\,|t-xy|_{-\varepsilon_2}^{s_2-1},$$

and if we introduce the relevant normalizing factors then this equation is seen to be equivalent to

$$\Big(\Delta+\frac{\partial}{\partial t}(\mathbb{E}_x-s_2)\Big)T(s_1,s_2,\varepsilon_1,\varepsilon_2)=-T(s_1,s_2-1,\varepsilon_1,-\varepsilon_2). \tag{2-2}$$

This is precisely the second identity in the statement. By adding (2-1) and (2-2), we obtain

$$\Big(\Delta+\frac{\partial}{\partial t}(\mathbb{E}_x+s_1+d)\Big)T(s_1,s_2,\varepsilon_1,\varepsilon_2)=T(s_1-1,s_2,-\varepsilon_1,\varepsilon_2), \tag{2-3}$$

which is equivalent to the first identity in the statement. Thus these identities are valid provided that $\operatorname{re} s_1$ and $\operatorname{re} s_2$ are sufficiently large. We may rewrite them in the form

$$T(s_1, s_2, \varepsilon_1, \varepsilon_2) = \square_{z_0+s_1+1} T(s_1+1, s_2, -\varepsilon_1, \varepsilon_2),$$
$$T(s_1, s_2, \varepsilon_1, \varepsilon_2) = -\square_{-(z_0+s_2+1)} T(s_1, s_2+1, \varepsilon_1, -\varepsilon_2),$$

and in this form they permit the analytic continuation of $T(s_1, s_2, \varepsilon_1, \varepsilon_2)$ in the usual way. Once the continuation is effected, the identity principle implies that the identities remain valid for the continued distributions. □

We define

$$S(z, \varepsilon_1, \varepsilon_2) = \square_z T(z - z_0, -(z + z_0), \varepsilon_1, \varepsilon_2),$$

and note that, by Proposition 2.1, we have

$$S(z, \varepsilon_1, \varepsilon_2) = T(z - z_0 - 1, -(z + z_0), -\varepsilon_1, \varepsilon_2), \tag{2-4}$$
$$S(z, \varepsilon_1, \varepsilon_2) = -T(z - z_0, -(z + z_0 + 1), \varepsilon_1, -\varepsilon_2). \tag{2-5}$$

Lemma 2.2. *We have*

$$tT(s_1, s_2, \varepsilon_1, \varepsilon_2) = (s_1+1)(s_1+s_2+d+1) T(s_1+1, s_2, -\varepsilon_1, \varepsilon_2)$$

and

$$(t - xy) T(s_1, s_2, \varepsilon_1, \varepsilon_2) = (s_2+1)(s_1+s_2+d+1) T(s_1, s_2+1, \varepsilon_1, -\varepsilon_2).$$

Proof. The identities follow directly from the definitions in the region where $\operatorname{re} s_1$ and $\operatorname{re} s_2$ are nonnegative. They follow in general by continuation. □

Proposition 2.3. *For all $z \in \mathbb{C}$ and all $\varepsilon_1, \varepsilon_2 \in \{\pm\}$ we have*

$$tS(z, \varepsilon_1, \varepsilon_2) = 0 \quad \textit{and} \quad (xy) S(z, \varepsilon_1, \varepsilon_2) = 0.$$

Proof. By (2-4) and Lemma 2.2, we have

$$\begin{aligned} tS(z, \varepsilon_1, \varepsilon_2) &= tT(z - z_0 - 1, -(z + z_0), -\varepsilon_1, \varepsilon_2) \\ &= (z - z_0)(-2z_0 - 1 + d + 1) T(z - z_0, -(z + z_0), \varepsilon_1, \varepsilon_2) = 0, \end{aligned}$$

since $2z_0 + 1 = d + 1$. A similar argument using (2-5) and Lemma 2.2 shows that we also have $(t - xy) S(z, \varepsilon_1, \varepsilon_2) = 0$, and the second identity follows from this and the first identity. □

Let $V \subset N$ be the hyperplane defined by $t = 0$. A consequence of Proposition 2.3 is that there is a unique tempered distribution $D(z, \varepsilon_1, \varepsilon_2)$ on V such that

$$S(z, \varepsilon_1, \varepsilon_2)(\Phi) = D(z, \varepsilon_1, \varepsilon_2)\big(\Phi|_V\big)$$

for all $\Phi \in \mathcal{S}(N)$. Moreover, we have $(xy) D(z, \varepsilon_1, \varepsilon_2) = 0$ and, in particular, $D(z, \varepsilon_1, \varepsilon_2)$ is supported on the cone $xy = 0$. Our next aim is to study the symmetry properties of $D(z, \varepsilon_1, \varepsilon_2)$. The key point is that the symmetry group of this distribution is much larger than the conformal symmetry group of the operator $\Box_z$ from which it was constructed.

We denote the result of applying $r \in \mathrm{GL}(N)$ to $n \in N$ by $r \cdot n$. This action induces an action of $\mathrm{GL}(N)$ on $\mathcal{S}(N)$ by $(r \cdot \Phi)(n) = \Phi(r^{-1} \cdot n)$ and on $\mathcal{S}'(N)$ by $(r \cdot T)(\Phi) = T(r^{-1} \cdot \Phi)$. If $r \in \mathrm{GL}(N)$ happens to stabilize V then all these actions are compatible with restriction and corestriction to V. Thus if $r(V) = V$ and $r \cdot S(z, \varepsilon_1, \varepsilon_2) = cS(z, \varepsilon_1, \varepsilon_2)$ for some constant c then

$$(r|_V) \cdot D(z, \varepsilon_1, \varepsilon_2) = cD(z, \varepsilon_1, \varepsilon_2).$$

Henceforth, we shall not generally distinguish between r and $r|_V$ in such situations. For $g \in \mathrm{GL}(d, \mathbb{R})$ we define $\boldsymbol{r}_g \in \mathrm{GL}(N)$ by

$$\boldsymbol{r}_g(x, y, t) = (xg^{-1}, gy, t).$$

For a skew-symmetric d-by-d matrix A we define $\boldsymbol{u}_A \in \mathrm{GL}(N)$ by

$$\boldsymbol{u}_A(x, y, t) = (x, y + Ax^\top, t),$$

and $\bar{\boldsymbol{u}}_A \in \mathrm{GL}(N)$ by

$$\bar{\boldsymbol{u}}_A(x, y, t) = (x + y^\top A, y, t).$$

For $a \in \mathbb{R}^\times$, we define $\boldsymbol{p}_a \in \mathrm{GL}(N)$ by

$$\boldsymbol{p}_a(x, y, t) = (ax, y, at),$$

and, finally, we define $\boldsymbol{s} \in \mathrm{GL}(N)$ by

$$\boldsymbol{s}(x, y, t) = \big((y_1, x_2, \dots, x_{m-2}), (x_1, y_2, \dots, y_{m-2})^\top, t\big).$$

Each of these elements stabilizes V. The restrictions of $\boldsymbol{r}_g$, $\boldsymbol{u}_A$, and $\bar{\boldsymbol{u}}_A$ to V generate the group $\mathrm{SO}(P)$ of the quadratic form $P = xy$. The element $\boldsymbol{s}$ lies in $\mathrm{O}(P) - \mathrm{SO}(P)$ and hence by adding this element we obtain a generating set for $\mathrm{O}(P)$. Finally, $\boldsymbol{p}_a$ is a similitude of P with multiplier a and so including these elements along with the others yields a generating set for $\mathrm{GO}(P)$. Let $H \subset \mathrm{GL}(N)$ be the group generated by all $\boldsymbol{r}_g$, $\boldsymbol{u}_A$, $\bar{\boldsymbol{u}}_A$, $\boldsymbol{p}_a$, and $\boldsymbol{s}$. The restriction map to V is an isomorphism from H to $\mathrm{GO}(P)$. Let σ be either the similitude character $\sigma : \mathrm{GO}(P) \to \mathbb{R}^\times$ or its pullback to H, depending on context.

Lemma 2.4. *For all $s_1, s_2 \in \mathbb{C}$, $\varepsilon_1, \varepsilon_2 \in \{\pm\}$, and $h \in H$ we have*

$$h \cdot T(s_1, s_2, \varepsilon_1, \varepsilon_2) = |\sigma(h)|_{\varepsilon_1\varepsilon_2}^{-(s_1+s_2+d+1)} T(s_1, s_2, \varepsilon_1, \varepsilon_2).$$

Proof. If f is a locally integrable function of moderate growth on N and T_f is the tempered distribution associated to f then the change-of-variable formula for integrals implies that

$$h \cdot T_f = |\det_N(h)|^{-1} T_{h \cdot f}$$

for all $h \in \mathrm{GL}(N)$. We may apply this formula to the function f on N given by $f(x, y, t) = |t|_{\varepsilon_1}^{s_1} |t - xy|_{\varepsilon_2}^{s_2}$ in the region where $\operatorname{re} s_1 \geq 0$ and $\operatorname{re} s_2 \geq 0$. The result will then follow in general by continuation. One may check that

$$|\det_N(h)| = |\sigma(h)|^{d+1}$$

for all $h \in H$. Thus the required formula follows from $h \cdot f = |\sigma(h)|_{\varepsilon_1\varepsilon_2}^{-(s_1+s_2)} f$ for all $h \in H$. It is sufficient to check this last claim for each of the generators of H that we enumerated above, and this is easily done. □

Proposition 2.5. *Let $z \in \mathbb{C}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Then, for all $h \in H$, we have*

$$h \cdot S(z, \varepsilon_1, \varepsilon_2) = |\sigma(h)|_{-\varepsilon_1\varepsilon_2}^{0} S(z, \varepsilon_1, \varepsilon_2).$$

Proof. This follows immediately from (2-4) and Lemma 2.4. □

Corollary 2.6. *Let $z \in \mathbb{C}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Then, for all $h \in \mathrm{GO}(P)$, we have*

$$h \cdot D(z, \varepsilon_1, \varepsilon_2) = |\sigma(h)|_{-\varepsilon_1\varepsilon_2}^{0} D(z, \varepsilon_1, \varepsilon_2).$$

Proof. This follows from Proposition 2.5 and the definition of $D(z, \varepsilon_1, \varepsilon_2)$. □

We must now recall some facts about $\mathrm{O}(P)$-invariant distributions supported on the cone $xy = 0$ in V. Let μ denote Lebesgue measure on V and set

$$V_+ = \{(x, y) \in V \mid xy > 0\}, \quad V_0 = \{(x, y) \in V \mid xy = 0\}, \quad V_- = \{(x, y) \in V \mid xy < 0\}.$$

We define tempered distributions $M_\pm$ and $F_\pm$ on V by

$$M_\pm(\Phi) = \int_{V_\pm} \Phi(x, y)\, d\mu(x, y) \quad \text{and} \quad F_\pm(\Phi) = \int_{V_\pm} \Phi(x, y) \log|P(x, y)|\, d\mu(x, y),$$

and let δ_0 denote the Dirac distribution at 0. It is evident that δ_0, $M_\pm$, and $F_\pm$ are $\mathrm{O}(P)$-invariant distributions on V. The same is true for the distributions obtained by applying Δ^n (for $n \geq 0$) to δ_0, $M_\pm$, or $F_\pm$. Let us denote by $\mathrm{GO}^+(P)$ the set of all $h \in \mathrm{GO}(P)$ such that $\sigma(h) > 0$. A calculation based on the definitions and the change-of-variable formula for integrals shows that

$$h \cdot M_\pm = \sigma(h)^{-d} M_\pm \tag{2-6}$$

and

$$h \cdot F_\pm = \sigma(h)^{-d}(F_\pm - \log(\sigma(h)) M_\pm) \tag{2-7}$$

for all $h \in \mathrm{GO}^+(P)$. We also note that

$$\boldsymbol{p}_{-1} \cdot M_\pm = M_\mp \tag{2-8}$$

and

$$\boldsymbol{p}_{-1} \cdot F_\pm = F_\mp. \tag{2-9}$$

We require some additional facts about $M_\pm$, $F_\pm$, and Δ. In the present situation, these are due to de Rham, but we shall use [Folland 1998] as a convenient and accessible reference for them. To begin with, it is clear that

$$\Delta M_+ = -\Delta M_-. \tag{2-10}$$

There is a nonzero constant c_1 (which depends on d) such that

$$\Delta^d M_\pm = \begin{cases} 0 & \text{if } d \text{ is even,} \\ \pm c_1 \delta_0 & \text{if } d \text{ is odd.} \end{cases} \tag{2-11}$$

Moreover, $\Delta^j M_\pm$ is nonzero for $0 \le j \le d-1$. These claims follow from Proposition 3 and the subsequent corollary in [Folland 1998]. (Note that there is a misprint in the statement of this proposition; the negative power on Δ in the list of invariant distributions annihilated by Δ should be replaced by its absolute value. Also, the first statement in the remark that follows the corollary is inaccurate when $p = q = 1$.) There is a nonzero constant c_2 (which depends on d) such that

$$\Delta^d (F_+ + F_-) = \begin{cases} c_2 \delta_0 & \text{if } d \text{ is even,} \\ 0 & \text{if } d \text{ is odd.} \end{cases} \tag{2-12}$$

Moreover, the set $\{\delta_0, \Delta^d F_+\}$ is linearly independent. These facts follow from Proposition 3 and the remark after Proposition 6 in [Folland 1998]. (Special note should be taken of Equation (14) in that reference when calculating $\Delta^d(F_+ + F_-)$ in the case where d is even, since it introduces an extra sign change between Folland's notation and ours.)

Proposition 2.7. *The space of* $\mathrm{GO}^+(P)$*-invariant tempered distributions supported on* V_0 *in* V *is one-dimensional if* d *is odd and two-dimensional if* d *is even. If* d *is odd then* $\{\delta_0\}$ *is a basis for this space and every* $\mathrm{GO}^+(P)$*-invariant tempered distribution supported on* V_0 *is* $\mathrm{GO}(P)$*-invariant. If* d *is even then* $\{\delta_0, \Delta^d F_+\}$ *is a basis for this space. In this case,* $\boldsymbol{p}_{-1}$ *has eigenvalues* ± 1 *in this space of distributions. In fact,* δ_0 *is a* 1*-eigenvector for* $\boldsymbol{p}_{-1}$ *and* $\Delta^d F_+ - \frac{1}{2} c_2 \delta_0$ *is a* (-1)*-eigenvector for* $\boldsymbol{p}_{-1}$.

Proof. By Proposition 6 in [Folland 1998], the space of $\mathrm{O}(P)$-invariant tempered distributions on V that are supported on V_0 is spanned by the distributions $\Delta^n \delta_0$ for $n \ge 0$, $\Delta^n M_+$ for $n \ge 1$, and $\Delta^{n+d} F_+$ for $n \ge 0$. It is easy to verify that $h \circ \Delta \circ h^{-1} = \sigma(h)\Delta$ for all $h \in \mathrm{GO}(P)$. In light of these facts, and (2-6) and (2-7), it is clear that the space of $\mathrm{GO}^+(P)$-invariant tempered distributions on V that are supported on V_0 is contained in the space spanned by the distributions δ_0, $\Delta^d M_+$, and $\Delta^d F_+$.

Suppose that d is odd. Then $\Delta^d M_+ = c_1\delta_0$ for a nonzero constant c_1 by (2-11), and

$$h \cdot \Delta^d F_+ = \Delta^d F_+ - c_1 \log(\sigma(h))\delta_0$$

for all $h \in \mathrm{GO}^+(P)$, by (2-7). Thus the space of $\mathrm{GO}^+(P)$-invariant distributions has $\{\delta_0\}$ as a basis. This confirms all the claims when d is odd.

Suppose that d is even. Then $\Delta^d M_+ = 0$ by (2-11) and $\Delta^d F_+$ is invariant under $\mathrm{GO}^+(P)$ by this and (2-7). Thus $\{\delta_0, \Delta^d F_+\}$ is a basis for the space of $\mathrm{GO}^+(P)$-invariant distributions in this case. Now

$$\begin{aligned} \boldsymbol{p}_{-1} \cdot \Delta^d F_+ &= (-1)^d \Delta^d \big(\boldsymbol{p}_{-1} \cdot F_+\big) \\ &= \Delta^d F_- = -\Delta^d F_+ + \Delta^d \big(F_+ + F_-\big) = -\Delta^d F_+ + c_2\delta_0, \end{aligned}$$

where we have used (2-9) from the first line to the second, and (2-12) for the last step. This equality is equivalent to

$$\boldsymbol{p}_{-1} \cdot \big(\Delta^d F_+ - \tfrac{1}{2}c_2\delta_0\big) = -\big(\Delta^d F_+ - \tfrac{1}{2}c_2\delta_0\big),$$

and this verifies the final claim. □

The inversion operator $\mathbb{I}$ is defined on $\mathcal{S}(N)$ by

$$(\mathbb{I}\Phi)(x, y, t) = \Phi(-x, -y, xy - t).$$

It is evident that $\mathbb{I}$ is a continuous operator and this allows us to define an inversion operator on $\mathcal{S}'(N)$ by $\mathbb{I}T(\Phi) = T(\mathbb{I}\Phi)$.

Lemma 2.8. *For all $s_1, s_2 \in \mathbb{C}$ and $\varepsilon_1, \varepsilon_2 \in \{\pm\}$, we have*

$$\mathbb{I}T(s_1, s_2, \varepsilon_1, \varepsilon_2) = \varepsilon_1\varepsilon_2 T(s_2, s_1, \varepsilon_2, \varepsilon_1).$$

Proof. When the real parts of s_1 and s_2 are sufficiently large, we have

$$\begin{aligned} \int_N |t|_{\varepsilon_1}^{s_1} &|t - xy|_{\varepsilon_2}^{s_2} (\mathbb{I}\Phi)\big(n(x, y, t)\big)\, d\mu(x, y, t) \\ &= \int_N |t|_{\varepsilon_1}^{s_1} |t - xy|_{\varepsilon_2}^{s_2} \Phi\big(n(-x, -y, xy - t)\big)\, d\mu(x, y, t) \\ &= \int_N |xy - t|_{\varepsilon_1}^{s_1} |-t|_{\varepsilon_2}^{s_2} \Phi\big(n(x, y, t)\big)\, d\mu(x, y, t) \\ &= \varepsilon_1\varepsilon_2 \int_N |t|_{\varepsilon_2}^{s_2} |t - xy|_{\varepsilon_1}^{s_1} \Phi\big(n(x, y, t)\big)\, d\mu(x, y, t). \end{aligned}$$

The normalizing factor is the same on both sides, and so we obtain

$$\mathbb{I}T(s_1, s_2, \varepsilon_1, \varepsilon_2)(\Phi) = \varepsilon_1\varepsilon_2 T(s_2, s_1, \varepsilon_2, \varepsilon_1)(\Phi)$$

when the real parts of s_1 and s_2 are sufficiently large. As usual, the claim follows in general by continuation. □

Theorem 2.9. *Let $\varepsilon \in \{\pm\}$. There is an entire function a_ε such that*

$$D(z,\varepsilon,-\varepsilon) = a_\varepsilon(z)\delta_0$$

for all $z \in \mathbb{C}$. We have $a_{-\varepsilon}(-z) = -a_\varepsilon(z)$ for all $z \in \mathbb{C}$. If d is odd then

$$D(z,\varepsilon,\varepsilon) = 0$$

for all $z \in \mathbb{C}$. If d is even then there is an entire function $c_\varepsilon(z)$ such that

$$D(z,\varepsilon,\varepsilon) = c_\varepsilon(z)\big(\Delta^d F_+ - \tfrac{1}{2}c_2\delta_0\big)$$

for all $z \in \mathbb{C}$.

Proof. We have already observed that the tempered distribution $D(z,\varepsilon_1,\varepsilon_2)$ on V is supported on V_0. It follows from Corollary 2.6 that $D(z,\varepsilon,-\varepsilon)$ is also invariant under $\mathrm{GO}(P)$. Proposition 2.7 then implies that there is a constant $a_\varepsilon(z)$ such that $D(z,\varepsilon,-\varepsilon) = a_\varepsilon(z)\delta_0$ for all $z \in \mathbb{C}$. If we choose a function $\Phi \in \mathcal{S}(V)$ such that $\Phi(0) = 1$ then we obtain

$$a_\varepsilon(z) = D(z,\varepsilon,-\varepsilon)(\Phi).$$

It follows that a_ε is entire, since we have proved that $z \mapsto D(z,\varepsilon,-\varepsilon)$ is an entire family. The corestriction of δ_0 from V to N is again δ_0 and so we may write

$$S(z,\varepsilon,-\varepsilon) = a_\varepsilon(z)\delta_0.$$

By (2-4), this is equivalent to

$$T(z - z_0 - 1, -(z + z_0), -\varepsilon, -\varepsilon) = a_\varepsilon(z)\delta_0,$$

and, by applying $\mathbb{I}$ to both sides and noting that $\mathbb{I}\delta_0 = \delta_0$, we obtain

$$T(-z - z_0, -(-z + z_0 + 1), -\varepsilon, -\varepsilon) = a_\varepsilon(z)\delta_0.$$

This, in turn, is equivalent to

$$-S(-z,-\varepsilon,\varepsilon) = a_\varepsilon(z)\delta_0$$

by (2-5). We also have

$$S(-z,-\varepsilon,\varepsilon) = a_{-\varepsilon}(-z)\delta_0,$$

and it follows that $a_{-\varepsilon}(-z) = -a_\varepsilon(z)$, as claimed.

Suppose that d is odd. By Corollary 2.6, $D(z,\varepsilon,\varepsilon)$ is invariant under $\mathrm{GO}^+(P)$, but satisfies $\boldsymbol{p}_{-1} \cdot D(z,\varepsilon,\varepsilon) = -D(z,\varepsilon,\varepsilon)$. By Proposition 2.7, this is impossible unless $D(z,\varepsilon,\varepsilon) = 0$. Finally, suppose that d is even. It is still the case that $D(z,\varepsilon,\varepsilon)$ is invariant under $\mathrm{GO}^+(P)$ and anti-invariant under $\boldsymbol{p}_{-1}$. According to Proposition 2.7, this implies that $D(z,\varepsilon,\varepsilon)$ is proportional to $\Delta^d F_+ - \frac{1}{2}c_2\delta_0$ as stated. □

3. Determination of a_ε and c_ε

Let $\Phi_0 \in \mathscr{S}(N)$ be defined by

$$\Phi_0(x, y, t) = e^{-(t^2+\|x\|^2+\|y\|^2)},$$

where $\|x\|^2 = xx^\top$ and similarly for y. As we saw above, the function a_ε considered in Theorem 2.9 is given by

$$a_\varepsilon(z) = T(z - z_0 - 1, -(z + z_0), -\varepsilon, -\varepsilon)(\Phi_0). \tag{3-1}$$

This leads us to consider the entire function

$$a(s_1, s_2) = T(s_1, s_2, +, +)(\Phi_0)$$

in general.

For $s \in \mathbb{C}$ with $\operatorname{re} s > 0$ and $t \in \mathbb{R}$, we define

$$Z(s, t) = \int_{\mathbb{R}^d \oplus (\mathbb{R}^d)^\top} |xy|_+^s e^{-(t^2(xy)^2+\|x\|^2+\|y\|^2)}\, d\mu(x, y), \tag{3-2}$$

where, as before, μ denotes Lebesgue measure. We shall have to investigate the properties of $Z(s, t)$ in some detail. It happens that $Z(s, t)$ can be evaluated in terms of known special functions. In fact,

$$Z(s, t) = \pi^d |t|_+^{-(s+d)} \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \Psi\left(\frac{s+d}{2}, \frac{d+1}{2}, \frac{1}{t^2}\right), \tag{3-3}$$

where Ψ denotes the classical confluent hypergeometric function of the second kind. (The reader may consult Sections 9.10–9.13 in [Lebedev 1972] for an excellent account of the basic properties of this function.) However, we prefer not to rely on this fact, since we can obtain what is needed directly from (3-2) by using methods that are applicable in more general situations, where the analogue of (3-3) is unknown.

Let $b(s) = (s + 1)(s + d)$ be the b-function of the polynomial xy.

Lemma 3.1. *For all $s \in \mathbb{C}$ with $\operatorname{re} s > 0$ we have*

$$b(s)Z(s, 0) = 4Z(s + 2, 0).$$

Proof. We have $\Delta|xy|_-^{s+1} = b(s)|xy|_+^s$ and the identity follows from this and the fact that

$$\Delta e^{-(\|x\|^2+\|y\|^2)} = 4(xy)e^{-(\|x\|^2+\|y\|^2)}$$

by integration by parts. □

Lemma 3.2. *For all $s \in \mathbb{C}$ with* $\operatorname{re} s > 0$ *we have*

$$Z(s,0) = \pi^d \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}.$$

Proof. Apart from the normalization, this is a consequence of Proposition 6.3.1 in [Igusa 2000]. □

Lemma 3.3. *The integral in* (3-2) *converges absolutely*, *uniformly in t*, *and locally uniformly in s on the region where* $\operatorname{re} s > -1$.

Proof. The integrand of $Z(s,t)$ is bounded in absolute value by that of $Z(\operatorname{re} s, 0)$. From the usual argument, based on Landau's lemma, the evaluation of this integral given in Lemma 3.2 implies what is claimed. □

Note that Lemma 3.3 implies that $Z(s,t)$ is a holomorphic function of s in the region where $\operatorname{re} s > -1$. For $k \geq 0$, let

$$Z^{(k)}(s,t) = \frac{\partial^k}{\partial t^k} Z(s,t).$$

By differentiating under the integral sign in (3-2), we obtain

$$Z^{(1)}(s,t) = -2t Z(s+2,t). \tag{3-4}$$

Lemma 3.4. *For $k \geq 1$, we have*

$$Z^{(k)}(s,t) = \sum_{j=1}^{k} c_j^k(t) Z(s+2j,t),$$

where c_j^k is a polynomial of degree at most j.

Proof. This follows by induction from (3-4). □

Lemma 3.5. *Suppose that f is a C^1 function on $\mathbb{R}$. Then, for $l \geq 0$, we have*

$$\frac{\partial}{\partial t}\left(t^{2l+4}\int_0^1 (1-v)^{l+1} f(t\sqrt{v})\,dv\right) = 2(l+1)t^{2l+3}\int_0^1 (1-v)^l f(t\sqrt{v})\,dv.$$

Proof. The key point is the identity

$$t\frac{\partial}{\partial t} f(t\sqrt{v}) = 2v\frac{\partial}{\partial v} f(t\sqrt{v}). \tag{3-5}$$

Given this, the statement follows by differentiating under the integral sign, introducing (3-5), integrating by parts, and simplifying the result. □

Proposition 3.6. *For each $t \in \mathbb{R}$ and each $k \geq 0$, the function*

$$s \mapsto \frac{Z^{(k)}(s,t)}{Z(s,0)}$$

extends to an entire function. The extension is smooth as a function of t and the processes of continuation and differentiation in t commute; *that is, for all* $s \in \mathbb{C}$,

$$\frac{\partial}{\partial t}\frac{Z^{(k)}(s,t)}{Z(s,0)} = \frac{Z^{(k+1)}(s,t)}{Z(s,0)}.$$

Moreover, *we have* $\dfrac{Z(-d,t)}{Z(-d,0)} = 1$ *and* $\dfrac{Z(-1,t)}{Z(-1,0)} = 1$ *for all* t.

Proof. We begin with the case $k = 0$. For $l \ge 0$, we have

$$e^u = \sum_{j=0}^{l} \frac{u^j}{j!} + \frac{u^{l+1}}{l!}\int_0^1 (1-v)^l e^{uv}\,dv.$$

By introducing this identity into (3-2) with $u = -t^2(xy)^2$ and interchanging the order of the integration (which is easily justified), we obtain

$$\text{(3-6)}\quad Z(s,t) = \sum_{j=0}^{l} \frac{(-1)^j}{j!} t^{2j} Z(s+2j,0) + \frac{(-1)^{l+1}}{l!} t^{2l+2} \int_0^1 (1-v)^l Z(s+2l+2, t\sqrt{v})\,dv.$$

It follows from Lemma 3.2 that the function $s \mapsto 1/Z(s,0)$ is entire. By multiplying (3-6) by this entire function and using Lemma 3.2, we find that

$$\text{(3-7)}\quad \frac{Z(s,t)}{Z(s,0)} = \sum_{j=0}^{l} \frac{(-1)^j}{j!} t^{2j} \Big(\frac{s+1}{2}\Big)_j \Big(\frac{s+d}{2}\Big)_j + \frac{(-1)^{l+1}}{l!Z(s,0)} t^{2l+2} \int_0^1 (1-v)^l Z(s+2l+2, t\sqrt{v})\,dv.$$

In (3-7), the first summand on the right-hand side is a polynomial in s and t. The second summand is holomorphic in s and smooth in t on the region where $\operatorname{re} s > -2l-3$. This expression thus serves to continue the ratio $Z(s,t)/Z(s,0)$ holomorphically to the region $\operatorname{re} s > -2l-3$. Since l was arbitrary, the first claim is established when $k = 0$.

Next we must show that the function $Z^{(1)}(s,t)/Z(s,0)$ has an entire continuation and that it is equal to the derivative of the continuation of $Z(s,t)/Z(s,0)$ with respect to t. Note that there is an issue here at $t = 0$, because (3-7) implies that the Maclaurin series of $Z(s,t)$ is

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^{2j} \Big(\frac{s+1}{2}\Big)_j \Big(\frac{s+d}{2}\Big)_j,$$

and the radius of convergence of this series is zero for most s. Thus, although there is an argument based on the identity principle for the required equality at nonzero values of t, it fails at $t = 0$. The equality

$$\frac{Z^{(1)}(s,t)}{Z(s,0)} = -2t\frac{Z(s+2,t)}{Z(s,0)} = -2t\cdot\frac{s+1}{2}\cdot\frac{s+d}{2}\cdot\frac{Z(s+2,t)}{Z(s+2,0)}, \tag{3-8}$$

together with what we have already done, implies that $s \mapsto Z^{(1)}(s,t)/Z(s,0)$ continues to an entire function. To verify the second statement, we begin by writing out (3-7) with l replaced by $l+1$. In light of (3-8), it suffices to show that the derivative of the resulting expression is equal to the factor $-t(s+1)(s+d)/2$ times (3-7) with s replaced by $s+2$. This is easily done for the first term, so we concentrate on the second term. The required equality turns out to be equivalent to

$$\frac{\partial}{\partial t}\Big(t^{2l+4}\int_0^1 (1-v)^{l+1} Z(s+2l+4, t\sqrt{v})\,dv\Big) = 2(l+1)t^{2l+3}\int_0^1 (1-v)^{l} Z(s+2l+4, t\sqrt{v})\,dv,$$

and this follows from Lemma 3.5. This completes the proof for $k=0$. The general case follows by combining Lemma 3.4, the case $k=0$, and the observation that $Z(s+2j,0)/Z(s,0)$ is a polynomial in s for $j \ge 0$.

For the last claim, we substitute $s = -d$ into (3-7) with l chosen large enough so that $-d+2l+2 > 0$. Since $1/Z(-d,0) = 0$, the second summand vanishes. In addition, all terms but the first in the first summand vanish and the first term is 1. This gives the required conclusion at $s = -d$. A similar argument succeeds when $s = -1$. □

Proposition 3.7. *Let $\alpha < \beta$ and $\gamma > 0$, and define p by*

$$p = \begin{cases} 1 & \text{if } d = 1 \text{ and } \alpha > -1, \\ 2 & \text{if } d = 1 \text{ and } \alpha \le -1, \\ 0 & \text{if } d > 1 \text{ and } \alpha > -1, \\ 1 & \text{if } d > 1 \text{ and } \alpha \le -1. \end{cases}$$

Then there is a constant $K_{\alpha,\beta,\gamma}$ such that

$$\left|\frac{Z(s,t)}{Z(s,0)}\right| \le K_{\alpha,\beta,\gamma}|t|^{-(\operatorname{re} s+1)}(1+\log|t|)^p$$

for all (s,t) such that $\alpha \le \operatorname{re} s \le \beta$, $|\operatorname{im} s| \le \gamma$, and $|t| \ge 1$.

Proof. For $c \in \mathbb{R}$, let (c) denote the contour $\tau \mapsto c + i\tau$. We have

$$e^{-z} = \frac{1}{2\pi i}\oint_{(c)} \Gamma(w) z^{-w}\,dw$$

for $c > 0$ and $\operatorname{re} z > 0$. By introducing this into (3-2) and changing the order of integration we obtain

$$Z(s,t) = \frac{1}{2\pi i}\oint_{(c)} Z(s-2w,0)\Gamma(w)|t|^{-2w}\,dw, \tag{3-9}$$

when $\operatorname{re} s > -1+2c$ and $t \neq 0$. Since $Z(s,t)$ and the right-hand side of the proposed inequality are even in t, it will suffice to derive the inequality when $t \geq 1$. We henceforth assume that this is so. We also assume for the moment that $d > 1$, $\alpha > -1$, and $\beta - \alpha < 1$. Then we may choose c_1 and c_2 such that

$$0 < c_1 < \frac{\alpha+1}{2} < \frac{\beta+1}{2} < c_2 < \frac{\alpha+2}{2}.$$

If $\alpha \leq \operatorname{re} s \leq \beta$ then

$$c_1 < \frac{\operatorname{re} s+1}{2} < c_2 < \frac{\operatorname{re} s+2}{2}.$$

It follows from Lemma 3.2 and (3-9) that

$$\frac{Z(s,t)}{Z(s,0)} = \frac{1}{2\pi i}\oint_{(c_1)} \frac{\Gamma\left(\frac{s+1}{2}-w\right)\Gamma\left(\frac{s+d}{2}-w\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+d}{2}\right)}\Gamma(w)t^{-2w}\,dw, \tag{3-10}$$

and from this, Cauchy's formula, and the standard estimate on the gamma function in vertical strips, we obtain

$$\frac{Z(s,t)}{Z(s,0)} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{s+d}{2}\right)}t^{-(s+1)} + \frac{1}{2\pi i}\oint_{(c_2)} \frac{\Gamma\left(\frac{s+1}{2}-w\right)\Gamma\left(\frac{s+d}{2}-w\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+d}{2}\right)}\Gamma(w)t^{-2w}\,dw. \tag{3-11}$$

This is valid for all s such that $\alpha \leq \operatorname{re} s \leq \beta$. If we also impose a bound of the form $|\operatorname{im} s| \leq \gamma$ on s then (3-11) implies a uniform bound of the form

$$\left|\frac{Z(s,t)}{Z(s,0)}\right| \ll t^{-(\operatorname{re} s+1)} + t^{-2c_2},$$

and this gives what is required since $2c_2 > \operatorname{re} s + 1$. Every vertical strip of finite width may be covered by a finite number of vertical strips of width at most $1/2$ and this allows us to remove the restriction that $\beta - \alpha < 1$ from this conclusion. The case where $d = 1$ is handled similarly. The salient difference is that, in (3-10), the integrand has a double pole at $w = (s+1)/2$ instead of a simple pole. Thus the residue term in (3-11) includes a factor of $\log t$, and this accounts for the value of p in this case. We have now obtained the required estimate provided that $\alpha > -1$.

To obtain the required estimate for $\alpha \leq -1$, observe that we have

$$\frac{Z(s,t)}{Z(s,0)} = \frac{Z(s,1)}{Z(s,0)} - 2\left(\frac{s+1}{2}\right)\left(\frac{s+d}{2}\right)\int_1^t \tau\,\frac{Z(s+2,\tau)}{Z(s+2,0)}\,d\tau. \tag{3-12}$$

This equality follows from Lemma 3.1, (3-4), Proposition 3.6, and the fundamental theorem of calculus. Equation (3-12) allows us to obtain the required estimate for $\alpha > -3$ from the estimate for $\alpha > -1$. We may then proceed inductively to establish it for any α. □

The verification of (3-3) could be based upon the following result, although we shall use it for a different purpose. The next few results are the least easily generalizable part of the argument, not because a differential operator such as that considered in Lemma 3.8 does not exist in general, but rather because it will be of higher order and hence harder to handle.

Lemma 3.8. *Let $\delta = t\dfrac{\partial}{\partial t}$ be the Euler operator in t. Then the differential operator*

$$t^2(\delta+s+1)(\delta+s+d)+2\delta$$

annihilates $Z(s,t)/Z(s,0)$ for all $s \in \mathbb{C}$.

Proof. Since $Z(s,t)$ is an even function of t, it will suffice to verify this for $t > 0$. Also, we may assume that $\operatorname{re} s > 0$, since the result will follow in general by continuation, in light of Proposition 3.6. Under these assumptions, we have from (3-9) that

$$Z(s,t) = \frac{1}{2\pi i}\oint_{(1/2)} Z(s-2w,0)\Gamma(w)t^{-2w}\,dw.$$

Now $b(s+\delta)t^{-2w} = b(s-2w)t^{-2w}$ and so

$$\begin{aligned} b(s+\delta)Z(s,t) &= \frac{1}{2\pi i}\oint_{(1/2)} b(s-2w)Z(s-2w,0)\Gamma(w)t^{-2w}\,dw \\ &= \frac{1}{2\pi i}\oint_{(1/2)} 4Z(s+2-2w,0)\Gamma(w)t^{-2w}\,dw = 4Z(s+2,t) \end{aligned}$$

by Lemma 3.1. On the other hand,

$$\delta Z(s,t) = t\frac{\partial}{\partial t}Z(s,t) = -2t^2 Z(s+2,t) = -\tfrac{1}{2}t^2 b(s+\delta)Z(s,t)$$

by (3-4), and this is equivalent to the claim. □

Proposition 3.9. *Suppose that d is odd. Then we have*

$$\frac{\partial}{\partial s}\Big(\frac{Z(s,t)}{Z(s,0)}\Big)\Big|_{s=-d} = \Big(\frac{d-1}{2}\Big)!\sum_{j=0}^{(d-3)/2}\frac{1}{(d-1-2j)\,j!}\,t^{d-1-2j}.$$

This should be interpreted as zero when $d = 1$.

Proof. Let $f(s,t) = Z(s,t)/Z(s,0)$ and $F(s,t) = \partial/\partial s f(s,t)$. We seek to evaluate $F(-d,t)$. From Lemma 3.8, we have

$$\bigl(t^2(\delta+s+1)(\delta+s+d)+2\delta\bigr) f(s,t) = 0.$$

By differentiating this relation with respect to s, we obtain

$$\bigl(t^2(\delta+s+1)(\delta+s+d)+2\delta\bigr) F(s,t) + t^2(2\delta+2s+d+1) f(s,t) = 0.$$

We showed in Proposition 3.6 that $f(-d,t) = 1$ for all t. We evaluate the previous relation at $s = -d$ and use this fact to obtain

$$\bigl(t^2(\delta+1-d)\delta+2\delta\bigr) F(-d,t) + t^2(2\delta+1-d)1 = 0,$$

or

$$\bigl(t^2(\delta+1-d)+2\bigr)\bigl(\delta F(-d,t)\bigr) = (d-1)t^2.$$

Let $u = \delta F(-d,t)$. Then u satisfies the differential equation

$$\frac{\partial u}{\partial t} + \Bigl(\frac{1-d}{t} + \frac{2}{t^3}\Bigr) u = \frac{d-1}{t}$$

for $t > 0$ and remains bounded as $t \to 0^+$. It is routine to solve this equation by the method of integrating factors to find that the unique solution that has the required boundedness is $u = 0$ if $d = 1$ and

$$u = \Bigl(\frac{d-1}{2}\Bigr)! \sum_{j=0}^{(d-3)/2} \frac{1}{j!}\, t^{d-1-2j}$$

if $d \geq 3$. Since $\delta F(-d,t) = u$, we conclude that $F(-d,t) = F(-d,0)$ when $d = 1$ and

$$F(-d,t) = F(-d,0) + \Bigl(\frac{d-1}{2}\Bigr)! \sum_{j=0}^{(d-3)/2} \frac{1}{(d-1-2j)\,j!}\, t^{d-1-2j}$$

when $d \geq 3$. However, it is apparent from the definition that $F(-d,0) = 0$, and this concludes the evaluation. □

The same argument as in the proof of Proposition 3.9 would also succeed when d is even. However, in this case the result would not be an elementary function.

Lemma 3.10. *For all* $s_1, s_2 \in \mathbb{C}$ *with* $\operatorname{re} s_1 \geq 0$ *and* $\operatorname{re} s_2 \geq 0$ *we have*

$$a(s_1,s_2) = \frac{Z(s_1+s_2+1,0)}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+m-1)} \int_{-\infty}^{\infty} |t|^{s_1} |t-1|^{s_2} \frac{Z(s_1+s_2+1,t)}{Z(s_1+s_2+1,0)}\, dt.$$

Proof. By definition,

$$a(s_1,s_2) = \frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+m-1)} \int_N |t|^{s_1}\, |t-xy|^{s_2} e^{-(t^2+\|x\|^2+\|y\|^2)}\, d\mu(x,y,t).$$

By excluding the set where $xy=0$ (which is of measure zero) and replacing t by $t(xy)$ in the integral, we obtain

$$a(s_1,s_2)=\frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+m-1)}\int_{-\infty}^{\infty}|t|^{s_1}\,|t-1|^{s_2}Z(s_1+s_2+1,t)dt.$$

This is equivalent to what is stated. □

We shall use the standard notation $\psi(w)=\dfrac{\Gamma'(w)}{\Gamma(w)}$ for the logarithmic derivative of the gamma function.

Lemma 3.11. *Let* $w_1, w_2\in\mathbb{C}$ *with* $\operatorname{re} w_1>-1$ *and* $\operatorname{re}(w_1+w_2)<-1$. *Then*

$$\int_0^\infty t^{w_1}(1+t)^{w_2}\,dt=\frac{\Gamma(1+w_1)\Gamma(-(w_1+w_2+1))}{\Gamma(-w_2)},$$

$$\int_0^\infty t^{w_1}(1+t)^{w_2}\log t\,dt=\big(\psi(1+w_1)-\psi(-(w_1+w_2+1))\big)\times\frac{\Gamma(1+w_1)\Gamma(-(w_1+w_2+1))}{\Gamma(-w_2)},$$

$$\int_0^\infty t^{w_1}(1+t)^{w_2}\log(1+t)\,dt=\big(\psi(-w_2)-\psi(-(w_1+w_2+1))\big)\times\frac{\Gamma(1+w_1)\Gamma(-(w_1+w_2+1))}{\Gamma(-w_2)}.$$

Proof. The first formula follows from entry 3.194.3 in [Gradshteyn and Ryzhik 2000]. The other two result from the first by differentiation. □

Theorem 3.12. *We have*

$$a_+(z)=\begin{cases}2^{d+1}\pi^{d-1}\sin(\pi z) & \text{if } d \text{ is even},\\ 2^{d+1}\pi^{d-1}\big(\sin(\pi z)-(-1)^{(d-1)/2}\big) & \text{if } d \text{ is odd},\end{cases}$$

and

$$a_-(z)=\begin{cases}2^{d+1}\pi^{d-1}\sin(\pi z) & \text{if } d \text{ is even},\\ 2^{d+1}\pi^{d-1}\big(\sin(\pi z)+(-1)^{(d-1)/2}\big) & \text{if } d \text{ is odd}.\end{cases}$$

Proof. We shall calculate $a_-(z)$. Since we know that $a_+(z)=-a_-(-z)$ by Theorem 2.9, this will suffice. From (3-1), our goal is to make the specialization $s_1=z-\frac{1}{2}d-1$ and $s_2=-z-\frac{1}{2}d$ in $a(s_1,s_2)$. For this purpose, we use the expression for $a(s_1,s_2)$ given in Lemma 3.10. Of course, this is not directly possible, since s_1 and s_2 cannot both have nonnegative real parts simultaneously. The strategy is to break $a(s_1,s_2)$ into pieces each of which may be evaluated for some z and whose continuation may thereby be determined. The reason this strategy can succeed is that, for the desired specialization, we have $s_1+s_2+1=-d$ and hence

$$\frac{Z(s_1+s_2+1,t)}{Z(s_1+s_2+1,0)}=1$$

by the last part of Proposition 3.6. This leaves relatively elementary integrals to be evaluated. We begin by writing

$$a(s_1, s_2) = a_1(s_1, s_2) + a_2(s_1, s_2) + a_3(s_1, s_2),$$

with

$$a_1(s_1, s_2) = \frac{Z(s_1+s_2+1, 0)}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+d+1)} \int_{-\infty}^{0} |t|^{s_1} |t-1|^{s_2} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\, dt,$$

and $a_2(s_1, s_2)$ and $a_3(s_1, s_2)$ being given by similar expressions with the range of integration being 0 to 1 for $a_2(s_1, s_2)$ and 1 to ∞ for $a_3(s_1, s_2)$.

We have

$$a_1(s_1, s_2) = \frac{Z(s_1+s_2+1, 0)}{\Gamma(s_2+1)\Gamma(s_1+s_2+d+1)} A_1(s_1, s_2),$$

with

$$A_1(s_1, s_2) = \frac{1}{\Gamma(s_1+1)} \int_{-\infty}^{0} |t|^{s_1} |t-1|^{s_2} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\, dt.$$

By Proposition 3.7, the integral in the definition of $A_1(s_1, s_2)$ converges provided that $\operatorname{re} s_1 > -1$. In fact, $A_1(s_1, s_2)$ extends to an entire function. To see this, assume that $\operatorname{re} s_1 > -1$. Then we have

$$\begin{aligned} A_1(s_1, s_2) &= \frac{1}{\Gamma(s_1+2)} \int_{-\infty}^{0} \Big(\frac{\partial}{\partial t} |t|_-^{s_1+1}\Big) |t-1|_+^{s_2} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\, dt \\ &= -\frac{1}{\Gamma(s_1+2)} \int_{-\infty}^{0} |t|_-^{s_1+1} \frac{\partial}{\partial t}\Big(|t-1|_+^{s_2} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\Big)\, dt \\ &= -\frac{s_2}{\Gamma(s_1+2)} \int_{-\infty}^{0} |t|_-^{s_1+1} |t-1|_-^{s_2-1} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\, dt \\ &\qquad\qquad + \frac{2}{\Gamma(s_1+2)} \int_{-\infty}^{0} |t|_+^{s_1+2} |t-1|_+^{s_2} \frac{Z(s_1+s_2+3, t)}{Z(s_1+s_2+1, 0)}\, dt \\ &= -\frac{s_2}{\Gamma(s_1+2)} \int_{-\infty}^{0} |t|_+^{s_1+1} |t-1|_+^{s_2-1} \frac{Z(s_1+s_2+1, t)}{Z(s_1+s_2+1, 0)}\, dt \\ &\qquad + \frac{(s_1+2)b(s_1+s_2+1)}{2\Gamma(s_1+3)} \int_{-\infty}^{0} |t|_+^{s_1+2} |t-1|_+^{s_2} \frac{Z(s_1+s_2+3, t)}{Z(s_1+s_2+3, 0)}\, dt \\ &= -s_2 A_1(s_1+1, s_2-1) + \tfrac{1}{2}(s_1+2) b(s_1+s_2+1) A_1(s_1+2, s_2). \end{aligned}$$

The main steps in this calculation rely on Lemma 3.1 and on (3-4). This recurrence relation establishes that $A_1(s_1, s_2)$ extends to an entire function.

Now suppose that $s_1 + s_2 + 1 = -d$ with $\operatorname{re} s_1 > -1$. Then, by the last part of Proposition 3.6, we have

$$A_1(s_1,s_2)=\frac{1}{\Gamma(s_1+1)}\int_{-\infty}^{0}|t|^{s_1}\,|t-1|^{s_2}\,dt=\frac{1}{\Gamma(s_1+1)}\int_{0}^{\infty}t^{s_1}(t+1)^{s_2}\,dt$$
$$=\frac{1}{\Gamma(s_1+1)}\frac{\Gamma(s_1+1)\,\Gamma(-s_1-s_2-1)}{\Gamma(-s_2)}=\frac{\Gamma(d)}{\Gamma(-s_2)},$$

where we have used Lemma 3.11 to evaluate the integral. Notice that this evaluation, initially obtained under the assumption that $\operatorname{re} s_1>-1$, is valid on the whole affine plane $s_1+s_2+1=-d$, since both sides are known to be entire on this plane. Remarks such as this will be taken for granted henceforth.

When d is odd, we also need to determine the partial derivative $\partial A_1/\partial s_1$ at a point on the affine plane $s_1+s_2+1=-d$. Assume for the moment that d is odd. In the region of convergence of the original definition of $A_1(s_1,s_2)$, we may differentiate under the integral to obtain

$$\frac{\partial A_1}{\partial s_1}(s_1,s_2)=-\psi(s_1+1)A_1(s_1,s_2)$$
$$+\frac{1}{\Gamma(s_1+1)}\int_{-\infty}^{0}|t|^{s_1}\,|t-1|^{s_2}\log|t|\frac{Z(s_1+s_2+1,t)}{Z(s_1+s_2+1,0)}\,dt$$
$$+\frac{1}{\Gamma(s_1+1)}\int_{-\infty}^{0}|t|^{s_1}\,|t-1|^{s_2}\frac{\partial}{\partial s}\left(\frac{Z(s,t)}{Z(s,0)}\right)\Big|_{s=s_1+s_2+1}dt.$$

Using Proposition 3.9, it follows that at a point on the affine plane $s_1+s_2+1=-d$ with $\operatorname{re} s_1>-1$, we have

$$\frac{\partial A_1}{\partial s_1}(s_1,s_2)=-\frac{\psi(s_1+1)\Gamma(d)}{\Gamma(-s_2)}+\frac{1}{\Gamma(s_1+1)}\int_{0}^{\infty}t^{s_1}(1+t)^{s_2}\log t\,dt$$
$$+\left(\frac{d-1}{2}\right)!\,\frac{1}{\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{1}{(d-1-2j)\,j!}\int_{0}^{\infty}t^{s_1+d-1-2j}(1+t)^{s_2}dt.$$

The integrals that appear here may be evaluated using Lemma 3.11 and we find that when d is odd and (s_1,s_2) is a point on the affine plane $s_1+s_2+1=-d$ then we have

$$\frac{\partial A_1}{\partial s_1}(s_1,s_2)=-\frac{\psi(d)\Gamma(d)}{\Gamma(-s_2)}+\left(\frac{d-1}{2}\right)!\frac{1}{\Gamma(-s_2)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\frac{\Gamma(s_1+d-2j)}{\Gamma(s_1+1)}.$$

Note that the form in which this partial derivative has been written makes it clear that it is entire on the affine plane.

Next we consider $a_3(s_1,s_2)$. We have

$$a_3(s_1,s_2)=\frac{Z(s_1+s_2+1,0)}{\Gamma(s_1+1)\Gamma(s_1+s_2+d+1)}A_3(s_1,s_2),$$

with

$$A_3(s_1,s_2) = \frac{1}{\Gamma(s_2+1)} \int_1^\infty |t|^{s_1}\,|t-1|^{s_2} \frac{Z(s_1+s_2+1,t)}{Z(s_1+s_2+1,0)}\,dt.$$

By Proposition 3.7, the integral converges when $\operatorname{re} s_2 > -1$. We may derive a recurrence relation for $A_3(s_1,s_2)$, just as we did for $A_1(s_1,s_2)$, and thus conclude that $A_3(s_1,s_2)$ extends to an entire function. This done, we prefer to replace t by $1/t$ in the integral defining $A_3(s_1,s_2)$ so as to write

$$A_3(s_1,s_2) = \frac{1}{\Gamma(s_2+1)} \int_0^1 t^{-s_1-s_2-2}(1-t)^{s_2} \frac{Z(s_1+s_2+1,t^{-1})}{Z(s_1+s_2+1,0)}\,dt.$$

From this, we deduce as before that if (s_1,s_2) is a point on the affine plane $s_1+s_2+1=-d$ then

$$A_3(s_1,s_2) = \frac{1}{\Gamma(s_2+1)} \int_0^1 t^{d-1}(1-t)^{s_2}\,dt = \frac{\Gamma(d)}{\Gamma(-s_1)},$$

where we have used Euler's beta integral.

When d is odd, we also need to determine the partial derivative $\partial A_3/\partial s_1$ at a point on the affine plane $s_1+s_2+1=-d$. In the region of convergence, we have

$$\begin{aligned}\frac{\partial A_3}{\partial s_1}(s_1,s_2) = &-\frac{1}{\Gamma(s_2+1)} \int_0^1 t^{-s_1-s_2-2}(1-t)^{s_2}\log(t) \frac{Z(s_1+s_2+1,t^{-1})}{Z(s_1+s_2+1,0)}\,dt \\ &+ \frac{1}{\Gamma(s_2+1)} \int_0^1 t^{-s_1-s_2-2}(1-t)^{s_2} \frac{\partial}{\partial s}\Big(\frac{Z(s,t^{-1})}{Z(s,0)}\Big)\Big|_{s=s_1+s_2+1}\,dt.\end{aligned}$$

Using Proposition 3.9, it follows that at a point on the affine plane $s_1+s_2+1=-d$ with $\operatorname{re} s_2 > -1$, we have

$$\begin{aligned}\frac{\partial A_3}{\partial s_1}(s_1,s_2) = &-\frac{1}{\Gamma(s_2+1)} \int_0^1 t^{d-1}(1-t)^{s_2}\log t\,dt \\ &+ \Big(\frac{d-1}{2}\Big)!\,\frac{1}{\Gamma(s_2+1)} \sum_{j=0}^{(d-3)/2} \frac{1}{(d-1-2j)\,j!} \int_0^1 t^{2j}(1-t)^{s_2}\,dt.\end{aligned}$$

By taking the derivative of Euler's beta integral, we obtain

$$\int_0^1 t^{w_1}(1-t)^{w_2}\log t\,dt = \big(\psi(w_1+1)-\psi(w_1+w_2+2)\big)\frac{\Gamma(w_1+1)\Gamma(w_2+1)}{\Gamma(w_1+w_2+2)}$$

under the same convergence restrictions that apply to the beta integral itself. Thus if d is odd and (s_1,s_2) is a point on the affine plane $s_1+s_2+1=-d$ then

$$\frac{\partial A_3}{\partial s_1}(s_1,s_2) = \frac{(\psi(-s_1)-\psi(d))\Gamma(d)}{\Gamma(-s_1)} + \Big(\frac{d-1}{2}\Big)! \sum_{j=0}^{(d-3)/2} \frac{(2j)!}{(d-1-2j)\,j!}\,\frac{1}{\Gamma(s_2+2j+2)}.$$

We are now ready to evaluate $a_{13}(s_1, s_2) = a_1(s_1, s_2) + a_3(s_1, s_2)$ at a point on the affine plane $s_1 + s_2 + 1 = -d$. We shall eventually have to distinguish cases based on the parity of d. The reason for grouping a_1 and a_3 together will become apparent when d is odd. We have

$$a_{13}(s_1, s_2) = \frac{Z(s_1+s_2+1, 0)}{\Gamma(s_1+s_2+d+1)} A_{13}(s_1, s_2),$$

where we define $A_{13}(s_1, s_2)$ to be

$$A_{13}(s_1, s_2) = \frac{1}{\Gamma(s_2+1)} A_1(s_1, s_2) + \frac{1}{\Gamma(s_1+1)} A_3(s_1, s_2).$$

By using Lemma 3.2, the duplication formula for the gamma function, and the evaluation $\Gamma(1/2) = \sqrt{\pi}$, we find that

$$\text{(3-13)} \qquad \frac{Z(s_1+s_2+1, 0)}{\Gamma(s_1+s_2+d+1)} = \frac{\pi^d}{2^{s_1+s_2+d}\Gamma\left(\frac{d}{2}\right)} \cdot \frac{\Gamma\left(\frac{s_1+s_2+2}{2}\right)}{\Gamma\left(\frac{s_1+s_2+d+2}{2}\right)}.$$

At a point (s_1, s_2) on the affine plane $s_1+s_2+1 = -d$, we use the above evaluations of $A_1(s_1, s_2)$ and $A_3(s_1, s_2)$ to see that

$$A_{13}(s_1, s_2) = \frac{\Gamma(d)}{\Gamma(-s_2)\Gamma(s_2+1)} + \frac{\Gamma(d)}{\Gamma(-s_1)\Gamma(s_1+1)}.$$

The reflection formula for the gamma function allows us to reexpress this as

$$A_{13}(s_1, s_2) = -\frac{\Gamma(d)}{\pi}(\sin(\pi s_1) + \sin(\pi s_2)),$$

and a trigonometric identity lets us write it in the more convenient form

$$\text{(3-14)} \qquad A_{13}(s_1, s_2) = \frac{2\Gamma(d)}{\pi} \sin\left(\tfrac{\pi}{2}(d+1)\right) \cos\left(\tfrac{\pi}{2}(s_1 - s_2)\right).$$

Assume now that the integer d is even and that (s_1, s_2) is a point on the affine plane $s_1 + s_2 + 1 = -d$. By combining (3-13) and (3-14), we obtain

$$a_{13}(s_1, s_2) = (-1)^{d/2} \frac{4\pi^{d-1}\Gamma(d)\Gamma\left(\frac{1-d}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} \cos\left(\tfrac{\pi}{2}(s_1 - s_2)\right).$$

A calculation making use of the reflection and duplication formulas for the gamma function reveals that this simplifies to

$$\text{(3-15)} \qquad a_{13}(s_1, s_2) = 2^{d+1}\pi^{d-1} \cos\left(\tfrac{\pi}{2}(s_1 - s_2)\right).$$

This completes the evaluation of this term when d is even.

Assume now that d is odd. Then (3-14) reveals that $A_{13} = 0$ on the affine plane $s_1+s_2+1 = -d$. Also, from (3-13), the factor $Z(s_1+s_2+1, 0)/\Gamma(s_1+s_2+d+1)$ has a simple polar divisor along this plane. Thus, in order to evaluate $a_{13}(s_1, s_2)$

at a point (s_1, s_2) on the affine plane $s_1+s_2+1=-d$, we must reexpress it in the form

$$a_{13}(s_1, s_2) = L_1(s_1, s_2)L_2(s_1, s_2),$$

where

$$L_1(s_1, s_2) = \lim_{(w_1,w_2)\to(s_1,s_2)} \Big((w_1+w_2+d+1)\cdot \frac{Z(w_1+w_2+1, 0)}{\Gamma(w_1+w_2+d+1)}\Big),$$

$$L_2(s_1, s_2) = \lim_{(w_1,w_2)\to(s_1,s_2)} \frac{A_{13}(w_1, w_2)}{w_1+w_2+d+1}.$$

By using (3-13) and the reflection and duplication formulas for the gamma function, one finds that

$$L_1(s_1, s_2) = (-1)^{(d-1)/2}\frac{2^{d+1}\pi^{d-1}}{\Gamma(d)}.$$

On the other hand, since A_{13} vanishes along the affine plane $s_1+s_2+1=-d$, we have

$$L_2(s_1, s_2) = \frac{\partial A_{13}}{\partial s_1}(s_1, s_2).$$

From the definition of A_{13}, we have

$$\frac{\partial A_{13}}{\partial s_1}(s_1, s_2) = \frac{1}{\Gamma(s_2+1)}\frac{\partial A_1}{\partial s_1}(s_1, s_2) - \frac{\psi(s_1+1)}{\Gamma(s_1+1)}A_3(s_1, s_2) + \frac{1}{\Gamma(s_1+1)}\frac{\partial A_3}{\partial s_1}(s_1, s_2), \tag{3-16}$$

and each of the terms in this expression has been evaluated above. Since the computation is slightly involved, we shall simplify the result of substituting these evaluations into (3-16) in three pieces. The first of these accounts for terms that do not appear in the scope of the summation signs in the evaluation of $\partial A_1/\partial s_1$ and $\partial A_3/\partial s_1$ above; it equals

$$\begin{aligned}
-\frac{1}{\Gamma(s_2+1)}\frac{\psi(d)\Gamma(d)}{\Gamma(-s_2)} - \frac{\psi(s_1+1)}{\Gamma(s_1+1)}\frac{\Gamma(d)}{\Gamma(-s_1)} + \frac{1}{\Gamma(s_1+1)}\frac{\big(\psi(-s_1)-\psi(d)\big)\Gamma(d)}{\Gamma(-s_1)}& \\
= -\psi(d)\Big(\frac{\Gamma(d)}{\Gamma(s_2+1)\Gamma(-s_2)} + \frac{\Gamma(d)}{\Gamma(s_1+1)\Gamma(-s_1)}\Big)& \\
+\Gamma(d)\frac{\psi(-s_1)-\psi(s_1+1)}{\Gamma(s_1+1)\Gamma(-s_1)}& \\
= -\psi(d)A_{13}(s_1, s_2) - \Gamma(d)\frac{\psi(s_1+1)-\psi(-s_1)}{\Gamma(s_1+1)\Gamma(-s_1)}& \\
= -\Gamma(d)\frac{\psi(s_1+1)-\psi(-s_1)}{\Gamma(s_1+1)\Gamma(-s_1)} = -\Gamma(d)\cos(\pi s_1),&
\end{aligned}$$

since $A_{13}(s_1, s_2)=0$ and we find that

$$\frac{\psi(1-w)-\psi(w)}{\Gamma(w)\Gamma(1-w)} = \cos(\pi w)$$

by differentiating the reciprocal of the reflection formula. The second piece is the result of substituting the summation from $\partial A_1/\partial s_1$ above. It is

$$\Big(\frac{d-1}{2}\Big)!\,\frac{1}{\Gamma(-s_2)\Gamma(s_2+1)\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\Gamma(s_1+d-2j).$$

Recall that we are considering a point (s_1, s_2) such that $s_1+s_2+1=-d$. Thus $s_1+d=-s_2-1$ and so this piece of the sum is

$$\begin{aligned}
&\Big(\frac{d-1}{2}\Big)!\frac{1}{\Gamma(-s_2)\Gamma(s_2+1)\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\Gamma(-s_2-1-2j)\\
&=-\Big(\frac{d-1}{2}\Big)!\frac{\sin(\pi s_2)}{\pi\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\Gamma\big(1-(s_2+2j+2)\big)\\
&=-\Big(\frac{d-1}{2}\Big)!\frac{\sin(\pi s_2)}{\pi\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\frac{\pi}{\Gamma(s_2+2j+2)\sin(\pi(s_2+2j+2))}\\
&=-\Big(\frac{d-1}{2}\Big)!\frac{\sin(\pi s_2)}{\pi\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\frac{\pi}{\Gamma(s_2+2j+2)\sin(\pi s_2)}\\
&=-\Big(\frac{d-1}{2}\Big)!\frac{1}{\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\frac{1}{\Gamma(s_2+2j+2)},
\end{aligned}$$

where we have used the reflection formula twice. The last piece of the sum arises from the summation in the above evaluation for $\partial A_3/\partial s_1$. It is

$$\Big(\frac{d-1}{2}\Big)!\,\frac{1}{\Gamma(s_1+1)}\sum_{j=0}^{(d-3)/2}\frac{(2j)!}{(d-1-2j)\,j!}\frac{1}{\Gamma(s_2+2j+2)},$$

hence it cancels the second piece of the sum. Thus $L_2(s_1, s_2) = -\Gamma(d)\cos(\pi s_1)$ and it follows that when the integer d is odd and (s_1, s_2) is a point on the affine plane $s_1+s_2+1=-d$, we have

$$a_{13}(s_1,s_2) = (-1)^{(d+1)/2}2^{d+1}\pi^{d-1}\cos(\pi s_1). \tag{3-17}$$

This completes the evaluation of this term when d is odd.

It remains to consider the term $a_2(s_1, s_2)$, which is given by

$$a_2(s_1,s_2)=\frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+d+1)}\int_0^1 t^{s_1}(1-t)^{s_2}Z(s_1+s_2+1,t)\,dt,$$

provided that $\operatorname{re} s_1 > -1$ and $\operatorname{re} s_2 > -1$. Let us write $s = s_1 + s_2 + 1$ and choose l so that $-d + 2l + 2 > -1$. By (3-6), we have

$$Z(s,t)=\sum_{j=0}^{l}\frac{(-1)^j}{j!}t^{2j}Z(s+2j,0)+\frac{(-1)^{l+1}}{l!}t^{2l+2}\int_0^1(1-v)^l Z(s+2l+2,t\sqrt{v})\,dv,$$

and so

$$a_2(s_1,s_2)=\frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s+d)}\sum_{j=0}^{l}\frac{(-1)^j}{j!}Z(s+2j,0)\times\int_0^1 t^{s_1+2j}(1-t)^{s_2}\,dt+F(s_1,s_2),$$

where

$$F(s_1,s_2)=\frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s+d)}\times\frac{(-1)^{l+1}}{l!}\int_0^1\int_0^1 t^{s_1+2l+2}(1-t)^{s_2}(1-v)^l Z(s+2l+2,t\sqrt{v})\,dv\,dt.$$

The function $s \mapsto Z(s+2l+2, t\sqrt{v})$ is holomorphic where $\operatorname{re} s > -2l-3$. Thus F is holomorphic on the domain where $\operatorname{re} s_1 > -2l-3$ and $\operatorname{re} s_2 > -1$. (Note that these inequalities imply that $\operatorname{re} s > -2l-3$ also.) The set of (s_1, s_2) such that $\operatorname{re} s_1 > -2l-3$ and $\operatorname{re} s_2 > -1$ contains a nonempty open subset of the affine plane $s = -d$. On this open subset of the affine plane $s = -d$ we have $F(s_1, s_2) = 0$, because of $\Gamma(s+d)$ factor in the above expression for $F(s_1, s_2)$. It suffices to evaluate $a_2(s_1, s_2)$ on this open set, and it follows that we may ignore F in doing so. By using the beta integral and Lemma 3.2, we find that the potentially significant part of $a_2(s_1, s_2)$ is equal to

$$\begin{aligned}
&\frac{\pi^d}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\sum_{j=0}^{l}\frac{(-1)^j}{j!}\frac{\Gamma((s+2j+1)/2)}{\Gamma(s+2j+1)}\frac{\Gamma((s+d)/2+j)}{\Gamma(s+d)}\frac{\Gamma(s_1+2j+1)}{\Gamma(s_1+1)}\\
&\quad=\frac{\pi^d}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\sum_{j=0}^{l}\frac{(-1)^j}{j!}\left(\frac{s+d}{2}\right)_j(s_1+1)_{2j}\frac{\Gamma((s+2j+1)/2)}{\Gamma(s+2j+1)}\frac{\Gamma((s+d)/2)}{\Gamma(s+d)}\\
&\quad=\frac{2^{1-2s-d}\pi^{d+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\sum_{j=0}^{l}\frac{(-1)^j2^{-2j}}{j!}\left(\frac{s+d}{2}\right)_j(s_1+1)_{2j}\frac{1}{\Gamma\left(\frac{s+2j+2}{2}\right)\Gamma\left(\frac{s+d+1}{2}\right)},
\end{aligned}$$

where we have used the duplication formula in the form

$$\frac{\Gamma\left(\frac{w}{2}\right)}{\Gamma(w)}=\frac{2^{1-w}\sqrt{\pi}}{\Gamma\left(\frac{w+1}{2}\right)}$$

in order to reach the last line. When we evaluate this at $s = -d$, all terms but the $j = 0$ term vanish, because of the $((s+d)/2)_j$ factor. Thus at $s = -d$ the expression becomes

$$\frac{2^{d+1}\pi^{d+1}}{\Gamma\left(\frac{1}{2}\right)^2\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}=2^{d+1}\pi^{d-1}\sin\Bigl(\frac{\pi d}{2}\Bigr)$$

by the reflection formula. Thus if d is even and (s_1, s_2) is a point on the affine plane $s_1+s_2+1=-d$ then $a_2(s_1, s_2)=0$, whereas if d is odd and (s_1, s_2) is a point on the affine plane $s_1+s_2+1=-d$ then

$$a_2(s_1, s_2)=(-1)^{(d-1)/2}2^{d+1}\pi^{d-1}.$$

This leads to

$$a(s_1, s_2)=\begin{cases}2^{d+1}\pi^{d-1}\cos\left(\frac{\pi}{2}(s_1-s_2)\right) & \text{if } d \text{ is even,}\\ (-1)^{(d-1)/2}2^{d+1}\pi^{d-1}\bigl(1-\cos(\pi s_1)\bigr) & \text{if } d \text{ is odd.}\end{cases}$$

Now $a_-(z)=a(z-\frac{1}{2}d-1, -z-\frac{1}{2}d)$ and so

$$a_-(z)=\begin{cases}2^{d+1}\pi^{d-1}\sin(\pi z) & \text{if } d \text{ is even,}\\ 2^{d+1}\pi^{d-1}\bigl(\sin(\pi z)+(-1)^{(d-1)/2}\bigr) & \text{if } d \text{ is odd.}\end{cases}$$

Now recall that $a_+(z)=-a_-(-z)$ to obtain the other result. □

Our final goal in this section is to show that the entire functions c_+ and c_- that appear in Theorem 2.9 are, in fact, identically zero. We shall thus assume henceforth that $d \geq 2$ is even.

Lemma 3.13. *Let $\Phi \in \mathcal{S}(V)$ be given by*

$$\Phi(x, y)=(xy)^{d-1}e^{-(\|x\|^2+\|y\|^2)}.$$

Then

$$(\Delta^d F_+)(\Phi)=-2\binom{d-1}{d/2}d!\,(d-2)!\left(\frac{\pi}{2}\right)^d.$$

Proof. Note that $(|xy|_+^0+|xy|_-^0)/2$ is the characteristic function of V_+. From this we obtain

$$(\Delta^d F_+)(\Phi)=\frac{1}{2}\,\frac{d}{ds}\Big|_{s=0}\int_V(\Delta^d\Phi)(x, y)\bigl(|xy|_+^s+|xy|_-^s\bigr)\,d\mu(x, y)$$

for any $\Phi \in \mathcal{S}(V)$. By integration by parts and the b-function relation, we obtain

$$\int_V(\Delta^d\Phi)(x, y)\bigl(|xy|_+^s+|xy|_-^s\bigr)\,d\mu(x, y)$$
$$=\Bigl(\prod_{j=1}^{d}b(s-j)\Bigr)\int_V\Phi(x, y)\bigl(|xy|_+^{s-d}+|xy|_-^{s-d}\bigr)\,d\mu(x, y),$$

still for any $\Phi \in \mathcal{S}(V)$. On specializing to the Φ in the statement, we obtain

$$\begin{aligned}\int_V \Phi(x, y)\big(|xy|_+^{s-d} + |xy|_-^{s-d}\big)\, d\mu(x, y) \\ &= \int_V \big(|xy|_-^{s-1} + |xy|_+^{s-1}\big) e^{-(\|x\|^2+\|y\|^2)}\, d\mu(x, y) \\ &= \int_V |xy|_+^{s-1} e^{-(\|x\|^2+\|y\|^2)}\, d\mu(x, y) = Z(s-1, 0),\end{aligned}$$

so that

$$(\Delta^d F_+)(\Phi) = \frac{1}{2}\frac{d}{ds}\Big|_{s=0}\Big(Z(s-1, 0)\cdot \prod_{j=1}^{d} b(s-j)\Big).$$

Now

$$Z(s-1, 0)\cdot \prod_{j=1}^{d} b(s-j) = 4Z(s+1, 0)\cdot s(s+1-d)\cdot \prod_{j=2}^{d-1} b(s-j),$$

from which it follows that

$$(\Delta^d F_+)(\Phi) = 2Z(1, 0)(1-d)\prod_{j=2}^{d-1} b(-j).$$

By writing out the product more explicitly and recalling that d is even, it is easy to see that

$$(1-d)\prod_{j=2}^{d-1} b(-j) = -(d-1)!(d-2)!.$$

From Lemma 3.2,

$$Z(1, 0) = \pi^d \frac{\Gamma(1)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)} = \frac{\pi^d}{\left(\frac{d-2}{2}\right)!}\left(\frac{1}{2}\right)_{d/2} = \frac{\pi^d d!}{2^d\left(\frac{d-2}{2}\right)!\left(\frac{d}{2}\right)!},$$

and combining all these factors gives the stated result. □

It will be helpful to introduce the abbreviation

$$\kappa_d = -2\binom{d-1}{d/2} d!(d-2)!\left(\frac{\pi}{2}\right)^d$$

for the value found in Lemma 3.13. All that actually matters about κ_d is that it is not zero.

Theorem 3.14. *The functions c_+ and c_- are identically zero.*

Proof. Let $\Psi \in \mathcal{S}(N)$ be the function

$$\Psi(x, y, t) = (xy)^{d-1} e^{-(t^2+\|x\|^2+\|y\|^2)}$$

and define

$$c(s_1, s_2, \varepsilon) = T(s_1, s_2, -\varepsilon, \varepsilon)(\Psi),$$

where $\varepsilon \in \{\pm\}$. The function $(s_1, s_2) \mapsto c(s_1, s_2, \varepsilon)$ is entire and we have

$$\begin{aligned} c\bigl(z - \tfrac{1}{2}d - 1, -z - \tfrac{1}{2}d, \varepsilon\bigr) &= T\bigl(z - \tfrac{1}{2}d - 1, -z - \tfrac{1}{2}d, -\varepsilon, \varepsilon\bigr)(\Psi) \\ &= S(z, \varepsilon, \varepsilon)(\Psi) = D(z, \varepsilon, \varepsilon)\bigl(\Psi|_V\bigr) \\ &= c_\varepsilon(z)\bigl(\Delta^d F_+ - \tfrac{1}{2}c_2\delta_0\bigr)(\Phi) \\ &= c_\varepsilon(z)(\Delta^d F_+)(\Phi) = \kappa_d c_\varepsilon(z), \end{aligned}$$

where $\Phi = \Psi|_V$ is the Schwartz function used in Lemma 3.13. Here we have used (2-4) from the first line to the second, the definition of D and Theorem 2.9 from the second line to the third, and then Lemma 3.13. It follows that

$$c_\varepsilon(z) = \kappa_d^{-1} c\bigl(z - \tfrac{1}{2}d - 1, -z - \tfrac{1}{2}d, \varepsilon\bigr), \tag{3-18}$$

and this identity will be the basis for evaluating $c_\varepsilon(z)$. By a calculation similar to that used to prove Lemma 3.10, we find that if $\operatorname{re} s_1 \geq 0$ and $\operatorname{re} s_2 \geq 0$ then

$$c(s_1, s_2, \varepsilon) = \frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_1+s_2+d+1)} \int_{-\infty}^{\infty} |t|_{-\varepsilon}^{s_1} |t-1|_{\varepsilon}^{s_2} Z(s_1+s_2+d, t)\, dt.$$

As before, we break this integral into three pieces $c_1(s_1, s_2, \varepsilon)$, $c_2(s_1, s_2, \varepsilon)$, and $c_3(s_1, s_2, \varepsilon)$ with the ranges of integration being from $-\infty$ to 0, from 0 to 1, and from 1 to ∞, respectively. The pieces can be analyzed separately.

We have

$$c_1(s_1, s_2, \varepsilon) = \frac{Z(s, 0)}{\Gamma(s_2+1)\Gamma(s+1)} C_1(s_1, s_2, \varepsilon),$$

with

$$C_1(s_1, s_2, \varepsilon) = \frac{1}{\Gamma(s_1+1)} \int_{-\infty}^{0} |t|_{-\varepsilon}^{s_1} |t-1|_{\varepsilon}^{s_2} \frac{Z(s, t)}{Z(s, 0)}\, dt,$$

where we have introduced the abbreviation $s = s_1 + s_2 + d$ that will be used throughout the proof. As in the proof of Theorem 3.12, the integral defining $C_1(s_1, s_2, \varepsilon)$ converges provided that $\operatorname{re} s_1 > -1$ and $C_1(s_1, s_2, \varepsilon)$ extends from this region to be an entire function. At a point on the affine plane $s = -1$ with $\operatorname{re} s_1 > -1$ we have

$$\begin{aligned} C_1(s_1, s_2, \varepsilon) &= -\frac{1}{\Gamma(s_1+1)} \int_0^{\infty} t^{s_1}(1+t)^{s_2}\, dt \\ &= -\frac{1}{\Gamma(s_1+1)} \cdot \frac{\Gamma(s_1+1)\Gamma(d)}{\Gamma(-s_2)} = -\frac{\Gamma(d)}{\Gamma(-s_2)}, \end{aligned}$$

since $Z(-1, t)/Z(-1, 0) = 1$ for all t by Proposition 3.6. Note that we have used Lemma 3.11 to evaluate the integral. By using the duplication and reflection formulas for the gamma function as before, we conclude that if (s_1, s_2) is a point on the affine plane $s = -1$ then

$$c_1(s_1, s_2, \varepsilon) = 2\pi^{d-1} \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sin(\pi s_2).$$

Similarly,

$$c_3(s_1, s_2, \varepsilon) = \frac{Z(s,0)}{\Gamma(s_1+1)\Gamma(s+1)} C_3(s_1, s_2, \varepsilon),$$

with

$$C_3(s_1, s_2, \varepsilon) = \frac{1}{\Gamma(s_2+1)} \int_1^\infty t^{s_1}(t-1)^{s_2} \frac{Z(s,t)}{Z(s,0)}\, dt.$$

At a point on the affine plane $s = -1$, we obtain the evaluation

$$c_3(s_1, s_2, \varepsilon) = -2\pi^{d-1} \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sin(\pi s_1)$$

from this and Euler's beta integral. At a point (s_1, s_2) on the affine plane $s = -1$, we have

$$\sin(\pi s_2) - \sin(\pi s_1) = 2\sin\left(\tfrac{\pi}{2}(s_2 - s_1)\right)\cos\left(\tfrac{\pi}{2}(d+1)\right) = 0$$

since d is even, and so at such a point

$$c_1(s_1, s_2, \varepsilon) + c_3(s_1, s_2, \varepsilon) = 0.$$

It remains to show that $c_2(s_1, s_2, \varepsilon)$ is also zero on the affine plane $s = -1$. As before, it is sufficient to show that it is zero on some nonempty open subset of this affine plane. Provided that $\operatorname{re} s_1 > -1$ and $\operatorname{re} s_2 > -1$, we have

$$c_2(s_1, s_2, \varepsilon) = \frac{\varepsilon}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s+1)} \int_0^1 t^{s_1}(1-t)^{s_2} Z(s,t)\, dt.$$

Let l be a large natural number. By introducing (3-6) into this identity, we obtain

$$\text{(3-19)} \quad c_2(s_1, s_2, \varepsilon) = \frac{\varepsilon}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s+1)} \sum_{j=0}^{l} \frac{(-1)^j}{j!} Z(s+2j, 0) \int_0^1 t^{s_1+2j}(1-t)^{s_2} dt + \varepsilon F(s_1, s_2),$$

where

$$F(s_1, s_2) = \frac{(-1)^{l+1}}{l!\,\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s+1)} \int_0^1 \int_0^1 t^{s_1+2l+2}(1-t)^{s_2}(1-v)^l Z(s+2l+2, t\sqrt{v})\, dv\, dt.$$

Suppose that $\operatorname{re} s_1 > -2l-3$ and $\operatorname{re} s_2 > -1$. Then $\operatorname{re}(s+2l+2) > d-2 \geq 0$ and so the function $s \mapsto Z(s+2l+2, t\sqrt{v})$ is holomorphic on this region. It follows that the double integral in the expression for F converges and is a holomorphic function of (s_1, s_2) on this region. Provided that l is large enough, this region contains an open subset of the affine plane $s = -1$ and, on this open set, $F(s_1, s_2) = 0$ because of the $\Gamma(s+1)$ factor in the expression for F. Thus, on this open subset of the affine plane, $c_2(s_1, s_2, \varepsilon)$ coincides with the first summand in (3-19). From the beta integral and Lemma 3.2, this summand is equal to

$$\frac{\varepsilon\pi^d}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\sum_{j=0}^{l}\frac{(-1)^j}{j!}\left(\frac{s+1}{2}\right)_j(s_1+1)_{2j}\Gamma\left(\frac{s+d}{2}+j\right)\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma(s+1)}\frac{1}{\Gamma(s_1+s_2+2j+2)}.$$

The reason for arranging the factors in the terms in this expression in this particular way is that, so arranged, each factor is regular at $s = -1$. Moreover, the $((s+1)/2)_j$ factor vanishes when $s = -1$ unless $j = 0$. When $j = 0$ and $s = -1$, the integer

$$s_1 + s_2 + 2j + 2 = s + 2 - d = 1 - d$$

is negative, and so the last factor in the $j = 0$ term vanishes when $s = -1$. Thus all terms vanish on the affine plane $s = -1$. We conclude that $c_2(s_1, s_2, \varepsilon)$ is identically zero on a nonempty open subset of the affine plane $s = -1$. It follows that $c(s_1, s_2, \varepsilon)$ is also identically zero on this subset. Since the restriction of $c(s_1, s_2, \varepsilon)$ to this affine plane is entire, we conclude that it vanishes on the whole affine plane. This and (3-18) yield the required conclusion. □

In the following result, we summarize the results obtained in this section.

Corollary 3.15. *Let* $z_0 = d/2$, $z \in \mathbb{C}$, *and* $\varepsilon \in \{\pm\}$. *Then*

$$\Box_z T(z - z_0, -(z + z_0), \varepsilon, \varepsilon) = 0$$

and

$$\Box_z T(z - z_0, -(z + z_0), \varepsilon, -\varepsilon) = a_\varepsilon(z)\delta_0,$$

where $a_\varepsilon(z)$ *is given in* Theorem 3.12.

Note that when d is odd we have

$$a_-(z) - a_+(z) = (-1)^{(d-1)/2}2^{d+2}\pi^{d-1},$$

so that

$$(-1)^{(d-1)/2}2^{-(d+2)}\pi^{1-d}\big(T(z-z_0, -(z+z_0), -, +) - T(z-z_0, -(z+z_0), +, -)\big)$$

is always a fundamental solution for $\Box_z$. When d is even,

$$2^{-(d+1)}\pi^{1-d}\csc(\pi z)T(z - z_0, -(z + z_0), \varepsilon, -\varepsilon)$$

is a fundamental solution for $\Box_z$ for both $\varepsilon = +$ and $\varepsilon = -$ provided that z is not an integer. We have not succeeded in determining any fundamental solution for $\Box_z$

when d is even and z is an integer. Of course, as the referee has pointed out, it is possible that no fundamental solution exists in some or all of these cases. Since it would take us somewhat far afield, we do not attempt to decide this question here.

References

[Barchini et al. 2009] L. Barchini, A. C. Kable, and R. Zierau, "Conformally invariant systems of differential operators", *Adv. Math.* **221**:3 (2009), 788–811. MR 2010d:22020 Zbl 1163.22007

[Folland 1998] G. B. Folland, "Hermite distributions associated to the group $O(p, q)$", *Proc. Amer. Math. Soc.* **126**:6 (1998), 1751–1763. MR 98g:33036 Zbl 0964.33012

[Folland and Stein 1974] G. B. Folland and E. M. Stein, "Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group", *Comm. Pure Appl. Math.* **27** (1974), 429–522. MR 51 #3719 Zbl 0293.35012

[Gradshteyn and Ryzhik 2000] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 6th ed., edited by A. Jeffrey, Academic Press, San Diego, CA, 2000. MR 2001c:00002 Zbl 0981.65001

[Igusa 2000] J.-i. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics **14**, American Mathematical Society, Providence, RI, 2000. MR 2001j:11112 Zbl 0959.11047

[Kable 2011a] A. C. Kable, "K-finite solutions to conformally invariant systems of differential equations", *Tohoku Math. J.* (2) **63**:4 (2011), 539–559. MR 2872955 Zbl 1236.22011

[Kable 2011b] A. C. Kable, "The Heisenberg ultrahyperbolic equation: K-finite and polynomial solutions", preprint, 2011. To appear in *Kyoto J. Math.*

[Lebedev 1972] N. N. Lebedev, *Special functions and their applications*, edited by R. A. Silverman, Dover, New York, 1972. MR 50 #2568 Zbl 0271.33001

Received August 14, 2011. Revised December 11, 2011.

ANTHONY C. KABLE
DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER, OK 74078
UNITED STATES
akable@math.okstate.edu

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 258 No. 1 July 2012

0030-8730(201209)258:1;1-2