Pacific Journal of Mathematics

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We provide a uniqueness theorem for CR and conformal mappings that generate compact sequences of iteration.

1. Introduction

The primary aim of this paper is to prove a version of uniqueness theorem for CR and conformal mappings.

Let M be a C^∞-smooth manifold and let $\mathscr{C}$ be a class of smooth mappings from M into itself containing the identity map. We say that the pair $(M, \mathscr{C})$ satisfies *the Cartan uniqueness property* (or simply *Cartan uniqueness*) at $p \in M$ if an element $f \in \mathscr{C}$ coincides with the identity map whenever $f(p) = p$, $df_p = \mathrm{Id}_{T_pM}$ and

$$\{f^k : k \in \mathbb{Z}_+\}$$

is compact, where Id_{T_pM} is the identity transform of tangent space T_pM of M at p, $\mathbb{Z}_+$ is the set of nonnegative integers, and f^k is the k-time composition of f, namely,

$$f^k = \underbrace{f \circ \cdots \circ f}_{k}.$$

In this definition, the compactness of a subclass $\mathscr{C}'$ of $\mathscr{C}$ means that every sequence in $\mathscr{C}'$ contains a subsequence that is uniformly Cauchy on every compact subset of M. (In other words, $\mathscr{C}'$ is relatively compact with respect to the compact-open topology.) One of the most important examples of this property in complex analysis is the uniqueness theorem of H. Cartan.

Theorem 1.1 (The Cartan uniqueness theorem). *Let Ω be a domain in $\mathbb{C}^n$ and let $\mathscr{H}$ be the class of holomorphic mappings from Ω into itself. Then $(\Omega, \mathscr{H})$ satisfies the Cartan uniqueness property at every point of Ω. In particular, if Ω is bounded, then $f \in \mathscr{H}$ coincides with the identity map if $f(p) = p$ and $df_p = \mathrm{Id}_{T_p\Omega}$ for some $p \in \Omega$, since the sequence $\{f^k : k \in \mathbb{Z}_+\}$ is automatically compact by the Montel theorem.*

It is not trivial to determine under what conditions a similar uniqueness theorem holds if the reference point p lies on the domain boundary. For biholomorphic mappings we have the following theorems:

Theorem 1.2 [Krantz 1987]. *Suppose that Ω is a strongly pseudoconvex domain that is not biholomorphic to the ball. Let $f : \Omega \to \Omega$ be a biholomorphic mapping and let $p \in \partial\Omega$. If $f(z) = z + o(|z - p|)$, then $f \equiv \mathrm{Id}$.*

Theorem 1.3 [Huang 1993]. *Suppose that Ω is a bounded pseudoconvex domain satisfying Condition R. Let $f : \Omega \to \Omega$ be a biholomorphic map and let $p \in \partial\Omega$. If $f(z) = z + o(|z - p|)$ and $\{f^k : k \in \mathbb{Z}\}$ is compact, then $f \equiv \mathrm{Id}$.*

A crucial part in the proof of the Cartan uniqueness theorem are the Cauchy estimates for holomorphic mappings, which are consequences of the ellipticity of $\bar{\partial}$. In contrast, the operator $\bar{\partial}$ does not enjoy ellipticity or even subellipticity on the boundary of a domain. Huang [1993] exploited Bell's theorem [1987] on the boundary behavior of biholomorphic mappings in the C^∞ smooth sense, which is obtained from an analysis of the transformation formula of the Bergman kernel function.

In this paper, we prove a CR version of the uniqueness theorem:

Theorem 1.4 (CR case). *Let M be either a real hypersurface in $\mathbb{C}^{n+1}$ that does not contain any analytic hypersurface or a compact real hypersurface that bounds a domain. Let $\mathcal{H}_b$ be the class of all CR mappings from M into itself. Then $(M, \mathcal{H}_b)$ satisfies the Cartan uniqueness property at every strongly pseudoconvex point.*

The main interest of this theorem is that we assume neither a global type condition on M nor global injectivity of the mappings in the class $\mathcal{H}_b$. Therefore, we may regard this theorem as a generalization of Theorem 1.2 in the CR case. The proof of Theorem 1.4 is based on the method of derivative estimates of CR diffeomorphisms by the local solvability of the CR Yamabe equation, which was mainly developed in [Schoen 1995; Fischer-Colbrie and Schoen 1980].

By considering the conformal Yamabe equation instead of the CR Yamabe equation, we can have a conformal version of Theorem 1.4. For given two Riemannian manifolds (M, g) and (N, h), a diffeomorphism f from M to N is said to be *conformal* if $f^*h = u\,g$ for some positive function u on M. We define a little bit wider class of mappings as follows.

Definition 1.5. Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \to N$ is said to be *semiconformal* if $f^*h = \lambda g$ for some smooth function λ. In this definition, we assume neither that f is 1-1 nor $\lambda > 0$.

Theorem 1.6 (Conformal case). *Let (M, g) be a Riemannian manifold of dimension $n > 2$, and let $\mathcal{S}$ be the class of all semiconformal mappings from M into itself. Then $(M, \mathcal{S})$ satisfies the Cartan uniqueness property at every point of M.*

We present the proof of Theorem 1.6 in Section 2 and then prove Theorem 1.4 in Section 3. Each section contains fundamental definitions of corresponding geometric objects — Yamabe equation, CR and pseudohermitian structures, CR mappings and so on.

2. Proof of Theorem 1.6

We start this section by recalling the Yamabe equation and problem. Let (M, g) be a Riemannian manifold of real dimension $n \geq 3$. Let $\tilde{g} = e^{2f} g$ be a conformal change of a Riemannian metric g, where f is smooth real-valued function on M. If we denote by S and $\tilde{S}$ the scalar curvatures of g and $\tilde{g}$, respectively, it turns out that they satisfy the transformation law

$$\tilde{S} = e^{-2f}\left(S - 2(n-1)\Delta_g f - (n-1)(n-2)|\nabla_g f|^2\right),$$

where $\Delta_g f$ denotes the Laplacian — the trace of the second covariant derivative — of f and $\nabla_g f$ its covariant derivative for the metric g. Let ϕ be the positive function satisfying $e^{2f} = \phi^{p_n - 2}$, where $p_n = 2n/(n-2)$. Then the equation above turns into the following nonlinear equation for ϕ:

$$(2\text{-}1) \qquad\qquad -a_n \Delta_g \phi + S\phi = \tilde{S}\phi^{p_n - 1},$$

where $a_n = 4(n-1)/(n-2)$. This is called the *Yamabe equation* and the linear operator $L_g = -a_n \Delta_g + S$ is called the *conformal Laplacian* for g. When we mention the *Yamabe problem*, we mean the problem of finding a positive solution ϕ of (2-1) that makes $\tilde{S}$ constant. This problem was first introduced in [Yamabe 1960], and its solvability has been intensively investigated there and elsewhere [Trudinger 1968; Aubin 1976a; 1976b; Schoen 1984; 1995].

For our purposes, a local scalar flattening argument is needed rather than the global solvability of the Yamabe problem:

Theorem 2.1 [Fischer-Colbrie and Schoen 1980]. *Let (M, g) be a Riemannian manifold and let Q be a smooth function on M. For $x \in M$ and $R > 0$, we denote by $B_R(x)$ the geodesic ball centered at x of radius R. If the minimum eigenvalue*

$$\lambda(B_R(x)) = \inf\left\{\int_{B_R(x)} (|\nabla f|^2 + Qf^2)\, dV : \text{Support}(f) \subset B_R(x),\ \int_{B_R(x)} f^2 = 1\right\}$$

of $\Delta_g - Q$ on $B_R(x)$ is positive, then there exists a positive function ϕ on M such that

$$(\Delta_g - Q)\phi = 0$$

on $B_R(x)$.

Let us return back to the situation of Theorem 1.6. Let p be a fixed point of M and let $f : M \to M$ be a semiconformal map satisfying that $f(p) = p$, $df_p = \mathrm{Id}_{T_p M}$ and that $\{f^k : k \in \mathbb{Z}_+\}$ is compact. Then we can choose a neighborhood $U = B_R(p)$ of p such that

(i) f is one-to-one on U.

Choosing R small enough, we may also assume that the minimum eigenvalue of $-L_g$ is positive on U by the Poincaré inequality (see [Gilbarg and Trudinger 1983], for example), and that

$$\|u\|_q \le C R \, \|\nabla u\|_q$$

for every $u \in C_0^\infty(U)$ and $1 \le q < \infty$, where C is a constant depending only on the dimension n. Therefore, there exists a positive function ϕ on M such that $L_g \phi = 0$ on U by Theorem 2.1. Replacing the metric g by $\phi^{4/(n-2)} g$, then we may assume that

(ii) the metric g is scalar flat on U

by the Yamabe equation (2-1).

Thanks to the assumption that $\{f^k : k \in \mathbb{Z}_+\}$ is compact, there exists a neighborhood V of p that is relatively compact in U, such that $f^k(V) \subset U$ for every $k = 1, 2, \ldots$. By (i), f^k is a conformal transformation from V to $f^k(V) \subset U$. Therefore, there exists a positive function u_k on V such that

$$(f^k)^* g = (u_k)^{4/(n-2)} g$$

for every $k = 1, 2, \ldots$. We denote u_1 by u.

Since g and $(f^k)^* g$ are scalar flat on V, u_k satisfies the homogeneous Yamabe equation

$$\Delta_g u_k = 0$$

on V by (2-1).

Since f^k is a one-to-one map from V into U, it follows that

$$\int_V u_k^{2n/(n-2)} dV_g = \mathrm{Vol}_{(f^k)^* g} V = \mathrm{Vol}_g f^k(V) \le \mathrm{Vol}_g U < \infty.$$

By the elliptic mean value inequality, there exists $C > 0$ such that $u_k < C$ for every k on a neighborhood V' of p that is relatively compact in V.

Let V'' be a neighborhood of p that is relatively compact in V'. By the elliptic estimate for Δ_g [Gilbarg and Trudinger 1983], there is a C_j independent of k such that

(2-2) $$\|u_k\|_{C^j(V'')} < C_j.$$

Note that

$$(2\text{-}3) \qquad u_k(x) = u(f^{k-1}(x)) \cdot \cdots \cdot u(f(x)) \cdot u(x)$$

for every positive integer k. Let $x = (x^1, \ldots, x^n)$ be local coordinates on V centered at p and let

$$u(x) = 1 + h_j(x) + O(|x|^{j+1}),$$

where h_j is a j-th degree homogeneous polynomial. Since $f(x) = x + o(|x|)$ by the hypothesis, we see that

$$u_k(x) = 1 + k h_j(x) + O(|x|^{j+1})$$

from (2-3). Therefore, if h_j does not vanish, then the j-th order differential of u_k at $p = 0$ diverges as $k \to \infty$, which contradicts the inequality (2-2). This means that h_j vanishes identically on V'', hence $v = u - 1$ vanishes at p up to infinite order. By the unique continuation principle [Garofalo and Lin 1987; Kazdan 1988], we have $v = 0$ on V'', namely, $u = 1$ on V''. This implies that f is an isometry on V''. Therefore, f coincides with the identity map on V'', since every local geodesic passing through p should be preserved by f.

Let $F = \{x \in M : f(x) = x\}$ and let F° the interior of F. Since $V'' \subset F^\circ$, we see that F° is nonempty. If x_0 is a limit point of F°, then obviously f satisfies that $f(x_0) = x_0$ and $df_{x_0} = \mathrm{Id}_{T_{x_0} M}$. Repeating all the arguments above, we conclude that $x_0 \in F^\circ$. Thus F° is closed. This completes the proof of Theorem 1.6. $\qquad \square$

Corollary 2.2. *Let f be a conformal transformation of M such that $f(p) = p$ and $df_p = \mathrm{Id}_{T_p M}$. Then either $f \equiv \mathrm{Id}$ or M is conformally equivalent to the unit sphere S^n and f can be transformed into the conformal transformation ϕ_a of S^n that fixes $p_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ for some $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, where $\phi_a = \Phi_a/|\Phi_a|$ and Φ_a is the affine transformation of $\mathbb{R}^{n+1}$ defined by*

$$y_0 = \frac{|a|^2}{4} + \left(1 - \frac{|a|^2}{4}\right) x^0 + \frac{1}{\sqrt{2}} \sum_{j=1}^n a_j x^j,$$

$$y^j = x^j + \frac{a_j}{\sqrt{2}} (1 - x^0)$$

for $j = 1, \ldots, n$.

Proof. By Theorem 1.6, we may assume that $\{f^k : k \in \mathbb{Z}\}$ is noncompact if f is not the identity transform. Then Schoen's theorem [1995] implies that M is conformally equivalent to either $\mathbb{R}^n$ or S^n. Since $\mathbb{R}^n$ is conformally equivalent to $S^n \setminus \{p_\infty = (-1, 0, \ldots, 0)\}$, we may also assume that f is a conformal transformation of S^n fixing $p_0 = (1, 0, \ldots, 0)$. By Obata's theorem [1970], either f has two fixed points or it has only one fixed point. Moreover, if f has two fixed points, then each fixed

point is a contracting fixed point of f or f^{-1}. This contradicts the hypothesis that $df_{p_0} = \mathrm{Id}_{T_{p_0}M}$. In particular, M is conformally equivalent to S^n, since if M were conformally equivalent to $\mathbb{R}^n$, then f should fix two points p_0 and p_∞ as a conformal transform of S^n. If f has only one fixed point p_0 and $df_{p_0} = \mathrm{Id}$, then $f = \Phi_a / |\Phi_a|$ for some $a \in \mathbb{R}^n$ by the algebraic characterization of the conformal transformation group of S^n via projectivization. See [Obata 1970] for more details.

□

3. Proof of Theorem 1.4

We first review some definitions related to CR manifolds and CR mappings.

Let M be a smooth manifold of real dimension $2n+1$ and let H be a subbundle of TM with fiber dimension $2n$. Let J be a smooth section of the endomorphism bundle of H satisfying $J^2 = -\mathrm{Id}_H$. Then the triple (M, H, J) is called a *CR manifold* if it satisfies the integrability condition

$$[\Gamma(H_{1,0}), \Gamma(H_{1,0})] \subset \Gamma(H_{1,0}),$$

where $H_{1,0}$ is the subbundle of $\mathbb{C} \otimes H$ on which $J = i$, and $\Gamma(H_{1,0})$ is the space of smooth sections of $H_{1,0}$. A typical example of a CR manifold is a real hypersurface of a complex manifold. Let (M, H, J) and (M', H', J') be two CR manifolds. A smooth map $f : M \to M'$ is called a *CR mapping* if $df(H_{1,0}) \subset H'_{1,0}$. For a CR manifold (M, H, J), let θ be a nonvanishing real 1-form on M that vanishes on H. *The Levi form L_θ* is the symmetric bilinear form on H defined by

$$(L_\theta)_x(X, Y) = d\theta_x(X, JY)$$

for every $x \in M$ and $X, Y \in H_x$.

A CR manifold (M, H, J) is said to be *strongly pseudoconvex* if the Levi form L_θ for some θ is positive definite on H. In this case, the 1-form becomes a contact form. The quadruple (M, H, J, θ) is called a *pseudohermitian manifold;* see [Webster 1978]. We abbreviate this by (M, θ).

For a pseudohermitian manifold (M, θ), let ξ be the vector field on M defined by $\theta(\xi) = 1$ and $\xi \lrcorner d\theta = 0$. Then for every $x \in M$, $T_x M = [\xi_x] \oplus H_x$, where $[\xi_x]$ denotes the space generated by ξ_x. This decomposition defines a natural projection $\pi : T_x M \to H_x$. We define a Riemannian metric g on M by

(3-1) $g_x(X, Y) = \theta_x(X)\,\theta_x(Y) + (L_\theta)_x(\pi(X), \pi(Y))$

for every $X, Y \in T_x M$. The *Tanaka–Webster connection* ∇ on (M, θ) is an affine connection for which g, V and J are parallel. This connection is determined uniquely under suitable conditions on the torsion tensor. See [Tanaka 1975; Webster 1978]. By differentiating the Tanaka–Webster connection form, the *pseudohermitian*

curvature tensor can be defined. The trace of this tensor is the Ricci tensor, and the trace of the Ricci tensor is the *Webster scalar curvature R_θ*. Obviously, the Webster scalar curvature depends the choice of the contact form θ.

Let $\tilde{\theta} = u^{2/n}\theta$ be another choice of contact form, where u is a positive smooth function, and let $\widetilde{R}$ and R be the Webster scalar curvatures for $\tilde{\theta}$ and θ, respectively. Then it is known that

$$(3\text{-}2) \qquad\qquad -b_n \Delta_\theta u + Ru = \widetilde{R}u^{p-1},$$

where

$$b_n = \frac{2(2n+1)}{n+1}, \quad p = 2 + \frac{2}{n},$$

and Δ_θ is the sublaplacian for θ. Equation (3-2) is called the *CR Yamabe equation*. The *CR Yamabe problem* is to find a positive solution of (3-2) that makes $\widetilde{R}$ constant. One may refer to [Jerison and Lee 1987; 1989; Lee 1986] for the properties of the CR Yamabe equation and the solvability of the CR Yamabe problem.

Now let us consider the situation of Theorem 1.4. Let M be a smooth real hypersurface in $\mathbb{C}^{n+1}$, $p \in M$ be a strongly pseudoconvex point and let $f : M \to M$ be a CR mapping satisfying that $f(p) = p$, $df_p = \mathrm{Id}_{T_pM}$ and that the iteration sequence $\{f^k : k \in \mathbb{Z}_+\}$ is compact. Let Γ be the connected component of the set of strongly pseudoconvex points in M that contains p. One should notice that Theorem 2.1 is still valid for subelliptic cases, since the proof depends only on the Fredholm alternative theorem. Therefore, by a similar argument as in the conformal case, we can choose a neighborhood U of p in Γ and a contact 1-form θ on Γ such that

- f is one-to-one, and
- the Webster scalar curvature for θ vanishes on U.

Take a relatively compact neighborhood V of p in U such that $f^k(V) \subset U$ for every $k = 1, 2, \dots$. Then an iteration argument as in the conformal case and the subellipticity of the sublaplacian yield that $u - 1$ vanishes at p up to infinite order, where u is the positive function on V defined by

$$(3\text{-}3) \qquad\qquad f^*\theta = u^{2/n}\theta.$$

Remark 3.1. Although $\Delta_\theta u = 0$ on V, we cannot conclude that $u \equiv 1$ at this stage, since the unique continuation principle for subelliptic operator has not been completely solved. In fact, an example in [Bahouri 1986] shows that a continuation theorem of Garofalo–Lin–Kazdan type cannot hold in the 3-dimensional Heisenberg group. Some partial results on the unique continuation principle for the sublaplacian were obtained in [Garofalo and Lanconelli 1990] for the Heisenberg group and in [Niu and Wang 2010] for more general nilpotent groups.

Let g be the Riemannian metric on Γ defined by (3-1). Then (3-3) yields that

$$\tilde{g} := f^*g = \lambda^2 \theta \otimes \theta + \lambda L_\theta,$$

where $\lambda = u^{2/n}$. Since $u - 1$ vanishes at p up to infinite order, so does $\lambda - 1$.

Let $(x^0, \ldots, x^{2n})$ be local coordinates centered at p such that $g_{ij}(0) = \delta_{ij}$. Since $\lambda - 1$ vanishes at $p = 0$ up to infinite order, the Taylor coefficients of g_{ij} and $\tilde{g}_{ij}$ at $p = 0$ coincide. Since derivatives of f at 0 of order ≥ 2 are completely determined by differences between the Taylor coefficients of g_{ij} and $\tilde{g}_{ij}$, we see that f coincides with the identity at 0 up to infinite order. Note that f is a CR diffeomorphism from V onto $f(V)$. Since every local CR diffeomorphism on a strongly pseudoconvex CR manifold is uniquely determined by its finite order jet at the fixed point [Chern and Moser 1974; Kim and Zaitsev 2005], we conclude that $f \equiv \mathrm{Id}$ on V.

If M is a compact real hypersurface that bounds a domain D, then the CR mapping f extends continuously to a holomorphic map F on D by the Bochner–Hartogs extension theorem. Since $F - \mathrm{Id}_D$ vanishes on an open piece V of the boundary $\partial D = M$, we see that F and hence f coincide with the identity map. Now suppose that M is a real hypersurface in $\mathbb{C}^{n+1}$ containing no analytic hypersurface. Let

$$F = \{x \in M : f(x) = x\}$$

and let F° be the interior of F, which is nonempty by the argument above. Let x_0 be a limit point of F°. By Trépreau's theorem [1986], there exists a neighborhood Ω of x_0 in $\mathbb{C}^{n+1}$ such that $\Omega \setminus M = \Omega_+ \cup \Omega_-$ and such that f extends continuously to a holomorphic map F defined on Ω_+. Since $F \equiv \mathrm{Id}$ on a nonempty open piece $F^\circ \cap \Omega$ in $\partial \Omega_+$, we see that $F \equiv \mathrm{Id}$ on Ω_+. Therefore, $x_0 \in F^\circ$. This yields the conclusion. $\qquad \square$

Corollary 3.2. *Let M be a real hypersurface in $\mathbb{C}^{n+1}$ containing no analytic hypersurface. Let p be a strongly pseudoconvex point of M. If $f : M \to M$ is a CR automorphism that $f(p) = p$ and $df_p = \mathrm{Id}$, then either $f = \mathrm{Id}$ or M is CR equivalent to the sphere S^{2n+1}, and f can be transformed into the CR transformation $\phi_{a,r}$ of S^{2n+1} fixing $p_0 = (1, 0, \ldots, 0) \in \mathbb{C}^{n+1}$ for some $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $r \in \mathbb{R}$, where $\phi_{a,r} = \Phi_{a,r}/\Psi_{a,r}$ and $\Phi_{a,r}$ is the affine transformation of $\mathbb{C}^{n+1}$ defined by*

$$w^0 = \left(1 - \tfrac{1}{2}(|a|^2 + ir)\right)z^0 - i \sum_{j=1}^{n} \bar{a}^j z^j + \tfrac{1}{2}(|a|^2 + ir),$$

$$w^j = z^j + ia^j(1 - z^0)$$

for $j = 1, \ldots, n$, and $\Psi_{a,r}$ is the $\mathbb{C}$-valued function defined by

$$\Psi_{a,r}(z) = \tfrac{1}{2}\left(|a|^2 + 2 + ir\right) - i \sum_{j=1}^{n} \bar{a}^j z^j - \tfrac{1}{2}\left(|a|^2 + ir\right)z^0.$$

Proof. By Theorem 1.4, we may assume that $\{f^k : k \in \mathbb{Z}\}$ is noncompact if f is not the identity map. Let Γ be the connected component of the set of strongly pseudoconvex points that contains p. Since f is a CR diffeomorphism of M onto M, it preserves Γ. Moreover, the group $\{f_{|\Gamma}^k\}$ is also noncompact, since otherwise, $f_{|\Gamma} \equiv \mathrm{Id}$ by Theorem 1.4 and this implies that $f \equiv \mathrm{Id}$ on M. Therefore, Schoen's theorem [1995] implies that Γ is CR equivalent to either the Heisenberg group $\mathbf{H}^n$ or the standard unit sphere in $\mathbb{C}^{n+1}$. Since S^{2n+1} is the one point compactification of $\mathbf{H}^n$, we also may assume that f is a CR transformation of S^{2n+1} fixing $p_0 = (1, 0, \ldots, 0)$. By a result in [Webster 1977], either f has two fixed points or it has only one fixed point. If f has two fixed points, then each fixed point is a contracting fixed point of f or f^{-1}. This contradicts the hypothesis that $df_{p_0} = \mathrm{Id}$. Hence f has only one fixed point p_0. In particular, Γ cannot be equivalent to the Heisenberg group. Since S^{2n+1} is a boundary-free compact manifold, we can conclude that $M = \Gamma$ and that M is CR equivalent to the unit sphere S^{2n+1}.

To obtain explicit formulas for CR transformations of S^{2n+1} it is useful to imbed S^{2n+1} into the complex projective $(n+1)$-space $\mathbb{C}\mathrm{P}^{n+1}$ in the following manner: Let $\mathbb{C}^{n+1}$ be a complex Euclidean $(n + 1)$-space with a coordinate system $(z^0, \ldots, z^n)$, and let S^{2n+1} be given by the equation

$$|z^0|^2 + |z^1|^2 + \cdots + |z^n|^2 = 1\,.$$

We also let $\mathbb{C}^{n+2}$ be a complex Euclidean $(n + 2)$-space with a coordinate system $(Z^0, Z^1, \ldots, Z^{n+1})$, and let $\mathbb{C}\mathrm{P}^{n+1}$ be the projective $(n + 1)$-space with a homogeneous coordinate system $[Z^0, Z^1, \ldots, Z^{n+1}]$. We define a holomorphic embedding of $\mathbb{C}^{n+1}$ into $\mathbb{C}\mathrm{P}^{n+1}$ by the equations

$$(3\text{-}4) \qquad Z^0 = 1 + z^0, \quad Z^j = z^j \ (j = 1, \ldots, n), \quad Z^{n+1} = i(1 - z^0),$$

and the image of S^{2n+1} in $\mathbb{C}\mathrm{P}^{n+1}$ under this embedding is the real hypersurface Q that is defined by

$$|Z^1|^2 + \cdots + |Z^n|^2 + \frac{i}{2}\big(Z^{n+1}\overline{Z}^0 - Z^0\overline{Z}^{n+1}\big) = 0.$$

The special unitary group $\mathrm{SU}(n + 1, 1)$ is the group of the linear transformations of $\mathbb{C}^{n+2}$ leaving the quadratic form

$$|Z^1|^2 + \cdots + |Z^n|^2 + \frac{i}{2}\big(Z^{n+1}\overline{Z}^0 - Z^0\overline{Z}^{n+1}\big)$$

invariant, and whose determinant has absolute value 1. We can regard the CR transformation group of S^{2n+1} as $\mathrm{SU}(n + 1, 1)$. The Lie algebra of $\mathrm{SU}(n + 1, 1)$

consists of $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} \lambda & -2i^t\bar{a} & r \\ -\frac{1}{2i}\bar{b} & B & a \\ q & {}^t b & -\bar{\lambda} \end{pmatrix},$$

where B is a skew-hermitian $n \times n$ matrix, a and b are column n-vectors with complex entries, and r, q are real numbers.

In particular, the Lie algebra of the isotropy group $\mathrm{SU}_{p_0}(n+1, 1)$ at the point p_0 consists of the matrices of the form

$$\begin{pmatrix} \lambda & -2i^t\bar{a} & r \\ 0 & B & a \\ 0 & 0 & -\bar{\lambda} \end{pmatrix}.$$

If λ is not purely imaginary, then f has two fixed points, by [Webster 1977]. This contradicts our hypothesis. Write $\lambda = i\theta$ for some $\theta \in \mathbb{R}$. The isotropy group $\mathrm{SU}_{p_0}(n+1, 1)$ itself, as a subgroup of $\mathrm{SU}(n+1, 1)$, consists of the matrices of the form

$$(3\text{-}5) \qquad \begin{pmatrix} e^{i\theta} & -2ie^{i\theta}({}^t\bar{a}) & -ie^{i\theta}|a|^2 + re^{i\theta} \\ 0 & T & Ta \\ 0 & 0 & e^{i\theta} \end{pmatrix},$$

where $T \in \mathrm{SU}(n)$ and r is a real number. Hence we can consider f as a linear transformation of the form (3-5). Since $df_{p_0}|_{T_{p_0} S^{2n+1}} = \mathrm{Id}$, T is the identity map. So f is represented by the matrix

$$(3\text{-}6) \qquad \begin{pmatrix} e^{i\theta} & -2ie^{i\theta}({}^t\bar{a}) & -ie^{i\theta}|a|^2 + re^{i\theta} \\ 0 & \mathrm{Id} & a \\ 0 & 0 & e^{i\theta} \end{pmatrix}.$$

By Webster [1977], the CR transformation represented by (3-6) has only one fixed point.

Since $[Z^0, \ldots, Z^{n+1}]$ is the homogeneous coordinate of $\mathbb{C}P^{n+1}$, the inverse map of (3-4) is given by

$$(3\text{-}7) \qquad z^0 = \frac{iZ^0 - Z^{n+1}}{iZ^0 + Z^{n+1}} \quad \text{and} \quad z^j = \frac{2iZ^j}{iZ^0 + Z^{n+1}}$$

for $j = 1, \ldots, n$. If $e^{i\theta}$ is not 1, then the map represented by the matrix (3-6) does not satisfy that $df_{p_0}|_{T_{p_0} S^{2n+1}} = \mathrm{Id}$. This implies that $e^{i\theta} = 1$. Using (3-6) and (3-7) we can conclude that $f = \Phi_{a,r}/\Psi_{a,r}$ for some $a \in \mathbb{C}^n$ and $r \in \mathbb{R}$. $\qquad \square$

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