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ON THE GEOMETRIC FLOWS SOLVING KÄHLERIAN INVERSE σ_k EQUATIONS

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Here we extend our previous work on the inverse σ_k problem. The inverse σ_k problem is a fully nonlinear geometric PDE on compact Kähler manifolds. Given a proper geometric condition, we prove that a large family of nonlinear geometric flows converges to the desired solution of the given PDE.

1. Introduction

We study general flows for the inverse σ_k-curvature problem in Kähler geometry. This is a continuation of our previous work [Fang et al. 2011].

Geometric curvature flow has been a central topic in the recent development of geometric analysis. The σ_k-curvature problems and inverse σ_k-curvature problems, fully nonlinear in nature, have appeared in several geometric settings. Andrews [1994; 2007] studies the curvature flow of embedded convex hypersurfaces in the Euclidean space. Several authors study the σ_k-equation in conformal geometry; see, for example, [Viaclovsky 2000; Chang et al. 2002; Guan and Wang 2003; Brendle 2005] and references therein. It is thus interesting to explore the corresponding problem in Kähler geometry.

In Kähler geometry, special cases of the σ_k-problem have appeared in earlier literature. Among them, one important example is Yau's seminal work on the complex Monge–Ampère equations in the Calabi conjecture. The general case has been studied recently in [Hou 2009; Hou et al. 2010]. There exist, however, some analytical difficulties in completely solving this problem for $k < n$.

Another important example is Donaldson's J-flow [1999], which gives rise to an inverse σ_1-type equation. J-flow is fully studied in [Chen 2000; 2004; Weinkove 2004; 2006; Song and Weinkove 2008]. The general case is described and treated in [Fang et al. 2011], via a specific geometric flow. In contrast to the σ_k-problem, we can pose nice geometric conditions to overcome the analytical difficulties in the inverse σ_k-problem. Here we construct more general geometric flows to solve this problem.

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We now describe the problem in more detail.

Let (M, ω) be a compact Kähler manifold without boundary. Let χ be a Kähler metric in the class $[\chi]$ other than $[\omega]$. For a fixed integer $1 \le k \le n$, we define

$$\sigma_k(\chi) = \binom{n}{k} \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

It is easy to see that $\sigma_k(\chi)$ is a global defined function on M, and pointwise it is the k-th elementary symmetric polynomial on the eigenvalues of χ with respect to ω. Define

$$c_k := \frac{\int_M \sigma_{n-k}(\chi)}{\int_M \sigma_n(\chi)} = \binom{n}{k} \frac{[\chi]^{n-k} \cdot [\omega]^k}{[\chi]^n},$$

a topological constant depending only on cohomology classes $[\chi]$ and $[\omega]$.

Problem [Fang et al. 2011]. Let (M, ω), χ and c_k be given as above. Is there a metric $\tilde{\chi} \in [\chi]$ satisfying

$$c_k \tilde{\chi}^n = \binom{n}{k} \tilde{\chi}^{n-k} \wedge \omega^k ? \tag{1-1}$$

To tackle this problem, we consider the geometric flow

$$\begin{cases} \dfrac{\partial}{\partial t} \varphi = c_k^{1/k} - \left(\dfrac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)} \right)^{1/k}, \\ \varphi(0) = 0 \end{cases} \tag{1-2}$$

in the space of Kähler potentials of χ:

$$\mathcal{P}_\chi := \left\{ \varphi \in C^\infty(M) \;\middle|\; \chi_\varphi := \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0 \right\}.$$

It is easy to see that the stationary point of the flow corresponds to the solution of (1-1).

When $k = 1$, Equation (1-2) is Donaldson's J-flow [1999], defined in the setting of the moment map; see [Chen 2000]. In this case, Song and Weinkove [2008] provide a necessary and sufficient condition for the flow to converge to the critical metric. For general k, this problem is solved in [Fang et al. 2011] with an analogous condition, which we now describe.

We define $\mathscr{C}_k(\omega)$ to be

$$\begin{aligned} \mathscr{C}_k(\omega) = \Big\{ [\chi] > 0 \;\Big|\; & \text{there exists } \chi' \in [\chi] \text{ such that} \\ & n c_k \chi'^{n-1} - \binom{n}{k} (n-k) \chi'^{n-k-1} \wedge \omega^k > 0 \Big\}. \end{aligned} \tag{1-3}$$

Here the inequality indicates that the left-hand side is a positive $(n-1, n-1)$ form.

For $k = n$, condition (1-3) holds for any Kähler class. Hence $\mathscr{C}_n(\omega)$ is the entire Kähler cone of M.

The need for the cone condition (1-3) is easy to see once we write (1-1) locally as

$$\frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)} = \sigma_k(\chi^{-1}) = c_k.$$

Here χ^{-1} denotes the inverse matrix of χ under local coordinates. Since $\chi^{-1} > 0$, we necessarily have, for all i,

$$\sigma_k(\chi^{-1} \mid i) < c_k.$$

This condition is equivalent to the cone condition (1-3). See [Fang et al. 2011, Proposition 2.4].

In this note, we generalize the following result:

Theorem 1.1 [Fang et al. 2011]. *Let (M, ω) be a compact Kähler manifold. Let k be a fixed integer $1 \le k \le n$. Assume $\chi \in [\chi]$ is another Kähler form and $[\chi] \in \mathscr{C}_K(\omega)$; then the flow*

$$\frac{\partial}{\partial t}\varphi = c_k^{1/k} - \left(\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)}\right)^{1/k}, \tag{1-4}$$

with any initial value $\chi_0 \in [\chi]$, has long-time existence and converges to a unique smooth metric $\tilde{\chi} \in [\chi]$ satisfying

$$c_k \tilde{\chi}^n = \binom{n}{k} \tilde{\chi}^{n-k} \wedge \omega^k. \tag{1-5}$$

Specifically, we study an abstract flow on M of the form

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi = F(\chi_\varphi) - C, \\ \varphi(0) = 0, \end{cases} \tag{1-6}$$

where, for $f \in C^\infty(\mathbb{R}_{>0}, \mathbb{R})$,

$$F(\chi_\varphi) = f\left[\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)}\right], \quad C = f(c_k).$$

Note that (1-2) is a special case of (1-6) for $f(x) = -x^{1/k}$.

Abusing notation, we also regard F as a symmetric function on

$$\Gamma_n := \{\chi \in \mathbb{R}^n \mid \chi_1 > 0,\ \chi_2 > 0,\ \dots,\ \chi_n > 0\}$$

by writing $F(\chi_\varphi) = F(\chi_1, \dots, \chi_n)$, where (χ_i) are eigenvalues of χ_φ with respect to ω. Then by carefully examining the proof of Theorem 1.1 in [Fang et al. 2011], we observed that the following structure conditions on F are necessary:

- Ellipticity: $F_i > 0$.
- Concavity: $F_{ij} \le 0$.
- Strong concavity: $F_{ij} + (F_i/\chi_j)\delta_{ij} \le 0$.

Here $F_i = \partial F/\partial \chi_i$ and $F_{ij} = \partial^2 F/\partial \chi_i \partial \chi_j$. Concavity of F follows from strong concavity and ellipticity of F.

It is easy to check that $F(\chi_1, \dots, \chi_n) := -\big(\sigma_{n-k}(\chi)/\sigma_n(\chi)\big)^{1/k}$ satisfies these conditions.

We prove the following:

Theorem 1.2 (Main theorem). *Let (M, ω) be a compact Kähler manifold and let k be a fixed integer, $1 \le k \le n$. Let χ be another Kähler metric such that $[\chi] \in \mathscr{C}_k$. Assume that $f \in C^\infty(\mathbb{R}_{>0}, \mathbb{R})$ satisfies the conditions*

$$f' < 0, \qquad f'' \ge 0, \qquad f'' + \frac{f'}{x} \le 0. \tag{1-7}$$

Then the flow (1-6) with any initial value $\chi_0 \in [\chi]$ has long-time existence and the metric χ_φ converges in C^∞-norm to the critical metric $\tilde{\chi} \in [\chi]$ that is the unique solution of (1-1).

Remark 1.3. The novelty of our theorem is that there exists a large family of nonlinear geometric flows that yields the convergence towards the solution of the inverse σ_k problem (1-1). For example, the function f can be chosen as $f(x) = -\ln x$ or $f(x) = -x^p$, for $0 < p \le 1$. For the special case $f(x) = -\ln x$ and $k = n$, we get an analogue of the Kähler–Ricci flow. For $f(x) = -x$ and $k = n$, a similar flow was studied in [Cao and Keller 2011].

Remark 1.4. Theorem 1.2 is inspired by, and can be viewed as a Kähler analogue of, Andrews' result [2007] on pinching estimates of evolutions of convex hypersurfaces. In fact, our structure conditions are very similar to his.

This paper is organized as follows: in Section 2, we discuss the conditions on f and strong concavity of F; in Section 3, we give the proof of the main result.

2. Strong concavity

Here we explore concavity properties for functions involving the quotient of elementary symmetric polynomials.

Proposition 2.1. *Let $\chi \in \Gamma_n$ and $f : \mathbb{R}_{>0} \to \mathbb{R}$, define*

$$\rho(\chi_1, \dots, \chi_n) = f\big(\sigma_{n-k}(\chi)/\sigma_n(\chi)\big),$$

and suppose f satisfies the conditions

$$f' < 0, \qquad f'' \ge 0, \qquad f'' + \frac{f'}{x} \le 0. \tag{2-1}$$

Then ρ satisfies:

- *Ellipticity*: $\rho_i > 0$ *for all* i.
- *Concavity*: $\rho_{ij} \leq 0$.
- *Strong concavity*: $\rho_{ij} + (\rho_i/\chi_j)\delta_{ij} \leq 0$.

We refer to the conditions in (2-1) as the structure conditions on f.

The proof is based on the following two propositions:

Proposition 2.2. *Let* $g(\chi_1, \dots, \chi_n) = \log \sigma_k(\chi)$ *and* $\chi \in \Gamma_n$. *Then*

- $g_i > 0$,
- $g_{ij} \leq 0$, *and*
- $g_{ij} + (g_i/\chi_j)\delta_{ij} \geq 0$.

Proposition 2.3. *Let* $h(\chi_1, \dots, \chi_n) := -g(1/\chi_1, \dots, 1/\chi_n) = -\log \sigma_k(\chi^{-1})$ *and* $\chi \in \Gamma_n$. *Then*

- $h_i > 0$,
- $h_{ij} \leq 0$, *and*
- $h_{ij} + (h_i/\chi_j)\delta_{ij} \leq 0$.

We refer the reader to the appendix of [Fang et al. 2011] for a detailed proof of Propositions 2.2 and 2.3.

Proof of Proposition 2.1. Direct computation shows

$$\rho_i = -f'\sigma_{k-1}(\chi^{-1} \mid i)\frac{1}{\chi_i^2} > 0.$$

Concavity of ρ follows from strong concavity and $\rho_i > 0$, and hence it suffices to show that

$$\rho_{ij} + \frac{\rho_i}{\chi_j}\delta_{ij} \leq 0.$$

Direct computation yields

$$\rho_{ij} + \frac{\rho_i}{\chi_j}\delta_{ij} = f''\sigma_{k-1}(\chi^{-1} \mid i)\sigma_{k-1}(\chi^{-1} \mid j)\frac{1}{\chi_i^2}\frac{1}{\chi_j^2} + f'\sigma_{k-2}(\chi^{-1} \mid i, j)\frac{1}{\chi_i^2}\frac{1}{\chi_j^2}(1-\delta_{ij}) + \sigma_{k-1}(\chi^{-1} \mid i)\frac{1}{\chi_i^3}\delta_{ij}. \tag{2-2}$$

Since $f'' + f'/x \leq 0$ and $f'' \geq 0$, we have

$$\rho_{ij} + \frac{\rho_i}{\chi_j}\delta_{ij} \leq f''\biggl\{\frac{\sigma_{k-1}(\chi^{-1} \mid i)\sigma_{k-1}(\chi^{-1} \mid j)}{\chi_i^2\chi_j^2} - \sigma_k(\chi^{-1})\biggl[\frac{\sigma_{k-2}(\chi^{-1} \mid i, j)}{\chi_i^2\chi_j^2}(1-\delta_{ij}) + \frac{\sigma_{k-1}(\chi^{-1} \mid i)}{\chi_i^3}\delta_{ij}\biggr]\biggr\} \leq 0. \tag{2-3}$$

The last inequality follows from Proposition 2.3 and the equality

$$h_{ij} + \frac{h_i}{\chi_j}\delta_{ij} = \frac{1}{\sigma_k(\chi^{-1})^2}\Bigg\{\frac{\sigma_{k-1}(\chi^{-1} \mid i)\sigma_{k-1}(\chi^{-1} \mid j)}{\chi_i^2\chi_j^2} - \sigma_k(\chi^{-1})\Bigg[\frac{\sigma_{k-2}(\chi^{-1} \mid i, j)}{\chi_i^2\chi_j^2}(1-\delta_{ij}) + \frac{\sigma_{k-1}(\chi^{-1} \mid i)}{\chi_i^3}\delta_{ij}\Bigg]\Bigg\}. \tag{2-4}$$ □

For a hermitian matrix $A = (a_{i\bar{j}})$, let its eigenvalues be $\chi = (\chi_1, \dots, \chi_n)$. For $f \in C^\infty(\mathbb{R}_{>0}, \mathbb{R})$, we define

$$F(A) := \rho(\chi_1, \dots, \chi_n) = f\Big(\frac{\sigma_{n-k}(\chi)}{\sigma_n(\chi)}\Big).$$

Define

$$F^{i\bar{j}} := \frac{\partial F}{\partial a_{i\bar{j}}}, \qquad F^{i\bar{j},k\bar{l}} := \frac{\partial^2 F}{\partial a_{i\bar{j}} a_{k\bar{l}}}.$$

It is a classical result that the properties of $F(A)$ follow from those of $\rho(\chi)$; see, for example, [Spruck 2005, Theorem 1.4]. In particular, Proposition 2.1 leads to the following:

Proposition 2.4. *Let $F(A)$ be defined as above, and let $f \in C^\infty(\mathbb{R}_{>0}, \mathbb{R})$ satisfy (2-1). Then F satisfies*:

- *Ellipticity*: $F^{i\bar{j}} > 0$.
- *Concavity*: $F^{i\bar{j},k\bar{l}} \le 0$.
- *Strong concavity*: *at* $A = \operatorname{diag}(\chi_1, \dots, \chi_n)$, *we have* $F^{i\bar{i},j\bar{j}} + (F^{i\bar{i}}/\chi_j)\delta_{ij} \le 0$.

3. Proof of the main theorem

Long-time existence. Differentiating the flow (1-6), we get

$$\frac{\partial}{\partial t}\Big(\frac{\partial \varphi}{\partial t}\Big) = F^{i\bar{j}}(\chi)\partial_i\partial_{\bar{j}}\Big(\frac{\partial \varphi}{\partial t}\Big).$$

By Proposition 2.4, $\partial\varphi/\partial t$ satisfies a parabolic equation. By the maximum principle, we have

$$\min_{t=0} \frac{\partial \varphi}{\partial t} \le \frac{\partial \varphi}{\partial t} \le \max_{t=0} \frac{\partial \varphi}{\partial t},$$

and thus

$$\min F(\chi_0) \le F(\chi_\varphi) = f(\sigma_k(\chi_\varphi^{-1})) \le \max F(\chi_0).$$

By the monotonicity of f, there exist two universal positive constants λ_1 and λ_2 such that

$$\lambda_1 \le \sigma_k(\chi_\varphi^{-1}) \le \lambda_2. \tag{3-1}$$

This implies that χ_φ remains Kähler; that is, $\chi_\varphi > 0$. Also, with the bound (3-1), regarding the estimate aspect, f, f', and f'' are all bounded.

Concerning the behavior of the flow (1-6) for arbitrary triple data (M, ω, χ), we have:

Theorem 3.1. *Let (M, ω, χ) be given as above; the general inverse σ_k flow (1-6) has long-time existence.*

Proof. Following [Chen 2004], we derive time-dependent C^2-estimates for the potential φ. Since $\chi_\varphi > 0$, it suffices to derive an upper bound for $G := tr_\omega \chi_\varphi = g^{p\bar{q}}\chi_{p\bar{q}}$. By a straightforward computation, we get

$$\begin{aligned}\text{(3-2)}\quad \frac{\partial G}{\partial t} &= g^{p\bar{q}} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},p}\chi_{k\bar{l},\bar{q}} + g^{p\bar{q}} F^{i\bar{j}} \chi_{i\bar{j},p\bar{q}} \\ &= F^{i\bar{j}}(g^{p\bar{q}}\chi_{p\bar{q}})_{i\bar{j}} + g^{p\bar{q}} F^{i\bar{j},k\bar{l}} \chi_{i\bar{j},p}\chi_{k\bar{l},\bar{q}} + g^{p\bar{q}} F^{i\bar{j}}(\chi_{m\bar{q}} R^m_{pi\bar{j}} - \chi_{m\bar{j}} R^m_{pi\bar{q}}).\end{aligned}$$

The second term is nonpositive by the concavity of F. For the last term, by choosing normal coordinates, it is easy to see that

$$g^{p\bar{q}} F^{i\bar{j}}(\chi_{m\bar{q}} R^m_{pi\bar{j}} - \chi_{m\bar{j}} R^m_{pi\bar{q}}) \le C_3 + C_4 G, \tag{3-3}$$

for two universal positive constants.

Now the upper bound of G follows from the standard maximum principle. Consequently, we have long-time existence for the flow (1-6). □

In what follows, we give the proof of the main theorem. Following [Fang et al. 2011], we first derive a partial C^2-estimate for the potential φ depending on the C^0-norm of φ when the condition $[\chi] \in \mathscr{C}_k(\omega)$ holds. Then we follow the method developed in [Song and Weinkove 2008] to get a uniform C^0-estimate and the convergence of the flow.

Partial C^2-estimate. Without loss of generality, we can assume the initial metric χ_0 is the metric χ' in $[\chi]$ satisfying cone condition (1-3). Since different initial data differ by a fixed potential function, the same estimates carry over. Again, since $\chi_\varphi > 0$, it suffices to bound χ_φ from above. Take $G(x, t, \xi) := \log(\chi_{i\bar{j}}\xi^i\xi^{\bar{j}}) - A\varphi$, for $x \in M$ and $\xi \in \mathbf{T}^{(1,0)}_x M$ with $g_{i\bar{j}}\xi^i\xi^{\bar{j}} = 1$. A is a constant to be determined. Assume G attains its maximum at $(x_0, t_0) \in M \times [0, t]$, along the direction ξ_0. Choose normal coordinates of ω at x_0, such that $\xi_0 = \partial/\partial z_1$ and $(\chi_{i\bar{j}})$ is diagonal at x_0. By the definition of G, it is easy to see that $\chi_{1\bar{1}} = \chi_1$ is the largest eigenvalue of $\{\chi_{i\bar{j}}\}$ at x_0. We can assume $t_0 > 0$; otherwise we would be done. Thus, locally, we consider $H := \log \chi_{1\bar{1}} - A\varphi$ instead, which also achieves its maximum at (x_0, t_0).

For simplicity, we write $\chi = \chi_\varphi$. At x_0, assume that $\chi = \mathrm{diag}(\chi_1, \dots, \chi_n)$ with $\chi_1 \ge \chi_2 \cdots \ge \chi_n > 0$. We use χ to denote the hermitian matrix $(\chi_{i\bar{j}})$ or the set of the eigenvalues of χ_φ interchangeably when no confusion arises.

We compute the evolution of H:

$$\frac{\partial H}{\partial t}=\frac{\chi_{1\bar{1},t}}{\chi_{1\bar{1}}}-A\frac{\partial\varphi}{\partial t}=\frac{F^{i\bar{j}}\chi_{i\bar{j},1\bar{1}}+F^{i\bar{j},k\bar{l}}\chi_{i\bar{j},1}\chi_{k\bar{l},\bar{1}}}{\chi_{1\bar{1}}}-A\frac{\partial\varphi}{\partial t},$$

$$H_{i\bar{i}}=\frac{\chi_{1\bar{1},i\bar{i}}}{\chi_{1\bar{1}}}-\frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2}-A\varphi_{i\bar{i}}.$$

By the maximum principle, at (x_0,t_0) we have

$$0\le\frac{\partial H}{\partial t}-\sum_{i=1}^{n}F^{i\bar{i}}H_{i\bar{i}}=\frac{1}{\chi_{1\bar{1}}}F^{i\bar{i}}(\chi_{i\bar{i},1\bar{1}}-\chi_{1\bar{1},i\bar{i}})-A\frac{\partial\varphi}{\partial t}+AF^{i\bar{i}}\varphi_{i\bar{i}}+B,\tag{3-4}$$

where

$$B=\frac{1}{\chi_{1\bar{1}}}\sum_{1\le i,j,k,l\le n}F^{i\bar{j},k\bar{l}}\chi_{i\bar{j},1}\chi_{k\bar{l},\bar{1}}+\sum_{i=1}^{n}F^{i\bar{i}}\frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2}$$

is the collection of all terms involving third-order derivatives.

We claim that $B\le 0$; the proof is presented at the end of this section. Assuming that, (3-4) leads to

$$\frac{1}{\chi_{1\bar{1}}}F^{i\bar{i}}(\chi_{i\bar{i},1\bar{1}}-\chi_{1\bar{1},i\bar{i}})\ge A\frac{\partial\varphi}{\partial t}-AF^{i\bar{i}}\varphi_{i\bar{i}}.\tag{3-5}$$

We simplify the left-hand side of (3-5) by the Ricci identity:

$$\begin{aligned}\text{LHS}&=\frac{1}{\chi_{1\bar{1}}}\sum_{i=1}^{n}F^{i\bar{i}}(\chi_{i\bar{i}}R_{i\bar{i}1\bar{1}}-\chi_{1\bar{1}}R_{1\bar{1}i\bar{i}})\\&\le\frac{C_1\sum_{i=1}^{n}F^{i\bar{i}}\chi_i}{\chi_{1\bar{1}}}-\sum_{i=1}^{n}F^{i\bar{i}}R_{1\bar{1}i\bar{i}}\le\frac{C_0}{\chi_{1\bar{1}}}+C_2\sum_{i=1}^{n}F^{i\bar{i}}.\end{aligned}\tag{3-6}$$

For the bound on $\sum_{i=1}^{n}F^{i\bar{i}}\chi_i$, we used (3-1) and the following computation:

$$\begin{aligned}\sum_{i=1}^{n}F^{i\bar{i}}\chi_i&=-f'\sum_{i=1}^{n}\sigma_{k-1}(\chi^{-1}|i)\frac{1}{\chi_i^2}\chi_i\\&=-f'\sum_{i=1}^{n}\sigma_{k-1}(\chi^{-1}|i)\frac{1}{\chi_i}=-kf'\sigma_k(\chi^{-1})\le C.\end{aligned}\tag{3-7}$$

To deal with the right-hand side of (3-5), we divide into two cases:

Case 1: $k<n$. In this case, we have the following technical lemma due to the cone condition.

Lemma 3.2. *For $k<n$, assume that $\chi_0=\chi'\in[\chi]$ is a Kähler form satisfying the cone condition (1-3), and that $C_1\le\sigma_k(\chi^{-1})\le C_2$ for two universal constants.*

Then there exists a universal constant N such that if $\chi_1/\chi_n \geq N$, then there exists a universal constant $\theta > 0$ such that

$$\sigma_k^{1/k}\left(\frac{\chi_{0i\bar{i}}}{\chi_i^2}\right) \geq (1+\theta)c_k^{-1/k}\sigma_k^{2/k}(\chi^{-1}). \tag{3-8}$$

We refer the reader to [Fang et al. 2011, Theorem 2.8] for a proof.

Case 1a: $\chi_1/\chi_n \geq N$, where N is given in Lemma 3.2. Applying Lemma 3.2, we claim that there exists a universal constant $\epsilon > 0$ such that

$$\frac{\partial\varphi}{\partial t} - F^{i\bar{i}}\chi_{i\bar{i}} + (1-\epsilon)F^{i\bar{i}}\chi_{0i\bar{i}} \geq 0. \tag{3-9}$$

Indeed, by direct computation, we have

$$\begin{aligned} \sum_{i=1}^n F^{i\bar{i}}\chi_{0i\bar{i}} &= -f'\sum_{i=1}^n \sigma_{k-1}(\chi^{-1} \mid i)\frac{\chi_{0i\bar{i}}}{\chi_i^2} \\ &\geq -kf'\sigma_k^{1-1/k}(\chi^{-1})\sigma_k^{1/k}\left(\frac{\chi_{0i\bar{i}}}{\chi_i^2}\right) \\ &\geq -kf'\sigma_k^{1-1/k}(\chi^{-1})(1+\theta)c_k^{-1/k}\sigma_k^{2/k}(\chi^{-1}). \end{aligned} \tag{3-10}$$

The first inequality follows from Gårding's inequality.

Therefore, by taking ϵ such that $(1-\epsilon)(1+\theta) = 1$, Equation (3-9) is reduced to

$$\frac{\partial\varphi}{\partial t} - F^{i\bar{i}}\chi_{i\bar{i}} - kf'\sigma_k^{1+1/k}(\chi^{-1})c_k^{-1/k} \geq 0. \tag{3-11}$$

By scaling, we can assume $c_k = 1$, and modifying f by adding a constant, we can further assume that $f(1) = 0$. Plugging in $F^{i\bar{i}}$ and letting $x = \sigma_k(\chi^{-1})$, (3-11) is equivalent to

$$f(x) + kf'(x)x - kf'(x)x^{1+1/k} \geq 0. \tag{3-12}$$

The inequality above holds provided $f'' + f'/x \leq 0$ and $f(1) = 0$.

Combining (3-5), (3-6) and (3-9), we have

$$A\epsilon\sum_{i=1}^n F^{i\bar{i}}\chi_{0i\bar{i}} \leq \frac{C_1}{\chi_1} + C_2\sum_{i=1}^n F^{i\bar{i}}. \tag{3-13}$$

Since χ_0 is a fixed form, there exists a universal constant $\lambda > 0$ such that

$$A\lambda\sum_{i=1}^n F^{i\bar{i}} \leq A\epsilon\sum_{i=1}^n F^{i\bar{i}}\chi_{0i\bar{i}}.$$

Hence, in (3-13), taking A such that $A\lambda - C_2 = 1$, an upper bound for χ_1 will follow once we have shown $\sum_{i=1}^n F^{i\bar{i}}$ is bounded from below. For that we have

$$\sum_{i=1}^{n} F^{i\bar{i}} = -f' \sum \sigma_{k-1}(\chi^{-1} \mid i)\frac{1}{\chi_i^2} \geq -kf'\sigma_k^{1-1/k}(\chi^{-1})\sigma_k^{1/k}\Big(\frac{1}{\chi_i^2}\Big) \geq \tilde{C}\sigma_k^{1+1/k}(\chi^{-1}) \geq C. \tag{3-14}$$

Case 1b: $\chi_1/\chi_n \leq N$. In this case, the upper bound for χ_1 follows directly from the lower bound (3-1) on $\sigma_k(\chi^{-1})$. Since

$$\lambda_1 \leq \sigma_k(\chi^{-1}) \leq \binom{n}{k}\frac{1}{\chi_n^k}, \tag{3-15}$$

we get an upper bound for χ_n, and thus an upper bound for χ_1, because $\chi_1 \leq N\chi_n$.

Case 2: $k = n$. In this case, we continue on (3-5) directly. Since we are only concerned with f on the closed interval $[\lambda_1, \lambda_2]$, we can assume that f is positive by adding a constant. By (3-6), we have that

$$\text{LHS of (3-5)} \leq \frac{C_0}{\chi_1} + C_2\sum_{i=1}^{n} F^{i\bar{i}} \leq C_3\sum_{i=1}^{n}\frac{1}{\chi_i}. \tag{3-16}$$

For the right-hand side, we have

$$\text{RHS of (3-5)} \geq A\big(-f(c_k) + nf'\sigma_n(\chi^{-1})\big) + A\epsilon C_4\sum_{i=1}^{n}\frac{1}{\chi_i}. \tag{3-17}$$

Combining (3-16) and (3-17) and taking A such that $A\epsilon C_4 - C_3 = 1$, we find there exists a universal constant C such that

$$\sum_{i=1}^{n}\frac{1}{\chi_i} \leq C. \tag{3-18}$$

Consequently, we have a lower bound on χ_i for all i, and thus an upper bound for χ_1 by (3-1).

Thus we have proved that there exists a universal constant C such that

$$\chi_1 \leq C.$$

This leads to:

Theorem 3.3. *Let the notation be as above*; *we have*

$$|\partial\bar{\partial}\varphi|_{C^0} \leq Ce^{A\varphi - \inf_{M\times[0,t]}\varphi}$$

for two universal constants A and C and any time interval $[0, t]$.

Finally, we prove the claim that

$$B=\frac{1}{\chi_{1\bar{1}}}\sum_{1\le i,j,k,l\le n}F^{i\bar{j},k\bar{l}}\chi_{i\bar{j},1}\chi_{k\bar{l},\bar{1}}+\sum_{i=1}^{n}F^{i\bar{i}}\frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2}\le 0.$$

We divide B into three groups:

$$X=\frac{1}{\chi_{1\bar{1}}}\sum_{1\le i,j\le n}F^{i\bar{i},j\bar{j}}\chi_{i\bar{i},1}\chi_{j\bar{j},\bar{1}}+F^{1\bar{1}}\frac{|\chi_{1\bar{1},1}|^2}{\chi_{1\bar{1}}^2}.$$

That X is nonpositive follows from the strong concavity of F in Proposition 2.4.

$$Y=\frac{1}{\chi_{1\bar{1}}}\sum_{i=2}^{n}F^{i\bar{1},1\bar{i}}\chi_{i\bar{1},1}\chi_{1\bar{i},\bar{1}}+\sum_{i=2}^{n}F^{i\bar{i}}\frac{|\chi_{1\bar{1},i}|^2}{\chi_{1\bar{1}}^2}.$$

One sees by direct computation that $F^{i\bar{1},1\bar{i}}+F^{i\bar{i}}/\chi_1\le 0$ for all i, and thus $Y\le 0$.

$$Z=\frac{1}{\chi_{1\bar{1}}}\sum_{i\ne j,\,j>1,\,k\ne l,\,k>1}F^{i\bar{j},k\bar{l}}\chi_{i\bar{j},1}\chi_{k\bar{l},\bar{1}}.$$

Again by direct computation, each term is nonpositive. We have thus finished the proof of the claim.

C^0-estimate and convergence of the flow. Following the method in [Song and Weinkove 2008], we introduce two functionals. The monotonic behavior of these functionals along the flow (1-6) yields the C^0-estimate and convergence of the flow. Define functionals in $\mathcal{P}_{\chi_0}$ by

$$\mathcal{F}_{k,\chi_0}(\phi)=\mathcal{F}_k(\phi)=\int_0^1\int_M\dot{\phi}_t\chi_{\phi_t}^k\wedge\omega^{n-k}\,dt,\tag{3-19}$$

where ϕ_t is an arbitrary smooth path in $\mathcal{P}_{\chi_0}$ connecting 0 and ϕ, and $\dot{\phi}_t$ denotes a time derivative. One can readily check that this definition is independent of the choice of the path φ_t. Moreover, define

$$\mathcal{F}_{k,n}(\phi)=\binom{n}{k}\mathcal{F}_k(\phi)-c_{n-k}\mathcal{F}_n(\phi).\tag{3-20}$$

The first variation of $\mathcal{F}_{n-k,n}$ is

$$\frac{d}{dt}\mathcal{F}_{n-k,n}(\phi)=\int_M\dot{\phi}_t\left(\binom{n}{k}\chi_{\phi_t}^{n-k}\wedge\omega^k-c_k\chi_{\phi_t}^n\right).$$

It follows that the Euler–Lagrange equation of $\mathcal{F}_{n-k,n}$ is precisely the critical equation (1-1):

$$c_k\chi_\phi^n=\binom{n}{k}\chi_\phi^{n-k}\wedge\omega^k.$$

We have the following properties, the first of which is shown in [Fang et al. 2011, Theorem 4.1].

Proposition 3.4 (uniqueness). *The solution to the critical equation (1-1) is unique up to a constant.*

Proposition 3.5 (monotonicity of $\mathscr{F}_{n-k,n}$). *The functional $\mathscr{F}_{n-k,n}$ is decreasing along the flow (1-6).*

Proof. By direct computation, we have

$$\frac{d}{dt}\mathscr{F}_{n-k,n}(\varphi_t) = \int_M \dot{\varphi}_t\left(\binom{n}{k}\chi_\varphi^{n-k}\wedge\omega^k - c_k\chi_\varphi^n\right) = \int_M \big(f(\sigma_k(\chi_\varphi^{-1})) - f(c_k)\big)(\sigma_k(\chi_\varphi^{-1}) - c_k)\chi_\varphi^n < 0. \tag{3-21}$$

The integrand is of the form $(f(a) - f(b))(a - b)$, which is negative because $f' < 0$. □

Proposition 3.6 (monotonicity of $\mathscr{F}_{n-k}$). *The functional $\mathscr{F}_{n-k}$ is nonincreasing along the flow (1-6).*

Proof. First define $g(x) = f(1/x)$. It follows that g is concave if and only if $f'' + f'/x \le 0$. Then by Jensen's inequality, we have

$$\begin{aligned}\frac{1}{\int_M \chi^{n-k}\wedge\omega^k}\int_M f(\sigma_k(\chi^{-1}))\chi^{n-k}\wedge\omega^k &= \frac{1}{\int_M \chi^{n-k}\wedge\omega^k}\int_M g\left(\frac{\sigma_n(\chi)}{\sigma_{n-k}(\chi)}\right)\chi^{n-k}\wedge\omega^k \\ &\le g\left(\frac{1}{\int_M \chi^{n-k}\wedge\omega^k}\int_M \frac{\sigma_n(\chi)}{\sigma_{n-k}(\chi)}\chi^{n-k}\wedge\omega^k\right) \\ &= g\left(\frac{1}{c_k}\right) = f(c_k).\end{aligned} \tag{3-22}$$

Hence

$$\frac{\partial}{\partial t}\mathscr{F}_{n-k} = \int_M \big(f(\sigma_k(\chi_\varphi^{-1})) - f(c_k)\big)\chi_\varphi^{n-k}\wedge\omega^k \le 0. \tag{3-23}$$ □

Finally, we single out the essential steps for the rest of the proof. By [Fang et al. 2011, Theorem 4.5], we have uniform bounds for the oscillation of φ_t, that is,

$$\|\sup\varphi_t - \inf\varphi_t\| \le C.$$

Then using the functional $\mathscr{F}_{n-k}$, we obtain a suitable normalization $\hat{\varphi}_t$ of φ_t, for which we can get uniform C^0-estimates, and thus uniform C^2-estimates by Theorem 3.3. Higher-order estimates follow from the Evans–Krylov and Schauder estimates. The corresponding metric thus converges to the critical metric solving the inverse σ_k problem (1-1).

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