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In this paper, we prove a sharp lower bound of the first (nonzero) eigenvalue of the anisotropic Laplacian with the Neumann boundary condition. Equivalently, we prove an optimal anisotropic Poincaré inequality for convex domains, which generalizes the classical result of Payne and Weinberger. A lower bound of the first (nonzero) eigenvalue of the anisotropic Laplacian with the Dirichlet boundary condition is also proved.

1. Introduction

In this paper we are interested in studying the eigenvalues of the anisotropic Laplacian Q, which is a natural generalization of the ordinary Laplacian Δ. We say that F is a *norm* on $\mathbb{R}^n$ if $F : \mathbb{R}^n \to [0, +\infty)$ is a convex function of class $C^1(\mathbb{R}^n \setminus \{0\})$, which is even and positively 1-homogeneous, that is,

$$F(t\xi) = |t| F(\xi) \quad \text{for any } t \in \mathbb{R},\ \xi \in \mathbb{R}^n,$$

and

$$F(\xi) > 0 \quad \text{for any } \xi \neq 0.$$

A typical norm on $\mathbb{R}^n$ is $F(\xi) = (\sum_{i=1}^n |\xi_i|^q)^{1/q}$ for $q \in (1, \infty)$. The *anisotropic Laplacian* (or *Finsler–Laplacian*) of $u : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$Qu(x) := \sum_{i=1}^n \frac{\partial}{\partial x_i}\big(F(\nabla u(x)) F_{\xi_i}(\nabla u(x))\big), \quad x \in \mathbb{R}^n, \tag{1}$$

where

$$F_{\xi_i}(\xi) = \frac{\partial F}{\partial \xi_i}(\xi) \quad \text{and} \quad \nabla u(x) = \Big(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_n}(x)\Big).$$

When $F(\xi) = |\xi| = (\sum_{i=1}^n |\xi_i|^2)^{1/2}$, the anisotropic Laplacian $Q = \Delta$, the usual Laplacian. Note that, in this paper, we use $\xi \in \mathbb{R}^n$ for F and $x \in \mathbb{R}^n$ for functions u.

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The anisotropic Laplacian has been studied by many mathematicians, in the context of both Finsler geometry (see, for example, [Amar and Bellettini 1994; Ge and Shen 2001; Ohta 2009; Ohta and Sturm 2011; Shen 2001]) and quasilinear PDE (see, for example, [Alvino et al. 1997; Belloni et al. 2003; Ferone and Kawohl 2009; Wang and Xia 2011b; 2011a; 2012]). Particularly, many problems related to the first eigenvalue of the anisotropic Laplacian have already been considered in [Belloni et al. 2003; Ge and Shen 2001; Kawohl 2011; Ohta 2009; Wang and Xia 2011a]. In this paper we investigate the estimates of the first eigenvalue of the anisotropic Laplacian.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and ν the outward normal unit vector of its boundary $\partial\Omega$. The first (nonzero) eigenvalue λ_1 of the anisotropic Laplacian Q is defined by the smallest positive constant such that there exists a nonconstant function u satisfying

$$-Qu = \lambda_1 u \quad \text{in } \Omega \tag{2}$$

with the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{3}$$

or the Neumann boundary condition

$$\langle F_\xi(\nabla u), \nu\rangle = 0 \quad \text{on } \partial\Omega. \tag{4}$$

We call λ_1 the *first Dirichlet eigenvalue* (respectively the *first Neumann eigenvalue*) and denote it by λ_1^D (respectively λ_1^N). Here $\langle F_\xi(\nabla u), \nu\rangle = \sum_{i=1}^n F_{\xi_i}(\nabla u)\nu^i$ and $\nu = (\nu^1, \dots, \nu^n)$. Equation (4) is a natural Neumann boundary condition for the anisotropic Laplacian. When $F(\xi) = |\xi|$, $\langle F_\xi(\nabla u), \nu\rangle = \partial u/\partial\nu$.

The first (nonzero) Dirichlet (respectively Neumann) eigenvalue can be formulated as a variational problem by

$$\lambda_1^D(\Omega) = \inf\left\{ \frac{\int_\Omega F^2(\nabla u)\,dx}{\int_\Omega u^2\,dx} \;\middle|\; 0 \neq u \in W_0^{1,2}(\Omega) \right\}. \tag{5}$$

$$\lambda_1^N(\Omega) = \inf\left\{ \frac{\int_\Omega F^2(\nabla u)\,dx}{\int_\Omega u^2\,dx} \;\middle|\; 0 \neq u \in W^{1,2}(\Omega),\ \int_\Omega u\,dx = 0 \right\}. \tag{6}$$

Therefore obtaining a sharp estimate of first eigenvalue is equivalent to obtaining the best constant in Poincaré type inequalities.

We remark that Equation (2) should be understood in a weak sense, that is,

$$\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\left(\tfrac{1}{2}F^2\right)(\nabla u)\varphi_i\,dx = \int_\Omega \lambda_1 u\varphi\,dx \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Finding a lower bound for the first eigenvalue is always an interesting problem. In [Belloni et al. 2003; Ge and Shen 2001], the authors proved the Faber–Krahn type inequality for the first Dirichlet eigenvalue of the anisotropic Laplacian. A Cheeger type estimate for the first eigenvalue of the anisotropic Laplacian involving the isoperimetric constant was also obtained there. In this paper, we are interested in the Payne–Weinberger type sharp estimate [Payne and Weinberger 1960] of the first eigenvalue in terms of some geometric quantity, such as the diameter with respect to F.

Before stating our main result, we need to introduce some concepts and definitions. We say that $\partial\Omega$ is *weakly convex* if the second fundamental form of $\partial\Omega$ with respect to the inward normal is nonnegative definite. We say that $\partial\Omega$ is *F-mean convex* if the F-mean curvature H_F is nonnegative. For the definition of F-mean curvature, see Section 2.

There is another convex function F^0 related to F, which is defined to be the support function of $K := \{x \in \mathbb{R}^n : F(x) < 1\}$, namely,

$$F^0(x) := \sup_{\xi\in K}\langle x, \xi\rangle.$$

It is easy to verify that $F^0 : \mathbb{R}^n \mapsto [0, +\infty)$ is also a convex, even, 1-positively, homogeneous function. Actually F^0 is dual to F (see, for instance, [Alvino et al. 1997]) in the sense that

$$F^0(x) = \sup_{\xi\neq 0}\frac{\langle x, \xi\rangle}{F(\xi)} \quad\text{and}\quad F(x) = \sup_{\xi\neq 0}\frac{\langle x, \xi\rangle}{F^0(\xi)}.$$

Hence the Cauchy–Schwarz inequality holds in the sense that

$$\langle \xi, \eta\rangle_{\mathbb{R}^n} \le F(\xi)F^0(\eta). \tag{7}$$

We call $\mathcal{W}_r(x_0) := \{x \in \mathbb{R}^n \mid F^0(x - x_0) \le r\}$ a *Wulff ball* of radius r with center at x_0. We say $\gamma : [0, 1] \to \Omega$ a *minimal geodesic* from x_1 to x_2 if

$$d_F(x_1, x_2) := \int_0^1 F^0(\dot\gamma(t))\,dt = \inf\int_0^1 F^0(\dot{\tilde\gamma}(t))\,dt,$$

where the infimum takes on all C^1 curves $\tilde\gamma(t)$ in Ω from x_1 to x_2. In fact γ is a straight line and $d_F(x_1, x_2) = F^0(x_2 - x_1)$. We call $d_F(x_1, x_2)$ the *F-distance* between x_1 and x_2.

Now we can define the *diameter* d_F of Ω with respect to the norm F on $\mathbb{R}^n$ as

$$d_F := \sup_{x_1,x_2\in\overline{\Omega}} d_F(x_1, x_2).$$

In the same spirit we define the *inscribed radius* i_F of Ω with respect to the norm F on $\mathbb{R}^n$ as the radius of the biggest Wulff ball that can be enclosed in $\overline{\Omega}$.

Our main result is the following.

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ a norm on $\mathbb{R}^n$. Let λ_1^N be the first Neumann eigenvalue of the anisotropic Laplacian* (1). *Assume that $\partial\Omega$ is weakly convex. Then λ_1^N satisfies*

$$\lambda_1^N \geq \frac{\pi^2}{d_F^2}. \tag{8}$$

Moreover, *equality holds in* (8) *if and only if $n = 1$*, *and hence Ω is a segment.*

Estimate (8) for the Neumann boundary problem is optimal. This is in fact a generalization of the classical result of Payne and Weinberger [1960] on an optimal estimate of the first Neumann eigenvalue of the ordinary Laplacian. See also [Bebendorf 2003]. There are many interesting generalizations. Here we just mention its generalization to Riemannian manifolds, since we will use the methods developed there. It should also be interesting to ask if the methods of [Payne and Weinberger 1960] and [Bebendorf 2003] work to reprove our result, since there are lots of motivations in computational mathematics.

For a smooth compact n-dimensional Riemannian manifold (M, g) with nonnegative Ricci curvature and diameter d, possibly with boundary, the first Neumann eigenvalue λ_1 of the Laplace operator Δ is defined to be the smallest positive constant such that there is a nonconstant function u satisfying

$$-\Delta u = \lambda_1 u \quad \text{in } M$$

with

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial M,$$

if ∂M is not empty, where ν denotes the outward normal of ∂M. The fundamental work in [Li 1979; Li and Yau 1980; Zhong and Yang 1984] gives us the following optimal estimate

$$\lambda_1 \geq \frac{\pi^2}{d^2}, \tag{9}$$

where d is the diameter of M with respect to g. Li and Yau [1980] derived a gradient estimate for the eigenfunction u and proved that $\lambda_1 \geq \pi^2/(4d^2)$, and Li [1979] used another auxiliary function to obtain a better estimate $\lambda_1 \geq \pi^2/(2d^2)$. Finally, Zhong and Yang [1984] were able to use a more precise auxiliary function to get the sharp estimate $\lambda_1 \geq \pi^2/d^2$, which is optimal in the sense that the lower bound is achieved by a circle or a segment. Recently Hang and Wang [2007] proved that equality (9) holds if and only if M is a circle or a segment. For related work see [Kröger 1992; Chen and Wang 1997; Bakry and Qian 2000]. These results were generalized to the p-Laplacian in [Valtorta 2012] and to the Laplacian on Alexandrov spaces in [Qian et al. 2012].

For the Dirichlet problem we have the following.

Theorem 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ a norm on $\mathbb{R}^n$. Assume that λ_1^D is the first Dirichlet eigenvalue of the anisotropic Laplacian* (1). *Assume further that $\partial\Omega$ is F-mean convex. Then λ_1^N satisfies*

$$\lambda_1^D \geq \frac{\pi^2}{4i_F^2}. \tag{10}$$

Estimate (10) is by no means optimal.

Our idea to prove the result on the Dirichlet eigenvalue is based on the gradient estimate technique for eigenfunctions from [Li 1979; Li and Yau 1980]. This idea also works for the first Neumann eigenvalue to get a rough estimate, say $\lambda_1^N \geq \pi^2/(2d_F^2)$. However, for getting the sharp estimate of the first Neumann eigenvalue (8), the method of Zhong and Yang seems hard to apply. Instead, we adopt the technique based on gradient comparison with a one dimensional model function, which was developed in [Kröger 1992] and improved in [Chen and Wang 1997; Bakry and Qian 2000]. Surprisingly, we find that the one dimensional model coincides with that for the Laplacian case. In fact, this must be the case because when we consider F in $\mathbb{R}$, it can only be $F(x) = c|x|$ with $c > 0$, a multiple of the standard Euclidean norm. In order to get the gradient comparison theorem, we need a Bochner type formula (13), a Kato type inequality (14), and a refined inequality (15), which was referred to as the "extended curvature-dimension inequality" in the context of [Bakry and Qian 2000]. Interestingly, the proof of these inequalities sounds more "natural" than the proof of their counterpart for the usual Laplace operator. These inequalities may have their own interest. Another difficulty we encounter is handling the boundary maximum due to the different representation of the Neumann boundary condition (4). We find a suitable vector field V (see its explicit construction in Section 3) to avoid this difficulty. With the gradient comparison theorem, we are able to follow step by step the argument in [Bakry and Qian 2000] to get the sharp estimate. The proof for the rigidity part of Theorem 1.1 closely follows [Hang and Wang 2007]. Here we need to pay more attention to the points with vanishing $|\nabla u|$.

A natural question arises of whether one can generalize Theorem 1.1 to manifolds. The anisotropic Laplacian with the norm F does not have a direct generalization to Riemannian manifolds. However, it has a (natural) generalization to Finsler manifolds. In fact, $\mathbb{R}^n$ with F can be viewed as a special Finsler manifold. On a general Finsler manifold, there is a generalized anisotropic Laplacian; see for instance [Ge and Shen 2001; Ohta 2009; Shen 2001]. A Lichnerowicz type result for the first eigenvalue of this Laplacian was obtained in [Ohta 2009] under a condition on some kind of new Ricci curvature Ric_N, $N \in [n, \infty]$. A Li–Yau–Zhong–Yang type sharp estimate, that is, a generalization of Theorem 1.1 for this generalized Laplacian on Finsler manifolds would be a challenging problem. We will study this problem in a forthcoming paper.

The paper is organized as follows. In Section 2, we give some preliminary results on 1-homogeneous convex functions and the F-mean curvature, and prove useful inequalities. In Section 3 we prove the sharp estimate for the first Neumann eigenvalue and classify the equality case. We handle the first Dirichlet eigenvalue in Section 4.

2. Preliminary

Without loss of generality, we may assume that $F \in C^3(\mathbb{R}^n \setminus \{0\})$ and F is a *strongly convex norm* on $\mathbb{R}^n$, that is, F satisfies

$$\mathrm{Hess}(F^2) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}.$$

In fact, for any norm $F \in C^1(\mathbb{R}^n \setminus \{0\})$, there exists a sequence $F_\varepsilon \in C^3(\mathbb{R}^n \setminus \{0\})$ such that the strongly convex norm $\widetilde{F}_\varepsilon := \sqrt{F_\varepsilon^2 + \varepsilon |x|^2}$ converges to F uniformly in $C^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. Then the corresponding first eigenvalue λ_1^ε of the anisotropic Laplacian with respect to $\widetilde{F}_\varepsilon$ converges to λ_1 as well. Here $|\cdot|$ denotes the Euclidean norm. Therefore, in the following sections, we assume that $F \in C^3(\mathbb{R}^n \setminus \{0\})$ and F is a strongly convex norm on $\mathbb{R}^n$. Thus (2) is degenerate elliptic among Ω and uniformly elliptic in $\Omega \setminus \mathscr{C}$, where $\mathscr{C} := \{x \in \Omega \mid \nabla u(x) = 0\}$ denotes the set of degenerate points. The standard regularity theory for degenerate elliptic equations (see, for example, [Belloni et al. 2003; Tolksdorf 1984]) implies that $u \in C^{1,\alpha}(\Omega) \cap C^{2,\alpha}(\Omega \setminus \mathscr{C})$.

The following property is an obvious consequence of the 1-homogeneity of F.

Proposition 2.1. *Let $F : \mathbb{R}^n \to [0, +\infty)$ be a 1-homogeneous function. Then the following holds*:

(i) $\sum_{i=1}^n F_{\xi_i}(\xi)\xi_i = F(\xi)$;

(ii) $\sum_{j=1}^n F_{\xi_i \xi_j}(\xi)\xi_j = 0$ *for any* $i = 1, 2, \dots, n$. □

For simplicity, from now on we will follow the summation convention and frequently use the notations $F = F(\nabla u)$, $F_i = F_{\xi_i}(\nabla u)$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/(\partial x_i \partial x_j)$, and so on. Denote

$$\begin{aligned} a_{ij}(\nabla u)(x) &:= \frac{\partial^2}{\partial \xi_i \partial \xi_j}\left(\tfrac{1}{2}F^2\right)(\nabla u(x)) = (F_i F_j + F F_{ij})(\nabla u(x)), \\ a_{ijk}(\nabla u)(x) &:= \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k}\left(\tfrac{1}{2}F^2\right)(\nabla u(x)). \end{aligned} \tag{11}$$

In the following we simply write a_{ij} and a_{ijk} if no confusion appears. With these notations, we can rewrite the anisotropic Laplacian (1) as

$$Qu = a_{ij}u_{ij}. \tag{12}$$

For the function $\frac{1}{2}F^2(\nabla u)$ we have a Bochner type formula.

Lemma 2.1 (Bochner formula). *At a point where* $\nabla u \neq 0$, *we have*

$$\begin{aligned}a_{ij}\big(\tfrac{1}{2}F^2(\nabla u)\big)_{ij} \\ = a_{ij}a_{kl}u_{ik}u_{jl} + (Qu)_k \frac{\partial}{\partial \xi_k}\big(\tfrac{1}{2}F^2\big)(\nabla u) - a_{ijl}\frac{\partial}{\partial x_l}\big(\tfrac{1}{2}F^2(\nabla u)\big)u_{ij}.\end{aligned} \tag{13}$$

Proof. The formula is derived from a direct computation.

$$\begin{aligned} a_{ij}(\nabla u)\big(\tfrac{1}{2}F^2(\nabla u)\big)_{ij} &= a_{ij}\frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial \xi_k}\big(\tfrac{1}{2}F^2\big)(\nabla u)u_{ik}\Big) \\ &= a_{ij}\frac{\partial^2}{\partial \xi_k \partial \xi_l}\big(\tfrac{1}{2}F^2\big)(\nabla u)u_{ik}u_{jl} + a_{ij}\frac{\partial}{\partial \xi_k}\big(\tfrac{1}{2}F^2\big)(\nabla u)u_{ijk} \\ &= a_{ij}a_{kl}u_{ik}u_{jl} + \frac{\partial}{\partial \xi_k}\big(\tfrac{1}{2}F^2\big)(\nabla u)\Big(\frac{\partial}{\partial x_k}(a_{ij}u_{ij}) - \Big(\frac{\partial}{\partial x_k}a_{ij}\Big)u_{ij}\Big). \end{aligned}$$

Taking into account (12) and

$$\frac{\partial}{\partial \xi_k}\big(\tfrac{1}{2}F^2\big)\frac{\partial}{\partial x_k}a_{ij} = a_{ijl}\frac{\partial}{\partial x_l}\big(\tfrac{1}{2}F^2(\nabla u)\big),$$

we get (13). ☐

When $F(\xi) = |\xi|$, (13) is just the usual Bochner formula

$$\tfrac{1}{2}\Delta(|\nabla u|^2) = |D^2u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle.$$

We have a Kato type inequality for the square of the "anisotropic" norm of the Hessian.

Lemma 2.2 (Kato inequality). *At a point where* $\nabla u \neq 0$, *we have*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq a_{ij}F_kF_lu_{ik}u_{jl}. \tag{14}$$

Proof. It is clear that

$$a_{ij}a_{kl}u_{ik}u_{jl} - a_{ij}F_kF_lu_{ik}u_{jl} = a_{ij}FF_{kl}u_{ik}u_{jl} = FF_iF_jF_{kl}u_{ik}u_{jl} + F^2F_{ij}F_{kl}u_{ik}u_{jl}.$$

Since (F_{ij}) is positive definite, we know the first term

$$FF_iF_jF_{kl}u_{ik}u_{jl} = FF_{kl}(F_iu_{ik})(F_ju_{jl}) \geq 0.$$

The second term $F_{ij}F_{kl}u_{ik}u_{jl}$ is nonnegative as well. Indeed, we can write the matrix $(F_{kl})_{k,l} = O^T \Lambda O$ for some orthogonal matrix O and diagonal matrix $\Lambda = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n)$ with $\mu_i \geq 0$ for any $i = 1, 2, \dots, n$. Set $U = (u_{ij})_{i,j}$ and $\widetilde{U} = OUO^T = (\tilde{u}_{ij})_{i,j}$. Then we have

$$\begin{aligned} F_{ij}F_{kl}u_{lj}u_{ki} &= \mathrm{tr}(O^T\Lambda OUO^T\Lambda OU) = \mathrm{tr}(\Lambda OUO^T\Lambda OUO^T) \\ &= \mathrm{tr}(\Lambda\widetilde{U}\Lambda\widetilde{U}) = \mu_i\mu_j\tilde{u}_{ij}^2 \geq 0, \end{aligned}$$ □

When $F(\xi) = |\xi|$, (14) is the usual Kato inequality

$$|\nabla^2 u|^2 \geq |\nabla|\nabla u||^2.$$

The following inequality is crucial to apply the gradient comparison argument in Section 3.

Lemma 2.3. *At a point where* $\nabla u \neq 0$, *we have*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1}\left(\frac{a_{ij}u_{ij}}{n} - F_iF_ju_{ij}\right)^2 \tag{15}$$

Proof. Let

$$A = F_iF_ju_{ij} \quad \text{and} \quad B = FF_{ij}u_{ij}.$$

The right hand side of (15) equals

$$\frac{(A+B)^2}{n} + \frac{n}{n-1}\left(\frac{B}{n} - \frac{n-1}{n}A\right)^2 = A^2 + \frac{1}{n-1}B^2.$$

The left hand side of (15) is

$$A^2 + 2FF_iF_jF_{kl}u_{ik}u_{jl} + F^2F_{ij}F_{kl}u_{ik}u_{jl}.$$

Since (F_{ij}) is semipositively definite, we know

$$FF_iF_jF_{kl}u_{ik}u_{jl} = FF_{kl}(F_iu_{ik})(F_ju_{jl}) \geq 0.$$

Using the same notations as in the proof of Lemma 2.2, we have

$$F^2F_{ij}F_{kl}u_{ik}u_{jl} = F^2\mu_i\mu_j\tilde{u}_{ij}^2 = F^2\mu_i^2\tilde{u}_{ii}^2 + F^2\sum_{i\neq k}\mu_i\mu_k\tilde{u}_{ik}^2 \geq F^2\mu_i^2\tilde{u}_{ii}^2,$$

$$B = FF_{ij}u_{ij} = \mathrm{tr}(O^T\Lambda OU) = \mathrm{tr}(\Lambda OUO^T) = \mu_i\tilde{u}_{ii}.$$

We claim that (F_{ij}) is a matrix of rank $n-1$, that is, one of μ_i is zero. Firstly, $F_{ij}u_j = 0$. Secondly, for any nonzero $V \perp F_\xi(\nabla u)$, $F_{ij}V^iV^j = a_{ij}V^iV^j > 0$. The claim follows easily. Thus the Hölder inequality gives

$$F^2\mu_i^2\tilde{u}_{ii}^2 \geq \frac{1}{n-1}F^2(\mu_i\tilde{u}_{ii})^2 = \frac{1}{n-1}B^2.$$ □

When $F(\xi)=|\xi|$, (15) is

$$|\nabla^2 u|^2 \geq \frac{(\Delta u)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta u}{n} - \frac{u_i u_j u_{ij}}{|\nabla u|^2}\right)^2.$$

We now recall the definition of F-mean curvature. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain whose boundary $\partial\Omega$ is an $(n-1)$-dimensional, oriented, compact submanifold without boundary in $\mathbb{R}^n$. We denote by ν and $d\sigma$ the outward normal of $\partial\Omega$ and the area element, respectively. Let $\{e_\alpha\}_{\alpha=1}^{n-1}$ be a basis of the tangent space $T_p(\partial\Omega)$, and let $g_{\alpha\beta}=g(e_\alpha, e_\beta)$ and $h_{\alpha\beta}$ be the first and second fundamental forms, respectively. $\partial\Omega$ is called weakly convex if $(h_{\alpha\beta})$ is nonnegative definite. Moreover, let $(g^{\alpha\beta})$ be the inverse matrix of $(g_{\alpha\beta})$ and $\overline{\nabla}$ the covariant derivative in $\mathbb{R}^n$. The F-second fundamental form $h^F_{\alpha\beta}$ and the F-mean curvature H_F are defined by

$$h^F_{\alpha\beta} := \langle F_{\xi\xi} \circ \overline{\nabla}_{e_\alpha} \nu, e_\beta \rangle \quad \text{and} \quad H_F = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta} h^F_{\alpha\beta},$$

respectively. We call $\overrightarrow{H_F} = -H_F \nu$ the F-mean curvature vector (it is easy to check that all definitions are independent of the choice of coordinates). $\partial\Omega$ is called weakly F-convex (respectively F-mean convex) if $(h^F_{\alpha\beta})$ is nonnegative definite (respectively $H_F \geq 0$). It is well known that when we consider a variation of $\partial\Omega$ with variation vector field $\varphi \in C_0^\infty(\partial\Omega, \mathbb{R}^n)$, the first variation of the F-area functional $\mathcal{F}(X) := \int_{\partial\Omega} F(\nu)\, d\sigma$ reads as

$$\delta_\varphi \mathcal{F}(X) = -\int_{\partial\Omega} \langle \overrightarrow{H_F}, \varphi \rangle \, d\sigma.$$

It is easy to see from the convexity of F that $h^F_{\alpha\beta}$ being nonnegative definite is equivalent to the ordinary second fundamental form $h_{\alpha\beta}$ being nonnegative definite. In other words, there is no difference between weakly F-convex and weakly convex. However, F-mean convex is different from mean convex. For more properties of H_F, we refer to [Wang and Xia 2011b] and the references therein. Here we will use the following lemma, which interprets the relation between the anisotropic Laplacian and the F-mean curvature of level sets of functions.

Lemma 2.4 [Wang and Xia 2011b, Theorem 3]. *Let u be a C^2 function with a regular level set $S_t := \{x \in \overline{\Omega} \mid u = t\}$. Let $H_F(S_t)$ be the F-mean curvature of the level set S_t. We then have*

$$Qu(x) = -F H_F(S_t) + F_i F_j u_{ij} = -F H_F(S_t) + \frac{\partial^2 u}{\partial \nu_F^2}$$

for x with $u(x)=t$, where $\nu_F := F_\xi(\nu) = -F_\xi(\nabla u)$.

We point out that we have used the inward normal in [Wang and Xia 2011b] and there is an sign error in formula (5) there. Hence the term $FH_F(S_t)$ in formula (9) there should be read as $-FH_F(S_t)$.

3. Sharp estimate of the first Neumann eigenvalue

It is well-known that the existence of the first Neumann eigenfunction can be obtained from the direct method in the calculus of variations. We note that the first Neumann eigenfunction must change sign, for its average vanishes.

In this Section we first prove the following gradient comparison theorem, which is the most crucial part for the proof of the sharp estimate. For simplicity, we write λ_1 instead of λ_1^N throughout this section.

Theorem 3.1. *Let Ω, u, λ_1 be as in Theorem 1.1. Let v be a solution of the* 1*-dimensional model problem on some interval* (a, b):

$$v'' - Tv' = -\lambda_1 v, \quad v'(a) = v'(b) = 0, \quad v' > 0 \tag{16}$$

with $T(t) = -(n-1)/t$ or 0. *Assume that* $[\min u, \max u] \subset [\min v, \max v]$. *Then*

$$F(\nabla u)(x) \le v'\big(v^{-1}(u(x))\big). \tag{17}$$

Proof. First, since $\int u = 0$, we know that $\min u < 0$ while $\max u > 0$. We may assume that $[\min u, \max u] \subset (\min v, \max v)$ by multiplying u by a constant $0 < c < 1$. If we prove the result for this u, then, letting $c \to 1$, we have (17).

Under the condition $[\min u, \max u] \subset (\min v, \max v)$, v^{-1} is smooth on a neighborhood U of $[\min u, \max u]$.

Consider $P := \psi(u)(\frac{1}{2}F(\nabla u)^2 - \phi(u))$, where $\psi, \phi \in C^\infty(U)$ are two positive smooth functions to be determined later. We first assume that P attains its maximum at $x_0 \in \Omega$. Then we consider the case where $x_0 \in \partial\Omega$. If $\nabla u(x_0) = 0$, $P \le 0$ is obvious. Hence we assume $\nabla u(x_0) \neq 0$. From now on we compute at x_0. As in Section 2, we use the notation (11). Since x_0 is the maximum of P, we have

$$P_i(x_0) = 0, \tag{18}$$

$$a_{ij}(x_0)P_{ij}(x_0) \le 0. \tag{19}$$

Equality (18) gives

$$\frac{\partial}{\partial x_i}\big(\tfrac{1}{2}F^2(\nabla u) - \phi(u)\big) = -\frac{\psi(u))_i}{\psi^2}P, \quad F_iF_ju_{ij} = \phi' - \frac{\psi'}{\psi^2}P. \tag{20}$$

Then we compute $a_{ij}P_{ij}$.

$$\begin{aligned} a_{ij}P_{ij} = \frac{P}{\psi}a_{ij}(\psi(u))_{ij} + \psi a_{ij}\frac{\partial}{\partial x_i x_j}\big(\tfrac{1}{2}F^2(\nabla u) - (\phi(u))\big) \\ + 2a_{ij}(\psi(u))_i\frac{\partial}{\partial x_j}\big(\tfrac{1}{2}F^2(\nabla u) - \phi(u)\big). \end{aligned}$$

It is easy to see from Proposition 2.1 that

$$\frac{\partial}{\partial \xi_i}\big(\tfrac{1}{2}F^2\big)(\nabla u)u_i = F^2(\nabla u), \quad a_{ij}u_iu_j = F^2(\nabla u), \quad a_{ijk}u_k = 0. \tag{21}$$

By using (20), (21), the Bochner formula (13), and eigenvalue equation (2), we get

$$\begin{aligned}(22)\quad & a_{ij}P_{ij} \\ &= \left(-\lambda_1 u\frac{\psi'}{\psi} + F^2\frac{\psi''}{\psi} - 2F^2\frac{\psi'^2}{\psi^2}\right)P + \psi(a_{ij}a_{kl}u_{ik}u_{jl} - \lambda_1 F^2) + \psi(\lambda_1 u\phi' - F^2\phi'').\end{aligned}$$

Applying Lemma 2.3 to (22), replacing F^2 by $2P/\psi + \phi$, and using (20), (2), and (19), we deduce

$$\begin{aligned}(23)\quad 0 &\geq a_{ij}P_{ij} \\ &\geq \left(-\lambda_1 u\frac{\psi'}{\psi} + F^2\frac{\psi''}{\psi} - 2F^2\frac{\psi'^2}{\psi^2}\right)P + \psi(\lambda_1 u\phi' - F^2\phi'') \\ &\qquad + \psi\left(\frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1}\left(\frac{a_{ij}u_{ij}}{n} - F_iF_ju_{ij}\right)^2 - \lambda_1 F^2\right) \\ &= \frac{1}{\psi}\left[2\frac{\psi''}{\psi} - (4 - \frac{n}{n-1})\frac{\psi'^2}{\psi^2}\right]P^2 \\ &\quad + \left[2\phi\left(\frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2}\right) - \frac{n+1}{n-1}\frac{\psi'}{\psi}\lambda_1 u - \frac{2n}{n-1}\frac{\psi'}{\psi}\phi' - 2\lambda_1 - 2\phi''\right]P \\ &\qquad + \psi\left[\frac{1}{n-1}\lambda_1^2u^2 + \frac{n+1}{n-1}\lambda_1 u\phi' + \frac{n}{n-1}\phi'^2 - 2\lambda_1\phi - 2\phi\phi''\right] \\ &:= a_1P^2 + a_2P + a_3.\end{aligned}$$

We are lucky to observe that the coefficients a_i, $i = 1, 2, 3$, coincide with those appearing in the ordinary Laplacian case; see, for example, [Bakry and Qian 2000, Lemma 1]. The next step is to choose suitable positive functions ψ and ϕ such that $a_1, a_2 > 0$ everywhere and $a_3 = 0$, which has already be done in [Bakry and Qian 2000]. For completeness, we sketch the main idea here.

Choose $\phi(u) = \frac{1}{2}v'(v^{-1}(u))^2$, where v is a solution of the 1-dimensional problem (16). One can compute that

$$\phi'(u) = v''(v^{-1}(u)), \quad \phi''(u) = \frac{v'''}{v'}(v^{-1}(u)).$$

Setting $t = v^{-1}(u)$ and $u = v(t)$, we have

$$\begin{aligned}\frac{a_3(t)}{\psi} &= \frac{1}{n-1}\lambda_1^2v^2 + \frac{n+1}{n-1}\lambda_1 vv'' + \frac{n}{n-1}v''^2 - \lambda_1 v'^2 - v'v''' \\ &= -v'(v'' - Tv' + \lambda_1 v)' + \frac{1}{n-1}(v'' - Tv' + \lambda_1 v)(nv'' + Tv' + \lambda_1 v) = 0.\end{aligned}$$

Here we have used the fact that T satisfies $T' = T^2/(n-1)$. For a_1, a_2, we introduce

$$X(t) = \lambda_1 \frac{v(t)}{v'(t)}, \quad \psi(u) = \exp\left(\int h(v(t))\right), \quad f(t) = -h(v(t))v'(t).$$

With these notations, we have

$$\begin{aligned} f' &= -h'v'^2 + f(T - X), \\ v'|^2_{v^{-1}} a_1 \psi &= 2f(T-X) - \frac{n-2}{n-1} f^2 - 2f' := 2(Q_1(f) - f'), \\ a_2 = f\Big(\frac{3n-1}{n-1}T - 2X\Big) &- 2T\Big(\frac{n}{n-1}T - X\Big) - f^2 - f' := Q_2(f) - f'. \end{aligned}$$

We may now use [Bakry and Qian 2000, Corollary 3], which says that there exists a bounded function f on $[\min u, \max u] \subset (\min v, \max v)$ such that $f' < \min\{Q_1(f), Q_2(f)\}$.

In view of (23), we know that, by our choice of ψ and ϕ, $P(x_0) \le 0$, and hence $P(x) \le 0$ for every $x \in \Omega$, which leads to (17).

Now we consider the case $x_0 \in \partial\Omega$. Suppose that P attains its maximum at $x_0 \in \partial\Omega$. We introduce a new vector field $V(x) = (V^i(x))_{i=1}^n$ defined on $\partial\Omega$ by

$$V^i(x) = \sum_{j=1}^n a_{ij}(\nabla u(x))\nu^j(x).$$

Because (a_{ij}) is positive, $V(x)$ must point outward. Hence

$$\frac{\partial P}{\partial V}(x_0) \ge 0.$$

On the other hand, we see, from the Neumann boundary condition and homogeneity of F, that

$$\frac{\partial u}{\partial V}(x_0) = u_i a_{ij}(\nabla u(x))\nu^j = FF_j\nu^j = 0.$$

Thus we have

$$0 \le \frac{\partial P}{\partial V}(x_0) = \psi F F_i u_{ij} a_{jk} \nu^k. \tag{24}$$

We now choose a local coordinate $\{e_i\}_{i=1,\dots,n}$ around x_0 such that $e_n = \nu$ and $\{e_\alpha\}_{\alpha=1,\dots,n-1}$ is the orthonormal basis of the tangent space of $\partial\Omega$. Denote by $h_{\alpha\beta}$ the second fundamental form of $\partial\Omega$. By the assumption that $\partial\Omega$ is weakly convex, we know the matrix $(h_{\alpha\beta}) \ge 0$.

The Neumann boundary condition implies that

$$F_i\nu^i(x_0) = F_n(x_0) = 0. \tag{25}$$

By taking the tangential derivative of (25), we get

$$D_{e_\beta}\left(\sum_{i=1}^{n} F_i \nu^i\right)(x_0) = 0$$

for any $\beta = 1, \ldots, n-1$. Computing $D_{e_\beta}(\sum_{i=1}^{n} F_i \nu^i)(x_0)$ explicitly, we have

$$\begin{aligned}
0 = D_{e_\beta}\left(\sum_{i=1}^{n} F_i \nu^i\right)(x_0) &= \sum_{i,j=1}^{n} F_{ij} u_{j\beta} \nu^i + \sum_{i=1}^{n} F_i \nu^i_\beta \\
&= \sum_{i,j=1}^{n} F_{ij} u_{j\beta} \nu^i + \sum_{i=1}^{n}\sum_{\gamma=1}^{n-1} F_i h_{\beta\gamma} e^i_\gamma \\
&= \sum_{j=1}^{n} F_{nj} u_{j\beta} + \sum_{\gamma=1}^{n-1} F_\gamma h_{\beta\gamma}.
\end{aligned} \tag{26}$$

In the last equality, we used $\nu_n = 1$, and $\nu_\beta = 0$ for $\beta = 1, \ldots, n-1$ in the chosen coordinate.

Combining (24), (25), and (26), we obtain

$$\begin{aligned}
0 \le \frac{\partial P}{\partial V}(x_0) &= \sum_{i,j,k=1}^{n} \psi F F_i u_{ij} a_{jk} \nu^k = \psi F \sum_{\alpha=1}^{n-1}\sum_{j=1}^{n} F_\alpha u_{\alpha j} a_{jn} \\
&= \psi F \sum_{\alpha=1}^{n-1}\sum_{j=1}^{n} F_\alpha u_{\alpha j} F_{jn} = -\psi F \sum_{\alpha,\gamma=1}^{n-1} F_\alpha F_\gamma h_{\alpha\gamma} \le 0.
\end{aligned}$$

Therefore we obtain that $(\partial P/\partial V)(x_0) = 0$. Since the tangent derivatives of P also vanish, we have $\nabla P(x_0) = 0$. It is also the case that (19) holds. Thus the previous proof for an interior maximum also works in this case. This finishes the proof of Theorem 3.1. □

Following the idea from [Bakry and Qian 2000], besides the gradient comparison with the 1-dimensional models, in order to prove the sharp estimate on the first eigenvalue of the anisotropic Laplacian, we need to study many properties of the 1-dimensional models, such as the difference $\delta(a) = b(a) - a$ as a function of $a \in [0, +\infty]$, where $b(a)$ is the first number for which $v'(b(a)) = 0$ (Note that $v' > 0$ in $(a, b(a))$). As we already saw in Theorem 3.1, the 1-dimensional model (16) appears the same as that in the Laplacian case. Therefore, we can directly use the results of [Bakry and Qian 2000] on the properties of 1-dimensional models. Here we use some simpler statement from [Valtorta 2012].

We define $\delta(a)$ as a function of $a \in [0, +\infty]$ as follows. On the one hand, we denote $\delta(\infty) = \pi/\sqrt{\lambda_1}$. This number comes from the 1-dimensional model (16) with $T = 0$. In fact, it is easy to see that solutions of (16) with $T = 0$ can be explicitly written as

$$v(t) = \sin \sqrt{\lambda_1} t$$

up to dilations. Hence in this case, $b(a) - a = \pi/\sqrt{\lambda_1}$ for any $a \in \mathbb{R}$. On the other hand, we denote $\delta(a) = b(a) - a$ as a function of $a \in [0, +\infty)$ relative to the 1-dimensional model (16) with $T = -(n-1)/x$.

We have the following property of $\delta(a)$.

Lemma 3.1 [Bakry and Qian 2000; Valtorta 2012, Theorem 5.3, Corollary 5.4]. *The function $\delta(a) : [0, \infty] \to \mathbb{R}^+$ is a continuous function such that*

$$\delta(a) > \frac{\pi}{\sqrt{\lambda_1}} \quad \textit{and} \quad \delta(\infty) = \frac{\pi}{\sqrt{\lambda_1}}.$$

$m(a) := v(b(a)) < 1$, $\lim_{a\to\infty} m(a) = 1$, *and* $m(a) = 1$ *if and only if* $a = \infty$.

In order to prove the main result, we also need the following comparison theorem on the maximum values of eigenfunctions. This theorem is obtained as a consequence of a standard property of the volume of small balls with respect to some invariant measure; see [Bakry and Qian 2000, Section 6].

Lemma 3.2. *Let Ω, u, λ_1 be as in Theorem 1.1. Let v be a solution of the* 1-*dimensional model problem on some interval* $(0, \infty)$:

$$v'' = -\frac{n-1}{t} v' - \lambda_1 v, \quad v(0) = -1, \quad v'(0) = 0.$$

Let b be the first number after 0 *with $v'(b) = 0$ and denote $m = v(b)$. Then* $\max u \geq m$.

The proof of Lemma 3.2 is similar to that of [Bakry and Qian 2000, Theorem 11]. The essential part is the gradient comparison theorem 3.1. We omit it here.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let u be an eigenfunction with eigenvalue λ_1. Since $\int u = 0$, we may assume $\min u = -1$ and $0 \leq k = \max u \leq 1$. Given a solution v to (16), denote $m(a) = v(b(a))$ with $b(a)$ the first number with $v'(b(a)) = 0$ after a.

Lemmas 3.1 and 3.2 imply that for any eigenfunction u, there exists a solution v to (16) such that $\min v = \min u = -1$ and $\max v = \max u = k \leq 1$.

We now get the expected estimate by using Theorem 3.1. Choosing $x_1, x_2 \in \overline{\Omega}$ with $u(x_1) = \min u = -1$, $u(x_2) = \max u = k$ and $\gamma(t) : [0, 1] \to \overline{\Omega}$ the minimal geodesic from x_1 to x_2. Consider the subset I of [0,1] such that $(d/dt)u(\gamma(t)) \geq 0$.

By the gradient comparison estimate (17) and Lemma 3.1, we have

$$
\begin{aligned}
d_F &\geq \int_0^1 F^0(\dot{\gamma}(t))\,dt \geq \int_I F^0(\dot{\gamma}(t))\,dt \\
&\geq \int_0^1 \frac{1}{F(\nabla u)}\langle \nabla u, \dot{\gamma}(t)\rangle\,dt = \int_{-1}^k \frac{1}{F(\nabla u)}\,du \\
&\geq \int_{-1}^k \frac{1}{v'(v^{-1}(u))}\,du = \int_a^{b(a)} dt = \delta(a) \geq \frac{\pi}{\sqrt{\lambda_1}},
\end{aligned}
$$

which leads to

$$\lambda_1 \geq \frac{\pi^2}{d_F^2}.$$

It remains to prove the equality case. The idea of the proof follows from [Hang and Wang 2007]. Here we must pay more attention to the points with vanishing ∇u.

Assume that $\lambda_1 = \pi^2/d_F^2$. It can be easily seen from the proof of Theorem 1.1 that $a = \infty$, which leads to $\max u = \max v = 1$ by Lemma 3.1. We will prove that Ω is in fact a segment in $\mathbb{R}$. We divide the proof into several steps.

Step 1. $S := \{x \in \overline{\Omega} \mid u(x) = \pm 1\} \subset \partial\Omega$.

Let $\mathcal{P} = F(\nabla u)^2 + \lambda_1 u^2$. After a simple calculation using the Bochner formula (13) and the Kato inequality (14), we obtain

$$
\begin{aligned}
\tfrac{1}{2}a_{ij}\mathcal{P}_{ij} &= a_{ij}a_{kl}u_{ik}u_{jl} - \tfrac{1}{2}a_{ijl}u_{ij}\mathcal{P}_l - \lambda_1^2 u^2 \\
&\geq a_{ij}F_kF_l u_{ik}u_{jl} - \tfrac{1}{2}a_{ijl}u_{ij}\mathcal{P}_l - \lambda_1^2 u^2 \\
&= -\tfrac{1}{2}a_{ijl}u_{ij}\mathcal{P}_l + \frac{1}{4F^2}(a_{ij}\mathcal{P}_i\mathcal{P}_j - 4\lambda_1 u u_i \mathcal{P}_i)
\end{aligned}
$$

on $\Omega \setminus \mathscr{C}$. Namely,

$$\tfrac{1}{2}a_{ij}\mathcal{P}_{ij} + b_i\mathcal{P}_i \geq 0 \tag{27}$$

on $\Omega \setminus \mathscr{C}$ for some $b_i \in C^0(\Omega)$. If $\mathcal{P}$ attains its maximum on $x_0 \in \partial\Omega$, then arguing as in Theorem 3.1, we have $\nabla\mathcal{P}(x_0) = 0$. However, from the Hopf Theorem, $\nabla\mathcal{P}(x_0) \neq 0$, a contradiction. Hence $\mathcal{P}$ attains its maximum at $\mathscr{C}$, and therefore,

$$\mathcal{P} \leq \lambda_1. \tag{28}$$

Take any two points $x_1, x_2 \in S$ with $u(x_1) = -1$, $u(x_2) = 1$. Let

$$\gamma(t) = \Bigl(1 - \frac{t}{F^0(x_2 - x_1)}\Bigr)x_1 + \frac{t}{F^0(x_2 - x_1)}x_2 : [0, l] \to \overline{\Omega}$$

be the straight line from x_1 to x_2, where $l := F^0(x_2 - x_1)$ is the distance from x_1 to x_2 with respect to F. Denote $f(t) := u(\gamma(t))$. It is easy to see $F^0(\dot{\gamma}(t)) = 1$. It follows from (28) and the Cauchy–Schwarz inequality (7) that

$$|f'(t)| = |\nabla u(\gamma(t)) \cdot \dot{\gamma}(t)| \le F(\nabla u)(\gamma(t)) \le \sqrt{\lambda_1(1-f(t)^2)}. \tag{29}$$

Here we have used the Cauchy–Schwarz inequality (7) again. Hence

$$\begin{aligned} d_F \ge l \ge \int_{\{0\le t\le l,\, f'(t)>0\}} dt &\ge \int_0^l \frac{1}{\sqrt{\lambda_1}} \frac{f'(t)}{\sqrt{1-f(t)^2}}\, dt \\ &= \frac{1}{\sqrt{\lambda_1}} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{\sqrt{\lambda_1}}. \end{aligned} \tag{30}$$

Since $d_F = \pi/\sqrt{\lambda_1}$, we must have $d_F = l$, which means $S \subset \partial\Omega$.

Step 2. $\mathcal{P} = F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ in $\overline{\Omega}$. Hence $S \equiv \mathcal{C}$.

From Step 1, we know that $\Omega^* := \overline{\Omega} \setminus S$ is connected. Let $E := \{x \in \Omega^* : \mathcal{P} = \lambda_1\}$. It is clear that E is closed. In view of (27), thanks to the strong maximum principle, we know that E is also open. We now show that E is nonempty. Indeed, from the fact that all inequalities in (29) and (30) are equalities, we obtain $f(t) = u(\gamma(t)) = -\cos\sqrt{\lambda_1}t$ for $t \in (0, l)$. Hence

$$\mathcal{P}(\gamma(t)) = f'(t)^2 + \lambda_1 f(t)^2 = \lambda_1.$$

Thus E is nonempty, open, and closed in Ω^*. Therefore, we obtain $\mathcal{P} \equiv \lambda_1$ in $\overline{\Omega}$ (for $x \in S$, $\mathcal{P} = \lambda_1$ is obvious).

Step 3: Define $X = \nabla u / F(\nabla u)$ in Ω^* and X^* the cotangent vector given by $X^*(Y) = \langle X, Y\rangle$ for any tangent vector Y. Then, in Ω^*, we claim that

$$D^2 u = -\lambda_1 u X^* \otimes X^*, \tag{31}$$

and, moreover, $X = \vec{c}$ for some constant vector $\vec{c}$.

First, taking the derivative of $F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ gives

$$F_i F_j u_{ij} = -\lambda_1 u. \tag{32}$$

On the other hand, since $\mathcal{P} \equiv \lambda_1$, the proof of (27) leads to

$$a_{ij} a_{kl} u_{ik} u_{jl} = \lambda_1^2 u^2 = (F_i F_j u_{ij})^2. \tag{33}$$

Equation (33) in fact gives that

$$F_{ij} F_{kl} u_{ik} u_{jl} = 0. \tag{34}$$

Set $X^\perp := \{V \in \mathbb{R}^n \mid V \perp X\}$. $X^\perp$ is an $(n-1)$-dimensional vector subspace. Note that (F_{ij}) is exactly a matrix of rank $n-1$ (see the proof of Lemma 2.3) and $F_{ij} X^j = 0$. It follows from this fact and (34) that

$$u_{ij} V^i V^j = 0 \quad \text{for any } V \in X^\perp. \tag{35}$$

Equations (32) and (35) imply (31), which in turn implies

$$u_{ij} = \frac{-\lambda_1 u u_i u_j}{F^2(\nabla u)}. \tag{36}$$

By differentiating X, we obtain from (36) that

$$\nabla_i X^j = \frac{u_{ij}}{F(\nabla u)} - \frac{u_j}{F^2(\nabla u)} F_k u_{ki} = 0.$$

Thus $X = \vec{c}$ in Ω^*.

Step 4: The maximum point and the minimum point are unique.

We already knew that $f(t) = u(\gamma(t)) = -\cos\sqrt{\lambda_1}t$ and $\nabla u(\gamma(t)) \neq 0$ for $t \in (0, l)$. Hence u is C^2 along $\gamma(t)$ for $t \in (0, l)$, and it follows that

$$D^2 u(\dot\gamma(t), \dot\gamma(t))\,|_{\gamma(t)} = \lambda_1 \cos t \quad \text{for any } t \in (0, l). \tag{37}$$

On the other hand, we deduce from (31) that

$$D^2 u(\dot\gamma(t), \dot\gamma(t))\,|_{\gamma(t)} = -\lambda_1 u(\gamma(t)) \langle X, \dot\gamma(t)\rangle^2. \tag{38}$$

Combining (37) and (38), and taking $t \to 0$, we get

$$|\langle X, \dot\gamma(t)\rangle| = 1 = F(X) F^0(\dot\gamma(t)),$$

which means equality in the Cauchy–Schwarz inequality (7) holds. Hence $X = \pm F^0_\xi(\dot\gamma(t))$. Noting that $\dot\gamma(t) = x_2 - x_1 / F^0(x_2 - x_1)$, we have

$$X = F^0_\xi(x_2 - x_1).$$

Suppose there is some point x_3 with $u(x_3) = 1$. Using the same argument, we obtain $X = F^0_\xi(x_3 - x_1)$. In view of $F^0(x_3 - x_1) = F^0(x_2 - x_1)$, we conclude that $x_3 = x_2$. Therefore, there is only one maximum point and only one minimum point.

Step 5: $n = 1$ and Ω is a segment.

From Step 4, we have $\nabla u \neq 0$ for most points of $\partial\Omega$, and at these points $X = \nabla u / F(\nabla u)$ lies in the tangent spaces due to the Neumann boundary condition. This is impossible unless $n = 1$, because X is a constant vector. This completes the proof. □

4. Estimate of the first Dirichlet eigenvalue

As in Section 3, for simplicity, we write λ_1 instead of λ_1^D throughout this section.

It is well-known that the existence of first Dirichlet eigenfunction can be easily proved by using the direct method in the calculus of variations. Moreover, by the assumption that F is even, the first Dirichlet eigenfunction u does not change sign; see [Belloni et al. 2003, Theorem 3.1]. We may assume u is nonnegative. By multiplying u by a constant, we can also assume that $\sup_\Omega u = 1$ and $\inf_\Omega u = 0$ without loss of generality.

For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, $\beta^2 > \sup(\alpha + u)^2$, consider the function

$$P(x) = \frac{F^2(\nabla u)}{2(\beta^2-(\alpha+u)^2)}.$$

Suppose that $P(x)$ attains its maximum at $x_0 \in \overline{\Omega}$.

With the assumption that Ω is F-mean convex, we first exclude the possibility $x_0 \in \partial\Omega$ with $\nabla u(x_0) \neq 0$. Indeed, suppose we have $x_0 \in \partial\Omega$ with $\nabla u(x_0) \neq 0$. Define

$$\nu_F := F_\xi(\nu)$$

on $\partial\Omega = \{x \in \overline{\Omega} \mid u(x) = 0\}$. In view of $\langle \nu_F, \nu\rangle = F(\nu) > 0$, ν_F must point outward. From the Dirichlet boundary condition, we know that

$$\nu = -\nabla u/|\nabla u|$$

for $\nabla u \neq 0$. Hence $\nu_F = -F_\xi(\nabla u)$. Since P attains its maximum at x_0, we have

$$0 \leq \frac{\partial P}{\partial \nu_F}(x_0) = \frac{F F_i u_{ij} \nu_F^j}{\beta^2 - (\alpha + u)^2} + F^2 \frac{\alpha(\partial u/\partial \nu_F)}{(\beta^2 - (\alpha + u)^2)^2}$$

Hence

$$-\frac{\partial^2 u}{\partial \nu_F^2} + \frac{F\alpha(\partial u/\partial \nu_F)}{\beta^2 - \alpha^2} \geq 0.$$

Note that $\partial u/\partial \nu_F = -F(\nabla u)$. Since $\partial\Omega$ itself is a level set of u, we can apply Lemma 2.4 to obtain

$$\frac{\partial^2 u}{\partial \nu_F^2} = Qu + F H_F.$$

In view of $Qu(x_0) = -\lambda_1 u(x_0) = 0$, we obtain that

$$-F H_F - F^2 \frac{\alpha}{\beta^2-\alpha^2} \geq 0.$$

This contradicts the fact that $H_F(\partial\Omega) \geq 0$.

On the other hand, if $\nabla u(x_0) = 0$, $F(\nabla u)(x_0) = 0$ and $P(x_0) = 0$, which implies $F(\nabla u) = 0$, that is, u is constant, a contradiction.

Therefore we may assume $x_0 \in \Omega$ and $\nabla u(x_0) \neq 0$. Since a_{ij} is positively definite on $\overline{\Omega} \setminus \mathscr{C}$, where $\mathscr{C} := \{x \mid \nabla u(x) = 0\}$, it follows from the maximum principle that

$$P_i(x_0) = 0, \tag{39}$$

$$a_{ij}(x_0) P_{ij}(x_0) \leq 0. \tag{40}$$

From now on we will compute at the point x_0. Equality (39) gives

$$\frac{\partial}{\partial x_i}\bigl(\tfrac{1}{2}F^2(\nabla u)\bigr) = -\frac{F^2(\nabla u)(\alpha+u)u_i}{\beta^2-(\alpha+u)^2}. \tag{41}$$

Then we compute $a_{ij}(x_0)P_{ij}(x_0)$.

$$\begin{aligned} a_{ij}(x_0)P_{ij}(x_0) &= \frac{1}{\beta^2-(\alpha+u)^2}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\bigl(\tfrac{1}{2}F^2(\nabla u)\bigr) \\ &\qquad + 2a_{ij}\frac{\partial}{\partial x_i}\bigl(\tfrac{1}{2}F^2(\nabla u)\bigr)\frac{\partial}{\partial x_j}\Bigl(\frac{1}{\beta^2-(\alpha+u)^2}\Bigr) \\ &\qquad\qquad + a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\Bigl(\frac{1}{\beta^2-(\alpha+u)^2}\Bigr)\tfrac{1}{2}F^2(\nabla u) \\ &= I + II + III. \end{aligned}$$

By using (41), (21), the Bochner formula (13), and Equation (2), we obtain

$$I = \frac{1}{\beta^2-(\alpha+u)^2}[a_{ij}a_{kl}u_{ik}u_{jl} - \lambda_1 F^2], \tag{42}$$

$$II = -\frac{4F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^3}, \tag{43}$$

$$III = \frac{F^4}{(\beta^2-(\alpha+u)^2)^2} + \frac{4F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^3} - \frac{\lambda_1 F^2 u(\alpha+u)}{(\beta^2-(\alpha+u)^2)^2}. \tag{44}$$

We now apply Lemma 2.2 to (42) and obtain

$$\begin{aligned} a_{ij}a_{kl}u_{ik}u_{jl} &\geq a_{ij}F_kF_lu_{ik}u_{jl} \\ &= \frac{1}{F^2}a_{ij}\frac{\partial}{\partial x_i}\bigl(\tfrac{1}{2}F^2(\nabla u)\bigr)\frac{\partial}{\partial x_j}\bigl(\tfrac{1}{2}F^2(\nabla u)\bigr) \\ &= \frac{F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^2}. \end{aligned}$$

Here we have used (41) and (21) again in the last equality. Therefore, we have

$$I \geq \frac{F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^3} - \frac{\lambda_1 F^2}{\beta^2-(\alpha+u)^2}. \tag{45}$$

Combining (40), (43), (44), and (45), we obtain

$$0 \geq a_{ij}P_{ij} \geq \frac{F^4\beta^2}{(\beta^2-(\alpha+u)^2)^3} - \frac{\lambda_1 F^2}{\beta^2-(\alpha+u)^2} - \frac{\lambda_1 F^2 u(\alpha+u)}{(\beta^2-(\alpha+u)^2)^2}.$$

It follows that

$$\frac{F^2(\nabla u)}{\beta^2-(\alpha+u)^2}(x_0) \leq \frac{\lambda_1}{\beta^2}(\beta^2-\alpha(\alpha+u)). \tag{46}$$

Noting that $\sup_\Omega u = 1$, we choose $\alpha > 0$ and $\beta = \alpha + 1$. Then estimate (46) becomes

$$\frac{F^2(\nabla u)}{(\alpha+1)^2-(\alpha+u)^2}(x_0) \le \lambda_1\Big(1-\frac{\alpha(\alpha+u)}{(\alpha+1)^2}\Big) \le \lambda_1.$$

Hence we conclude that

$$\frac{F^2(\nabla u)}{(\alpha+1)^2-(\alpha+u)^2} \le \lambda_1. \tag{47}$$

for any $x \in \overline{\Omega}$.

Choose $x_1 \in \Omega$ with $u(x_1) = \sup u = 1$ and $x_2 \in \partial\Omega$ with

$$d_F(x_1, x_2) = d_F(x_1, \partial\Omega) \le i_F$$

and $\gamma(t) : [0, 1] \to \overline{\Omega}$ the minimal geodesic connecting x_1 with x_2. Using the gradient estimates (47), we have

$$\begin{aligned}\frac{\pi}{2} - \arcsin\Big(\frac{\alpha}{\alpha+1}\Big) &= \int_0^1 \frac{1}{\sqrt{(\alpha+1)^2-(\alpha+u)^2}}\,du \le \sqrt{\lambda_1}\int_0^1 \frac{1}{F(\nabla u)}\,du\\ &\le \sqrt{\lambda_1}\int_0^1 \frac{1}{F(\nabla u(\gamma(t)))}\langle \nabla u(\gamma(t)), \dot\gamma(t)\rangle\,dt\\ &\le \sqrt{\lambda_1}\int_0^1 F^0(\dot\gamma(t))\,dt \le \sqrt{\lambda_1}\,i_F.\end{aligned}$$

Here we have used the Cauchy–Schwarz inequality (7). Letting $\alpha \to 0$, we obtain

$$\lambda_1 \ge \frac{\pi^2}{4i_F^2}.$$

Thus we finish the proof of Theorem 1.2.

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