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CONVERGENCE OF AXIALLY SYMMETRIC VOLUME-PRESERVING MEAN CURVATURE FLOW

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We study the convergence of axially symmetric hypersurfaces evolving by volume-preserving mean curvature flow. Assuming the surfaces do not develop singularities along the axis of rotation at any time during the flow, and without any additional conditions, as for example on the curvature, we prove that the flow converges to a hemisphere, when the initial hypersurface has a free boundary and satisfies Neumann boundary data, and to a sphere when it is compact without boundary.

1. Introduction

Consider n-dimensional hypersurfaces M_t, defined by a one-parameter family of smooth immersions $\boldsymbol{x}_t : M^n \to \mathbb{R}^{n+1}$. The hypersurfaces M_t are said to *move by mean curvature* if $\boldsymbol{x}_t = \boldsymbol{x}(\,\cdot\,, t)$ satisfies

$$\frac{d}{dt}\boldsymbol{x}(p,t) = -H(p,t)\nu(p,t), \quad p \in M^n, t > 0, \tag{1-1}$$

where $\nu(p,t)$ denotes a smooth choice of unit normal on M_t at $\boldsymbol{x}(p,t)$ (outer normal in case of compact surfaces without boundary), and $H(p,t)$ the mean curvature with respect to this normal.

If in addition the evolving compact surfaces M_t are assumed to enclose a prescribed volume V, the corresponding evolution equation is

$$\frac{d}{dt}\boldsymbol{x}(p,t) = -\big(H(p,t) - h(t)\big)\nu(p,t), \quad p \in M^n, t > 0, \tag{1-2}$$

where $h(t)$ is the average of the mean curvature,

$$h(t) = \frac{\int_{M_t} H\,dg_t}{\int_{M_t} dg_t},$$

and g_t denotes the metric on M_t. As under the flow (1-1), the surface area $|M_t|$ of the hypersurface is known to decrease under (1-2), while the enclosed volume remains constant in the latter; see [Athanassenas 1997].

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We are interested in an axially symmetric surface, enclosing a given volume V, and which has a nonempty boundary contained in a plane Π that is perpendicular to the axis of rotation. Motivated by the fact that the stationary solution to the associated variational problem satisfies a Neumann boundary condition, we also assume the surface to meet that plane Π at right angles along its boundary. Assuming the surface to be smooth, it will also intersect the axis of rotation orthogonally.

We consider the case where the evolving hypersurfaces do not develop singularities, in particular they do not pinch off along the axis of rotation during the flow, having only one intersection with that axis at the point that is the furthest from the supporting plane Π, and prove that the flow converges to a half-sphere.

The methods we use apply also in the case of evolving axially symmetric hypersurfaces without boundary having a similar lower height bound, and in that case we prove in Section 8 that the flow converges to a sphere.

The results in this paper make use of the axial symmetry, but no additional conditions on the curvature of the surface are assumed. Convergence to spheres has been previously proved for the volume flow in [Huisken 1987], for compact, uniformly convex initial surfaces, while Li [2009] assumes bounds on the traceless second fundamental form.

Our results can be seen as complementing the work in [Athanassenas 1997; 2003], and in the PhD dissertation [Kandanaarachchi 2011]: in the case of the surface behaving like a "bridge" between two parallel surfaces, if one were able to flow through singularities, the axially symmetric volume-preserving flow would converge to a number of spheres and (possibly) two hemispheres on the parallel planes, like beads strung along the axis of rotation.

2. Notation, definitions and assumptions

In the case of the surface M_t intersecting the obstacle Π, we will at different stages divide it into two parts as in [Altschuler et al. 1995]: one adjacent to the plane and one containing the (only) intersection with the axis of rotation.

Let $\Pi = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = 0\}$ and let M_t be contained in the right half-space, $M_t \subset \{x_1 > 0\}$. We use R_t as the generic notation for the part of the surface closest to the plane, and C_t for the rest — the cap that intersects the axis of rotation — and we will introduce various superscripts depending on the situation that will be made clear in the text.

We denote by $P(t) = (d(t), 0)$ the *pole*: the point of intersection of M_t with the axis of rotation. We assume that there are no singularities developing, so that $P(t)$ is the only point of intersection of M_t with the axis of rotation for all time. We are interested in those solutions where the generating curve of the initial hypersurface is smooth and can be written as a graph over the x_1 axis except at the pole.

We use the notation

$$\rho_t : [0, d(t)] \to \mathbb{R}$$

for the radius function of the surface of revolution.

Let $\boldsymbol{i}_1, \dots, \boldsymbol{i}_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$ and let $\boldsymbol{i}_1$ be the direction of the axis of rotation. We denote the quantities associated with the cap with a tilde $\tilde{}$, and in this context we work with the vertical graph equation [Altschuler et al. 1995].

Furthermore we define the following quantities on M_t:

Let $\omega = \hat{\boldsymbol{x}}/|\hat{\boldsymbol{x}}| \in \mathbb{R}^{n+1}$, $\hat{\boldsymbol{x}} = (0, x_2, \dots, x_{n+1})$, denote the outer unit normal to the cylinder intersecting M_t at the point $\boldsymbol{x}(p, t)$. We call $u = \langle \boldsymbol{x}, \omega \rangle$ the *height function* of M_t, and set $v = \langle \nu, \omega \rangle^{-1}$. Note that v corresponds to $\sqrt{1 + \dot{\rho}^2}$; it will be used to obtain gradient estimates.

The corresponding quantities on the cap C_t are the height measured from the plane Π, $\tilde{u} = \langle \boldsymbol{x}, \boldsymbol{i}_1 \rangle$ and $\tilde{v} = \langle \nu, \boldsymbol{i}_1 \rangle^{-1}$.

We divide the hypersurface into two regions using a plane $L_\alpha(t)$, which is parallel to Π and intersects the surface at points where $\langle \nu, \boldsymbol{i}_1 \rangle_{|L_\alpha(t) \cap M_t} = 1/\alpha$, with $\alpha > 1$ being a constant. We define the *cap*, determined by the inclination angle, as the connected component of M_t containing the pole P,

$$C_t^\alpha = \{\boldsymbol{x}(p, t) \in M_t : 1/\alpha < \langle \nu, \boldsymbol{i}_1 \rangle \le 1\},$$

and we call $R_t^\alpha = M_t \backslash C_t^\alpha$ the *cylindrical part* of the surface. Note that $L_\alpha(t)$ is chosen such that the specific inclination angle is achieved nowhere else between that plane and the pole $P(t)$. As long as the flow is smooth, C_t^α is by definition a graph over the x_1 axis except at the pole. We denote by $l_\alpha(t)$ the x_1 coordinate of $L_\alpha(t)$, so $L_\alpha(t) = \{x_1 = l_\alpha(t)\}$.

Assumption 2.1. *We assume that for any $\alpha > 1$ there exists a constant $c(\alpha) > 0$, depending only on α, such that $u_{|R_t^\alpha} > c(\alpha)$, that is, we assume a lower height bound in R_t^α, independent of time, dependent on α.*

Thus $P(t)$ is assumed to be the only point of intersection of M_t with the axis of rotation for all time. The assumption prevents singularities from developing on the axis of rotation.

For an axially symmetric surface the mean curvature is given by

$$H = -\frac{\ddot{\rho}}{(1+\dot{\rho}^2)^{3/2}} + \frac{n-1}{\rho(1+\dot{\rho}^2)^{1/2}},$$

while the principal curvatures are $k = -\ddot{\rho}/(1 + \dot{\rho}^2)^{3/2}$ and $p = 1/(\rho\sqrt{1 + \dot{\rho}^2})$.

We also introduce another quantity, $q = \langle \nu, \boldsymbol{i}_1 \rangle\, u^{-1}$; thus $p^2 + q^2 = u^{-2}$.

3. Height estimates

In this section we prove that M_t satisfies uniform height bounds: both the height function u defined above and the height when measured as distance from the obstacle Π, denoted by $\tilde{u}$, are bounded. That is then used to show that the length of the generating curve of the surface remains bounded.

Lemma 3.1. *The evolving surfaces M_t satisfy the uniform height bound*

$$u \leq R = \big(|M_0|/\omega_n\big)^{1/n}.$$

Proof. We follow a method used in [Athanassenas 1997] to get bounds for u. Given $R > 0$, assume that $u_{M_t} \geq R$ at some given time t. Since the surface area is decreasing under the flow, and by comparing to the projection of the surface onto the plane Π, we have

$$|M_0| \geq |M_t| > \omega_n R^n,$$

where ω_n is the volume of the n dimensional unit ball. Therefore $R > (|M_0|/\omega_n)^{1/n}$ would contradict the fact that the evolution decreases the surface area. □

Lemma 3.2. *There is a constant l such that the evolving surfaces M_t satisfy the height bound $\tilde{u} \leq l$, that is, the distance from the plane Π is uniformly bounded.*

Proof. Here $\alpha = 1/\cos\theta$,

$$C_t^\alpha = \{\boldsymbol{x}(p,t) \in M_t : 1/\alpha < \langle \nu, \boldsymbol{i}_1 \rangle \leq 1\},$$

and $R_t^\alpha = M_t \backslash C_t^\alpha$. From Assumption 2.1, we know that $u > c(\alpha)$ in R_t^α. As $u_{|\partial C_t^\alpha} \leq R$ by Lemma 3.1 and $|\dot{\rho}| \geq \tan(\frac{\pi}{2} - \theta) = 1/\sqrt{\alpha^2 - 1}$ in C_t^α, we have

$$d(t) - \tilde{u}_{|\partial C_t^\alpha} \leq R\tan\theta = R\sqrt{\alpha^2 - 1}.$$

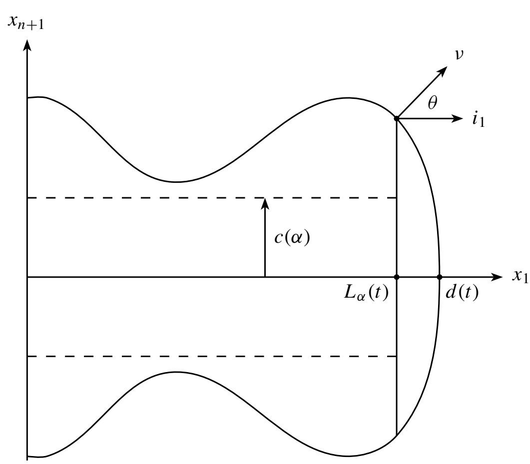

Figure 1. The cylinder of radius $c(\alpha)$.

Assume there exists a length l_1 such that $\tilde{u}_{|R_t^\alpha} > l_1$. Then

$$|M_0| \geq |M_t| > n\omega_n c^{n-1}(\alpha) l_1,$$

where now we compared $|M_t|$ to the surface area of an n dimensional cylinder of radius $c(\alpha)$ and length l_1. Having

$$l_1 > \frac{|M_0|}{n\omega_n c^{n-1}(\alpha)}$$

would contradict the fact that the evolution decreases the surface area. Therefore

$$\tilde{u} < \frac{|M_0|}{n\omega_n c^{n-1}(\alpha)} + R\sqrt{\alpha^2 - 1} =: l. \qquad \square$$

Next we show that the length of the generating curve is bounded.

Lemma 3.3. *Assume M_t to be smooth, axially symmetric hypersurfaces, evolving by (1-2) and with a radius function satisfying $\rho(x_1, t) > 0$ for $x_1 \in [0, d(t))$. Then there exists a constant c_*, independent of time, such that*

$$\int_0^{d(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq c_*.$$

Proof. Let us divide M_t into R_t^α and C_t^α for any $\alpha > 1$. As the surface area is decreasing under the flow, $|M_t| \leq |M_0|$, we have

$$2\pi \int_0^{d(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq |M_0|,$$

$$2\pi \int_0^{l_\alpha(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq 2\pi \int_0^{d(t)} \rho^{n-1} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq |M_0|.$$

From Assumption 2.1,

$$2\pi c^{n-1}(\alpha) \int_0^{l_\alpha(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq |M_0| \quad \text{and} \quad \int_0^{l_\alpha(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)}.$$

We can estimate the length of the generating curve of the cap C_t^α by $l + R$. Therefore

$$\int_0^{d(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)} + l + R =: c_*. \qquad \square$$

4. Estimates on *h*

We now derive *a priori* estimates for $h(t)$ for solutions of the graphical equation.

Lemma 4.1. *Assume M_t to be smooth, axially symmetric hypersurfaces, evolving by (1-2) and with a radius function satisfying $\rho(x_1, t) > 0$ for $x_1 \in [0, d(t))$. Then there is a constant c_1 such that $0 \leq h(t) \leq c_1$ throughout the flow.*

Proof. Following [Athanassenas 2003] we parametrize M_t by its radius function $\rho \in C^\infty([0, d(t)))$, then clearly

$$H = -\frac{\ddot{\rho}}{(1+\dot{\rho}^2)^{3/2}} + \frac{n-1}{\rho(1+\dot{\rho}^2)^{1/2}}.$$

From Lemma 3.3, we know that $\int_0^{d(t)} \sqrt{1+\dot{\rho}^2}\, dx_1 \le c_*$. Our proof follows the ideas of [Athanassenas 1997], the difference being the boundary term when integrating by parts. For the sake of completeness we include it here. For the second term of

$$h(t) = \frac{1}{|M_t|}\int_{M_t} \big(k + (n-1)p\big)\, dg_t, \quad t \in [0, T),$$

we have

$$0 \le \frac{n-1}{|M_t|}\int_0^{d(t)} \rho^{n-2}(x_1, t)\, dx_1 \le \frac{(n-1)R^{n-2} l}{|M_t|},$$

since $\rho \le R$ and $d(t) \le l$ by Lemmas 3.1 and 3.2.

For the first term note that $\ddot{\rho}/(1+\dot{\rho}^2) = \frac{d}{dx_1}(\arctan \dot{\rho})$. Therefore

$$\begin{aligned} \text{(4-1)} \quad \int_{M_t} k\, dg_t &= -\int_0^{d(t)} \frac{d}{dx_1}(\arctan \dot{\rho})\rho^{n-1} dx_1 \\ &= (\arctan \dot{\rho})\rho^{n-1}|_{x_1=0} - (\arctan \dot{\rho})\rho^{n-1}|_{x_1=d(t)} \\ &\qquad + (n-1)\int_0^{d(t)} (\arctan \dot{\rho})\dot{\rho}\rho^{n-2} dx_1 \\ &= (n-1)\int_0^{d(t)} (\arctan \dot{\rho})\dot{\rho}\rho^{n-2} dx_1, \end{aligned}$$

because $\arctan \dot{\rho} = 0$ when $x_1 = 0$, and we have $\rho(d(t)) = 0$ at the pole. Since $0 \le (\arctan \dot{\rho})\dot{\rho} \le \frac{\pi}{2}|\dot{\rho}| \le \frac{\pi}{2}\sqrt{1+\dot{\rho}^2}$, we obtain

$$\begin{aligned} 0 \le \frac{1}{|M_t|}\int_{M_t} k\, dg_t &\le \frac{(n-1)}{|M_t|}\frac{\pi}{2}\int_0^{d(t)} \sqrt{1+\dot{\rho}^2}\rho^{n-2} dx_1 \\ &\le \frac{(n-1)R^{n-2}}{|M_t|}\frac{\pi}{2}\int_0^{d(t)} \sqrt{1+\dot{\rho}^2}\, dx_1 \le \frac{(n-1)c_* R^{n-2}}{|M_t|}\frac{\pi}{2}, \end{aligned}$$

where we have used Lemma 3.3.

From the isoperimetric inequality and the fact that the flow decreases surface area we know that

$$V^{n/(n+1)} < c|M_t| \le c|M_0|.$$

Combining these arguments we conclude that $0 \le \dfrac{\int H dg}{\int dg} \le c_1$. □

5. Evolution equations and gradient estimates

The maximum principle for noncylindrical or time dependent domains is discussed in [Lumer and Schnaubelt 1999]. We use that version of the maximum principle in this paper.

Lemma 5.1. *For the flow (1-2) we have the following evolution equations*:

(i) $\left(\frac{d}{dt}-\Delta\right)u = h/v-(n-1)/u$.

(ii) $\left(\frac{d}{dt}-\Delta\right)\tilde{u} = h/\tilde{v}$.

(iii) $\left(\frac{d}{dt}-\Delta\right)v = -|A|^2v+(n-1)v/u^2-(2/v)|\nabla v|^2$.

(iv) $\left(\frac{d}{dt}-\Delta\right)\tilde{v} = -|A|^2\tilde{v}-(2/\tilde{v})|\nabla\tilde{v}|^2$.

(v) $\left(\frac{d}{dt}-\Delta\right)H = (H-h)|A|^2$.

(vi) $\left(\frac{d}{dt}-\Delta\right)|A|^2 = -2|\nabla A|^2+2|A|^4-2hC$.

(vii) $\left(\frac{d}{dt}-\Delta\right)p = |A|^2p+2q^2(k-p)-hp^2$.

(viii) $\left(\frac{d}{dt}-\Delta\right)k = |A|^2k-2(n-1)q^2(k-p)-hk^2$.

where $C=g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}$, with g^{ij} denoting the components of the inverse of the first fundamental form, *and h_{ij} those of the second fundamental form.*

Proof. (i) and (iii) are proved in [Athanassenas 1997]; (v) and (vi) in [Huisken 1987].

(ii) For $\tilde{u}=\langle \boldsymbol{x},\boldsymbol{i}_1\rangle$ we have

$$\frac{d}{dt}\tilde{u}=\left\langle\frac{d}{dt}\boldsymbol{x},\boldsymbol{i}_1\right\rangle=-(H-h)\langle\nu,\boldsymbol{i}_1\rangle \quad\text{and}\quad \Delta\tilde{u}=\langle\Delta\boldsymbol{x},\boldsymbol{i}_1\rangle=-H\langle\nu,\boldsymbol{i}_1\rangle,$$

so that

$$\left(\frac{d}{dt}-\Delta\right)\tilde{u}=h\langle\nu,\boldsymbol{i}_1\rangle.$$

(iv) For $\tilde{v}=\langle\nu,\boldsymbol{i}_1\rangle^{-1}$ we have

$$\frac{d}{dt}\tilde{v}=-\tilde{v}^2\left\langle\frac{d}{dt}\nu,\boldsymbol{i}_1\right\rangle=-\tilde{v}^2\langle\nabla H,\boldsymbol{i}_1\rangle\,.$$

The evolution equation follows from the well-known identity [Ecker and Huisken 1989]

$$\Delta\tilde{v}=-\tilde{v}^2\langle\nabla H,\boldsymbol{i}_1\rangle+\tilde{v}|A|^2+2\tilde{v}^{-1}\nabla\tilde{v}^2\,.$$

(vii) Using the same approach as in [Huisken 1990], we start with

$$\begin{aligned}\frac{d}{dt}p&=\frac{d}{dt}(u^{-2}-q^2)^{1/2}\\ &=\Delta p+p^{-1}|\nabla p|^2+p^{-1}|\nabla q|^2-3p^{-1}u^{-4}|\nabla u|^2+p^{-1}u^{-4}\\ &\qquad -qp^{-1}\left(|A|^2q+q(p^2-q^2-2kp)\right)-hu^{-2}+hq^2.\end{aligned}$$

Equation (vii) follows then from the relations

$$\text{(5-1)}\qquad \nabla_i u=\delta_{i1}qu,\quad \nabla_1\langle \nu,\boldsymbol{i}_1\rangle=kpu,\quad \nabla_i q=\delta_{i1}(q^2+kp),$$

$$\text{(5-2)}\quad \nabla_i p=\delta_{i1}q(p-k),\quad |A|^2=k^2+(n-1)p^2,\quad u^{-4}=p^4+2p^2q^2+q^4.$$

(viii) The evolution equation for H was derived in [Huisken 1987], and (viii) follows from (v), (vii), and the fact that $H=k+(n-1)p$. □

We proceed to obtain gradient estimates in the different parts of the surface: for the cap by using the vertical graph equation and part (iv) from Lemma 5.1 above, and for the cylindrical part away from the cap by using the evolution equation (iii) in Lemma 5.1.

The quantities $\tilde{u}$ and $\tilde{v}$ are used on the cap.

Lemma 5.2. *Assume M_t to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then the gradient estimate $\tilde{v}\le\alpha$ holds on the cap C_t^α. In addition, there is a constant $c_2(\alpha)$ such that $v\le c_2(\alpha)$ for the cylindrical part R_t^α.*

Proof. Note that

$$\left(\frac{d}{dt}-\Delta\right)\tilde{v}\le 0,$$

so that by the maximum principle $\tilde{v}\le\max(\max_{C_0^\alpha}\tilde{v},\max_{\partial C_t^\alpha}\tilde{v})$. By definition in C_t^α we have $\tilde{v}\le\alpha$, and this is supported by the evolution equation!

From the assumption we know that $u>c(\alpha)$ in R_t^α. As in [Athanassenas 1997, Proposition 4] we calculate

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)u^2v&=-|A|^2u^2v+(n-1)v+2uh-2(n-1)v-2v|\nabla u|^2-\frac{2}{v}\nabla v\nabla(u^2v)\\&\le 2hu-(n-1)v.\end{aligned}$$

If $v>2c_1R/(n-1)$ the right side is negative, and proceeding as in [Athanassenas 1997] we conclude $v\le c_2(\alpha)$ in R_t^α. It is important to note that on the boundary of R_t^α, either $v=1$ (along the intersection with Π), or $v=\alpha/\sqrt{\alpha^2-1}$. Thereby, we have bounds for v and $\tilde{v}$ in R_t^α and C_t^α respectively. □

Remark 5.3. (i) *The gradient bounds from Lemma 5.2 guarantee that R_t^α remains a graph. As C_t^α remains a graph for all $\alpha>1$, we see that $M_t\backslash P(t)$ remains a graph throughout the flow.*

(ii) *As the height of the graph is bounded we find a lower bound for the minimum $d(t)$ from*

$$V=\int_0^{d(t)}\omega_n\rho^n(x)\,dx_1\le\omega_nR^n\int_0^{d(t)}dx_1=\omega_nR^nd(t).$$

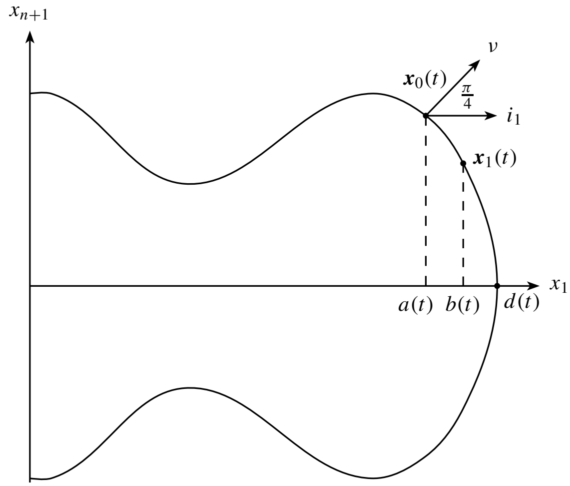

Figure 2. $\boldsymbol{x}_0(t)$ — the boundary point of $C_t^{\sqrt{2}}$.

Lemma 5.4. *Assume the M_t to be axially symmetric surfaces as described in* Section 2 *and to evolve by* (1-2). *Let $\boldsymbol{x}_0(t)$ be a boundary point of $C_t^{\sqrt{2}}$, which we can assume without loss of generality to lie on the generating curve and be such that $\langle \nu(\boldsymbol{x}_0(t)), \boldsymbol{i}_1\rangle = 1/\sqrt{2}$ (with some abuse of notation for the corresponding normal $\nu(\boldsymbol{x}_0(t))$). Then $H(\boldsymbol{x}_0(t)) \geq 0$ for $0 \leq t \leq T_{\max} \leq \infty$.*

Proof. Suppose $H(\boldsymbol{x}_0(t)) < 0$, then by continuity there is a connected region $C_t^{\sqrt{2},H^-} \subset C_t^{\sqrt{2}}$, with $\boldsymbol{x}_0(t) \in \partial C_t^{\sqrt{2},H^-}$, which clearly can be chosen to be axially symmetric, and such that $H_{|C_t^{\sqrt{2},H^-}} < 0$. Let $\boldsymbol{x}_1(t)$ denote the other boundary point along the generating curve in $C_t^{\sqrt{2},H^-} \subset C_t^{\sqrt{2}}$, and let $a(t) = \langle \boldsymbol{x}_0(t), \boldsymbol{i}_1\rangle$ and $b(t) = \langle \boldsymbol{x}_1(t), \boldsymbol{i}_1\rangle$ denote the x_1 coordinate of $\boldsymbol{x}_0(t)$, $\boldsymbol{x}_1(t)$, respectively. Then

$$0 > \int_{C_t^{\sqrt{2},H^-}} H\,dg = \int_{a(t)}^{b(t)} \Big(-\frac{\ddot{\rho}}{1+\dot{\rho}^2}\rho^{n-1} + (n-1)\rho^{n-2}\Big)\,dx_1\,.$$

The second term being positive means that the first is negative, and given the bounds on the radius we find

$$\int_{a(t)}^{b(t)} \Big(-\frac{\ddot{\rho}}{1+\dot{\rho}^2}\Big)\,dx_1 = \int_{a(t)}^{b(t)} \Big(-\frac{d}{dx_1}(\arctan\dot{\rho})\Big)\,dx_1 < 0\,.$$

This results in

$$\arctan\dot{\rho}(a(t)) < \arctan\dot{\rho}(b(t)) \quad \text{and} \quad -\frac{\pi}{4} < \arctan\dot{\rho}(b(t)),$$

by the choice of $a(t)$. But this is not possible in $C_t^{\sqrt{2}}$, where $-\frac{\pi}{2} \leq \arctan\dot{\rho} < -\frac{\pi}{4}$, contradicting our assumption and therefore $H(\boldsymbol{x}_0(t)) \geq 0$. □

6. Curvature estimates

Proposition 6.1. *Assume M_t to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then there is a constant c_2 depending only on the initial hypersurface, such that the principal curvatures k and p satisfy $k/p < c_2$, independently of time.*

Proof. We calculate from Lemma 5.1 that

$$\frac{d}{dt}\left(\frac{k}{p}\right)=\Delta\frac{k}{p}+\frac{2}{p}\nabla_i p\nabla_i\left(\frac{k}{p}\right)+2\frac{q^2}{p^2}(p-k)\big((n-1)p+k\big)+\frac{hk}{p}(p-k).$$

If $k/p\geq 1$ then $(hk/p)\,(p-k)<0$. This implies that

$$\frac{k}{p}\leq\max\Big(1,\max_{M_0}\frac{k}{p}\Big). \tag{6-1}$$

Note that for this consideration, the smooth function k/p is defined over the whole surface, and in view of the orthogonality on the boundary, via a reflection argument there are no boundary data involved. □

Proposition 6.2. *Assume M_t to be axially symmetric surfaces as described in Section 2 that evolve by (1-2) and let A be the second fundamental form. Then there exists a constant c_3, independent of time, such that $|A|^2\leq c_3$.*

Proof. We proceed as in [Ecker and Huisken 1991] and [Athanassenas 1997] and calculate the evolution equation for the product $g=|A|^2\varphi(v^2)$ in $R_t^{\sqrt{2}}$, where $\varphi(r)=r/(\lambda-\mu r)$, with $v=\langle\nu,\omega\rangle^{-1}$ and appropriately chosen constants $\lambda,\mu>0$. From the evolution equation of g we find the inequality

$$\begin{aligned}\Big(\frac{d}{dt}-\Delta\Big)g&\\ &\leq-2\mu g^2-2\lambda\varphi v^{-3}\nabla v\cdot\nabla g-\frac{2\lambda\mu}{(\lambda-\mu v^2)^2}|\nabla v|^2g-2hC\varphi(v^2)+\frac{2(n-1)}{u^2}v^2\varphi'|A|^2.\end{aligned}$$

We estimate the second last term as in [Athanassenas 1997] using Young's inequality and obtain

$$\begin{aligned}-2hC\varphi(v^2)&\leq 2h|A|^3\varphi(v^2)\\ &\leq\tfrac{3}{2}|A|^4\varphi(v^2)+\tfrac{1}{2}h^4\varphi^{-2}(v^2)=\tfrac{3}{2}g^2+\tfrac{1}{2}h^4\varphi^{-2}(v^2).\end{aligned}$$

We choose $\mu>\frac{3}{4}$ and $\lambda>\mu\max v^2$. As $\varphi'v^2=\dfrac{\lambda}{(\lambda-\mu v^2)^2}\,\varphi$ we have

$$\frac{2(n-1)}{u^2}v^2\varphi'|A|^2=\frac{2(n-1)\lambda}{u^2(\lambda-\mu v^2)}g.$$

As $u>c(1/\sqrt{2})=c_0$ in $R_t^{\sqrt{2}}$ we get

$$\frac{2(n-1)\lambda}{u^2(\lambda-\mu v^2)}g\leq c_4 g.$$

Therefore we have

$$\begin{aligned}\Big(\frac{d}{dt}-\Delta\Big)g &\le -c_5 g^2 + c_6 g - c_7 \nabla v\cdot\nabla g + c_8(h,\max v)\\ &\le -c_5\Big(g-\frac{c_6}{2c_5}\Big)^2 - c_7\nabla v\cdot\nabla g + c_9.\end{aligned}$$

The right side of this inequality is negative at a maximum of g, where

$$g > \frac{c_6}{2c_5} + \sqrt{\frac{c_9}{c_5}}.$$

On $\partial R_t^{\sqrt{2}}$ we have $H = k + (n-1)p \ge 0$ by Lemma 5.4. Also, as $k/p < c_2$, we get $|k|/p < c$ on this boundary and thus we have

$$|A|^2 = k^2 + (n-1)p^2 \le (c^2 + n - 1)p^2 \le C\rho^{-2} \le Cc_0^{-2}$$

on $\partial R_t^{\sqrt{2}}$. By the maximum principle,

$$g \le \max\Big(\max_{R_0^{\sqrt{2}}} g, \max_{\partial R_t^{\sqrt{2}}} |A|^2\varphi(v^2)\Big).$$

Since $v \le c_2(\sqrt{2})$ and $\varphi(v^2)$ is bounded, we have a bound for g in $R_t^{\sqrt{2}}$.

The evolution equation for $\tilde g = |A|^2\varphi(\tilde v^2)$ on $C_t^{\sqrt{2}}$ is the same as the one for g without the last term on the right side. Thereby we obtain a bound for $\tilde g$ in the same way as above. □

Proposition 6.3. *Assume M_t to be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then for each $m \ge 1$ there is a C_m such that*

$$|\nabla^m A|^2 \le C_m\,,$$

uniformly on M_t, for $0 \le t \le T_{\max} \le \infty$.

Proof. Having obtained uniform bounds on $|A|^2$ and h the proof is a repetition of that of Theorem 4.1 in [Huisken 1987]. □

Thus we have long-time existence for the flow:

Corollary 6.4. *Let M_t be axially symmetric surfaces as described in Section 2 that evolve by (1-2). Then*

$$T_{\max} = \infty.$$

7. Convergence to surfaces of constant mean curvature

Having long-time existence, Proposition 8 of [Athanassenas 1997] gives convergence to a constant mean curvature surface, which in our case is axially symmetric. By the classification of the Delaunay surfaces [1841] it has to be a half-sphere.

8. Other convergence results

Using the same estimates with very few changes one can show that a compact, axially symmetric surface without boundary, which encloses a volume V and intersects the axis only at two endpoints throughout the flow by (1-2), will converge to a sphere. We will only explain the parts that are different from the previous result.

We denote the surface again by M_t. We split it into a cylindrical part R_t^α and two caps $C_t^{(\alpha,i)}$, for $i=1,2$, in this case. The left side cap, $C_t^{(\alpha,1)}$, intersects the axis of rotation at $x_1=e(t)$, while the (only other) intersection on the right for $C_t^{(\alpha,2)}$ is located at $x_1=d(t)$. Assumption 2.1 holds on M_t.

Height estimates. The height estimates of Lemmas 3.1 and 3.2 change as follows:

Lemma 8.1. *Assume M_t to be axially symmetric, compact without boundary and evolving by (1-2). Then the height function u satisfies $u<R=(|M_0|/2\omega_n)^{1/n}$.*

Proof. Given $R>0$ assume that $u_{M_t}\geq R$ at some given time t. Take a plane perpendicular to the x_1-axis and intersecting the surface. This plane divides the surface into two parts, and by projecting both parts onto it we find

$$|M_0|\geq |M_t|>2\omega_n R^n.$$

Taking $R>\left(\dfrac{|M_0|}{2\omega_n}\right)^{1/n}$ would contradict the fact that the evolution decreases the surface area. □

The next lemma gives an estimate for the diameter of M_t in the x_1 direction.

Lemma 8.2. *Assume M_t to be smooth, axially symmetric, compact without boundary and evolving by (1-2). Then*

$$d(t)-e(t)<l=\frac{|M_0|}{n\omega_n c_0^{n-1}}+2R.$$

Proof. As in Lemma 3.2 let $\alpha=1/\cos\theta$. From Assumption 2.1 we know that $u>c(\alpha)$ in R_t^α. As $u_{|\partial C_t^{\alpha,i}}\leq R$ and $|\dot\rho|\geq\tan\left(\frac{\pi}{2}-\theta\right)$ in $C_t^{\alpha,i}$ for $i=1,2$, we have

$$d(t)-\tilde u_{|\partial C_t^{\alpha,1}}\leq R\tan\theta=R\sqrt{\alpha^2-1},$$

$$\tilde u_{|\partial C_t^{\alpha,2}}-e(t)\leq R\tan\theta=R\sqrt{\alpha^2-1}.$$

Assume there exists a length l_1 such that $\tilde u_{|R_t^\alpha}>l_1$. Then by the previous argument,

$$|M_0|\geq |M_t|>n\omega_n c^{n-1}(\alpha)l_1,$$

where now we compared $|M_t|$ to the surface area of an n dimensional cylinder of radius $c(\alpha)$ and length l_1. If $l_1>|M_0|/(n\omega_n c^{n-1}(\alpha))$ this would contradict the fact that the evolution decreases the surface area. Therefore

$$\tilde{u} < \frac{|M_0|}{n\omega_n c_0^{n-1}} + 2R\sqrt{\alpha^2 - 1}. \qquad \square$$

Again we can estimate the length of the generating curve throughout the flow:

Lemma 8.3. *Assume M_t to be smooth, axially symmetric, compact without boundary, evolving by (1-2) and with a radius function satisfying $\rho(x_1, t) > 0$ for x_1 in $(e(t), d(t))$. Then there exists a constant c_*, independent of time, such that*

$$\int_0^{d(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq c_*.$$

Proof. The proof is the same as that of Lemma 3.3 after taking into account the two caps on either side. Here we have

$$\int_{e(t)}^{d(t)} \sqrt{1 + \dot{\rho}^2}\, dx_1 \leq \frac{|M_0|}{2\pi c^{n-1}(\alpha)} + 2l + 2R =: c_*. \qquad \square$$

Lemma 8.4 (estimates on h). *Assume M_t to be smooth, axially symmetric, compact without boundary, evolving by (1-2) and with a radius function that satisfies $\rho(x_1, t) > 0$ for $x_1 \in (e(t), d(t))$. Then there is a constant c_1 such that $0 \leq h(t) \leq c_1$ throughout the flow.*

Proof. The only change to the proof of Lemma 4.1 is in the boundary values when integrating by parts in (4-1). Here the new boundary values are

$$(\arctan \dot{\rho})\rho^{n-1}|_{x_1 = a(t)} - (\arctan \dot{\rho})\rho^{n-1}|_{x_1 = b(t)}.$$

As $\rho(a(t)) = \rho(b(t)) = 0$, the boundary terms disappear and we get the same estimate for h. $\square$

Lemma 8.5 (gradient estimates). *Under the above assumptions, the gradient estimate $|\tilde{v}| \leq \alpha$ holds on the caps $C_t^{\alpha,i}$, $i = 1, 2$. In addition there is a constant c, such that $v \leq c$ for the cylindrical part R_t^{α}.*

Proof. The gradient estimates are as in Lemma 5.2, but in this setting instead of one cap C_t^{α} we have two caps on either side, and the same estimate holds for both caps. $\square$

Concluding this section, we remark that $H \geq 0$ at points where the caps $C_t^{\sqrt{2},i}$, $i = 1, 2$, meet the cylindrical part $R_t^{\sqrt{2}}$ of the surface. The proof is using the same arguments as the one for Lemma 5.4 after the appropriate adjustments of the sign of $\arctan \dot{\rho}$ for the cap on the left of the surface. The results on *curvature estimates* and the *convergence* to a limiting surface of constant mean curvature follow along the same lines as previously proved. In this case the limit surface is a sphere.

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