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RAYS AND SOULS IN VON MANGOLDT PLANES

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We study rays in von Mangoldt planes, which has applications to the structure of open complete manifolds with lower radial curvature bounds. We prove that the set of souls of any rotationally symmetric plane of nonnegative curvature is a closed ball, and if the plane is von Mangoldt we compute the radius of the ball. We show that each cone in $\mathbb{R}^3$ can be smoothed to a von Mangoldt plane.

1. Introduction

Let M_m denote $\mathbb{R}^2$ equipped with a smooth, complete, rotationally symmetric Riemannian metric given in polar coordinates as $g_m := dr^2 + m^2(r)\,d\theta^2$; let o denote the origin in $\mathbb{R}^2$. We say that M_m is a *von Mangoldt plane* if its sectional curvature $G_m := -m''/m$ is a nonincreasing function of r.

The Toponogov comparison theorem was extended in [Itokawa et al. 2003] to open complete manifolds with radial sectional curvature bounded below by the curvature of a von Mangoldt plane, leading to various applications in [Shiohama and Tanaka 2002; Kondo and Ohta 2007; Kondo and Tanaka 2011] and generalizations in [Mashiko and Shiohama 2006; Kondo and Tanaka 2010; Machigashira 2010].

A point q in a Riemannian manifold is called a *critical point of infinity* if each unit tangent vector at q makes angle $\leq \pi/2$ with a ray that starts at q. Let $\mathfrak{C}_m$ denote the set of critical points of infinity of M_m; clearly $\mathfrak{C}_m$ is a closed, rotationally symmetric subset that contains every pole of M_m, so that $o \in \mathfrak{C}_m$. One reason for studying $\mathfrak{C}_m$ is the following consequence of the generalized Toponogov theorem of [Itokawa et al. 2003].

Lemma 1.1. *Let $\hat{M}$ be a complete noncompact Riemannian manifold with radial curvature bounded below by the curvature of a von Mangoldt plane M_m, and let $\hat{r}$ and r denote the distance functions to the basepoints $\hat{o}$ and o of $\hat{M}$ and M_m, respectively. If $\hat{q}$ is a critical point of $\hat{r}$, then $\hat{r}(\hat{q})$ is contained in $r(\mathfrak{C}_m)$.*

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Combined with the critical point theory of distance functions [Grove 1993; Greene 1997, Lemma 3.1; Petersen 2006, §11.1], Lemma 1.1 implies the following.

Corollary 1.2. *In the setting of Lemma 1.1, for any c in $[a, b] \subset r(M_m - \mathfrak{C}_m)$,*

- *the $\hat{r}^{-1}$-preimage of $[a, b]$ is homeomorphic to $\hat{r}^{-1}(a) \times [a, b]$, and the $\hat{r}^{-1}$-preimages of points in $[a, b]$ are all homeomorphic,*
- *the $\hat{r}^{-1}$-preimage of $[0, c]$ is homeomorphic to a compact smooth manifold with boundary, and the homeomorphism maps $\hat{r}^{-1}(c)$ onto the boundary,*
- *if $K \subset \hat{M}$ is a compact smooth submanifold, possibly with boundary, such that $\hat{r}(K) \supset r(\mathfrak{C}_m)$, then $\hat{M}$ is diffeomorphic to the normal bundle of K.*

If M_m is von Mangoldt and $G_m(0) \leq 0$, then $G_m \leq 0$ everywhere, so every point is a pole, and hence $\mathfrak{C}_m = M_m$ so that Lemma 1.1 yields no information about the critical points of $\hat{r}$. Of course, there are other ways to get this information, as illustrated by classical Gromov's estimate: if M_m is the standard $\mathbb{R}^2$, then the set of critical points of $\hat{r}$ is compact; see, for example, [Greene 1997, p. 109].

The following theorem determines $\mathfrak{C}_m$ when $G_m \geq 0$ everywhere; note that the plane M_m in (i)–(iii) need not be von Mangoldt.

Theorem 1.3. *If $G_m \geq 0$, then:*

(i) *$\mathfrak{C}_m$ is the closed R_m-ball centered at o for some $R_m \in [0, \infty]$.*

(ii) *R_m is positive if and only if $\int_1^\infty m^{-2}$ is finite.*

(iii) *R_m is finite if and only if $m'(\infty) < \frac{1}{2}$.*

(iv) *If M_m is von Mangoldt and R_m is finite, then the equation $m'(r) = \frac{1}{2}$ has a unique solution ρ_m, and the solution satisfies $\rho_m > R_m$ and $G_m(r_m) > 0$.*

(v) *If M_m is von Mangoldt and R_m is finite and positive, then R_m is the unique solution of the integral equation*

$$\int_x^\infty \frac{m(x)\,dr}{m(r)\sqrt{m^2(r) - m^2(x)}} = \pi.$$

Here is a sample application of Theorem 1.3 (iv) and Corollary 1.2:

Corollary 1.4. *Let $\hat{M}$ be a complete noncompact Riemannian manifold with radial curvature from the basepoint $\hat{o}$ bounded below by the curvature of a von Mangoldt plane M_m. If $G_m \geq 0$ and $m'(\infty) < \frac{1}{2}$, then $\hat{M}$ is homeomorphic to the metric ρ_m-ball centered at $\hat{o}$, where ρ_m is the unique solution of $m'(r) = \frac{1}{2}$.*

Theorem 1.3 should be compared with the following results of Tanaka:

- The set of poles in any M_m is a closed metric ball centered at o of some radius R_p in $[0, \infty]$ [Tanaka 1992b, Lemma 1.1].

- $R_p > 0$ if and only if $\int_1^\infty m^{-2}$ is finite and $\liminf_{r\to\infty} m(r) > 0$ [Tanaka 1992a].
- If M_m is von Mangoldt, then R_p is a unique solution of an explicit integral equation [Tanaka 1992a, Theorem 2.1].

It is natural to wonder when the set of poles equals $\mathfrak{C}_m$, and we answer the question when M_m is von Mangoldt.

Theorem 1.5. *If M_m is a von Mangoldt plane, then*:

(a) *If R_p is finite and positive, then the set of poles is a proper subset of the component of $\mathfrak{C}_m$ that contains o.*

(b) *$R_p = 0$ if and only if $\mathfrak{C}_m = \{o\}$.*

Of course $R_p = \infty$ implies $\mathfrak{C}_m = M_m$, but the converse is not true: Theorem 1.11 ensures the existence of a von Mangoldt plane with $m'(\infty) = \frac{1}{2}$ and $G_m \geq 0$, and for this plane $\mathfrak{C}_m = M_m$ by Theorem 1.3, while R_p is finite by Remark 4.7.

We say that a ray γ in M_m *points away from infinity* if γ and the segment $[\gamma(0), o]$ make an angle $< \frac{\pi}{2}$ at $\gamma(0)$. Define $A_m \subset M_m - \{o\}$ as follows: $q \in A_m$ if and only if there is a ray that starts at q and points away from infinity; by symmetry, $A_m \subset \mathfrak{C}_m$.

Theorem 1.6. *If M_m is a von Mangoldt plane, then A_m is open in M_m.*

Any plane M_m with $G_m \geq 0$ has another distinguished subset, namely the set of souls, that is, points produced via the soul construction of Cheeger–Gromoll.

Theorem 1.7. *If $G_m \geq 0$, then $\mathfrak{C}_m$ is equal to the set of souls of M_m.*

Recall that the soul construction takes as input a basepoint in an open complete manifold N of nonnegative sectional curvature and produces a compact totally convex submanifold S without boundary, called a *soul*, such that N is diffeomorphic to the normal bundle to S. Thus if N is contractible, as happens for M_m, then S is a point. The soul construction also gives a continuous family of compact totally convex subsets that starts with S and ends with N, and according to [Mendonça 1997, Proposition 3.7] $q \in N$ is a critical point of infinity if and only if there is a soul construction such that the associated continuous family of totally convex sets drops in dimension at q. In particular, any point of S is a critical point of infinity, which can also be seen directly; see the proof of [Maeda 1974/1975, Lemma 1]. In Theorem 1.7 we prove conversely that every point of $\mathfrak{C}_m$ is a soul; for this M_m need not be von Mangoldt.

In regard to Theorem 1.3 (iii), it is worth mentioning $G_m \geq 0$ implies that m' is nonincreasing, so $m'(\infty)$ exists, and moreover, $m'(\infty) \in [0, 1]$ because $m \geq 0$. As we note in Remark A.5 for any von Mangoldt plane M_m, the limit $m'(\infty)$ exists as a number in $[0, \infty]$. It follows that if $G_m \geq 0$ or if M_m is von Mangoldt, then

M_m admits total curvature, which equals $2\pi(1 - m'(\infty))$ and hence takes values in $[-\infty, 2\pi]$; thus $m'(\infty) = \frac{1}{2}$ if and only if M_m has total curvature π. Standard examples of von Mangoldt planes of positive curvature are the one-parametric family of paraboloids, all satisfying $m'(\infty) = 0$ [Shiohama et al. 2003, Example 2.1.4], and the one-parametric family of two-sheeted hyperboloids parametrized by $m'(\infty)$, which takes every value in (0, 1) [Shiohama et al. 2003, Example 2.1.4].

A property of von Mangoldt planes, discovered in [Elerath 1980; Tanaka 1992b] and crucial to this paper, is that the cut locus of any $q \in M_m - \{o\}$ is a ray that lies on the meridian opposite q. (If M_m is not von Mangoldt, its cut locus is not fully understood, but it definitely can be disconnected [Tanaka 1992a, p. 266], and known examples of cut loci of compact surfaces of revolution [Gluck and Singer 1979; Sinclair and Tanaka 2006] suggest that it could be complicated.)

As we note in Lemma 3.14, if M_m is a von Mangoldt plane, and if $q \neq o$, then $q \in \mathfrak{C}_m$ if and only if the geodesic tangent to the parallel through q is a ray. Combined with Clairaut's relation this gives the following "choking" obstruction for a point q to belong to $\mathfrak{C}_m$ (see Lemma 3.3):

Proposition 1.8. *If M_m is von Mangoldt and $q \in \mathfrak{C}_m$, then $m'(r_q) > 0$ and $m(r) > m(r_q)$ for $r > r_q$, where r_q is the r-coordinate of q.*

The above proposition is immediate from Lemmas 3.3 and 3.14. We also show in Lemma 3.10 that if M_m is von Mangoldt and $\mathfrak{C}_m \neq o$, then there is ρ such that $m(r)$ is increasing and unbounded on $[\rho, \infty)$.

The following theorem collects most of what we know about $\mathfrak{C}_m$ for a von Mangoldt plane M_m with some negative curvature, where the case $\liminf_{r\to\infty} m(r) = 0$ is excluded because then $\mathfrak{C}_m = \{o\}$ by Proposition 1.8.

Theorem 1.9. *If M_m is a von Mangoldt plane with a point where $G_m < 0$ and such that* $\liminf_{r\to\infty} m(r) > 0$, *then*

(1) *M_m contains a line and has total curvature $-\infty$,*

(2) *if m' has a zero, then neither A_m nor $\mathfrak{C}_m$ is connected,*

(3) *$M_m - A_m$ is a bounded subset of M_m,*

(4) *the ball of poles of M_m has positive radius.*

In Example 6.1 we construct a von Mangoldt plane M_m to which Theorem 1.9 (2) applies. In Example 6.2 we produce a von Mangoldt plane M_m such that neither A_m nor $\mathfrak{C}_m$ is connected while $m' > 0$ everywhere. We do not know whether there is a von Mangoldt plane such that $\mathfrak{C}_m$ has more than two connected components.

Because of Lemma 1.1 and Corollary 1.2, one is interested in subintervals of $(0, \infty)$ that are disjoint from $r(\mathfrak{C}_m)$, as, for example, happens for any interval on which $m' \leq 0$, or for the interval (R_m, ∞) in Theorem 1.3. To this end we prove the following result, which is a consequence of Theorem 6.3.

Theorem 1.10. *Let M_n be a von Mangoldt plane with $G_n \geq 0$, $n(\infty) = \infty$, and such that $n'(x) < \frac{1}{2}$ for some x. Then for any $z > x$ there exists $y > z$ such that if M_m is a von Mangoldt plane with $n = m$ on $[0, y]$, then $r(\mathfrak{C}_m)$ and $[x, z]$ are disjoint.*

In general, if M_m and M_n are von Mangoldt planes with $n = m$ on $[0, y]$, then the sets $\mathfrak{C}_m$ and $\mathfrak{C}_n$ could be quite different. For instance, if M_n is a paraboloid, then $\mathfrak{C}_n = \{o\}$, but by Example 6.2 for any $y > 0$ there is a von Mangoldt M_m with some negative curvature such that $m = n$ on $[0, y]$, and by Theorem 1.9 the set $M_m - \mathfrak{C}_m$ is bounded and $\mathfrak{C}_m$ contains the ball of poles of positive radius.

Basic properties of von Mangoldt planes are described in Appendix A. In particular, in order to construct a von Mangoldt plane with prescribed G_m it suffices to check that 0 is the only zero of the solution of the Jacobi initial value problem (A.7) with $K = G_m$, where G_m is smooth on $[0, \infty)$. Prescribing values of m' is harder. It is straightforward to see that if M_m is a von Mangoldt plane such that m' is constant near infinity, then $G_m \geq 0$ everywhere and $m'(\infty) \in [0, 1]$. We do not know whether there is a von Mangoldt plane with $m' = 0$ near infinity, but all the other values in $(0, 1]$ can be prescribed:

Theorem 1.11. *For every $s \in (0, 1]$ there is $\rho > 0$ and a von Mangoldt plane M_m such that $m' = s$ on $[\rho, \infty)$.*

Thus each cone in $\mathbb{R}^3$ can be smoothed to a von Mangoldt plane, but we do not know how to construct a (smooth) capped cylinder that is von Mangoldt.

Structure of the paper. We collect notations and conventions in Section 2. Properties of von Mangoldt planes are reviewed in Appendix A, while Appendix B contains a calculus lemma relevant to continuity and smoothness of the turn angle. Section 3 contains various results on rays in von Mangoldt planes, including the proofs of Theorem 1.6 and Proposition 1.8. Planes of nonnegative curvature are discussed in Section 4, where Theorems 1.3 and 1.7 are proved. A proof of Theorem 1.11 is in Section 5, and the other results stated in the introduction are proved in Section 6.

2. Notations and conventions

All geodesics are parametrized by arclength. Minimizing geodesics are called *segments*. Let ∂_r and ∂_θ denote the vector fields dual to dr and $d\theta$ on $\mathbb{R}^2$. Given $q \neq o$, denote its polar coordinates by θ_q and r_q. Let γ_q, μ_q, and τ_q denote the geodesics defined on $[0, \infty)$ that start at q in the directions of ∂_θ, ∂_r, and $-\partial_r$, respectively. We refer to $\tau_q|_{(r_q, \infty)}$ as the *meridian opposite* q; note that $\tau_q(r_q) = o$. Also set $\kappa_{\gamma(s)} := \angle(\dot{\gamma}(s), \partial_r)$.

We write $\dot{r}$, $\dot{\theta}$, $\dot{\gamma}$, and $\dot{\kappa}$ for the derivatives of $r_{\gamma(s)}$, $\theta_{\gamma(s)}$, $\gamma(s)$, and $\kappa_{\gamma(s)}$ by s, and write m' for dm/dr; similar notations are used for higher derivatives.

Let $\hat{\kappa}(r_q)$ denote the maximum of the angles formed by μ_q and rays emanating from $q \neq o$; let ξ_q denote the ray with $\xi_q(0) = q$ for which the maximum is attained, that is, such that $\kappa_{\xi_q(0)} = \hat{\kappa}(r_q)$.

A geodesic γ in M_m–$\{o\}$ is called *counterclockwise* if $\dot{\theta} > 0$ and *clockwise* if $\dot{\theta} < 0$. A geodesic in M_m is clockwise, counterclockwise, or can be extended to a geodesic through o. If γ is clockwise, then it can be mapped to a counterclockwise geodesic by an isometric involution of M_m.

Convention. Unless stated otherwise, any geodesic in M_m that we consider is either tangent to a meridian or counterclockwise.

Due to this convention the Clairaut constant and the turn angle defined below are nonnegative, which will simplify notations.

3. Turn angle and rays in M_m

This section collects what we know about rays in M_m with emphasis on the cases when $G_m \geq 0$ or $G'_m \leq 0$. Let γ be a geodesic in M_m that does not pass through o, so that γ is a solution of the geodesic equations

$$\ddot{r} = mm'\dot{\theta}^2, \quad \dot{\theta}m^2 = c, \tag{3.1}$$

where c is called the *Clairaut constant of* γ. The equation $\dot{\theta}m^2 = c$ is the so-called *Clairaut's relation*, which, since γ is assumed counterclockwise, can be written as $c = m(r_{\gamma(s)}) \sin \kappa_{\gamma(s)}$. Thus $0 \leq c \leq m(r_\gamma(s))$ where $c = m(r_\gamma(s))$ only at points where γ is tangent to a parallel, and $c = 0$ when γ is tangent to a meridian.

A geodesic is called *escaping* if its image is unbounded; for example, any ray is escaping.

Fact 3.2. (1) A parallel through q is a geodesic in M_m if and only if $m'(r_q) = 0$ [Shiohama et al. 2003, Lemma 7.1.4].

(2) A geodesic γ in M_m is tangent to a parallel at $\gamma(s_0)$ if and only if $\dot{r}_{\gamma(s_0)} = 0$.

(3) If γ is a geodesic in M_m and $\dot{r}_{\gamma(s)}$ vanishes more than once, then γ is invariant under a rotation of M_m about o [Shiohama et al. 2003, Lemma 7.1.6] and hence not escaping.

Lemma 3.3. *If γ_q is escaping, then $m(r) > m(r_q)$ for $r > r_q$, and $m'(r_q) > 0$.*

Proof. Since γ_q is escaping, the image of $s \to r_{\gamma_q}(s)$ contains $[r_q, \infty)$, and q is the only point where γ_q is tangent to a parallel. The Clairaut constant of γ_q is $c = m(r_q)$, hence $m(r) > m(r_q)$ for all $r > r_q$. It follows that $m'(r_q) \geq 0$. Finally, $m'(r_q) \neq 0$ else γ_q would equal the parallel through q. □

Lemma 3.4. *If γ is an escaping geodesic that is tangent to the parallel P_q through q, then $\gamma \setminus \{q\}$ lies in the unbounded component of $M_m \setminus P_q$.*

Proof. By reflectional symmetry and uniqueness of geodesics, γ locally stays on the same side of the parallel P_q through q, that is, γ is the union of γ_q and its image under the reflecting fixing $\mu_q \cup \tau_q$. If γ could cross to the other side of P_q at some point $\gamma(s)$, then $|r_{\gamma(s)} - r_q|$ would attain a maximum between $\gamma(s)$ and q, and at the maximum point γ would be tangent to a parallel. Since γ is escaping, it cannot be tangent to parallels more than once, hence γ stays on the same side of P_q at all times, and since γ is escaping, it stays in the unbounded component of $M_m \setminus P_q$. □

For a geodesic $\gamma : (s_1, s_2) \to M_m$ that does not pass through o, we define the *turn angle T_γ of γ* as

$$T_\gamma := \int_\gamma d\theta = \int_{s_1}^{s_2} \dot{\theta}_{\gamma(s)}\, ds = \theta_{\gamma(s_2)} - \theta_{\gamma(s_1)}.$$

Clairaut's relation reads $\dot{\theta} = c/m^2 \geq 0$ so the above integral T_γ converges to a number in $[0, \infty]$. Since γ is unit speed, we have $(\dot{r})^2 + m^2\dot{\theta}^2 = 1$. Combining this with $\dot{\theta} = c/m^2$ gives

$$\dot{r} = \operatorname{sign}(\dot{r})\sqrt{1 - \frac{c^2}{m^2}},$$

which yields a useful formula for the turn angle: if γ is not tangent to a meridian or a parallel on (s_1, s_2), so that $\operatorname{sign}(\dot{r}_{\gamma(s)})$ is a nonzero constant, then

$$\text{(3.5)} \qquad \frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \operatorname{sign}(\dot{r}_{\gamma(s)})F_c(r) \quad \text{where} \quad F_c := \frac{c}{m\sqrt{m^2 - c^2}},$$

and thus if $r_i := r_{\gamma(s_i)}$, then

$$\text{(3.6)} \qquad T_\gamma = \operatorname{sign}(\dot{r}) \int_{r_1}^{r_2} F_c(r)\, dr.$$

Since $c^2 \leq m^2$, this integral is finite except possibly when some r_i is in the set $\{m^{-1}(c), \infty\}$. The integral (3.6) converges at $r_i = m^{-1}(c)$ if and only if $m'(r_i) \neq 0$. Convergence of (3.6) at $r_i = \infty$ implies convergence of $\int_1^\infty m^{-2}\, dr$, and the converse holds under the assumption $\liminf_{r\to\infty} m(r) > c$; this assumption is true when $G_m \geq 0$ or $G'_m \leq 0$, as follows from Lemma 3.10.

Example 3.7. If γ is a ray in M_m that does not pass through o, then $T_\gamma \leq \pi$ else there is s with $|\theta_{\gamma(s)} - \theta_{\gamma(0)}| = \pi$, and by symmetry the points $\gamma(s)$ and $\gamma(0)$ are joined by two segments, so γ would not be a ray.

Example 3.8. If T_{γ_q} is finite, then $m'(r_q) \neq 0$ and m^{-2} is integrable on $[1, \infty)$, as follows immediately from the discussion preceding Example 3.7.

Lemma 3.9. *If $\gamma : [0, \infty) \to M_m$ is a geodesic with finite turn angle, then γ is escaping.*

Proof. Note that γ is tangent to parallels in at most two points, for otherwise γ is invariant under a rotation about o, and hence its turn angle is infinite. Thus after cutting off a portion of γ we may assume it is never tangent to a parallel, so that $r_{\gamma(s)}$ is monotone. By assumption $\theta_{\gamma(s)}$ is bounded and increasing. By Clairaut's relation $m(r_{\gamma(s)})$ is bounded below, so that $m(0) = 0$ implies that $r_{\gamma(s)}$ is bounded below. If γ were not escaping, then $r_{\gamma(s)}$ would also be bounded above, so there would exist a limit of $(r_{\gamma(s)}, \theta_{\gamma(s)})$ and hence the limit of $\gamma(s)$ as $s \to \infty$, contradicting the fact that γ has infinite length. □

Lemma 3.10. *If m^{-2} is integrable on $[1, \infty)$, then*

(1) *the function $(r \log r)^{-\frac{1}{2}} m(r)$ is unbounded,*

(2) *if $G_m \geq 0$, then $m' > 0$ for all r,*

(3) *if M_m is von Mangoldt, then $m' > 0$ for all large r,*

(4) *if either $G_m \geq 0$ or $G'_m \leq 0$, then $m(\infty) = \infty$.*

Proof. Since m^{-2} is integrable, the function $(r \log r)^{-\frac{1}{2}} m(r)$ is unbounded, and in particular, m is unbounded. If $G_m \geq 0$ everywhere, then m' is nonincreasing with $m'(0) = 1$, and the fact that m is unbounded implies that $m' > 0$ for all r. If M_m is von Mangoldt, and $G_m(\rho_0) < 0$, then $G_m < 0$ for $r \geq \rho_0$, that is, m' is nondecreasing on $[\rho_0, \infty)$. Since m is unbounded, there is $\rho > \rho_0$ with $m(\rho) > m(\rho_0)$ so that $\int_{\rho_0}^{\rho} m' = m(\rho) - m(\rho_0) > 0$. Hence m' is positive somewhere on (ρ_0, ρ), and therefore on $[\rho, \infty)$. Finally, since m is an unbounded increasing function for large r, the limit $\lim_{r\to\infty} m(r) = m(\infty)$ exists and equals ∞. □

Lemma 3.11. *If γ_q is escaping, then $\liminf_{r\to\infty} m(r) > m(r_q)$ if and only if there is a neighborhood U of q such that γ_u is escaping for each $u \in U$.*

Proof. First, recall that $m(r) > m(r_q)$ for $r > r_q$ and $m'(r_q) > 0$ by Lemma 3.3. We shall prove the contrapositive: $\liminf_{r\to\infty} m(r) = m(r_q)$ if and only if there is a sequence $u_i \to q$ such that γ_{u_i} is not escaping.

If there is a sequence $z_i \in M_m$ with $r_{z_i} \to \infty$ and $m(r_{z_i}) \to m(r_q)$, then there are points $u_i \to q$ on μ_q with $m(r_{u_i}) = m(r_{z_i})$. If γ_{u_i} is escaping, then it meets the parallel through z_i, so Clairaut's relation implies that γ_{u_i} is tangent to the parallels through u_i and z_i, which cannot happen for an escaping geodesic.

Conversely, suppose there are $u_i \to q$ such that $\gamma_i := \gamma_{u_i}$ is not escaping. Let R_i be the radius of the smallest ball about o that contains γ_i, and let P_i be its boundary parallel. Note that $R_i \to \infty$ as γ_i converges to γ_q on compact sets and γ_q is escaping, and hence $\liminf_{r\to\infty} m(r) = \lim_{r\to\infty} m(R_i)$. For each i there is a sequence $s_{i,j}$ such that the r-coordinates of $\gamma_i(s_{i,j})$ converge to R_i, which implies

$\kappa_{\gamma_i(s_{i,j})} \to \pi/2$ as $j \to \infty$ and i is fixed. (Note that if γ_i is tangent to P_i, then $s_{i,j}$ is independent of j, namely, $\gamma(s_{i,j})$ is the point of tangency.) By Clairaut's relation, $m(R_i) = m(r_{u_i})$, hence $\liminf_{r\to\infty} m(r) = m(r_q)$. □

Lemma 3.12. *If M_m is von Mangoldt, then a geodesic $\gamma : [0, \infty) \to M_m \setminus \{o\}$ is a ray if and only if $T_\gamma \le \pi$.*

Proof. The "only if" direction holds even when M_m is not von Mangoldt by Example 3.7. Conversely, if γ is not a ray, then γ meets the cut locus of q, which by [Tanaka 1992b] is a subset of the opposite meridian $\tau_{\gamma(0)}|_{(r_{\gamma(0)},\infty)}$. Thus $T_\gamma > \pi$. □

Lemma 3.13. *If γ is a ray in a von Mangoldt plane, and if σ is a geodesic with $\sigma(0) = \gamma(0)$ and $\kappa_{\gamma(0)} > \kappa_{\sigma(0)}$, then σ is a ray and $T_\sigma \le T_\gamma$.*

Proof. Set $q = \gamma(0)$. If $\kappa_{\gamma(0)} = \pi$, then $\gamma = \tau_q$, so τ_q is a ray, which in a von Mangoldt plane implies that q is a pole [Shiohama et al. 2003, Lemma 7.3.1], so that σ is also a ray. If $\kappa_{\gamma(0)} < \pi$ and σ is not a ray, then σ is minimizing until it crosses the opposite meridian $\tau_q|_{(r_q,\infty)}$ [Tanaka 1992b]. Near q the geodesic σ lies in the region of M_m bounded by γ and μ_q hence before crossing the opposite meridian σ must intersect γ or μ_q, so they would not be rays. Finally, $T_\sigma \le T_\gamma$ holds as σ lies in the sector between γ and μ_q. □

Lemma 3.14. *If M_m is von Mangoldt and $q \ne o$, then γ_q is a ray if and only if $q \in \mathfrak{C}_m$.*

Proof. If γ_q is a ray, then $q \in \mathfrak{C}_m$ by symmetry. If $q \in \mathfrak{C}_m$, then either q is a pole and there is a ray in any direction, or q is not a pole. In the latter case τ_q is not a ray [Shiohama et al. 2003, Lemma 7.3.1], hence by the definition of $\mathfrak{C}_m$ there is a ray γ with $\kappa_{\gamma(0)} \ge \pi/2$, so γ_q is a ray by Lemma 3.13. □

Recall that $\hat{\kappa}(r_q)$ is the maximum of the angles formed by μ_q and rays emanating from $q \ne o$, and ξ_q is the ray for which the maximum is attained. It is immediate from definitions that $q \in \mathfrak{C}_m$ if and only if $\hat{\kappa}(r_q) \ge \pi/2$. Lemmas 3.15, 3.16, and 3.17 were suggested by the referee.

Lemma 3.15. $\mathfrak{C}_m \ne \{o\}$ *if and only if* $\liminf_{r\to\infty} m > 0$ *and* $\int_1^\infty m^{-2}$ *is finite.*

Proof. The "if" direction holds because by the main result of [Tanaka 1992a] the assumptions imply that the ball of poles has a positive radius. Conversely, if $q \in \mathfrak{C}_m - \{o\}$, then ξ_q is a ray different from μ_q. By [Tanaka 1992a, Lemma 1.3 and Proposition 1.7] if either $\liminf_{r\to\infty} m = 0$ or $\int_1^\infty m^{-2} = \infty$, then μ_q is the only ray emanating from q. □

Lemma 3.16. *The limit of the segments $[q, \tau_q(s)]$ as $s \to \infty$ is ξ_q.*

Proof. The segments $[q, \tau_q(s)]$ subconverge to a ray σ that starts at q. Since ξ_q is a ray, it cannot cross the opposite meridian $\tau_q|_{(r_q,\infty)}$. As $[q, \tau_q(s)]$ and ξ_q are minimizing, they only intersect at q, and hence the angle formed by μ_q and $[q, \tau_q(s)]$ is $\geq \hat{\kappa}(r_q)$. It follows that $\kappa_{\sigma(0)} \geq \hat{\kappa}(r_q)$, which must be an equality as $\hat{\kappa}(r_q)$ is a maximum, so $\sigma = \xi_q$. □

Lemma 3.17. *The function $r \to \hat{\kappa}(r)$ is left continuous and upper semicontinuous. In particular, the set $\{q : \hat{\kappa}(r_q) < \alpha\}$ is open for every α.*

Proof. If $\hat{\kappa}$ is not left continuous at r_q, then there exists $\varepsilon > 0$ and a sequence of points q_i on μ_q such that $r_{q_i} \to r_q-$ and either $\hat{\kappa}(r_{q_i}) - \hat{\kappa}(r_q) > \varepsilon$ or $\hat{\kappa}(r_q) - \hat{\kappa}(r_{q_i}) > \varepsilon$. In the former case ξ_{q_i} subconverge to a ray that makes a larger angle with μ_q than ξ_q, contradicting the maximality of $\hat{\kappa}(r_q)$. In the latter case, ξ_{q_i} intersects ξ_q for some i. Therefore, by Lemma 3.16 the segment $[q_i, \tau_q(s)]$ intersects $[q, \tau_q(s)]$ for large enough s at a point $z \neq \tau_q(s)$, so $\tau_q(s)$ is a cut point of z which cannot happen for a segment. This proves that $\hat{\kappa}$ is left continuous. A similar argument shows that

$$\limsup_{r_{q_i} \to r_q^+} \hat{\kappa}(r_{q_i}) \leq \hat{\kappa}(r_q),$$

so that $\hat{\kappa}$ is upper semicontinuous, which implies that $\{q : \hat{\kappa}(r_q) < \alpha\}$ is open for every α. □

Lemmas 3.12 and 3.14 imply that on a von Mangoldt plane $\hat{\kappa}(r_q) \geq \pi/2$ if and only if $T_{\gamma_q} \leq \pi$; the equivalence is sharpened in Theorem 3.24, whose proof occupies the rest of this section.

Lemma 3.18. *If σ is escaping and $0 < \kappa_{\sigma(0)} \leq \pi/2$, then $T_\sigma = \int_{r_q}^{\infty} F_c(r)\,dr$; moreover, if $\kappa_{\sigma(0)} = \pi/2$, then $c = m(r_q)$.*

Proof. This formula for T_σ is immediate from (3.6) once it is shown that $\sigma|_{(0,\infty)}$ is not tangent to a meridian or a parallel. If $\sigma|_{(0,\infty)}$ were tangent to a meridian, $\kappa_{\sigma(0)}$ would be 0 or π, which is not the case. Since σ is escaping, Fact 3.2 implies that σ is tangent to a parallel at most once; that is, $\dot{r}_\sigma$ has at most one zero. If $\kappa_{\sigma(0)} = \pi/2$, then σ is tangent to the parallel through $\sigma(0)$, and so $\sigma|_{(0,\infty)}$ is not tangent to a parallel. Finally, if $\kappa_{\sigma(0)} < \pi/2$, then σ is not tangent to a parallel, else it would be tangent to a parallel through u with $r_u > r_q$, which would imply $r_{\sigma(s)} \leq r_u$ for all s by Lemma 3.4, which cannot happen for an escaping geodesic. □

To better understand the relationship between $\hat{\kappa}(r_q)$ and T_{γ_q}, we study how T_σ depends on σ, or equivalently on $\sigma(0)$ and $\kappa_{\sigma(0)}$, when σ varies in a neighborhood of a ray γ_q.

Lemma 3.19. *If $G_m \geq 0$ or $G'_m \leq 0$, then the function $u \to T_{\gamma_u}$ is continuous at each point u where T_{γ_u} is finite.*

Proof. If T_{γ_u} is finite, then γ_u is escaping by Lemma 3.9, and hence $T_{\gamma_u} = \int_{r_u}^{\infty} F_{m(r_u)}$ by Lemma 3.18. We need to show that this integral depends continuously on r_u.

By Lemmas 3.3 and 3.10 and the discussion preceding Example 3.7, the assumptions on G_m and the finiteness of T_{γ_u} imply that $m(r) > m(r_u)$ for $r > r_u$, m^{-2} is integrable, $m'(r_u) > 0$, and $m(\infty) = \infty$. Hence there exists $\delta > r_u$ with $m'|_{[r_u,\delta]} > 0$, and $m(r) > m(\delta)$ for $r > \delta$; it is clear that small changes in u do not affect δ.

Write $\int_{r_u}^{\infty} F_{m(r_u)} = \int_{r_u}^{\delta} F_{m(r_u)} + \int_{\delta}^{\infty} F_{m(r_u)}$. On $[r_u, \delta]$ we can write $F_{m(r_u)} = h(r, r_u)(r - r_u)^{-1/2}$ for some smooth function h. Since $(r - r_u)^{-1/2}$ is the derivative of $2(r - r_u)^{1/2}$, one can integrate $F_{m(r_u)}$ by parts which easily implies continuous dependence of $\int_{r_u}^{\delta} F_{m(r_u)}$ on r_u.

Continuous dependence of $\int_{\delta}^{\infty} F_{m(r_u)}$ on r_u follows because $F_{m(r_u)}$ is continuous in r_u, and is dominated by Km^{-2} where K is a positive constant independent of small changes of r_u. □

Next we focus on the case when $\sigma(0)$ is fixed, while $\kappa_{\sigma(0)}$ varies near $\pi/2$. To get an explicit formula for T_σ we need the following.

Lemma 3.20. *If M_m is von Mangoldt, and γ_q is a ray, then there is $\varepsilon > 0$ such that every geodesic $\sigma : [0, \infty) \to M_m$ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [\pi/2, \pi/2 + \varepsilon]$ is tangent to a parallel exactly once, and if u is the point where σ is tangent to a parallel, then $m' > 0$ on $[r_u, r_q]$.*

Proof. If $\kappa_{\sigma(0)} = \pi/2$, then $\sigma = \gamma_q$, so it is tangent to a parallel only at q, as rays are escaping. If $\kappa_{\sigma(0)} > \pi/2$, then σ converges to γ_q on compact subsets as $\varepsilon \to 0$, so for a sufficiently small ε the geodesic σ crosses the parallel through q at some point $\sigma(s)$ such that $\kappa_{\sigma(s)} < \pi/2$. Since γ_q is a ray, rotational symmetry and Lemma 3.13 imply that $\sigma|_{[s,\infty)}$ is a ray, so σ is escaping. Thus σ is tangent to a parallel at a point u where $r_{\sigma(s)}$ attains a minimum, and is not tangent to a parallel at any other point by Fact 3.2. Finally, $r_u = \lim_{\varepsilon\to 0} r_q$, and since $m'(r_q) > 0$ by Proposition 1.8, we get $m' > 0$ on $[r_u, r_q]$ for small ε. □

Under the assumptions of Lemma 3.20 the Clairaut constant c of σ equals $m(r_u) = m(r_q) \sin \kappa_{\sigma(0)}$, and the turn angle of σ is given by

$$T_\sigma = \int_{r_q}^{\infty} F_{m(r_q)}(r)\, dr \quad \text{if} \quad \kappa_{\sigma(0)} = \frac{\pi}{2} \quad \text{and} \tag{3.21}$$

$$T_\sigma = \int_{r_u}^{\infty} F_c(r)\, dr - \int_{r_q}^{r_u} F_c(r)\, dr = \int_{r_q}^{\infty} F_c(r)\, dr + 2\int_{r_u}^{r_q} F_c(r)\, dr \tag{3.22}$$

if $\pi/2 < \kappa_{\sigma(0)} < \pi/2 + \varepsilon$. These integrals converge, that is, T_σ is finite, as follows from Example 3.8 and Lemmas 3.10 and 3.20.

Since any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [0, \pi/2 + \varepsilon]$ has finite turn angle, one can think of T_σ as a function of $\kappa_{\sigma(0)}$ where σ varies over geodesics with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [0, \pi/2 + \varepsilon]$.

Lemma 3.23. *If M_m is von Mangoldt, and γ_q is a ray, then there is $\delta > \pi/2$ such that the function $\kappa_{\sigma(0)} \to T_\sigma$ is continuous and strictly increasing on $[\pi/2, \delta]$, and continuously differentiable on $(\pi/2, \delta]$; moreover, the derivative of T_σ is infinite at $\pi/2$.*

Proof. The Clairaut constant c of σ equals $m(r_u) = m(r_q)\sin\kappa_{\sigma(0)}$, so the assertion is immediate from (elementary but nontrivial) Lemma B.2 about continuity and differentiability of the integrals (3.21) and (3.22). □

Theorem 3.24. *If M_m is von Mangoldt and $q \neq o$, then*

(1) *$\hat{\kappa}(r_q) > \pi/2$ if and only if $T_{\gamma_q} < \pi$,*

(2) *$\hat{\kappa}(r_q) = \pi/2$ if and only if $T_{\gamma_q} = \pi$.*

Proof. (1) If $\hat{\kappa}(r_q) > \pi/2$, then any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)} \in [\pi/2, \hat{\kappa}(r_q)]$ is a ray, and so has turn angle $\leq \pi$. By Lemma 3.23 the turn angle is increasing at $\pi/2$, so $T_{\gamma_q} < \pi$. Conversely, if $T_{\gamma_q} < \pi$, then by Lemma 3.23 the turn angle is continuous at $\pi/2$, so any geodesic σ with $\sigma(0) = q$ and $\kappa_{\sigma(0)}$ near $\pi/2$ has turn angle $< \pi$, and is therefore a ray, so $\hat{\kappa}(r_q) > \pi/2$.

(2) This follows from (1) and the fact that $\hat{\kappa}(r_q) \geq \pi/2$ if and only if $T_{\gamma_q} \leq \pi$. □

Proof of Theorem 1.6. By Theorem 3.24 we know that $q \in A_m$ if and only if $T_{\gamma_q} < \pi$, and by Lemma 3.19 the map $u \to T_{\gamma_u}$ is continuous at q, so the set $\{u \in M_m \mid T_{\gamma_u} < \pi\}$ is open, and hence so is A_m. □

Another proof of Theorem 1.6. Fix $q \in A_m$ so that $T_{\gamma_q} < \pi$ by Theorem 3.24. Fix $\varepsilon > 0$ such that $\varepsilon + T_{\gamma_q} < \pi$. Let P_q be the parallel through q. Then there is a ray γ with $\gamma(0) = q$ and $\kappa_{\gamma(0)} > \pi/2$ such that γ intersects P_q at points q and $\gamma(t)$, and the turn angle of $\gamma|_{(0,t)}$ is $< \varepsilon$.

For an arbitrary sequence $q_i \to q$ we need to show that $q_i \in A_m$ for all large i. Let $\gamma_i : [0, \infty) \to M_m$ be the geodesic with $\gamma_i(0) = q_i$ and $\kappa_{\gamma_i(0)} = \kappa_{\gamma(0)}$. Since γ_i converge to γ on compact sets, for large i there are $t_i > 0$ such that $\gamma_i(t_i) \in P_q$ and $t_i \to t$. The angle formed by γ and $\mu_{\gamma(t)}$ is $< \pi/2$. Rotational symmetry and Lemma 3.13 imply that if i is large, then $\gamma_i|_{[t_i,\infty)}$ is a ray whose turn angle is $\leq T_{\gamma_q}$. The turn angles of $\gamma_i|_{(0,t_i)}$ converge to the turn angle of $\gamma|_{(0,t)}$, which is $< \varepsilon$. Thus $T_{\gamma_i} < T_{\gamma_q} + \varepsilon < \pi$ for large i, so that γ_i is a ray by Lemma 3.12, and hence $q_i \in A_m$. □

4. Planes of nonnegative curvature

A key consequence of $G_m \geq 0$ is monotonicity of the turn angle and of $\hat{\kappa}$.

Proposition 4.1. *Suppose that M_m has $G_m \geq 0$. If $0 < r_u < r_v$ and γ_u has finite turn angle, then $T_{\gamma_u} \leq T_{\gamma_v}$ with equality if and only if G_m vanishes on $[r_u, \infty]$.*

Proof. The result is trivial when G is everywhere zero. Since γ_u has finite turn angle, m^{-2} is integrable, and hence m is a concave function with $m' > 0$ and $m(\infty) = \infty$ by Lemma 3.10.

Set $x := r_q$, so that the turn angle of γ_q is $\int_x^\infty F_{m(x)}$. As $m' > 0$, we can change variables by $t := m(r)/m(x)$ or $r = m^{-1}(tm(x))$ so that

$$\int_x^\infty F_{m(x)}(r)\, dr = \int_1^{m(\infty)/m(x)} \frac{dt}{l(t, x)t\sqrt{t^2-1}} = \int_1^\infty \frac{dt}{l(t, x)t\sqrt{t^2-1}},$$

where $l(t, x) := m'(r)$. Computing

$$\frac{\partial l(t, x)}{\partial x} = m''(r)\frac{\partial r}{\partial x} = \frac{m''(r)tm'(x)}{m'(r)} = -G(r)\frac{tm'(x)}{m'(r)} \le 0$$

we see that $l(t, x)$ is nonincreasing in x. Thus if $r_u < r_v$, then $l(t, r_u) \ge l(t, r_v)$ for all t implying $T_{\gamma_u} \le T_{\gamma_v}$. The equality occurs precisely when $l(t, x)$ is constant on $[1, \infty) \times [r_u, r_v]$, or equivalently, when $G(m^{-1}(tm(x)))$ vanishes on $[1, \infty) \times [r_u, r_v]$, which in turn is equivalent to $G = 0$ on $[r_u, \infty)$, because $tm(x)$ takes all values in $(m(r_u), \infty)$ so $m^{-1}(tm(x))$ takes all values in (r_u, ∞). □

Lemma 4.2. *If $G_m \ge 0$, then $\hat{\kappa}$ is nonincreasing in r.*

Proof. Let u_1, u_2, and v be points on μ_v with $0 < r_{u_1} < r_{u_2} < r_v$. By Lemma 3.16 the ray ξ_{u_i} is the limit of geodesic segments that join u_i with points $\tau_v(s)$ as $s \to \infty$. The segments $[u_1, \tau_v(s)]$ and $[u_2, \tau_v(s)]$ only intersect at the endpoint $\tau_v(s)$ for if they intersect at a point z, then z is a cut point for $\tau_v(s)$, so $[\tau_v(s), u_i]$ cannot be minimizing. Hence the geodesic triangle with vertices u_1, v, and $\tau_v(s)$ contains the geodesic triangle with vertices u_2, v, and $\tau_v(s)$. Since $G_m \ge 0$, the former triangle has larger total curvature, which is finite as M_m has finite total curvature. As m only vanishes at 0, concavity of m implies that m is nondecreasing.

If m is unbounded, Clairaut's relation implies that the angles at $\tau_v(s)$ tend to zero as $s \to \infty$. By the Gauss–Bonnet theorem $\kappa_{\xi_1(0)} - \kappa_{\xi_2(0)}$ equals the total curvature of the "ideal" triangle with sides ξ_1, ξ_2, and $[u_1, u_2]$. Thus $\hat{\kappa}(r_{u_1}) \ge \hat{\kappa}(r_{u_2})$ with equality if and only if G_m vanishes on $[r_{u_1}, \infty)$.

If m is bounded, then $\int_1^\infty m^{-2} = \infty$, so by [Tanaka 1992a, Proposition 1.7] the only ray emanating from q is μ_q so that $\hat{\kappa} = 0$ on $M_m \setminus \{o\}$. For future use note that in this case the angle formed by $\mu_q = \xi_q$ and $[q, \tau_q(s)]$ tends to zero as $s \to \infty$, so Clairaut's relation together with the boundedness of m imply that the angle at $\tau_q(s)$ in the bigon with sides $[q, \tau_q(s)]$ and τ_q also tends to zero as $s \to \infty$. □

Remark 4.3. By the above proof if $G_m \ge 0$ and m^{-2} is integrable on $[1, \infty)$, then $\hat{\kappa}(r_1) = \hat{\kappa}(r_2)$ for some $r_2 > r_1$ if and only if G_m vanishes on $[r_1, \infty)$.

Proof of Theorem 1.3. (i) Since rays converge to rays, $\mathfrak{C}_m$ is closed. As $o \in \mathfrak{C}_m$, rotational symmetry and Lemma 4.2 imply that $\mathfrak{C}_m$ is a closed ball.

(ii) Since m is concave and positive, it is nondecreasing, so $\liminf_{r\to\infty} m > 0$, and the claim follows from Lemma 3.15.

(iii) We prove the contrapositive that $M_m = \mathfrak{C}_m$ if and only if $m'(\infty) \geq \frac{1}{2}$. Note that the latter is equivalent to $c(M_m) \leq \pi$, where $c(Z)$ denotes the total curvature of a subset $Z \subseteq M_m$ which varies in $[0, 2\pi]$.

Suppose $c(M_m) \leq \pi$. Fix $q \neq o$, and consider the segments $[q, \tau_q(s)]$ that by Lemma 3.16 converge to ξ_q as $s \to \infty$. Consider the bigon bounded by $[q, \tau_q(s)]$ and its symmetric image under the reflection that fixes $\tau_q \cup \mu_q$. As in the proof of Lemma 4.2 we see that the angle at $\tau_q(s)$ goes to zero as $s \to \infty$, so the sum of angles in the bigon tends to $2(\pi - \hat{\kappa}(r_q))$, which by the Gauss–Bonnet theorem cannot exceed $c(M_m) \leq \pi$. We conclude that $\hat{\kappa}(r_q) \geq \pi/2$, so $q \in \mathfrak{C}_m$.

Conversely, suppose that $\mathfrak{C}_m = M_m$. Given $\varepsilon > 0$ find a compact rotationally symmetric subset $K \subset M_m$ with $c(K) > c(M_m) - \varepsilon$. Fix $q \neq o$ and consider the rays $\xi_{\mu_q(s)}$ as $s \to \infty$. If all these rays intersect K, then they subconverge to a line [Shiohama et al. 2003, Lemma 6.1.1], so by the splitting theorem M_m is the standard $\mathbb{R}^2$, and $c(M_m) = 0 < \pi$. Thus we can assume that there is v on the ray μ_q such that ξ_v is disjoint from K. Therefore, if s is large enough, then K lies inside the bigon bounded by $[v, \tau_v(s)]$ and its symmetric image under the reflection that fixes $\tau_q \cup \mu_q$. The sum of angles in the bigon tends to $2(\pi - \hat{\kappa}(r_v))$, and by the Gauss–Bonnet theorem it is bounded below by $c(K)$. Since $v \in \mathfrak{C}_m$, we have $\hat{\kappa}(r_v) \geq \pi/2$, and hence $c(K) \leq \pi$. Thus $c(M_m) < \pi + \varepsilon$, and since ε is arbitrary, we get $c(M_m) \leq \pi$, which completes the proof of (iii).

(iv) Since R_m is finite, $m'(\infty) < \frac{1}{2}$ by (iii). As $m'(0) = 1$, the equation $m'(x) = \frac{1}{2}$ has a solution ρ_m. As $G_m \geq 0$, the function m' is nonincreasing, so uniqueness of the solution is equivalent to positivity of $G_m(\rho_m)$. Since M_m is von Mangoldt, $G_m(\rho_m) > 0$ for otherwise G_m would have to vanish for $r \geq \rho_m$, implying $m'(\infty) = m'(\rho_m) = \frac{1}{2}$, so R_m would be infinite.

Now we show that $\rho_m > R_m$. This is clear if $R_m = 0$ because $\rho_m \geq 0$ and $m'(0) = 1 \neq \frac{1}{2} = m'(\rho_m)$. Suppose $R_m > 0$. Then m^{-2} is integrable by Lemma 3.15, so $m' > 0$ everywhere by the proof of Lemma 3.10. Hence for any $r_v \geq \rho_m$ we have $m(r_v) \geq m(\rho_m)$, which implies $tm(r_v) > m(\rho_m)$ for all $t > 1$. Thus $m^{-1}(tm(r_v)) > m^{-1}(m(\rho_m)) = \rho_m$. Applying m' to the inequality, we get in notations of Proposition 4.1 that $l(t, r_v) < m'(\rho_m) = \frac{1}{2}$, where the inequality is strict because $G_m(r_m) > 0$ by (iv). Now (4.5) below implies

$$T_{\gamma_v} = \int_1^\infty \frac{dt}{l(t, r_v)\, t\sqrt{t^2 - 1}} > \int_1^\infty \frac{2\, dt}{t\sqrt{t^2 - 1}} = \pi.$$

Since M_m is von Mangoldt, $v \notin \mathfrak{C}_m$ by Lemma 3.14. In summary, if $r_v \geq \rho_m$, then $v \notin \mathfrak{C}_m$, so $\rho_m > R_m$.

(v) Since R_m is positive and finite, and M_m is von Mangoldt, there are geodesics tangent to parallels whose turn angles are $\le \pi$ and $> \pi$. By Proposition 4.1, the turn angle is monotone with respect to r, so let r_q be the (finite) supremum of all x such that $\int_x^\infty F_{m(x)} < \pi$. Since $\mathfrak{C}_m$ is closed, $q \in \mathfrak{C}_m$ so that $T_{\gamma_q} \le \pi$. In fact, $T_{\gamma_q} = \pi$ for if $T_{\gamma_q} < \pi$, then r_q is not maximal because by Theorems 1.6 and 3.24 the set of points q with $T_{\gamma_q} < \pi$ is open in M_m. If $G_m(r_q) > 0$, then by monotonicity r_q is a unique solution of $T_{\gamma_q} = \pi$. If $G_m(r_q) = 0$, then $G_m|_{[r_q,\infty)} = 0$ as M_m is von Mangoldt, so (4.5) implies that the turn angle of each γ_v with $r_v \ge r_q$ equals $\pi/(2m'(r_q))$. So $m'(r_q) = \frac{1}{2}$ but this case cannot happen as R_m is infinite by (iii). □

In preparation for a proof of Theorem 1.7 we recall that the Cheeger–Gromoll soul construction with basepoint q, described, for example, in [Sakai 1996, Theorem V.3.4], starts by deleting the horoballs associated with all rays emanating from q, which results in a compact totally convex subset. The next step is to consider the points of this subset which are at maximal distance from its boundary, and these points in turn form a compact totally convex subset, and after finitely many iterations the process terminates in a subset with empty boundary, called a soul. As we shall see below, if $G_m \ge 0$, then the soul construction with basepoint $q \in \mathfrak{C}_m \setminus \{o\}$ takes no more than two steps; more precisely, deleting the horoballs for rays emanating from q results either in $\{q\}$ or in a segment with q as an endpoint. In the latter case the soul is the midpoint of the segment.

In what follows we let B_σ denote the (open) horoball for a ray σ with $\sigma(0) = q$, that is, the union over $t \in [0, \infty)$ of the metric balls of radius t centered at $\sigma(t)$. Let H_σ denote the complement of B_σ in the ambient complete Riemannian manifold.

Lemma 4.4. *Let σ be a ray in a complete Riemannian manifold M, and let $q = \sigma(0)$. Then for any nonzero $v \in T_q M$ that makes an acute angle with σ, the point $\exp_q(tv)$ lies in the horoball B_σ for all small $t > 0$.*

Proof. This follows from the definition of a horoball for if Υ denotes the image of $t \to \exp_q(tv)$, then

$$\lim_{s\to+0} \frac{d(\sigma(s), \Upsilon)}{d(\sigma(s), q)} = \sin \angle(v'(0), \sigma'(0)) < 1,$$

so B_σ contains a subsegment of $\Upsilon - \{q\}$ that approaches q. □

Proof of Theorem 1.7. For $q \in \mathfrak{C}_m$, let C_q denote the complement in M_m of the union of the horoballs for rays that start at q; note that C_q is compact and totally convex. If C_q equals $\{q\}$, then q is a soul. Otherwise, C_q has positive dimension and $q \in \partial C_q$. Set $\gamma := \xi_q$; thus γ is a ray.

Case 1. Suppose $\pi/2 < \hat{\kappa}(r_q) < \pi$. Let $\bar{\gamma}$ be the clockwise ray that is mapped to γ by the isometry fixing the meridian through q. We next show that q is the intersection of the complements of the horoballs for rays μ_q, γ, and $\bar{\gamma}$, implying that q is a soul for the soul construction that starts at q. As $\kappa_{\gamma(0)} > \pi/2$, any nonzero $v \in T_q M_m$ forms angle $< \pi/2$ with one of $\mu'(0)$, $\gamma'(0)$, or $\bar{\gamma}'(0)$, so $\exp_q(tv)$ cannot lie in the intersection of H_{μ_q}, H_γ, and $H_{\bar{\gamma}}$ for small t, and since the intersection is totally convex, it is $\{q\}$.

Case 2. Suppose $\hat{\kappa}(r_q) = \pi/2$, so that $\gamma = \gamma_q$, and suppose that G_m does not vanish along γ. By symmetry and Lemma 4.4, it suffices to show that every point of the segment $[o, q)$ near q lies in B_γ. Let α be the ray from o passing through q. The geodesic γ is orthogonal to α, and it suffices to show that there is a focal point w of α along γ (for this would imply that there is a family of geodesics of the same length that minimize the distance from w to α, and since the geodesics cannot minimize beyond the focal point, all points near q on α, except q, are in B_γ [Sakai 1996, Lemma III.2.11]).

Any α-Jacobi field along γ is of the form jn where n is a parallel nonzero normal vector field along γ and j solves $j''(t) + G_m(r_{\gamma(t)})j(t) = 0$, $j(0) = 1$, $j'(0) = 0$. Since $G_m \geq 0$, the function j is concave, so due to its initial values, j must vanish unless it is constant. The point where j vanishes is focal. If j is constant, then $G_m = 0$ along γ, which is ruled out by assumption.

Case 3. Suppose $\hat{\kappa}(r_q) = \pi$, that is, $\gamma = \tau_q$. For any vector $v \in T_q M_m$ pointing inside C_q, for small t the point $\exp_q(tv)$ is not in the horoballs for μ_q and τ_q, and hence v is tangent to a parallel, that is, C_q is a subsegment of the geodesic α tangent to the parallel through q. As C_q lies outside the horoballs for μ_q and τ_q, these rays there cannot contain focal points of α, implying that G_m vanishes along μ_q and τ_q, and hence everywhere, by rotational symmetry, so that M_m is the standard $\mathbb{R}^2$, and q is a soul.

Case 4. Suppose $\hat{\kappa}(r_q) = \pi/2$, so that $\gamma = \gamma_q$, and suppose that G_m vanishes along γ. By rotational symmetry $G_m(r) = 0$ for $r \geq r_q$, so $m(r) = ar + m(0)$ for $r \geq r_q$ where $a > 0$, as m only vanishes at 0. The turn angle of γ can be computed explicitly as

$$\text{(4.5)}\quad \int_x^\infty \frac{dr}{m(r)\sqrt{\dfrac{m(r)^2}{m(x)^2} - 1}} = \int_1^\infty \frac{dt}{at\sqrt{t^2 - 1}} = -\frac{1}{a}\operatorname{arccot}\big(\sqrt{t^2 - 1}\big)\Big|_1^\infty = \frac{\pi}{2a},$$

where $x := r_q$. Since γ is a ray, we deduce that $a \geq \frac{1}{2}$.

Let $z \leq x$ be the smallest number such that $m'|_{[z,\infty)} = a$; thus there is no neighborhood of z in $(0, \infty)$ on which G_m is identically zero.

Note that $m(r) = a(r - z) + m(z)$ for $r \geq z$, so the surface $M_m – B(o, z)$ is isometric to $C – B(\bar{o}, m(r_q)/a)$ where C is the cone with apex $\bar{o}$ such that cutting C

along the meridian from $\bar{o}$ gives a sector in $\mathbb{R}^2$ of angle $2\pi a$ with the portion inside the radius $m(r_q)/a$ removed.

Since γ_q is a ray, Lemma 4.4 implies the existence of a neighborhood U_q of q such that each point in U_p–$[o, q]$ lies in a horoball for a ray from q.

We now check that o lies in the horoball of γ_q. Concavity of m implies that the graph of m lies below its tangent line at z, so evaluating the tangent line at $r = 0$ and using $m(0) = 0$ gives $m(z)/a > z$. The Pythagorean theorem in the sector in $\mathbb{R}^2$ of angle $2\pi a$ implies that

$$d_{M_m}(\gamma_q(s), o) = \sqrt{s^2 + \left(x - z + \frac{m(z)}{a}\right)^2} + z - \frac{m(z)}{a},$$

which is $< s$ for large s, implying that o is in the horoball of γ_q.

To realize q as a soul, we need to look at the soul construction with arbitrary basepoint v, which starts by considering the complement in M_m of the union of the horoballs for all rays from v, which by the above is either v or a segment $[u, v]$ contained in $(o, v]$, where u is uniquely determined by v. It will be convenient to allow for degenerate segments for which $u = v$; with this convention the soul is the midpoint of $[u, v]$. Since z is the smallest such that $G_m|_{[z,\infty)} = 0$, the focal point argument of Case 2 shows that $u = v$ when $0 < r_v < z$. Set $y := r_v$, and let $e(y) := r_u$; note that $0 < e(y) \le y$, and the midpoint of $[u, v]$ has r-coordinate $h(y) := (y + e(y))/2$.

To realize each point of M_m as a soul, it suffices to show that each positive number is in the image of h. Since h approaches zero as $y \to 0$ and approaches infinity as $y \to \infty$, it is enough to show that h is continuous and then apply the intermediate value theorem.

Since $e(y) = y$ when $0 < y < z$, we only need to verify continuity of e when $y \ge z$. Let v_i be an arbitrary sequence of points on α converging to v, where as before α is the ray from o passing through q. Set $v_i := r_{v_i}$. Arguing by contradiction suppose that $e(y_i)$ does not converge to $e(y)$. Since $0 < e(y_i) \le y_i$ and $y_i \to y$, we may pass to a subsequence such that $e(y_i) \to e_\infty \in [0, y]$. Pick any w such that r_w lies between e_∞ and $e(y)$. Thus there is i_0 such that either $e(y_i) < r_w < e(y)$ for all $i > i_0$, or $e(y) < r_w < e(y_i)$ for all $i > i_0$. As $y \ge z$, we know that G_m vanishes along γ_v, so every α-Jacobi field along γ_v is constant. Therefore, the rays γ_{v_i} converge uniformly (!) to γ_v, as $v_i \to v$, and hence their Busemann functions b_i and b converge pointwise. Thus $b_i(w) \to b(w)$, but we have chosen w so that $b(w)$ and $b_i(w)$ are all nonzero, and $\operatorname{sign}(b(w)) = -\operatorname{sign}(b_i(w))$, which gives a contradiction proving the theorem. □

Remark 4.6. In Cases 1, 2, and 3 the soul construction terminates in one step, namely, if $q \in \mathfrak{C}_m$, then $\{q\}$ is the result of removing the horoballs for all rays that start at q. We do not know whether the same is true in Case 4 because the basepoint v needed to produce the soul q is found implicitly, via the intermediate value theorem, and it is unclear how v depends on q, and whether $v = q$.

Remark 4.7. Let M_m be as in Case 4 with $m'|_{[z,\infty)} = \frac{1}{2}$. If M_m is von Mangoldt, then no point q with $r_q \geq z$ is a pole because by (4.5) the turn angle of γ_q is π, which by Theorem 3.24 cannot happen for a pole.

5. Smoothed cones made von Mangoldt

Proof of Theorem 1.11. It is of course easy to find a von Mangoldt plane g_{m_x} that has zero curvature near infinity, but prescribing the slope of m' there takes more effort. We exclude the trivial case $x = 1$ in which $m(r) = r$ works.

For $u \in [0, \frac{1}{4}]$ set $K_u(r) = 1/(4(r+1)^2) - u$, and let m_u be the unique solution of (A.7) with $K = K_u$. Then g_{m_u} is von Mangoldt. For $u > 0$ let $z_u \in [0, \infty)$ be the unique zero of K_u; note that z_u is the global minimum of m'_u, and $z_u \to \infty$ as $u \to 0$.

Lemma 5.1. *The function $u \to m'_u(z_u)$ takes every value in* $(0, 1)$ *as u varies in* $(0, \frac{1}{4})$.

Proof. One verifies that $m_0(r) = \ln(r+1)\sqrt{r+1}$, that is, the right hand side solves (A.7) with $K = K_0$. Then $m'_0 = (2 + \ln(r+1))/(2\sqrt{r+1})$ is a positive function converging to zero as $r \to \infty$. By Sturm comparison $m_u \geq m_0 > 0$ and $m'_u \geq m'_0 > 0$.

We now show that $m'_u(z_u) \to 0$ as $u \to +0$. To this end fix an arbitrary $\varepsilon > 0$. Fix t_ε such that $m'_0(t_\varepsilon) < \varepsilon$. By continuous dependence on parameters (m_u, m'_u) converges to (m_0, m'_0) uniformly on compact sets as $u \to 0$. So for all small u we have $m'_u(t_\varepsilon) < \varepsilon$ and also $t_\varepsilon < z_u$. Since m'_u decreases on $(0, z_u)$, we conclude that $0 < m'_u(z_u) < m'_u(t_\varepsilon) < \varepsilon$, proving that $m'_u(z_u) \to 0$ as $u \to +0$.

On the other hand, $m'_{1/4}(z_{1/4}) = 1$ because $z_{1/4} = 0$ and by the initial condition $m'_{1/4}(0) = 1$. Finally, the assertion of the lemma follows from continuity of the map $u \to m'_u(z_u)$, because then it takes every value within $(0, 1)$ as u varies in $(0, \frac{1}{4})$. (To check continuity of the map fix u_*, take an arbitrary $u \to u_*$ and note that $z_u \to z_{u_*}$, so since m'_u converges to m'_{u_*} on compact subsets, it does so on a neighborhood of z_{u_*}, so $m'_u(z_u)$ converges to $m'_{u_*}(z_{u_*})$.) □

Continuing the proof of the theorem, fix an arbitrary $u > 0$. The continuous function $\max(K_u, 0)$ is decreasing and smooth on $[0, z_u]$ and equal to zero on $[z_u, \infty)$. So there is a family of nonincreasing smooth functions $G_{u,\varepsilon}$ depending on the small parameter ε such that $G_{u,\varepsilon} = \max(K_u, 0)$ outside the ε-neighborhood of z_u. Let $m_{u,\varepsilon}$ be the unique solution of (A.7) with $K = G_{u,\varepsilon}$; thus $m'_{u,\varepsilon}(r) = m'_{u,\varepsilon}(z_u + \varepsilon)$ for all $r \geq z_u + \varepsilon$. If ε is small enough, then $G_{u,\varepsilon} \leq K_0$, so $m_{u,\varepsilon} \geq m_0 > 0$ and

$m'_{u,\varepsilon} \geq m'_0 > 0$. By continuous dependence on parameters, the function $(u, \varepsilon) \to m'_{u,\varepsilon}$ is continuous, and moreover $m'_{u,\varepsilon}(z_u + \varepsilon) \to m'_u(z_u)$ as $\varepsilon \to 0$, and u is fixed.

Fix $x \in (0, 1)$. By Lemma 5.1 there are positive v_1 and v_2 such that $m'_{v_1}(z_{v_1}) < x < m'_{v_2}(z_{v_2})$. Letting u of the previous paragraph to be v_1, v_2, we find ε such that $m'_{v_1,\varepsilon}(z_{v_1} + \varepsilon) < x < m'_{v_2,\varepsilon}(z_{v_2} + \varepsilon)$, so by the intermediate value theorem there is u with $m'_{u,\varepsilon}(z_u + \varepsilon) = x$. Then the metric $g_{m_{u,\varepsilon}}$ has the asserted properties for $\rho = z_u + \varepsilon$. □

6. Other applications

Proof of Lemma 1.1. Assuming $\hat{r}(\hat{q}) \notin r(\mathfrak{C}_m)$ we will show that $\hat{q}$ is not a critical point of $\hat{r}$. Since $\hat{M}$ is complete and noncompact, there is a ray $\hat{\gamma}$ emanating from $\hat{q}$. Consider the comparison triangle $\Delta(o, q, q_i)$ in M_m for any geodesic triangle with vertices $\hat{o}$, $\hat{q}$, and $\hat{\gamma}(i)$. Passing to a subsequence, arrange so that the segments $[q, q_i]$ subconverge to a ray, which we denote by γ. Since $q \notin \mathfrak{C}_m$, the angle formed by γ and $[q, o]$ is $> \pi/2$, and hence for large i the same is true for the angles formed by $[q, q_i]$ and $[q, o]$. By comparison, $\hat{\gamma}$ forms angle $> \pi/2$ with any segment joining $\hat{q}$ to $\hat{o}$, that is, $\hat{q}$ is not a critical point of $\hat{r}$. □

Proof of Theorem 1.5. (a) Let P_m denote the set of poles; it is a closed metric ball [Tanaka 1992b, Lemma 1.1]. Moreover, P_m clearly lies in the connected component A_m^o of $A_m \cup \{o\}$ that contains o, and hence in the component of $\mathfrak{C}_m$ that contains o. By Theorem 1.6 A_m is open in M_m, so $A_m \cup \{o\}$ is locally path-connected, and hence A_m^o is open in M_m. If P_m were equal to A_m^o, the latter would be closed, implying $A_m^o = M_m$, which is impossible as the ball has finite radius.

(b) The "if" direction is trivial as $P_m \subset \mathfrak{C}_m$. Conversely, if $\mathfrak{C}_m \neq \{o\}$, then by Lemma 3.15 m^{-2} is integrable and $\liminf_{r\to\infty} m(r) > 0$, so $R_p > 0$ [Tanaka 1992a]. □

Proof of Theorem 1.9. By assumption there is a point of negative curvature, and since the curvature is nonincreasing, outside a compact subset the curvature is bounded above by a negative constant. As $\liminf_{r\to\infty} m(r) > 0$, m is bounded below by a positive constant outside any neighborhood of 0, so $\int_0^\infty m = \infty$. Hence the total curvature $2\pi \int_0^\infty G_m(r)m(r)\,dr$ is $-\infty$.

Hence there is a metric ball B of finite positive radius centered at o such that the total curvature of B is negative, and such that no point of $G_m \geq 0$ lies outside B. By [Shiohama et al. 2003, Theorem 6.1.1, p. 190], for any $q \in M_m$ the total curvature of the set obtained from M_m by removing all rays that start at q is in $[0, 2\pi]$. So for any q there is a ray that starts at q and intersects B.

If q is not in B, then the ray points away from infinity, so $q \in A_m$ and any point on this ray is in $\mathfrak{C}_m$. Thus $M_m - A_m$ lies in B. Since $\mathfrak{C}_m \neq \{o\}$, Theorem 1.5 implies that $R_p > 0$. Letting q run to infinity the rays subconverge to a line that intersects B; see, for example, [Shiohama et al. 2003, Lemma 6.1.1, p. 187].

If $m'(r_p) = 0$, the parallel through p is a geodesic but not a ray, so Lemma 3.14 implies that no point on the parallel through p is in $\mathfrak{C}_m$. Since $\mathfrak{C}_m$ contains o and all points outside a compact set, $\mathfrak{C}_m$ is not connected; the same argument proves that A_m is not connected. □

Example 6.1. Here we modify [Tanaka 1992b, Example 4] to construct a von Mangoldt plane M_m such that m' has a zero, and neither A_m nor $\mathfrak{C}_m$ is connected. Given $a \in (\pi/2, \pi)$ let $m_0(r) = \sin r$ for $r \in [0, a]$, and define m_0 for $r \geq a$ so that m_0 is smooth, positive, and $\liminf_{r\to\infty} m_0 > 0$. Thus $K_0 := -m_0''/m_0$ equals 1 on $[0, a]$. Let K be any smooth nonincreasing function with $K \leq K_0$ and $K|_{[0,a]} = 1$. Let m be the solution of (A.7); note that $m(r) = \sin(r)$ for $r \in [0, a]$ so that m' vanishes at $\pi/2$. By Sturm comparison $m \geq m_0 > 0$, and hence M_m is a von Mangoldt plane. Since $m'(a) < 0$ and $m > 0$ for all $r > 0$, the function m cannot be concave, so $K = G_m$ eventually becomes negative, and Theorem 1.9 implies that A_m and $\mathfrak{C}_m$ are not connected.

Example 6.2. Here we construct a von Mangoldt plane such that $m' > 0$ everywhere but A_m and $\mathfrak{C}_m$ are not connected. Let M_n be a von Mangoldt plane such that $G_n \geq 0$ and $n' > 0$ everywhere, and R_n is finite (where R_n is the radius of the ball $\mathfrak{C}_n$). This happens, for example, for any paraboloid, any two-sheeted hyperboloid with $n'(\infty) < \frac{1}{2}$, or any plane constructed in Theorem 1.11 with $n'(\infty) < \frac{1}{2}$. Fix $q \notin \mathfrak{C}_n$. Then γ_q has turn angle $> \pi$, so there is $R > r_q$ such that $\int_{r_q}^R F_{n(r_q)} > \pi$. Let G be any smooth nonincreasing function such that $G = G_n$ on $[0, R]$ and $G(z) < 0$ for some $z > R$. Let m be the solution of (A.7) with $K = G$. By Sturm comparison $m \geq n > 0$ and $m' \geq n' > 0$ everywhere; see Remark A.10. Since $m = n$ on $[0, R]$, on this interval we have $F_{m(r_q)} = F_{n(r_q)}$, so in the von Mangoldt plane M_m the geodesic γ_q has turn angle $> \pi$, which implies that no point on the parallel through q is in $\mathfrak{C}_m$. Now Theorem 1.9 (3) and (4) imply that A_m and $\mathfrak{C}_m$ are not connected.

Theorem 6.3. *Let M_m be a von Mangoldt plane such that $m'|_{[0,y]} > 0$ and $m'|_{[x,y]} < \frac{1}{2}$. Set $f_{m,x}(y) := m^{-1}(\cos(\pi b)m(y))$, where b is the maximum of m' on $[x, y]$. If $x \leq f_{m,x}(y)$, then $r(\mathfrak{C}_m)$ and $[x, f_{m,x}(y)]$ are disjoint.*

Proof. Set $f := f_{m,x}$. Arguing by contradiction assume there is $q \in \mathfrak{C}_m$ with $r_q \in [x, f(y)]$. Then γ_q has turn angle $\leq \pi$, so if $c := m(r_q)$, then

$$\pi \geq \int_{r_q}^{\infty} \frac{c\,dr}{m\sqrt{m^2 - c^2}} > \int_{r_q}^{y} \frac{c\,dr}{m\sqrt{m^2 - c^2}}$$
$$= \int_{c}^{m(y)} \frac{c\,dm}{m'(r)m\sqrt{m^2 - c^2}} \geq \int_{c}^{m(y)} \frac{c\,dm}{bm\sqrt{m^2 - c^2}} = \frac{1}{b}\arccos\left(\frac{c}{m(y)}\right),$$

so that $\pi b > \arccos(c/(m(y)))$, which is equivalent to $\cos(\pi b)m(y) < m(r_q)$.

On the other hand, $m(f(y))$ is in the interval $[0, m(y)]$ on which m^{-1} is increasing, so $f(y) < y$, and therefore m is increasing on $[x, f(y)]$. Hence $r_q < f(y)$ implies $m(r_q) < m(f(y)) = \cos(\pi b)m(y)$, which is a contradiction. □

Proof of Theorem 1.10. We use the notation of Theorem 6.3. The assumptions on n imply $n' > 0$, $n'|_{[x,\infty)} < \frac{1}{2}$, and $b = n'(x)$. Hence $f_{n,x}$ is an increasing smooth function of y with $f_{n,x}(\infty) = \infty$. In particular, if y is large enough, then $f_{n,x}(y) > z > x$; fix y that satisfies the inequality. Now if M_m is any von Mangoldt plane with $m = n$ on $[0, y]$, then $f_{m,x}(y) = f_{n,x}(y)$, so M_m satisfies the assumptions of Theorem 6.3, so $[x, z]$ and $r(\mathfrak{C}_m)$ are disjoint. □

Appendix A: Von Mangoldt planes

The purpose of this appendix is to discuss what makes von Mangoldt planes special among arbitrary rotationally symmetric planes.

For a smooth function $m : [0, \infty) \to [0, \infty)$ whose only zero is 0, let g_m denote the rotationally symmetric inner product on the tangent bundle to $\mathbb{R}^2$ that equals the standard Euclidean inner product at the origin and elsewhere is given in polar coordinates by $dr^2 + m(r)^2\, d\theta^2$. It is well known (see, for example, [Shiohama et al. 2003, §7.1]) that:

- Any rotationally symmetric complete smooth Riemannian metric on $\mathbb{R}^2$ is isometric to some g_m. (As before, M_m denotes $(\mathbb{R}^2, g_m)$.)
- If $\bar{m} : \mathbb{R} \to \mathbb{R}$ denotes the unique odd function such that $\bar{m}|_{[0,\infty)} = m$, then g_m is a smooth Riemannian metric on $\mathbb{R}^2$ if and only if $m'(0) = 1$ and $\bar{m}$ is smooth.
- If g_m is a smooth metric on $\mathbb{R}^2$, then g_m is complete, and the sectional curvature of g_m is a smooth function on $[0, \infty)$ that equals $-m''/m$.

It is easier to visualize M_m as a surface of revolution in $\mathbb{R}^3$, so we recall:

Lemma A.1. (1) *M_m is isometric to a surface of revolution in $\mathbb{R}^3$ if and only if $|m'| \le 1$.*

(2) *M_m is isometric to a surface of revolution $(r\cos\phi, r\sin\phi, g(r))$ in $\mathbb{R}^3$ if and only if $0 < m' \le 1$.*

Proof. (1) Consider a unit speed curve $s \to (x(s), 0, z(s))$ in $\mathbb{R}^3$ where $x(s) \ge 0$ and $s \ge 0$. Rotating the curve about the z-axis gives the surface of revolution

$$(x(s)\cos\phi,\ x(s)\sin\phi,\ z(s))$$

with metric $ds^2 + x(s)^2\, d\phi^2$. The meridians starting at the origin are rays, so for this metric to be equal to $ds^2 + m(s)^2\, d\phi^2$ we must have $m(s) = x(s)$. Since the curve has unit speed, $|x'(s)| \le 1$, so a necessary condition for writing the metric as a surface of revolution is $|m'(s)| \le 1$. It is also sufficient for if $|m'(s)| \le 1$, then we could let $z(s) := \int_0^s \sqrt{1-(m'(s))^2}\,ds$, so that now $(m(s), z(s))$ has unit speed.

(2) If, furthermore, $m' > 0$ for all s, then the inverse function of $m(s)$ makes sense, and we can write the surface of revolution $(m(s)\cos\phi, m(s)\sin\phi, z(s))$ as $(x\cos\phi, x\sin\phi, g(x))$ where $x := m(s)$ and $g(x) := z(m^{-1}(x))$. Conversely, given the surface $(x\cos\phi, x\sin\phi, g(x))$, the orientation-preserving arclength parametrization $x = x(s)$ of the curve $(x, 0, g(x))$ satisfies $x' > 0$. □

Example A.2. The standard $\mathbb{R}^2$ is the only von Mangoldt plane with $G_m \le 0$ that can be embedded into $\mathbb{R}^3$ as a surface of revolution because $m'(0) = 1$ and m' is nondecreasing afterwards.

Example A.3. If $G_m \ge 0$, then $m' \in [0, 1]$ because $m > 0$, m' is nonincreasing, and $m'(0) = 1$, so that M_m is isometric to a surface of revolution in $\mathbb{R}^3$. In fact, if $m'(s_0) = 0$, then $m|_{[s_0,\infty)} = m(s_0)$, that is, outside the s_0-ball about the origin M_m is a cylinder. Thus except for such surfaces M_m can be written as

$$(x\cos\phi, x\sin\phi, g(x)) \quad \text{for } g(x) = \int_0^{m^{-1}(x)} \sqrt{1-(m'(s))^2}\,ds.$$

Paraboloids and two-sheeted hyperboloids are von Mangoldt planes of positive curvature [Shiohama et al. 2003, p. 234–235] and are of the form $(x\cos\phi, x\sin\phi, g(x))$.

The defining property $G_m' \le 0$ of von Mangoldt planes clearly restricts the behavior of m'. Let $Z(G_m)$ denote the set where G_m vanishes; as M_m is von Mangoldt, $Z(G_m)$ is closed and connected, and hence it could be equal to the empty set, a point, or an interval, while m' behaves as follows.

(i) If $G_m > 0$, then m' is decreasing and takes values in $(0, 1]$.

(ii) If $G_m \le 0$, then m' is nondecreasing and takes values in $[1, \infty)$.

(iii) If $Z(G_m)$ is a positive number z, then m' decreases on $[0, z)$ and increases on (z, ∞), and m' may have two, one, or no zeros.

(iv) If $Z(G_m) = [a, b] \subset (0, \infty]$, then m' decreases on $[0, a)$, is constant on $[a, b]$, and increases on (b, ∞) if $b < \infty$. Also either $m'|_{[a,b]} = 0$ or else m' has two, or no zeros.

Remark A.4. All the above possibilities occur with one possible exception: in Cases (iii) and (iv) we are not aware of examples where m' vanishes on $Z(G_m)$.

Remark A.5. Thus if M_m is von Mangoldt, then m' is monotone near infinity, so $m'(\infty)$ exists; moreover, $m'(\infty) \in [0, \infty]$, for otherwise m would vanish on $(0, \infty)$. It follows that M_m admits total curvature, which equals

$$\int_0^{2\pi}\int_0^{\infty} G_m m\,dr\,d\theta = -2\pi \int_0^{\infty} m'' = 2\pi(1 - m'(\infty)) \in [-\infty, 2\pi].$$

Here the *total curvature of a subset* $A \subset M_m$ is the integral of G_m over A with respect to the Riemannian area form $m\,dr d\theta$, provided the integral converges to a number in $[-\infty, \infty]$, in which case we say that A *admits total curvature*.

Remark A.6. The zeros of m' correspond to parallels that are geodesics and are of interest. In contrast with restrictions on the zero set of m' for von Mangoldt planes, if M_m is not necessarily von Mangoldt, then any closed subset of $[0, \infty)$ that does not contain 0 can be realized as the set of zeros of m'. (Indeed, for any closed subset of a manifold there is a smooth nonnegative function that vanishes precisely on the subset [Bröcker and Jänich 1982, Whitney's Theorem 14.1]. It follows that if C is a closed subset of $[0, \infty)$ that does not contain 0, then there is a smooth function $g : [0, \infty) \to [0, \infty)$ that is even at 0, satisfies $g(0) = 1$, and is such that $g(s) = 0$ if and only if $s \in C$. If m is the solution of $m' = g$ and $m(0) = 0$, then M_m has the promised property.)

A common way of constructing von Mangoldt planes involves the Jacobi initial value problem

$$m'' + Km = 0, \quad m(0) = 0, \quad m'(0) = 1, \tag{A.7}$$

where K is smooth on $[0, \infty)$. It follows from the proof of [Kazdan and Warner 1974, Lemma 4.4] that g_m is a complete smooth Riemannian metric on $\mathbb{R}^2$ if and only if the following condition holds:

($\star$) the (unique) solution m of (A.7) is positive on $(0, \infty)$.

Remark A.8. A basic tool that produces solutions of (A.7) satisfying condition ($\star$) is the Sturm comparison theorem that implies that if m_1 is a positive function that solves (A.7) with $K = K_1$, and if K_2 is any nonincreasing smooth function with $K_2 \le K_1$, then the solution m_2 of (A.7) with $K = K_2$ satisfies $m_2 \ge m_1$, so that g_{m_2} is a von Mangoldt plane.

Example A.9. If K is a smooth function on $[0, \infty)$ such that $\max(K, 0)$ has compact support, then a positive multiple of K can be realized as the curvature G_m of some M_m; of course, if K is nonincreasing, then M_m is von Mangoldt. (Indeed, in [Kazdan and Warner 1974, Lemma 4.3] Sturm comparison was used to show that if $\int_t^\infty \max(K, 0) \le 1/(4t + 4)$ for all $t \ge 0$, then K satisfies ($\star$), and in particular, if $\max(K, 0)$ has compact support, then there is a constant $\varepsilon > 0$ such that the above inequality holds for εK.)

Remark A.10. A useful addendum to Remark A.8 is that the additional assumption $m_1' \ge 0$ implies $m_2' \ge m_1' > 0$. (Indeed, the function $m_1' m_2 - m_1 m_2'$ vanishes at 0 and has nonpositive derivative $(-K_1 + K_2) m_1 m_2$, so $m_1' m_2 \le m_1 m_2'$. As m_1, m_2, and m_1' are nonnegative, so is m_2'. Hence, $m_1 m_2' \le m_2 m_2'$, which gives $m_1' m_2 \le m_2 m_2'$, and the claim follows by canceling m_2.)

Question A.11. *Let* $m_0 : [r_0, \infty) \to (0, \infty)$ *be a smooth function such that* $r_0 > 0$ *and* $-m_0''/m_0$ *is nonincreasing. What are sufficient conditions for (or obstructions to) extending* m_0 *to a function* m *on* $[0, \infty)$ *such that* g_m *is a von Mangoldt plane?*

Appendix B: A calculus lemma

This appendix contains an elementary lemma on continuity and differentiability of the turn angle, which is needed for Theorem 3.24.

Given numbers $r_q > r_0 > 0$, let m be a smooth self-map of $(0, \infty)$ such that

- $m' > 0$ on $[r_0, r_q]$,
- $m(r) > m(r_q)$ for $r > r_q$,
- m^{-2} is integrable on $(1, \infty)$,
- $\liminf_{r\to\infty} m(r) > m(r_q)$.

Example B.1. Suppose $G_m \geq 0$ or $G_m' \leq 0$. If γ_q is a ray on M_m, and r_0 is sufficiently close to r_q, then m satisfies the above properties by Lemmas 3.3, 3.8, and 3.10.

Set $c_0 := m(r_0)$ and $c_q := m(r_q)$. Let $T = T(c)$ be the function given by the integral (3.21) for $c = c_q$, and by the sum of integrals (3.22) for $c_0 \leq c \leq c_q$, where F_c is given by (3.5) and $r_u := m^{-1}(c)$, where m^{-1} is the inverse of $m|_{[r_0, r_q]}$.

Lemma B.2. *Under the assumptions of the previous paragraph,* T *is continuous on* $(c_0, c_q]$, *continuously differentiable on* (c_0, c_q), *and* $T'(c)\sqrt{c_q^2 - c^2}$ *converges to* $-1/(m'(r_q)) < 0$ *as* $c \to c_q-$.

Proof. By definition T equals $\int_{r_q}^{\infty} F_c + \int_{r_u}^{r_q} F_c$ if $c \in [c_0, c_q)$ and $T = \int_{r_q}^{\infty} F_c$ if $c = c_q$. Step 1 shows that $\int_{r_q}^{\infty} F_c$ depends continuously on $c \in [c_0, c_q]$, while Step 2 establishes continuity of T at c_q. In Steps 3 and 4 we prove continuous differentiability and compute the derivatives of integrals $\int_{r_q}^{\infty} F_c$ and $\int_{r_u}^{r_q} F_c$ with respect to $c \in (c_0, c_q)$. Step 5 investigates the behavior of $T'(c)$ as $c \to c_q$.

Recall that the integral $\int_a^b H_c(r)\, dr$ depends continuously on c if for each $r \in (a, b)$ the map $c \to H_c(r)$ is continuous, and every c has a neighborhood U_0 in which $|H_c| \leq h_0$ for some integrable function h_0. If in addition each map $c \to H_c(r)$ is C^1, and every c has a neighborhood U_1 where $|\partial H_c/\partial c| \leq h_1$ for an integrable function h_1, then $\int_a^b H_c(r)\, dr$ is C^1 and differentiation under the integral sign is valid; the same conclusion holds when H_c and $\partial H_c/\partial c$ are continuous in the closure of $U_1 \times (a, b)$.

Step 1. The integrand F_c is smooth over (r_u, ∞), because the assumptions on m imply that $m(r) > c$ for $r > r_u$.

Since $0 < c \leq c_q$ we have $F_c \leq F_{c_q} = c_q/(m\sqrt{m^2 - c_q^2})$ which is integrable on (r_q, ∞). Indeed, fix $\delta > r_q$ and note that since m^{-2} is integrable on (δ, ∞), so is

F_{c_q}. To prove integrability of F_{c_q} on (r_q, δ), note that

$$h(r) := \frac{m(r) - m(r_q)}{r - r_q}$$

is positive on $[r_q, \infty)$, as $h(r_q) = m'(r_q) > 0$ and $m(r) > m(r_q)$ for $r > r_q$. Then F_{c_q} is the product of $(r - r_q)^{-1/2}$ and a function that is smooth on $[r_q, \delta]$, and hence F_{c_q} is integrable on (r_q, δ).

Thus the integrals $\int_{r_q}^{\delta} F_c(r)\, dr$ and $\int_{\delta}^{\infty} F_c(r)\, dr$ depend continuously on $c \in (0, c_q]$, and hence so does their sum $\int_{r_q}^{\infty} F_c(r)\, dr$.

Step 2. As $c \to c_q$, the integral $\int_{r_u}^{r_q} F_c$ converges to zero, for if K is the maximum of $(mm'\sqrt{m+c})^{-1}$ over the points with $r \in [r_0, r_q]$ and $c \in [c_0, c_q]$, then

$$\int_{r_u}^{r_q} F_c \le K \int_{r_u}^{r_q} \frac{m'\, dr}{\sqrt{m - c}} = K \int_0^{c_q - c} \frac{dt}{\sqrt{t}},$$

which goes to zero as $c \to c_q$. Thus T is continuous at $c = c_q$.

Step 3. To find an integrable function dominating $\partial F_c / \partial c$ on (r_q, ∞) locally in c, note that every $c \in (c_0, c_q)$ has a neighborhood of the form $(c_0, c_q - \delta)$ with $\delta > 0$, and over this neighborhood

$$\frac{\partial F_c}{\partial c} = \frac{m}{(m^2 - c^2)^{3/2}} \le \frac{m}{(m^2 - (c_q - \delta)^2)^{3/2}},$$

where the right hand side is integrable over $[r_q, \infty)$, as m^{-2} is integrable at ∞; thus

$$\frac{d}{dc} \int_{r_q}^{\infty} F_c = \int_{r_q}^{\infty} \frac{m}{(m^2 - c^2)^{3/2}}\, dr$$

is continuous with respect to $c \in (c_0, c_q)$. This integral diverges if $c = m(r_q)$.

Step 4. To check continuity of $\int_{r_u}^{r_q} F_c$ change variables via $t := m/c$ so that $r = m^{-1}(tc)$. Thus $dt = m'(r)\, dr/c = n(tc)\, dr/c$ where $n(r) := m'(m^{-1}(r))$, and

$$\int_{r_u}^{r_q} F_c(r)\, dr = \int_1^{c_q/c} \bar{F}_c(t)\, dt \quad \text{where} \quad \bar{F}_c(t) = \frac{1}{n(tc)t\sqrt{t^2 - 1}}.$$

Since $m' > 0$ on $[r_0, r_q]$ and $n(tc) = m'(r)$, the function $\bar{F}_c$ is smooth over $(1, c_q/c)$. To prove the continuity of $\int_1^{c_q/c} \bar{F}_c$, fix an arbitrary $(u, v) \subset (c_0, c_q)$. If $c \in (u, v)$ and $t \in (1, c_q/c)$, then $m^{-1}(tc)$ lies in the m^{-1}-image of $(u, (v/u)c_q)$, which by taking the interval (u, v) sufficiently small can be made to lie in an arbitrarily small neighborhood of $[r_0, r_q]$, so we may assume that $m' > 0$ on that neighborhood. It follows that the maximum K of $1/(n(tc))$ over $c \in [u, v]$ and $t \in [1, c_q/c]$ is finite, and $|\bar{F}_c| \le K/(t\sqrt{t^2 - 1})$ for $c \in (u, v)$, that is, $|F_c|$ is locally dominated by an integrable function that is independent of c; for the same reason the conclusion also holds for

$$\frac{\partial \overline{F}_c}{\partial c} = -\frac{n'(tc)}{n(tc)^2\sqrt{t^2-1}}.$$

Finally, given $c_* \in (c_0, c_q)$, fix $\delta \in (1, c_q/c_*)$ and write $\int_1^{c_q/c} \overline{F}_c = \int_1^{\delta} \overline{F}_c + \int_{\delta}^{c_q/c} \overline{F}_c$ for c varying near c_*. The first summand is C^1 at c_*, as the integrand and its derivative are dominated by the integrable function near c_*. The second summand is also C^1 at c_* as the integrand is C^1 on a neighborhood of $\{c_*\} \times [\delta, c_q/c]$. By the integral Leibniz rule

$$\frac{d}{dc}\int_1^{c_q/c} \overline{F}_c = -\frac{c_q}{c^2}\overline{F}_c\left(\frac{c_q}{c}\right) - \int_1^{c_q/c} \frac{n'(tc)\,dt}{n(tc)^2\sqrt{t^2-1}}.$$

The first summand equals $-\left(m'(r_q)\sqrt{c_q^2-c^2}\,\right)^{-1}$, and the second summand is bounded.

Step 5. Let us investigate the behavior of $\int_{r_q}^{\infty}(m/(m^2-c^2)^{3/2})\,dr$ from Step 3 as $c \to c_q-$. Fix $\delta > r_q$ such that $m' > 0$ on $[r_0, \delta]$ and write the above integral as the sum of the integrals over (r_q, δ) and (δ, ∞). The latter one is bounded. Integrate the former integral by parts as

$$\int_{r_q}^{\delta} \frac{mm'}{m'(m^2-c^2)^{3/2}}\,dr = -\int_{r_q}^{\delta} \frac{1}{m'}\,d\left(\frac{1}{\sqrt{m^2-c^2}}\right)$$
$$= \frac{1}{m'(r_q)\sqrt{c_q^2-c^2}} - \frac{1}{m'(\delta)\sqrt{\delta^2-c^2}} - \int_{r_q}^{\delta} \frac{m''\,dr}{(m')^2\sqrt{m^2-c^2}}.$$

Only the first summand is unbounded as $c \to c_q-$. The terms from Steps 4 and 5 enter into T' with coefficients 2 and 1, respectively, so as $c \to c_q-$

$$T'(c)\sqrt{c_q^2-c^2} \to -\frac{1}{m'(r_q)} < 0$$

as the bounded terms multiplied by $\sqrt{c_q^2-c^2}$ disappear in the limit. □

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