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ISOPERIMETRIC SURFACES WITH BOUNDARY, II

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Following our previous work with Dorff and Lawlor, we extend results for the so-called equitent problem of fixed boundary and fixed volume. We define sufficient conditions, which in $\mathbb{R}^2$ and $\mathbb{R}^3$ are also necessary, for local minima to be piecewise spherical, and we show that these are area-minimizing in their homotopy class. We also give new examples of these surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1. Introduction

Equitent problems, first introduced in the paper "Isoperimetric surfaces with boundary" [Dorff et al. 2011], ask what is the area-minimizing surface enclosing a given volume and spanning a given boundary. In this way, equitent problems represent a combination of isoperimetry and boundary conditions, such as in Steiner problems and minimal surfaces. Our previous approach, which we extend here, uses the technique of metacalibration. *Metacalibration* is a version of the calibration methods popularized by Harvey and Lawson [1982], adapted to use on isoperimetric problems. In particular, we use a combination of the mapping of [Gromov 1986], after [Knothe 1957], and the paired calibrations of [Lawlor and Morgan 1994].

In our original results, we construct various classes of surfaces bounded by the dual figures of uniform polytopes and enclosing a prescribed volume and prove that these surfaces are minimizing in their homotopy class. The results, however, turn out to be limited in scope, as shown in [Ross et al. 2011]. Consequently, there remains much ground to be covered.

In this paper, we will extend previous results by considering equitent systems generated by polytopes whose edges are all of a given length. This results in a much wider range of equitent surfaces than those bounded by uniform polytopes. We construct the conjectured minimizing surface using a refinement of previous methods and prove that this surface is indeed area-minimizing in its homotopy class.

Further, we show that any homotopically area-minimizing equitent surface with piecewise spherical faces and simplex vertex figures is equivalent to one generated

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by our construction. We conclude with a discussion of new equitent surfaces and a survey of open problems.

2. The surfaces

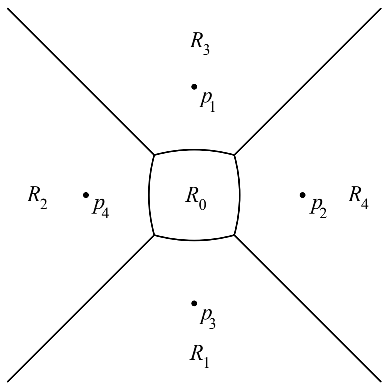
Let Γ be a convex polytope of dimension $m \le n$ with equal edge lengths, r, embedded in $\mathbb{R}^n$. Let $p_1, \dots, p_k$ be the vertices of Γ. For each p_i let R_i be the region farthest from p_i:

$$R_i = \{x \in \mathbb{R}^n : r < \|x - p_i\| \text{ and } \|x - p_j\| < \|x - p_i\| \text{ for all } j \neq i\}.$$

Note that if p_i and p_j share an edge in Γ, $\partial R_i \cap \partial R_j$ is a subset of the perpendicular bisecting hyperplane of that edge. Now, define

$$R_0 = \{x \in \mathbb{R}^n : \|x - p_i\| < r \text{ for } 1 \le i \le k\}.$$

See figure.



This region represents the enclosed volume. We suppose that $R_0 \neq \varnothing$ and $\mathcal{H}^{n-1}(\partial R_0 \cap \partial R_i) \neq 0$ for all $i > 0$. Let $V_0 = \mathcal{H}^n(R_0)$. Then let

$$M = \bigcup_{i=0}^{k} \partial R_i.$$

Notice that ∂R_0 is the portion of the surface that encloses the volume R_0. In order to have a nontrivial result, we require $R_0 \neq \varnothing$. The condition

$$\mathcal{H}^{n-1}(\partial R_0 \cap \partial R_i) \neq 0$$

for all $i > 0$ ensures that all smooth subsurfaces of M meet at 120 degree angles. For $m = 2$, the only viable generating figures, Γ, are equilateral triangles, rhombi with interior angles strictly greater than 60°, and small perturbations of regular pentagons. For $m = 3$, the valid generating figures include all but two of the eight convex deltahedra (polyhedra where all faces are equilateral triangles), as well as other polytopes with faces of higher degree. It is worth noting, however, that those generating figures, Γ, whose faces are not equilateral triangles produce surfaces which are locally minimal within their homotopy class, but not globally minimal. As will be seen in the proof, this construction gives sufficient conditions for minimizing surfaces to be piecewise spherical. Furthermore, due to the regularity properties of soap films proved by Taylor and Almgren [Taylor 1976], these are also necessary conditions in $\mathbb{R}^2$ and $\mathbb{R}^3$. In higher dimensions nonsimplicial vertex figures may be minimizing, but are not considered in this paper. See for example [Brakke 1991].

3. The minimization theorem

In this section we prove that the surfaces constructed are homotopically minimizing in the following sense: Let U be a bounded open set that contains $\overline{R_0}$, and let $M_0 = M \cap U$.

Theorem 1. *The surface M_0 is area-minimizing among all compact surfaces (rectifiable sets) in U with boundary $\partial U \cap M$ that enclose the fixed volume $\mathcal{H}^n(R_0)$ and are homotopically equivalent to M_0. This also holds with the weaker assumption that competitor surfaces are not necessarily homotopic to M_0 but separate space into the same regions as M_0 and these regions share boundary nontrivially (on a set of positive $\mathcal{H}^{n-1}$ measure) only if the corresponding regions do in M_0.*

Our proof uses a metacalibration argument that compares figures according to their flux on specially crafted vector fields. In particular, we use a paired-calibration approach with one vector field defined for each separated region.

Let N be any competitor surface and let S_i be the separated regions that correspond to each R_i respectively. (Then $\mathcal{H}^n(S_0) = \mathcal{H}^n(R_0)$ by the volume condition.) Define $v_i : S_i \to \mathbb{R}^n$ for $1 \le i \le m$ to be the constant vector field $-p_i/r$. Let $\phi : S_0 \to R_0$ be the Knothe–Rosenblatt rearrangement and let $v_0 : S_0 \to \mathbb{R}^n$ be given by $v_0 = \phi/r$.

At this point a few simple results would be useful:

Proposition 2. *If S_i and S_j share boundary nontrivially, then $v_i - v_j$ is a unit vector. If $N = M_0$ then $v_i - v_j$ is the unit normal to $\partial R_i \cap \partial R_j$.*

Proof. Note that S_i and S_j share boundary nontrivially if and only if p_i and p_j are adjacent in Γ. Thus $\|v_i - v_j\| = (1/r)\|p_i - p_j\| = 1$. Also if $N = M_0$, $v_i - v_j$ is the unit normal to $\partial R_i \cap \partial R_j$ since $\partial R_i \cap \partial R_j$ lies on the hyperplane equidistant to p_i and p_j. □

Proposition 3. *The matrix Dv_0 is triangular. If $N = M$ then v_0 is the identity scaled by $1/r$.*

Proof. Follows from the definition of v_0. See [Dorff et al. 2011] for details. □

Proposition 4. *For $i \neq 0$, $\int_{N\cap\partial S_i} v_i \cdot n \, d\mathcal{H}^{n-1} = \int_{M\cap\partial R_i} v_i \cdot n \, d\mathcal{H}^{n-1}$, where n is the unit normal to the surface of integration, outward pointing with respect to S_i or R_i.*

Proof. Follows from the divergence theorem since v_i is divergence free and $\partial(M \cap \partial R_i) = \partial(N \cap \partial S_i)$. □

Proof of Theorem 1. For any competitor surface N, let $G(N) = \sum_i \int_{N\cap\partial S_i} v_i \cdot n \, d\mathcal{H}^{n-1}$. Letting $P(N) = \sum_i \int_{N\cap\partial S_i} d\mathcal{H}^{n-1}$ be our objective function, we find that

$$\begin{aligned} G(N) &= \sum_i \int_{N\cap\partial S_i} v_i \cdot n \, d\mathcal{H}^{n-1} \\ &= \sum_{i\neq j} \int_{N\cap(\partial S_i\cap\partial S_j)} (v_i - v_j) \cdot n \, d\mathcal{H}^{n-1} \\ &\leq \sum_{i\neq j} \int_{N\cap(\partial S_i\cap\partial S_j)} \|v_i - v_j\| \, \|n\| \, d\mathcal{H}^{n-1} \\ &\leq \sum_{i\neq j} \int_{N\cap(\partial S_i\cap\partial S_j)} d\mathcal{H}^{n-1} \\ &= \sum_i \int_{N\cap\partial S_i} d\mathcal{H}^{n-1} = P(N), \end{aligned}$$

with equality if $N = M_0$.

Now also note that

$$\begin{aligned} \int_{N\cap\partial S_0} v_0 \cdot n \, d\mathcal{H}^{n-1} &= \int_{S_0} \operatorname{div} v_0 \, d\mathcal{H}^{n-1} \\ &= \int_{S_0} \frac{1}{r} \left(\frac{\partial\phi_1}{\partial x_1} + \frac{\partial\phi_2}{\partial x_2} + \cdots + \frac{\partial\phi_n}{\partial x_n} \right) d\mathcal{H}^{n-1} \\ &\geq \int_{S_0} \frac{n}{r} \sqrt[n]{\frac{\partial\phi_1}{\partial x_1} \frac{\partial\phi_2}{\partial x_2} \cdots \frac{\partial\phi_n}{\partial x_n}} \, d\mathcal{H}^{n-1} \\ &= \int_{S_0} \frac{n}{r} \sqrt[n]{1} \, d\mathcal{H}^{n-1} \\ &= \frac{n}{r} \mathcal{H}^{n-1}(S_0) = \frac{n}{r} \mathcal{H}^{n-1}(R_0), \end{aligned}$$

with equality if $N = M_0$. This follows from the AM-GM inequality and the equality

$$\frac{\partial\phi_1}{\partial x_1} \frac{\partial\phi_2}{\partial x_2} \cdots \frac{\partial\phi_n}{\partial x_n} = \det(D\phi) = 1,$$

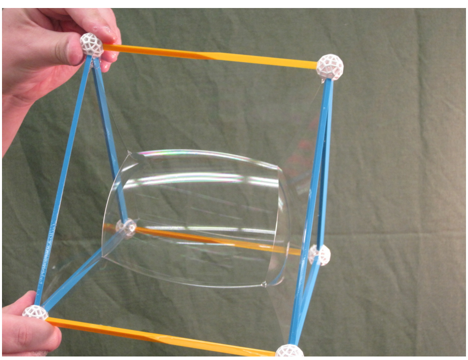
which is valid since ϕ is volume-preserving.

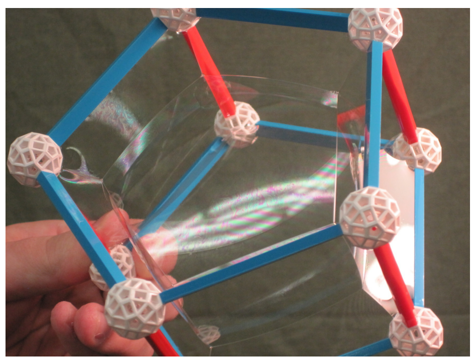
Combining these results we find

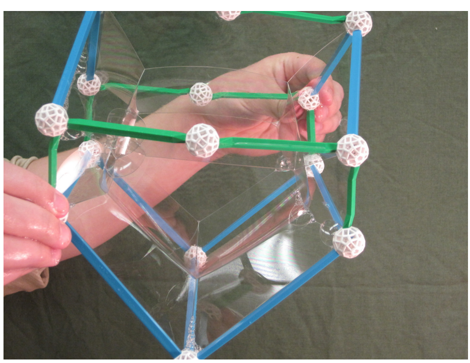
$$\begin{aligned} P(M_0) = G(M_0) &= \sum_{i\neq 0} \int_{M_0 \cap \partial R_i} v_i \cdot n \, d\mathcal{H}^{n-1} + \int_{M_0 \cap \partial R_0} v_0 \cdot n \, d\mathcal{H}^{n-1} \\ &= \sum_{i\neq 0} \int_{N \cap \partial S_i} v_i \cdot n \, d\mathcal{H}^{n-1} + \frac{n}{r} \mathcal{H}^{n-1}(R_0) \\ &\leq \sum_{i\neq 0} \int_{N \cap \partial S_i} v_i \cdot n \, d\mathcal{H}^{n-1} + \int_{N \cap \partial S_0} v_0 \cdot n \, d\mathcal{H}^{n-1} \\ &= G(N) \leq P(N). \end{aligned}$$ □

4. Soap films in $\mathbb{R}^3$

In [Dorff et al. 2011] and [Ross et al. 2011] we identified the regular tetrahedron, the regular octahedron, and the regular icosahedron as polytopes that generate realizable soap films. The only other three-dimensional polytopes, Γ, that generate surfaces realizable as soap films are the triangular dipyramid, the pentagonal dipyramid, and the snub disphenoid. The generated soap films are shown below.







This is due to the conditions proven by Jean Taylor [1976], namely that each face not intersecting with the boundary must meet with exactly two other faces in 120-degree angles. Thus, any generating figure, Γ, with nontriangular faces will not yield a surface realizable as a soap film. The remaining two deltahedra fail to meet the conditions of our construction because of their large circumradius.

Every surface generated by our construction will have piecewise spherical faces and simplicial vertex figures. The converse is also true. Given any area-minimizing equitent surface with piecewise spherical faces and simplicial vertex figures, we can recover the generating polytope, Γ, as the set of centers of each spherical face. In higher dimensions there may exist area-minimizing equitent surfaces with nonsimplicial vertex figures.

5. Conclusion

We have characterized all piecewise spherical equitent surfaces in two and three dimensions, and proven them to be area minimizing. Several interesting and open problems arise. An especially intriguing question deals with equitent surfaces that have negatively curved bubbles, meaning each face of the bubble region bends inward. Such a surface can be created using soap films, but our methods are not yet able to address this case. Similarly, equitent surfaces with nonspherical faces fall outside the scope of our approach. Finally, our construction generates surfaces whose vertices are cones over simplices. In spaces of dimension greater than three, however, minimal surfaces need not have simplicial vertex figures, and we may yet find interesting new equitent surfaces. Extensions of the metacalibration methods outlined in this paper show great promise in solving these open problems.

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